PRAGMATIC 2000

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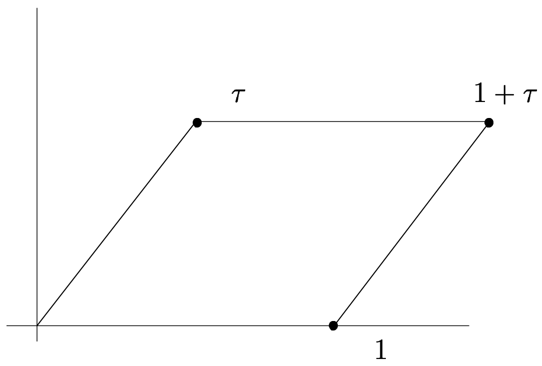
K. Hulek

I Elliptic curves and their moduli

We consider the *upper half plane*

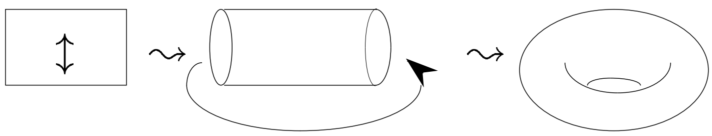
$$\mathbb{H}_1 := \{\tau \in \mathbb{C};\ \operatorname{Im} \tau > 0\}.$$



To every point $\tau \in \mathbb{H}_1$ one can associate the lattice

$$\Lambda_\tau = \mathbb{Z}\tau + \mathbb{Z};\ \text{resp. } E_\tau = \mathbb{C}/\Lambda_\tau.$$

The quotient E_τ is a *torus*.



It is a *compact Riemann surface*† (of *genus* 1) and at the same time an *algebraic curve (elliptic curve)*.

One knows from complex function theory that

$$E_\tau \cong E_{\tau'} \Leftrightarrow \text{exists} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}) \text{ such that } \tau' = \frac{a\tau + b}{c\tau + d}.$$

For this reason we introduce the following *action* of $\mathrm{SL}(2,\mathbb{Z})$ on $\mathbb{H}_1$:

$$\begin{array}{cccccc} \mathrm{SL}(2,\mathbb{Z}) & \times & \mathbb{H}_1 & \to & \mathbb{H}_1 \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} & : & \tau & \mapsto & \frac{a\tau+b}{c\tau+d}. \end{array}$$

Then by the above we have the following interpretation of the quotient

$$X^0(1) := \mathbb{H}_1/\mathrm{SL}(2,\mathbb{Z}) \overset{1:1}{\longleftrightarrow} \{E_\tau; E_\tau = \text{elliptic curve}\}/\text{isomorphism}.$$

We call $X^0(1)$ the (open) *modular curve*. The quotient $X^0(1)$ carries itself the structure of a Riemann surface. There is a unique function $j : \mathbb{H}_1 \to \mathbb{C}$ such that

$$\begin{array}{ccc} \mathbb{H}_1 & & \\ \downarrow \pi & \searrow^{j} & \\ X^0(1) = \mathbb{H}_1/\mathrm{SL}(2,\mathbb{Z}) & \overset{1:1}{\longrightarrow} & \mathbb{C}, \end{array}$$

i.e. there are "as many elliptic curves as complex numbers".

In this sense we can identify $X^0(1) = \mathbb{C}$. There is an obvious compactification

$$X(1) = \mathbb{P}^1 = \mathbb{C} \cup \{\infty\} = X^0(1) \cup \{\text{cusp}\}.$$

As a set we obtain this as follows. Let

$$\overline{\mathbb{H}}_1 = \mathbb{H}_1 \cup \mathbb{Q} \cup \{i\infty\}.$$

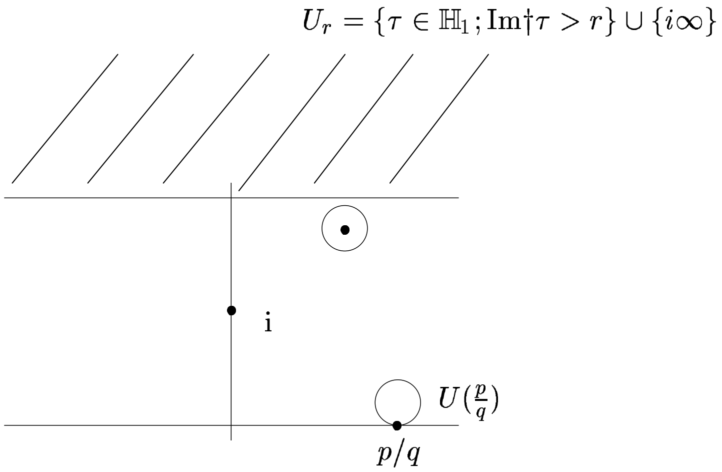
Then the action of $\mathrm{SL}(2,\mathbb{Z})$ on $\mathbb{H}_1$ extends to an action on $\overline{\mathbb{H}}_1$ and $\mathbb{Q} \cup \{i\infty\}$ is one orbit. Hence

$$X(1) := \overline{\mathbb{H}}_1/\mathrm{SL}(2,\mathbb{Z}) = \mathbb{H}_1/\mathrm{SL}(2,\mathbb{Z}) \cup (\mathbb{Q} \cup \{i\infty\})/\mathrm{SL}(2,\mathbb{Z}),$$

i.e.

$$X(1) = X^0(1) \cup \{\infty\}.$$

We can define a *topology* on $X(1)$, by defining the *horocyclic topology* on $\overline{\mathbb{H}}_1$ and then taking the quotient topology on $X(1)$. The open sets in the horocyclic topology on $\overline{\mathbb{H}}_1$ are the usual open sets plus the following



It is important for us to understand how we obtain an *analytic structure* on $X(1)$. For this purpose we consider the *stabilizer* of $i\infty$ in $\mathrm{SL}(2,\mathbb{Z})$. This is the group

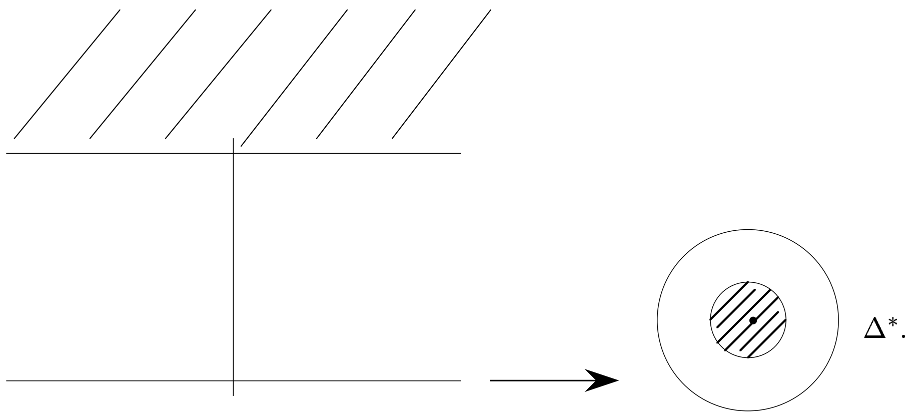
$$P(i\infty) = \left\{ \pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} ; n \in \mathbb{Z} \right\}.$$

This group acts on $\mathbb{H}_1$ as follows

$$\pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : \tau \mapsto \tau + n.$$

Taking the "partial quotient" with respect to $P(i\infty)$ we obtain

$$\begin{array}{ccl} \mathbb{H}_1 & \longrightarrow & \mathbb{H}_1/P(i\infty) = \Delta^* = \{z \in \mathbb{C}; 0 < |z| < 1\} \\ \tau & \mapsto & t = e^{2\pi i \dagger \tau}. \end{array}$$



Adding the point $\{i\infty\}$ then becomes adding the point 0 to the punctured disc Δ^*. We thus obtain

$$X(1) = X^0(1) \cup_{\Delta^*(\varepsilon)} \Delta(\varepsilon).$$

Level structures
Instead of $\Gamma = \mathrm{SL}(2,\mathbb{Z})$ one often considers subgroups of Γ. The group

$$\Gamma(n) := \{M \in \mathrm{SL}(2,\mathbb{Z});\ M \equiv \mathbf{1} \bmod n\}$$

is the *principal congruence subgroup* of *level* n. We define

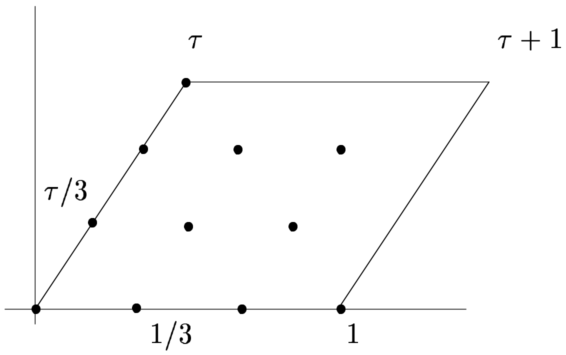
$$X^0(n) = \mathbb{H}_1/\Gamma(n).$$

This is the *(open) modular curve* of *level* n.

