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Algebraic Geometry
Sheet 4

You should assume that we are working over an algebraically closed field k.

Problem 1. *Let k be of prime characteristic p. The map*

$$F: \begin{array}{ccc} \mathbb{A}^n_k & \to & \mathbb{A}^n_k \\ (x_1, \ldots, x_n) & \mapsto & (x_1^p, \ldots, x_n^p) \end{array}$$

is known as the Frobenius map.

1. *Show that F is a bijective morphism.*
2. *Is F an isomorphism?*

Problem 2. *Consider $\ell_1 = V(x)$ and $\ell_2 = V(y)$ in $\mathbb{A}^2$.*

1. *Write down a morphism $\varphi : \ell_1 \to \ell_2$, such that $\varphi(0,0) = (0,0)$.*
2. *If $U_2 = \mathbb{A}^1 \setminus \{(0,0),(1,0)\} \subset \ell_2$, find $U_1 := \varphi^{-1}(U_2)$.*
3. *Determine $\mathcal{O}_{\ell_i}(U_i)$ for $i = 1,2$.*
4. *Write down the map $\varphi^* : \mathcal{O}_{\ell_2}(U_2) \to \mathcal{O}_{\ell_1}(U_1)$.*

Problem 3. *Consider $X_1 = V(x-y)$ and $X_2 = V(x^2 - y)$ in $\mathbb{A}^2$, and let $p = (0,0)$.*

1. *Determine $\mathcal{O}_{X_i,p}$ for $i = 1,2$.*
2. *Show explicitly that $\mathcal{O}_{X_1,p}$ and $\mathcal{O}_{X_2,p}$ have exactly one maximal ideal.*

Problem 4. *Let X be an affine variety and let G be a finite group which acts on X. Consider $A(X)^G$, the subalgebra of $A(X)$ consisting of all G-invariant functions on X (i.e. functions $f : X \to k$ such that $f(g.P) = f(P)$ for all $g \in G$ and $P \in X$). Assuming that it is finitely generated as a k-algebra, then there is an affine variety Y which has coordinate ring $A(X)^G$, and a morphism $\pi : X \to Y$ determined by the inclusion $A(X)^G \subset A(X)$. Show that Y can be considered as a quotient of X (denoted X/G) in the following sense:*

1. *π is surjective.*
2. *If $P, Q \in X$, then $\pi(P) = \pi(Q)$ if and only if there exists $g \in G$ such that $g.P = Q$.*

For a given group action, is the affine variety Y uniquely determined by these two properties?