Resolution of Singularities, characteristic $p > 0$ or mixed characteristic.

Problems with the tangent cone

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Let R be a regular local ring with maximal ideal $\mathfrak{M}$, fR the ideal of the singular scheme $\mathcal{X} \subset \operatorname{Spec} R$, and let x be the closed point and $m := m(x) = \mathrm{ord}_x(f)$, $m \geq 2$.

Let $F := \mathrm{in}_x(f) := f \bmod \mathfrak{M}^{m+1} \in \mathrm{gr}_{\mathfrak{M}} R$ be the initial form of f and let $\tau(x)$ be the least number of variables $(Y_1, \ldots, Y_{\tau(x)})$ needed to write the polynomial F. (Note that then $\mathrm{gr}_{\mathfrak{M}} R \cong k(x)[Y_1, \ldots, Y_{\tau(x)}, U_1, \ldots, U_d]$). In that case, Hironaka's first invariants for resolution of singularities are

$$(m(x), -\tau(x)).$$

The ideal $(Y_1, \ldots, Y_{\tau(x)}) \subset \mathrm{gr}_{\mathfrak{M}} R$ is uniquely determined: it is the ideal of the *directrix*, the vector space of translations which stabilize the tangent cone.

All these are very local considerations. Globally, in characteristic 0, Hironaka proved that

(i) the invariant $(m, -\tau)$ is upper semi-continuous when $\mathcal{X}$ is an hypersurface embedded in a regular algebraic variety over a field k with $\mathrm{char}(k) = 0$.

(ii) if you blow up the closed point x, any point x' above x, on the strict transform of $\mathcal{X}$ with $m(x') = m(x)$ is on the strict transform of $\mathbf{V}(y_1, \ldots, y_{\tau(x)})$, where $(y_1, \ldots, y_{\tau(x)})$ are elements in R for which $Y_j = y_i \bmod \mathfrak{M}^2$, $1 \leq j \leq \tau(x)$ and $\tau(x') \geq \tau(x)$.

A few subtle examples of H. Hironaka, T. Oda and O. Piltant show that (i) and (ii) are wrong when $\mathrm{char} k(x) = p > 0$. We will show how these major difficulties may be overcome when $\dim(X) \leq 3$.