We now want to understand the interpretation of $X^0(n)$. We consider the group of n-torsion points on an elliptic curve E

$$E^{(n)} := \{x \in E; nx = 0\} \ (\cong \dagger\mathbb{Z}_n \times \mathbb{Z}_n \text{ non-canonically}).$$

E.g. for $n = 3$:



There is a natural symplectic form, the *Weil pairing*

$$(\ ,\ ) : E^{(n)} \times E^{(n)} \to \mathbb{Z}_n.$$

(In our case we can define this by:

$$\left(\frac{\tau}{n}, \frac{1}{n}\right) = \overline{1}.)$$

Definition A *level-n structure* on E is a symplectic basis w_1, w_2 of $E^{(n)}$ i.e. a symplectic isomorphism $\alpha : E^{(n)} \to \mathbb{Z}_n \times \mathbb{Z}_n$.

We then have the following interpretation

$$\begin{array}{ccc} X^0(n) & \stackrel{1:1}{\longleftrightarrow} & \{\text{elliptic curves with a level-n structure}\}/_{\cong} \\ [\tau] & \longmapsto & \left(E_\tau; ([\frac{\tau}{n}], [\frac{1}{n}])\right). \end{array}$$

As before we have a compactification of $X^0(n)$ given by

$$X(n) = \overline{\mathbb{H}}_1/\Gamma(n) = X^0(n) \cup \{\text{cusps}\}.$$

The number of cusps depends on n.

Example $n = 7$; $X(7) = X^0(7) \cup \{24 \text{ cusps}\}$. One has $g(X(7)) = 3$ and in fact $X(7)$ is isomorphic to the Klein quartic

$$C := \{(x_0 : x_1 : x_2); x_0^3 x_1 - x_1^3 x_2 - x_2^3 x_0 = 0\}.$$

One has an exact sequence of groups

$$1 \to \Gamma(n) \to \Gamma = \mathrm{SL}(2, \mathbb{Z}) \to \mathrm{SL}(2, \mathbb{Z}_n) \to 0.$$

Since $\pm\mathbf{1}$ acts trivially on $\mathbb{H}_1$ this gives rise to a Galois covering

$$X(n) \to X(1) = X(n)/\mathrm{PSL}(2, \mathbb{Z}_n).$$

Example $\mathrm{PSL}(2, \mathbb{Z}_7) \cong G_{168}$ which is the symmetry group of C.

II Abelian varieties

Let $L \subset \mathbb{C}^g$ be a lattice of rank $2g$ i.e. $L \cong \mathbb{Z}^{2g}$ and L spans $\mathbb{C}^g$ as an $\mathbb{R}$-vector space.) Then

$$X = \mathbb{C}^g/L$$

is a compact g-dimensional *torus.*

Remark If $g \geq 2$ then X will in general not be a projective manifold e.g.

$$g = 2: \quad \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} i \\ 0 \end{pmatrix}, \begin{pmatrix} \sqrt{2} \\ i \end{pmatrix} \right\rangle.$$

Definition An *abelian variety* is a compact complex torus which is projective.

Definition A *Riemann form* with respect to a lattice L is a non-degenerate, alternating bilinear form

$$E : L \times L \to \mathbb{Z}$$

such that for its $\mathbb{R}$-linear extension

(i) $E(x, y) = E(ix, iy)$,

(ii) $E(ix, x) > 0$ for $x \neq 0$.

Then one has the following well-known result

X is an abelian variety $\Leftrightarrow$ there exists a Riemann form with respect to L.

In fact, this is easy to understand since

$$E \in \mathrm{Alt}(L \times L, \mathbb{Z}) = \mathrm{Hom}(\bigwedge^2 L, \mathbb{Z}) = H^2(X, \mathbb{Z}).$$

Then

(i) $\Leftrightarrow E \in H^2(X,\mathbb{Z}) \cap H^{1,1} = \mathrm{NS}(X)$,

(ii) $\Leftrightarrow$ If $\mathcal{L} \in \mathrm{Pic}\, X$ with $c_1(\mathcal{L}) = -E$, then $\mathcal{L} > 0$.

With respect to a suitable basis of L we have

$$E = \left(\begin{array}{ccc|ccc} & & & d_1 & & \\ & & & & \ddots & \\ & & & & & d_g \\ \hline -d_1 & & & & & \\ & \ddots & & & & \\ & & -d_g & & & \end{array}\right); \; d_i \in \mathbb{N}_{>0};\; d_1|d_2|\ldots|d_g.$$

Definition We say that E defines a *polarization* of *type* $(d_1, \ldots, d_g)$. If $(d_1, \ldots, d_g) = (1, \ldots, 1)$ we call this a *principal polarization.*

I.e. a polarization is a Riemann form or equivalently a class of an ample line bundle on X.

Example If $g = 1$, then $E = \begin{pmatrix} 0 & d \\ -d & 0 \end{pmatrix}$ and $d = \deg \mathcal{L}$.

Remark Let $c_1(\mathcal{L}) = -E$. Then we have a homomorphism (which only depends on $c_1(\mathcal{L})$):

$$\begin{array}{rcl} \lambda : X & \to & \mathrm{Pic}^0 X = \hat{X} \\ x & \mapsto & T_x^*\mathcal{L} \otimes \mathcal{L}^{-1} \end{array}$$

where $T_x : A \to A$ denotes translation by x. Then

$$\ker \lambda \cong (\mathbb{Z}/_{d_1} \times \ldots \times \mathbb{Z}/_{d_g})^2.$$

Again we want to consider *moduli* of abelian varieties. We shall first restrict ourselves to the *principally polarized case.*

The *Siegel space of genus g* (*Siegel upper half plane*) is defined by

$$\mathbb{H}_g = \{\tau \in \mathrm{Mat}(g \times g, \mathbb{C}); \tau = {}^t\tau; \mathrm{Im}\tau > 0\}.$$

To every point $\tau \in \mathbb{H}_g$ we associate the *lattice*

$$\begin{array}{rcl} L_\tau & = & \text{lattice spanned by the columns of } (\tau, \mathbf{1}_g) \\ A_\tau & = & \mathbb{C}^g/L_\tau. \end{array}$$

Then L_τ admits a Riemann form which belongs to a principal polarization. This form is given by

$$E(x,y) = x\mathrm{Im}(\tau)^t y.$$

This defines a map

$$\begin{array}{rcl} \mathbb{H}_g & \to & \{g\text{-dimensional p.p.a. varieties}\} \\ \tau & \mapsto & (A_\tau, \mathcal{L}_\tau) \quad (c_1(\mathcal{L}_\tau) = -E). \end{array}$$

Let

$$\mathrm{Sp}(2g,\mathbb{Z}) = \left\{ M \in \mathrm{SL}(2g,\mathbb{Z});\ M \begin{pmatrix} 0 & \mathbf{1}_g \\ -\mathbf{1}_g & 0 \end{pmatrix} {}^t M = \begin{pmatrix} 0 & \mathbf{1}_g \\ -\mathbf{1}_g & 0 \end{pmatrix} \right\}.$$

This is the *integer symplectic group*. The group $\mathrm{Sp}(2g,\mathbb{Z})$ acts on $\mathbb{H}_g$ by

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \tau \mapsto (A\tau + B)(C\tau + D)^{-1}.$$

Clearly this generalizes the usual action of $\mathrm{SL}(2,\mathbb{Z}) = \mathrm{Sp}(2,\mathbb{Z})$ on $\mathbb{H}_1$. Then

$$\boxed{\mathbb{H}_g/\mathrm{Sp}(2g,\mathbb{Z}) = \mathcal{A}_g = \{\text{ p.p.a. varieties}\}/\text{isomorphism.}}$$

The non-principally polarized case

We fix the type of the polarization:

$$\underline{d} = (d_1, \ldots, d_g), \quad (d_1|d_2|\ldots|d_g).$$

Then we define

$$E_{\underline{d}} = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_g \end{pmatrix}, \quad \Lambda_{\underline{d}} = \begin{pmatrix} 0 & E_{\underline{d}} \\ -E_{\underline{d}} & 0 \end{pmatrix}.$$

Again we can define a symplectic group

$$\mathrm{Sp}(\Lambda_{\underline{d}},\mathbb{Z}) = \{M \in \mathrm{SL}(2g,\mathbb{Z}); \quad M\Lambda_{\underline{d}}\, {}^t M = \Lambda_{\underline{d}}\}.$$

In this case $\mathrm{Sp}(\Lambda_{\underline{d}},\mathbb{Z})$ acts on $\mathbb{H}_g$ by

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \tau \mapsto (A\tau + BE_{\underline{d}})(C\tau + DE_{\underline{d}})^{-1}E_{\underline{d}}.$$

We then have

$$\boxed{\begin{array}{r} \mathbb{H}_g/\mathrm{Sp}(\Lambda_{\underline{d}},\mathbb{Z}) = \mathcal{A}_{\underline{d}} = \mathcal{A}_{d_1,\ldots,d_g} = \{(A,H);\ H \text{ is a polarization} \\ \text{of type } \underline{d}\}/\text{isomorphism} \end{array}}$$

Remark In this notation

$$\mathcal{A}_g = \mathcal{A}_{1,\dots,1}.$$

In both cases we can talk about *level structures*. This corresponds to taking suitable subgroups $\Gamma \subset \mathrm{Sp}(\Lambda_{\underline{d}}, \mathbb{Z})$. We shall return to this.

