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Algebraic Geometry
Sheet 5

You should assume that we are working over an algebraically closed field.

Problem 1. *Let* $\mathbb{Z}_n = \{\exp(\frac{2\pi ik}{n}) \mid k \in \mathbb{Z}\} \subset \mathbb{C}$ *be the group of* n*-th roots of unity. Let* $\mathbb{Z}_n$ *act on* $\mathbb{C}^m$ *by multiplication in each coordinate. Show that* $\mathbb{C}/\mathbb{Z}_n$ *is isomorphic to* $\mathbb{C}$ *for all* n*, but that* $\mathbb{C}^2/\mathbb{Z}_n$ *is not isomorphic to* $\mathbb{C}^2$ *for* $n \geq 2$*.*
Consider now the action of $\mathbb{Z}_3$ *on* $\mathbb{C}^2$ *given by* $\xi.(u,v) = (\xi u, \xi^2 v)$*. Is the quotient variety corresponding to this action isomorphic to the variety* $\mathbb{C}^2/\mathbb{Z}_3$ *considered above?*

Problem 2. *Consider a polynomial map*

$$f : \mathbb{C}^2 \to \mathbb{C}^2.$$

Show that the image of f *need not be algebraic. Then consider the general case*

$$f : \mathbb{C}^n \to \mathbb{C}^m$$

for $n, m \in \mathbb{N}$ *with* $n > 1$*.*

Problem 3. *Let* $V = V(x^2 + y^2 - 1) \subset \mathbb{A}^2$*. Determine where the following rational functions are regular*

$$f = \frac{y}{x}, \quad g = \frac{1-y}{x} \in K(V).$$

Problem 4. *We consider the local rings of the affine varieties*

$$V_1 = V(xy) \quad \text{and} \quad V_2 = V(y^2 - x^3 - x^2) \subset \mathbb{A}^2_{\mathbb{C}}.$$

- (i) *Show that the local ring of* V_1 *at a point* $P = (0,u)$ *with* $u \neq 0$ *is isomorphic to* $\mathbb{C}[t]_{(t)}$*. (As the latter is the local ring of a line at a point, the local ring* $\mathcal{O}_{V_1,P}$ *ignores the component* $V(y)$*.)*
- (ii) *Show that the local rings of* V_1 *and* V_2 *at* $P = (0,0)$ *have different integrality properties. (Although the varieties are locally analytically isomorphic, the local ring of* V_1 *at* P *detects the reducibility.)*