Questions (1) Are there good compactifications of the modular varieties $\mathcal{A}_{\underline{d}}$?
(2) Given such a compactification, can we then interprete the boundary points as degenerations of abelian varieties?

III Introduction to toric geometry

Fix an integer $r \geq 1$. We consider

$$\begin{array}{lcl} M & = & \text{free module of rank } r \text{ over } \dagger\mathbb{Z}\ (M \cong \mathbb{Z}^r), \\ N: & = & M^* := \mathrm{Hom}(M, \mathbb{Z}). \end{array}$$

Then we have a natural pairing

$$\langle\ ,\ \rangle : M \times M^* \to \mathbb{Z}.$$

We define

$$T := \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{C}^*) = M^* \otimes_{\mathbb{Z}} \mathbb{C}^*.$$

Then $T \cong (\mathbb{C}^*)^r$ is a *torus (algebraic torus)*. We have

$$\begin{array}{lclcl} M & = & \mathrm{Hom}(T, \mathbb{C}^*) & = & \text{group of characters of } T, \\ M^* & = & \mathrm{Hom}(\mathbb{C}^*, T) & = & \text{group of 1-parameter subgroups of } T. \end{array}$$

We also define

$$M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R};\ \ M^*_{\mathbb{R}} := M^* \otimes_{\mathbb{Z}} \mathbb{R}.$$

Definition (i) A *(rational polyhedral) cone* is a subset $\sigma \subset M^*_{\mathbb{R}}$ such that there exist elements $m^*_1, \dots, m^*_k \in M^*$ such that

$$\sigma = \mathbb{R}_{\geq 0} m^*_1 + \ldots + \mathbb{R}_{\geq 0} m^*_k.$$

A cone σ is called *strictly convex* if it does not contain a line.
(ii) If $\sigma \subset M^*_{\mathbb{R}}$ is a cone, then the *dual cone* is defined by

$$\sigma^\vee := \{x \in M_{\mathbb{R}}; \langle x, y\rangle \geq 0 \text{ for all } y \in \sigma\}.$$

Definition Assume $\sigma, \tau \subset M^*_{\mathbb{R}}$ are cones and $\tau \subsetneqq \sigma$. We say that τ is a *face* of $\sigma (\tau \prec \sigma)$, if there is an element $m \in \sigma^\vee$ such that

$$\tau = \sigma \cap m^\perp = \{y \in \sigma,\ \langle m, y\rangle = 0\}.$$

(I.e. τ is the intersection of σ with a hyperplane).

Definition A *fan (rational partial polyhedral decomposition)* in $M^*_{\mathbb{R}}$ is a collection Σ of strictly convex cones such that

(i) $\sigma \in \Sigma, \tau \prec \sigma \Rightarrow \tau \in \Sigma$,

(ii) $\sigma_1, \sigma_2 \in \Sigma \Rightarrow \sigma_1 \cap \sigma_2 \prec \sigma_1$.

Remark (i) If σ is a cone, then $\sigma^\vee$ is a cone.
(ii) If σ is strictly convex, then $\sigma^\vee$ has maximal dimension (i.e. is not contained in a hyperplane of $M_{\mathbb{R}}$).

If $\sigma \subset M_{\mathbb{R}}$ is a cone, then

$$H_\sigma := \sigma \cap M$$

is a semi-group. If σ is rational polyhedral, then H_σ is finitely generated, and hence $\mathbb{C}[H_\sigma]$ is a finitely generated $\mathbb{C}$-algebra. Note that if H is a semi-group, then the semi-group ring $\mathbb{C}[H]$ is the vector space generated by X^h; $h \in H$ with the ring structure $X^h \cdot X^{h'} = X^{h+h'}$. Then we can consider Spec $\mathbb{C}[H_\sigma]$. This is an affine variety.

Now let $\Sigma = \{\sigma\}$ be a fan in $M^*_{\mathbb{R}}$. Then every cone σ is strictly convex and hence its dual cone $\sigma^\vee$ is of maximal dimension. We then define

$$T_\sigma = \text{Spec } \mathbb{C}[\sigma^\vee \cap M] \quad (\dim T_\sigma = r).$$

For every cone $\sigma \in \Sigma$ we consider its dual cone $\sigma^\vee \subset M_{\mathbb{R}}$ and the affine variety T_σ. If $\tau \prec \sigma$ then $\sigma^\vee \subset \tau^\vee$ and hence

$$\mathbb{C}[\sigma^\vee \cap M] \hookrightarrow \mathbb{C}[\tau^\vee \cap M].$$

Dually this gives a morphism

$$T_\tau = \text{Spec } \mathbb{C}[\tau^\vee \cap M] \hookrightarrow T_\sigma = \text{Spec } [\sigma^\vee \cap M].$$

In particular $\{0\} \in \Sigma$ and

$$T_{\{0\}} = \text{Spec } \mathbb{C}[M] \cong \text{Spec } [T_1, T_1^{-1}, \ldots, T_r, T_r^{-1}].$$

I.e.

$$T_{\{0\}} (\cong (\mathbb{C}^*)^r) \hookrightarrow T_\sigma \quad (\sigma \in \Sigma).$$

Definition Let Σ be a fan in $M^*_{\mathbb{R}}$. The *torus embedding* T_Σ is defined by

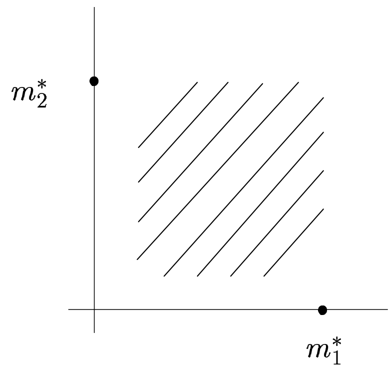
$$T_\Sigma = \coprod_{\sigma \in \Sigma} T_\sigma / \sim$$

where $x_1 \sim x_2$ if $x_1 \in T_{\sigma_1}, x_2 \in T_{\sigma_2}$ and there is a cone $\tau \subset \sigma_1 \cap \sigma_2$ such that $x_i \in T_\tau \cap T_{\sigma_i}$ and $x_1 = x_2$ in T_τ.

Remark (i) T_Σ is a *normal* and irreducible variety.
(ii) $T = T_{\{0\}} \subset T_\Sigma$ and the *action* of T on itself extends to T_Σ.
(iii) T_Σ is *smooth* $\Leftrightarrow$ T_σ is smooth for every $\sigma \in \Sigma$.
T_σ is *smooth* $\Leftrightarrow$ There is a basis $m_1^*, \ldots, m_r^*$ of M such that $\sigma = \mathbb{R}_{\geq 0} m_1 + \ldots + \mathbb{R}_{\geq 0} m_k, k \leq r$. In this case $T_\sigma \cong \mathbb{C}^k \times (\mathbb{C}^*)^{r-k}$.

Examples
(1) Single cones

(i)

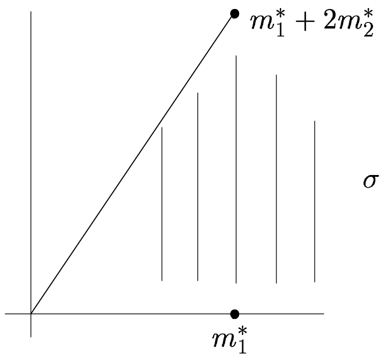


$$M_\mathbb{R}^* \cong \mathbb{R}^2$$

$$\sigma = \mathbb{R}_{\geq 0} m_1^* + \mathbb{R}_{\geq 0} m_2^*.$$

Then
$$\sigma^\vee = \mathbb{R}_{\geq 0} m_1 + \mathbb{R}_{\geq 0} m_2$$
and
$$T_\sigma = \text{Spec } \mathbb{C}[x_1, x_2] \cong \mathbb{C}^2.$$

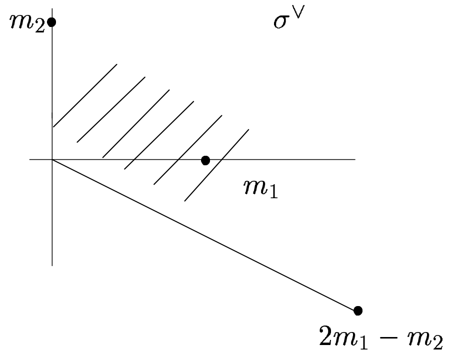
(ii)



$$\sigma = \mathbb{R}_{\geq 0} m_1^* + \mathbb{R}_{\geq 0} (m_1^* + 2m_2^*).$$

In this case the dual cone is given by

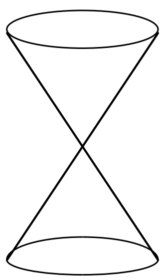
$$\sigma^\vee = \mathbb{R}_{\geq 0} m_1 + \mathbb{R}_{\geq 0} m_2 + \mathbb{R}_{\geq 0}(2m_1 - m_2).$$



$$H_{\sigma^\vee} = \sigma^\vee \cap M = \mathbb{Z}_{\geq 0} m_1 + \mathbb{Z}_{\geq 0} m_2 + \mathbb{Z}_{\geq 0}(2m_1 - m_2).$$

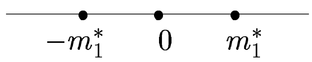
We find that T_σ is the quadric cone given by

$$T_\sigma = \text{Spec}\ \left(\mathbb{C}[X_1, X_2, X_3]/X_1^2 = X_2 X_3\right).$$



(2) Fans

(i) $\qquad M_{\mathbb{R}}^* = \mathbb{R}, \quad \Sigma = \{\sigma, -\sigma, \{0\}\}, \quad \sigma = \mathbb{R}_{\geq 0} m_1^*.$



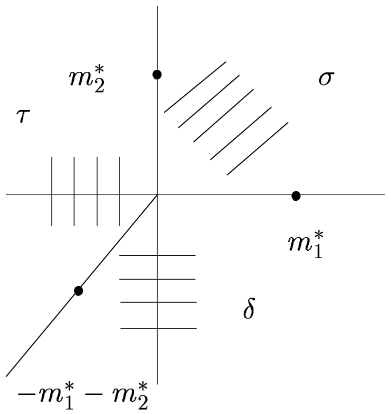
Then

$$\begin{array}{lclclcl} T_{\{0\}} & = & \text{Spec}\,\mathbb{C}[x_1, x_1^{-1}] & \cong & \mathbb{C}^* & & \\ T_\sigma & = & \text{Spec}\,\mathbb{C}[x_1] & \cong & \mathbb{C} \supset \mathbb{C}^* & = & T_{\{0\}} \\ T_{-\sigma} & = & \text{Spec}\,\mathbb{C}[x_1^{-1}] & \cong & \mathbb{C} \supset \mathbb{C}^* & = & T_{\{0\}}. \end{array}$$

We find that

$$T_\Sigma = T_\sigma \cup T_{-\sigma} \cup T_{\{0\}}/\sim \;\cong \mathbb{P}^1.$$

(ii) $M^*_{\mathbb{R}} = \mathbb{R}^2$

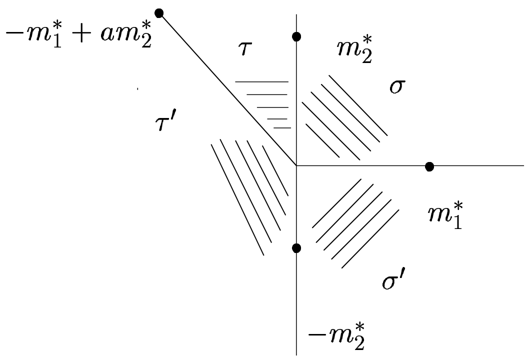


$\Sigma = \{\sigma, \tau, \delta\} +$ faces

$T_\sigma = \mathbb{C}^2, T_\tau \cong \mathbb{C}^2, T_\delta = \mathbb{C}^2$

$T_\Sigma = \mathbb{P}^2.$

(iii)



$X_\Sigma = F_a = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-a))$
(Hirzebruch surface)

The Voronoi decomposition ($g = 2$)

We consider the space

$$\mathrm{Sym}_2(\mathbb{R}) = \{M \in \mathrm{Mat}(2 \times 2, \mathbb{R});\;\; g = {}^t g\} (\cong \mathbb{R}^3).$$

In this we have the cones

$$\begin{array}{rcl}\mathrm{Sym}_2^{>0}(\mathbb{R}) & = & \{M \in \mathrm{Sym}_2(\mathbb{R}); M > 0\} \\ \mathrm{Sym}_2^{\geq 0}(\mathbb{R}) & = & \{M \in \mathrm{Sym}_2(\mathbb{R}); M \geq 0\}.\end{array}$$

The group $\mathrm{GL}(2,\mathbb{Z})$ acts on these spaces by

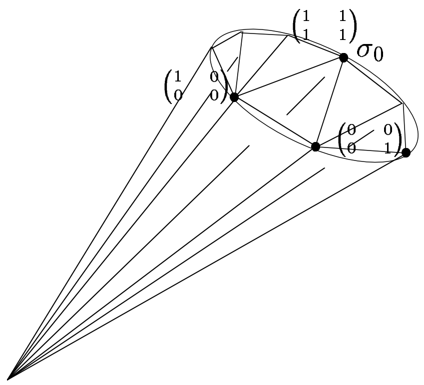
$$g:\ M \mapsto {}^t g^{-1} M g^{-1}.$$

We define the cone

$$\sigma_0 := \mathbb{R}_{\geq 0}\begin{pmatrix}1&0\\0&0\end{pmatrix} + \mathbb{R}_{\geq 0}\begin{pmatrix}1&1\\1&1\end{pmatrix} + \mathbb{R}_{\geq 0}\begin{pmatrix}0&0\\0&1\end{pmatrix}.$$

Definition The *Voronoi (Legendre) decomposition* (of the rational closure of $\mathrm{Sym}_2^{>0}(\mathbb{R})$) is defined by

$$\Sigma_L := \{g(\sigma_0); g \in \mathrm{GL}(2,\mathbb{Z})\} + \text{ faces}.$$



The union of these cones covers the entire cone $\mathrm{Sym}_2^{>0}(\mathbb{R})$. (This is a result in the reduction theory of quadratic forms.)

$$\begin{array}{lclcll}
\sigma & = & \{0\} & \Rightarrow & T_{\{0\}} & \cong & (\mathbb{C}^*)^3 \\
\dim\sigma & = & 1 & \Rightarrow & T_\sigma & \cong & \mathbb{C}^*\times\mathbb{C}^*\times\mathbb{C} \\
\dim\sigma & = & 2 & \Rightarrow & T_\sigma & \cong & \mathbb{C}^*\times\mathbb{C}\times\mathbb{C} \\
\dim\sigma & = & 3 & \Rightarrow & T_\sigma & \cong & \mathbb{C}^3.
\end{array}$$

The toric variety T_{Σ_L} is a smooth 3-dimensional variety.

IV Shioda modular surfaces

We had already encountered the modular curves $X(n)$ and the interpretation of these varieties as moduli spaces. We now want to construct a *universal elliptic curve* over these varieties. We shall assume $n \geq 3$. This implies that $\Gamma(n)$ acts freely. We consider

$$H(n) := \left\{ \begin{pmatrix}1&kn&ln\\0&a&b\\0&c&d\end{pmatrix}; k,l \in \mathbb{Z}; \begin{pmatrix}a&b\\c&d\end{pmatrix} \in \Gamma(n) \right\}.$$

This is an extension

$$1 \to \underset{\underset{L}{\|}}{n\mathbb{Z} \times n\mathbb{Z}} \to H(n) \to \Gamma(n) \to 1.$$

It acts on $\mathbb{C} \times \mathbb{H}_1$ by

$$\begin{pmatrix} 1 & kn & ln \\ 0 & a & b \\ 0 & c & d \end{pmatrix} : (z, \tau) \mapsto \left(\frac{z + kn\tau + ln}{c\tau + d}, \frac{a\tau + b}{c\tau + d} \right).$$

The quotient has the structure of a fibre space over $X^0(n)$

$$\begin{array}{ccc} S^0(n) = & \mathbb{C} \times \mathbb{H}_1 / H(n), & [z, \tau] \\ \downarrow & & \downarrow \\ X^0(n) = & \mathbb{H}_1 / \Gamma(n), & [\tau]. \end{array}$$

The fibre is given by

$$E_{[\tau]} = \mathbb{C}/(n\mathbb{Z}\tau + n\mathbb{Z}) \cong \mathbb{C}/(\mathbb{Z}\tau + \mathbb{Z}).$$

The elliptic curve $E_{[\tau]}$ can be given a level-n structure by

$$(w_1, w_2) := ([\tau], [1]) \in E_{[\tau]}^{(n)}.$$

Then $S^0(n)$ is the universal elliptic curve with a level-n structure.

Problem Compactify $S^0(n)$ over $X^0(n)$:

$$\begin{array}{ccc} S^0(n) & \subset & S(n) \\ \downarrow & & \downarrow \\ X^0(n) & \subset & X(n). \end{array}$$

The problem is then to find the right fibre over each cusp. For symmetry reasons it is enough to consider the cusp $\{\infty\} \in X(n)$.

We first consider the *stabilizer* at ∞, i.e. the group

$$P = \left\{ \begin{pmatrix} 1 & kn & ln \\ 0 & 1 & rn \\ 0 & 0 & 1 \end{pmatrix} ; k, l, r \in \mathbb{Z} \right\} \cong \mathbb{Z}^3.$$

We can write this as an extension

$$1 \to P' \to P \to P'' \to 1$$

where

$$P' = \left\{ \begin{pmatrix} 1 & 0 & ln \\ 0 & 1 & rn \\ 0 & 0 & 1 \end{pmatrix} ; l, r \in \mathbb{Z} \right\} \cong \mathbb{Z}^2$$

and

$$P'' \cong \left\{ \begin{pmatrix} 1 & kn & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} ; k \in \mathbb{Z} \right\} \cong \mathbb{Z}.$$

The group P' acts on a neighbourhood W of $\mathbb{C} \times \{\infty\}$ by

$$\begin{pmatrix} 1 & 0 & ln \\ 0 & 1 & rn \\ 0 & 0 & 1 \end{pmatrix} : (z, \tau) \mapsto (z + ln, \tau + rn).$$

We can take the partial quotient

$$\begin{array}{rcl} e : W & \to & (\mathbb{C}^*)^2 \\ (\tau, z) & \mapsto & (e^{2\pi i \tau / n}, e^{2\pi i z / n}) = (t, w). \end{array}$$

The group P'' then acts on $e(W)$, and indeed on $(\mathbb{C}^*)^2$, by

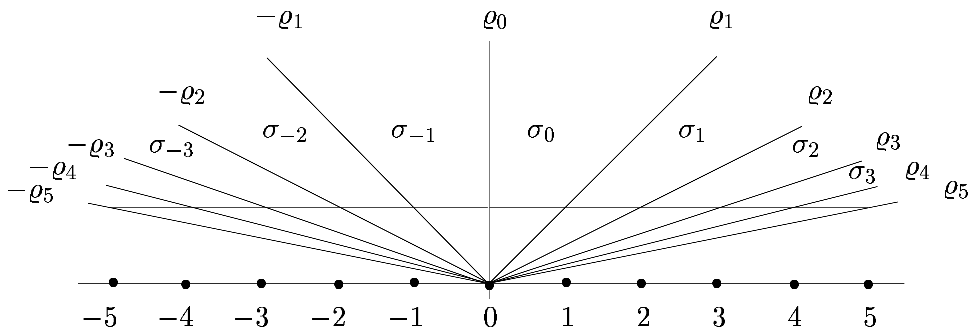
$$\begin{pmatrix} 1 & kn & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : (t, w) \mapsto (t, t^{nk} w).$$

Main idea. (1) Construct a suitable torus embedding $(\mathbb{C}^*)^2 \subset T_\Sigma$ in such a way that the action of P'' extends to T_Σ.
(2) Glue $S^0(n)$ and a suitable neighbourhood of $(T_\Sigma \backslash (\mathbb{C}^*)^2)/P''$ along the image $e(W)/P''$.

We set

$$M = \mathbb{Z}^2, \ N = M^* = \mathbb{Z}^2.$$

In $M^*_{\mathbb{R}}$ we consider the fan shown in the following figure.



We set

$$U := (1,0),\ V := (0,1)$$

with dual basis

$$U^* = (1,0),\ V^* = (0,1).$$

Let

$$U_k^* = (k+1,1),\ V_k^* = (k,1).$$

Then

$$\begin{aligned}\sigma_k &= \mathbb{R}_{\geq 0}U_k^* + \mathbb{R}_{\geq 0}V_k^*,\\ \rho_k &= \mathbb{R}_{\geq 0}V_k^*.\end{aligned}$$

The dual basis to U_k^*, V_k^* is given by

$$U_k = (1,-k),\ V_k(-1,k+1).$$

Then

$$\begin{aligned}\sigma_k^\vee &= \mathbb{R}_{\geq 0}U_k + \mathbb{R}_{\geq 0}V_k\\ \rho_k^\vee &= \mathbb{R}_{\geq 0}V_k + \mathbb{R}U_k = \mathbb{R}_{\geq 0}V_k + \mathbb{R}_{\geq 0}U_k + \mathbb{R}_{\geq 0}(-U_k).\end{aligned}$$

This shows that

$$T_{\sigma_k} \cong \mathbb{C}^2,\ T_{\rho_k} \cong \mathbb{C} \times \mathbb{C}^*.$$

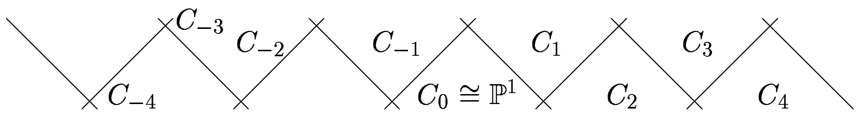
Let u, v be the natural coordinates on the torus T, and let u_k, v_k be the coordinates of $T_{\sigma_k} \cong \mathbb{C}^2$ corresponding to U_k, V_k. Then we have the inclusions

$$T_{\{0\}} \hookrightarrow T_{\sigma_k};\ (u,v) \mapsto (uv^{-k}, u^{-1}v^{k+1}) = (u_k, v_k).$$

We also have

$$\begin{aligned}T_{\rho_k} &\hookrightarrow T_{\sigma_k};\ (u_k, v_k) &&\mapsto (u_k, v_k)\\ T_{\rho_{k+1}} &\hookrightarrow T_{\sigma_k};\ (u_{k+1}, v_{k+1}) &&\mapsto (v_k^{-1}, u_k v_k^2).\end{aligned}$$

We then obtain the following picture: We have $T = (\mathbb{C}^*)^2 \hookrightarrow T_\Sigma$ and $T_\Sigma \backslash T$ looks as in the figure below.



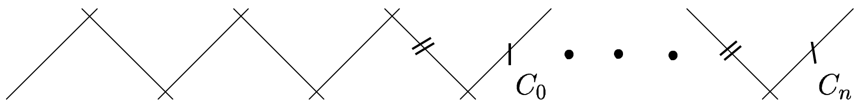
The next step in the construction is to define an action of the group P'' on T_Σ which extends that of P'' on $T = (\mathbb{C}^*)^2$. We had seen that

$$P'' = \left\{ \begin{pmatrix} 1 & kn & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} ;\ k \in \mathbb{Z} \right\} \cong \mathbb{Z}h, \quad h = \begin{pmatrix} 1 & n & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We define actions of the generator h on M, resp. M^* by

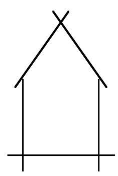
$$\begin{array}{rlcl} h: & M^* \longrightarrow M^*, \quad m^*\dagger & \mapsto & \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} m^* \\ {}^t h^{-1}: & M \longrightarrow M, \quad m & \mapsto & \begin{pmatrix} 1 & 0 \\ -n & 1 \end{pmatrix} m. \end{array}$$

Then this defines an action of P'' on T_Σ, where it acts on the chain of $\mathbb{P}^{1'}$'s as follows



$$h_i: \qquad C_i\dagger \mapsto C_{i+n}.$$

The quotient is an n-gon:



$(n = 5)$.

Finally we have to glue, i.e. we take

$$S_\infty(n) = S_0(n) \cup_{W/P} T_\Sigma / P''$$

where

$$X_\Sigma = \overset{\circ}{\overline{e(W)}},$$

i.e. the interior of the closure of $e(W)$ in T_Σ. We have then added an n-gon as the fibre over $\{\infty\}$. This n-gon consists of n copies of $C_i \cong \mathbb{P}^1$ with $C_i^2 = -2$.

V Compactifications of Siegel modular varieties

Recall that the group $\mathrm{Sp}(2g,\mathbb{Q})$ acts on Siegel space

$$\mathbb{H}_g = \{\tau \in \mathrm{Mat}(g \times g, \mathbb{C}); \tau = {}^t\tau, \mathrm{Im}\tau > 0\}$$

by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} : \tau \mapsto (A\tau + B)(C\tau + D)^{-1}.$$

Now let $\Gamma \subset \mathrm{Sp}(2g,\mathbb{Q})$ be an *arithmetic subgroup*, i.e.

$$[\Gamma, \Gamma \cap \mathrm{Sp}(2g,\mathbb{Z})] < \infty,\ [\mathrm{Sp}(2g,\mathbb{Z}), \Gamma \cap \mathrm{Sp}(2g,\mathbb{Z})] < \infty.$$

Then Γ acts *properly discontinuously* on $\mathbb{H}_g$. Let

$$\mathcal{A}(\Gamma) = \mathbb{H}_g/\Gamma \qquad (\textit{Siegel modular variety}).$$

Then one has the following properties of $\mathcal{A}(\Gamma)$:

- $\mathcal{A}(\Gamma)$ is a *V-manifold* (i.e. has only finite quotient singularities). (If $g = 1$, then $\mathcal{A}(\Gamma)$ is smooth.).
- $\mathcal{A}(\Gamma)$ is *normal.*
- $\dim \mathcal{A}(\Gamma) = 3g - 3 \ = \dim \mathbb{H}_g$.
- $\mathcal{A}(\Gamma)$ is *quasi-projective* (but not projective).

Problem Construct a "good" compactification of $\mathcal{A}_g$.

(1) Satake compactification $\overline{\mathcal{A}(-)}$
This is defined via modular forms.

- $\overline{\mathcal{A}(-)}$ is normal.
- $\overline{\mathcal{A}(-)}$ is a "minimal" compactification.
- $\overline{\mathcal{A}(-)}$ tends to have complicated singularities.
- $\overline{\mathcal{A}(-)}$ has no functorial interpretation. I.e. the points on the boundary $\overline{\mathcal{A}(-)} \backslash \mathcal{A}(\Gamma)$ do not correspond to degenerations of abelian varieties.

As a set $\overline{\mathcal{A}}_g = \mathcal{A}_g \coprod \mathcal{A}_{g-1} \coprod \cdots \coprod \mathcal{A}_1 \coprod \mathcal{A}_0$.

(2) Toroidal compactifications
Igusa constructed a partial resolution of $\overline{\mathcal{A}}_g$ by blowing up along the boundary. Igusa's compactification is an example of a *toroidal compactification.* Let $\mathcal{A}(\Gamma)^*$ be a toroidal compactification. Then one has:

- $\mathcal{A}(\Gamma)^*$ depends on the choice of certain fans.
- $\mathcal{A}(\Gamma)^*$ is *normal.*
- The fans can be chosen such that $\mathcal{A}(\Gamma)^*$ is a V-manifold.
- codim $\mathcal{A}(\Gamma)^* \backslash \mathcal{A}(\Gamma) = 1$.
- For good choices of the fan $\mathcal{A}(\Gamma)^*$ is projective.

The following result is crucial

Theorem V.0.1 (Alexeev-Nakamura) *Let $\mathcal{A}_g^{\mathrm{vor}}$ be the Voronoi compactification of $\mathcal{A}_g$. Then $\mathcal{A}_g^{\mathrm{vor}}$ has a modular interpretation in terms of degenerations of abelian varieties.*

Remark If $g = 1$, then all compactifications coincide.

Toroidal compactifications (of $\mathcal{A}_2$)

In this section I want to outline the construction of toroidal compactifications. As one can see all the essential steps already in the case $g = 2$, I will restrict to this case.

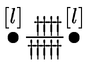
We had already encountered the *cusps* in the case $g = 1$. To enumerate all cusps in our situation we look at the following:

- lines $l \subset \mathbb{Q}^4$ (i.e. isotropic subspace of $\mathbb{Q}^4$ of dimension 1)
- isotropic planes $h \subset \mathbb{Q}^4$.

Given an arithmetic subgroup $\Gamma \subset \mathrm{Sp}(4, \mathbb{Q})$ we then define an *equivalence relation* as follows

$$\begin{aligned} l_1 \sim l_2 : \ &\Leftrightarrow \text{exists } g \in \Gamma \text{ with } g(l_1) = l_2 \\ h_1 \sim h_2 : \ &\Leftrightarrow \text{exists } g \in \Gamma \text{ with } g(h_1) = h_2. \end{aligned}$$

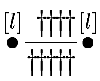
The *Titsbuilding* $\mathcal{T}(\Gamma)$ is the following graph:

- vertices: $[l]; \quad l \subset \mathbb{Q}^4, \quad \dim l = 1$
  $[h]; \quad h \subset \mathbb{Q}^4, \quad \dim h = 2, h$ is isotropic.
  Here $[\dagger]$ denotes the equivalence class of a line, resp. a plane.
- edges: 

$\Leftrightarrow l \subset h$ for some representatives.

In the general case one has to consider all isotropic subspaces $U \subset \mathbb{Q}^{2g}$.

Examples (1) $\Gamma = \mathrm{Sp}(4, \mathbb{Z})$. In this case the Tits building is

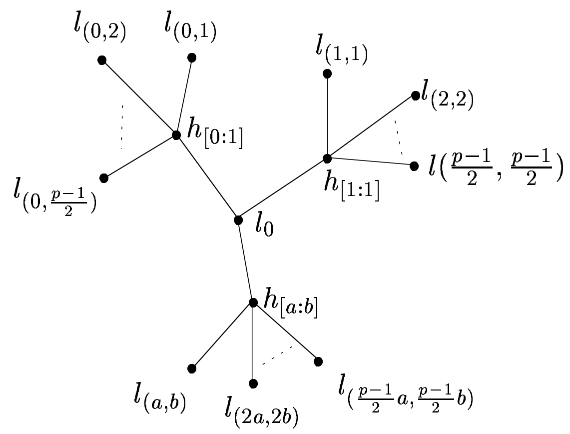


$; \quad l = \mathbb{Q}e_3; h = \mathbb{Q}e_3 + \mathbb{Q}e_4.$

(2) Let $p \geq 3$ be a prime number and let

$$\Gamma_{1,p}^{\text{lev}} := \{g \in \text{Sp}(4,\mathbb{Q})\ ;\ g - \mathbf{1} \in \begin{pmatrix} \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & p\mathbb{Z} \\ p\mathbb{Z} & p\mathbb{Z} & p\mathbb{Z} & p^2\mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & p\mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & p\mathbb{Z} \end{pmatrix} \}.$$

This group defines the Siegel modular variety $\mathcal{A}_{1,p}^{\text{lev}}$ which parametrises $(1,p)$-polarized abelian surfaces with a canonical level structure. In this case the Titsbuilding looks as in the figure below.



We have

$$l_0 = \mathbb{Q}(0,0,1,0); \quad l_{(a,b)} = \mathbb{Q}(0,a,0,b)$$

and

$$h_{[a:b]} = l_0 \wedge l_{(a,b)}$$

where

$$(a,b) \in (\mathbb{Z}_p \times \mathbb{Z}_p \backslash \{(0,0)\})/\pm 1; \quad [a:b] \in \mathbb{P}^1(\mathbb{Z}_p^2).$$

One then associates to each line l or plane h a *parabolic subgroup*, namely

$$P(l) = \{g \in \Gamma; g(l) = l\},$$

$$P(h) = \{g \in \Gamma; g(h) = h\}.$$

Example $\Gamma = \text{Sp}(4,\mathbb{Z})$

(i) For $l = \mathbb{Q}(l_0)$ we have

$$P(l) = \left\{ \begin{pmatrix} \varepsilon & m & s & n \\ 0 & a & * & b \\ 0 & 0 & \varepsilon & 0 \\ 0 & c & * & d \end{pmatrix}; \varepsilon = \pm 1; m,n,s, \in \mathbb{Z}; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2,\mathbb{Z}) \right\}.$$

(ii) For $h = \mathbb{Q}e_3 + \mathbb{Q}e_4$ we have

$$P(h) = \left\{ \begin{pmatrix} {}^tQ^{-1} & 0 \\ 0 & Q \end{pmatrix} \begin{pmatrix} \mathbf{1}_2 & S \\ 0 & \mathbf{1}_2 \end{pmatrix} ; Q \in \mathrm{GL}(2,\mathbb{Z}); S = {}^tS \in \mathrm{Sym}(2,\mathbb{Z}) \right\}.$$

We now "add a cusp" in the direction of each vertex of the Titsbuilding and then glue the various partial compactifications.

<u>Corank 1 boundary components</u>
There is an exact sequence

$$\begin{array}{ccccccc} 1 \to & P'(l) & \to P(l) \to P''(l) \to 1. \\ & || & \\ & \mathbb{Z} & \end{array}$$

In the case of $\Gamma = \mathrm{Sp}(4,\mathbb{Z})$ one has

$$P'(l) = \left\{ \begin{pmatrix} 1 & 0 & s & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} ; s \in \mathbb{Z} \right\} \cong \mathbb{Z}.$$

The group $P'(l)$ acts on $\mathbb{H}_2$ by

$$s : \quad \begin{pmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{pmatrix} \mapsto \begin{pmatrix} \tau_1 + s & \tau_2 \\ \tau_2 & \tau_3 \end{pmatrix}.$$

We can identify the quotient $P''(l)$ with

$$P''(l) = \left\{ \begin{pmatrix} \varepsilon & m & n \\ 0 & a & b \\ 0 & c & d \end{pmatrix} ; \ \varepsilon = \pm 1; m, n \in \mathbb{Z}; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) \right\}.$$

<u>1. Step:</u> We take the partial quotient with respect to $P'(l)$:

$$\begin{array}{ccc} e(l) : \mathbb{H}_2 & \longrightarrow & \mathbb{C}^* \times \mathbb{C} \times \mathbb{H}_1 \\ \begin{pmatrix} \tau_1 & \tau_2 \\ \dagger\tau_2 & \tau_3 \end{pmatrix} & \mapsto & \left(e^{2\pi i \tau_1}, \tau_2, \tau_3\right). \end{array}$$

Then we add the cusp $\{0\} \times \mathbb{C} \times \mathbb{H}_1$. The action of $P''(l)$ on $\mathbb{H}_2$ defines an action of $P''(l)$ on $e(l)(\mathbb{H}_2)$. This action extends to $\{0\} \times \mathbb{C} \times \mathbb{H}_1$ where it acts as in the construction of the Shioda modular surface. Let

$$X(l) := \overset{\circ}{\overline{e(l)(\mathbb{H}_2)}}.$$

Note that this contains $\{0\} \times \mathbb{C} \times \mathbb{H}_1$ and define

$$Y(l) := X(l)/P''(l).$$

This defines a neighbourhood of "the cusp l". What we have added is a surface $\mathbb{C} \times \mathbb{H}_1/P''(l)$ which is the open Shioda modular surface of level 1.

Corank 2 boundary components

Here too we have an exact sequence

$$\begin{array}{cccc} 1 \to & P'(h) & \to P(h) \to P(h'') \to 1. \\ & \dagger || \dagger & \\ & \mathbb{Z}^3 & \end{array}$$

In the case of $\Gamma = \mathrm{Sp}(4, \mathbb{Z})$ one has

$$\begin{array}{l} P'(h) = \left\{ \begin{pmatrix} \mathbf{1}_2 & S \\ 0 & \mathbf{1}_2 \end{pmatrix} ; S = {}^tS \in \mathrm{Sym}(2, \mathbb{Z}) \right\}, \\ P''(h) \cong \mathrm{GL}(2, \mathbb{Z}). \end{array}$$

The group $P'(h)$ acts on $\mathbb{H}_2$ as follows

$$\begin{pmatrix} s_1 & s_2 \\ s_2 & s_3 \end{pmatrix} : \begin{pmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{pmatrix} \mapsto \begin{pmatrix} \tau_1 + s_1 & \tau_2 + s_2 \\ \tau_2 + s_2 & \tau_3 + s_3 \end{pmatrix}.$$

1. Step: We take the partial quotient with respect to $P'(h)$. This defines a map

$$\begin{array}{cccl} e(h): & \mathbb{H}_2 & \dagger \longrightarrow & (\mathbb{C}^*)^3 = T \\ & \begin{pmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{pmatrix} & \longmapsto & (t_1, t_2, t_3) = (e^{2\pi i \tau_1}, e^{2\pi i \tau_2}, e^{2\pi i \tau_3}). \end{array}$$

We now consider a torus embedding of the torus T. In order to do this one must choose a fan Σ. This fan must cover the rational closure of $\mathrm{Sym}_2^+(\mathbb{R})$. Here we take the *Legendre (=second Voronoi)* fan. This gives us an embedding

$$T \dagger \subset T_\Sigma$$

Let

$$X(h) := X_\Sigma(h) := \overline{\overset{\circ}{e(h)(\mathbb{H}_2)}}.$$

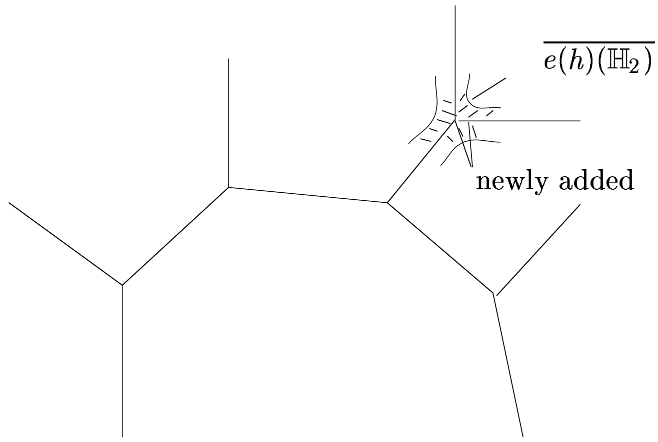
The group $P''(h)$ acts on $e(h)(\mathbb{H}_2)$ and this action extends to T_Σ and induces an action on $X(h)$. The crucial point here is that $P''(h) \cong \mathrm{GL}(2, \mathbb{Z})$ and this group acts on $\mathrm{Sym}_2(\mathbb{R})$ by

$$Q : M \mapsto {}^tQ^{-1} M Q^{-1}.$$

This action preserves the Voronoi decomposition. One then takes

$$Y(h) := X(h)/P''(h).$$

This gives a neighbourhood of "the cusp h".



Glueing process

The process of adding the cusps l and h is not independent. Assume that $l \subset h$. Then one has $P(l) \subset P(h)$ and we have a natural map

$$\pi(l,h): \; X(l) \to X(h)$$

which is compatible with the action of $P''(l)$ and $P''(h)$. It induces a map

$$\bar{\pi}(l,h): \; Y(l) \to Y(h).$$

We then identify

$$x \in Y(l) \sim x' \in Y(h) \Leftrightarrow x' = \bar{\pi}(l,h)(x).$$

The above process has to be done for all cusps.

This leads us to the *Voronoi compactification* $\mathcal{A}_g^{\mathrm{Vor}}$ of $\mathcal{A}_g$. $\mathcal{A}_g^{\mathrm{Vor}}$ is a projective variety (Alexeev).

VI Degenerations of abelian varieties: Preparations

The general question which we want to ask is the following: Let $\Gamma \subset \mathrm{Sp}(2g, \mathbb{Q})$ be some arithmetic subgroup and

$$\mathcal{A}(\Gamma) = \mathbb{H}_g/\Gamma = \text{ some moduli space of abelian varieties.}$$

Let $\mathcal{A}(\Gamma)^*$ be (some) compactification of $\mathcal{A}(\Gamma)$.

Question Can we interprete the "boundary points", i.e. the points $\mathcal{A}(\Gamma)^* \backslash \mathcal{A}(\Gamma)$ in terms of degenerate abelian varieties?

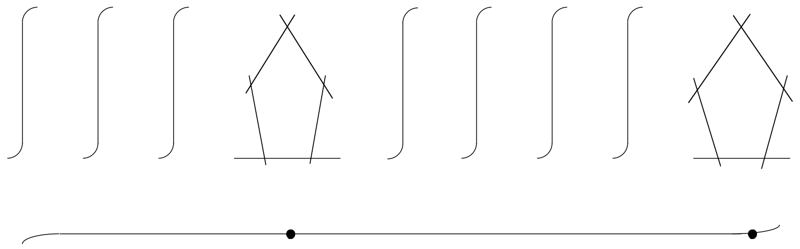
Example Shioda modular surfaces
Let

$$\Gamma(n) := \{M \in \mathrm{SL}(2,\mathbb{Z});\ M \equiv \mathbf{1} \mod n\}$$

and

$$X^0(n) := \mathbb{H}_1/\Gamma(n), X(n) = X^0(n) \cup \{\text{cusps}\}.$$

We have constructed the *Shioda modular surfaces* $S(n)$ (for $n \geq 3$) which are fibred over $X(n)$ and have n-gons over the cusps:



$S(5)$

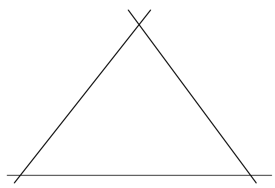
$X(5)$

The surfaces $S(n)$ have a group of sections isomorphic to $\mathbb{Z}_n \times \mathbb{Z}_n$. One can use these sections to define level-n structures on each smooth fibre of $S(n)$. In this way $S^0(n)$ becomes the *universal elliptic curve* with a *level-n structure.* The n-gons can be interpreted as degenerate elliptic curves with a level-n structure. This can be made precise by saying that $S(n)$ represents a suitably defined functor.

In the case $n = 3$ we consider the *Hesse-pencil*

$$C_\lambda : x_0^3 + x_1^3 + x_2^3 - 3\lambda x_0 x_1 x_2 = 0.$$

Then C_λ is smooth if and only if $\lambda \neq 1, \rho, \rho^2, \infty, (\rho = e^{2\pi i/3})$. If C_λ is singular, then it is a triangle:



C_∞

Let $P_1, \dots, P_9$ be the base points of the pencil C_λ. Blowing up $\mathbb{P}^2$ in these points, we obtain a surface $\tilde{\mathbb{P}}^2 = \tilde{\mathbb{P}}^2(P_1, \dots, P_9)$ which has a map to $\mathbb{P}^1$. Then

$$\begin{array}{ccccc} \tilde{C}_\lambda & \subset & \tilde{\mathbb{P}}^2 & \cong & S(3) \\ \downarrow & & \downarrow & & \downarrow \\ \lambda & \in & \mathbb{P}^1 & \cong & X(3) \end{array}$$

and the 4 singular members of the pencil C_λ are the 4 singular fibres in $S(3)$ over the cusps of $X(3)$.

In the case of $\Gamma = \mathrm{Sp}(2g,\mathbb{Z})$ we have the second Voronoi compactification $\mathcal{A}_g^{\mathrm{Vor}}$.

Theorem VI.0.2 (Alexeev, Nakamura) *$\mathcal{A}_g^{\mathrm{Vor}}$ is a coarse moduli scheme for the functor of PSQAV's (principally polarized stable quasi-abelian varieties).*

I want to describe the construction which leads to these degenerations. More generally Alexeev has introduced the concept of *stable semi-abelic varieties* (SSAV's). This leads to a projective variety $\mathcal{A}_g^S$ and $\mathcal{A}_g^{\mathrm{Vor}}$ is one component of $\mathcal{A}_g^S$.
Semi-abelian varieties
A crucial concept which we need is that of semi-abelian varieties.

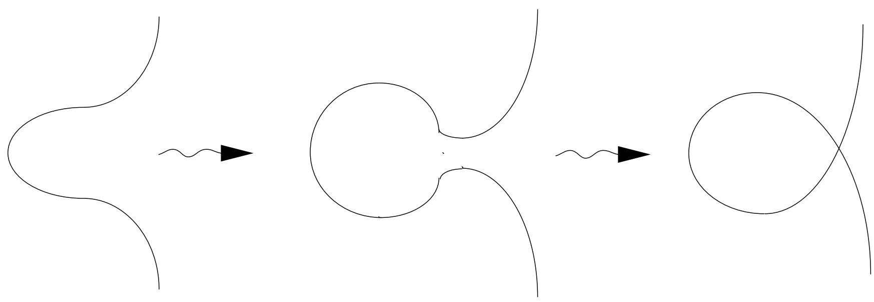
Definition A *semi-abelian variety* G is a group scheme which is an extension

$$1 \to T \to G \to A \to 1$$

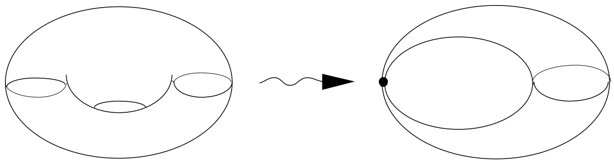
where T is a torus ($T \cong (\mathbb{C}^*)^r$) and A is an abelian variety.

Semi-abelian varieties appear naturally when one considers degenerations of abelian varieties.

(1) Everybody has seen the degenerations of a plane cubic to a *nodal curve*:



Topologically this corresponds to a "vanishing cycle":



$\mathbb{P}^1$with 2 points identified

In more algebraic terms

$$E = \mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau) = \mathbb{C}^*/\mathbb{Z} \quad (\mathbb{Z} \text{ acts by multiplication with } t^n; t = e^{2\pi i \tau}).$$

If $\tau \to i\infty$, then $t \to 0$ and one obtains $\mathbb{C}^* = \mathbb{P}^1 \backslash \{0, \infty\}$.
(2) We consider a genus 2 example. Let $\tau \in \mathbb{H}_2$ and consider the lattice L_τ spanned by the columns of the matrix

$$\begin{pmatrix} \tau_1 & \tau_2 & 1 & 0 \\ \tau_2 & \tau_3 & 0\dagger & 1 \end{pmatrix} \leadsto L_\tau = \mathbb{Z}e_1 + \mathbb{Z}e_2 + \mathbb{Z}e_3 + \mathbb{Z}e_4.$$

The associated abelian surface is

$$A_\tau = \mathbb{C}^2/L_\tau = \mathbb{C}^2/\mathbb{Z}^4 = (\mathbb{C}^*)^2/\mathbb{Z}^2.$$

Here we have divided by L_τ in two steps. The first step is

$$\begin{array}{ccccc} \mathbb{C}^2 & \to & (\mathbb{C}^*)^2 & = & \mathbb{C}^2/(\mathbb{Z}e_3 + \mathbb{Z}e_4), \\ (z_1, z_2) & \mapsto & (e^{2\pi i z_1}, e^{2\pi i z_2}) & = & (w_1, w_2). \end{array}$$

The group $\mathbb{Z}^2 = \mathbb{Z}e_1 + \mathbb{Z}e_2$ then acts on $(\mathbb{C}^*)^2$ by multiplication as follows:

$$\begin{array}{llll} e_1: & (w_1, w_2) & \mapsto & (t_1 w_1, t_2 w_2) \qquad (t_1 = e^{2\pi i \tau_1}, t_2 = e^{2\pi i \tau_2},\ t_3 = e^{2\pi i \tau_3}), \\ e_2: & (w_1, w_2) & \mapsto & (t_2 w_1, t_3 w_2). \end{array}$$

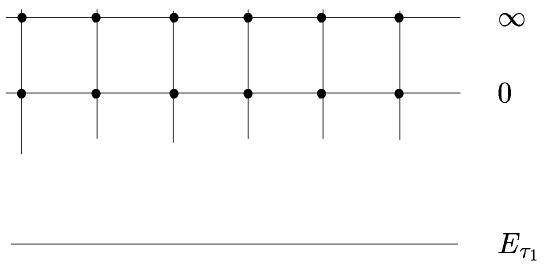
Now assume that $\tau_3 \to i\infty$. Then the action of e_2 is no longer free when $t_3 = 0$. But we can still consider

$$G = \mathbb{C}^2/(\mathbb{Z}e_1 + \mathbb{Z}e_3 + \mathbb{Z}e_4) = (\mathbb{C}^*)^2/\mathbb{Z}e_1.$$

This then has the following structure

$$\begin{array}{c} 1 \to \dagger\mathbb{C}^* \to G \to E_{\tau_1} = \mathbb{C}/(\mathbb{Z}\tau_1 + \mathbb{Z}) \to 1 \\ \qquad [w_1, w_2] \mapsto [w_1]. \end{array}$$

This is a *semi-abelian surface.* Geometrically it is a $\mathbb{C}^*$-bundle over E_{τ_1}:



By adding the 0-section and the section at infinity we can embed G as an open set in a $\mathbb{P}^1$-bundle over E_{τ_1}.

Classifying homomorphism

Let G be a semi-abelian variety

$$1 \to T \to G \to A \to 1$$

and let X be the character group of the torus T. Every such G is given by a classifiying homomorphism

$$c: \ X \to \widehat{A} = \mathrm{Pic}^0 A.$$

For $x \in X$ the point $c(x) \in \mathrm{Pic}^0 A$ is a line bundle on A which we denote by

$$\mathcal{O}_x := c(x).$$

Such a homomorphism defines G by

$$\boxed{G = \mathrm{Spec} \bigoplus_{x \in X} \mathcal{O}_x}$$

where we consider $\mathcal{O}_x$ as an $\mathcal{O}_A$-algebra.

Example Let us consider example (2). We define

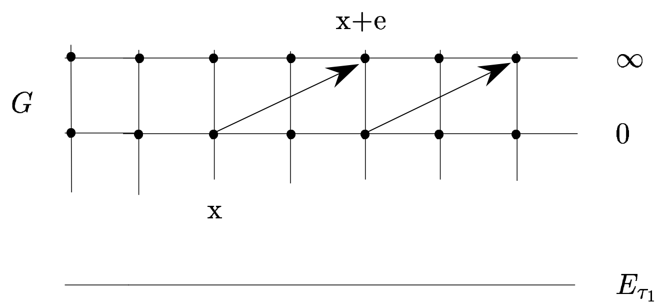
$$e := [\tau_2] \in E_{\tau_1} = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau_1).$$

Then the homomorphism c is given by

$$\begin{array}{rcl} c: \mathbb{Z} & \to & \mathrm{Pic}^0 E_{\tau_1} = E_{\tau_1} \\ 1 & \mapsto & \mathcal{O}(e - 0). \end{array}$$

Then one has $G \subset \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(e - 0))$.

Remark We shall then obtain a degenerate abelian surface as follows



$$X = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(e - O))/_{\sim} \supset G.$$

Delaunay and Voronoi cells

Let X be a lattice ($X \cong \mathbb{Z}^k$) and $X_{\mathbb{R}} = X \otimes_{\mathbb{Z}} \mathbb{R} (\cong \mathbb{R}^k)$.

Definition A *cell* δ is a compact convex polytype in $X_{\mathbb{R}}$ whose vertices lie in X.

Let

$$S : X \times X \to \mathbb{R}$$

be a symmetric, positive definite bilinear form. This induces a metric and hence a norm $|| \cdot || = || \cdot ||_s$ on $X_{\mathbb{R}}$.

Definition †

(i) Let $a \in X_{\mathbb{R}}$. We say that a lattice point $x \in X$ is a-*nearest* if

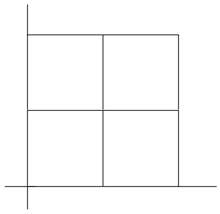
$$||x - a|| = \min\{||x' - a||;\ x' \in X\}.$$

(ii) A *Delaunay cell* with respect to S is the convex hull of all lattice points which are a-nearest to some point $a \in X_{\mathbb{R}}$.

The union of all Delaunay cells with respect to a given form S gives the Delauny decomposition Δ which is defined by S.

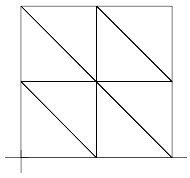
Examples $X = \mathbb{Z}^2$

(1)



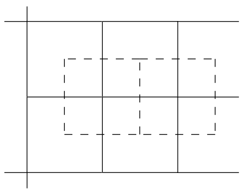
$$S = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

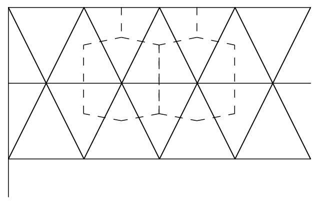
(2)



$$S = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

Definition Let δ be a Delaunay cell with respect to S. Then all points in $X_{\mathbb{R}}$ which define δ form again a cell $\hat{\delta}$. This is the *Voronoi cell* which is dual to the Delaunay cell.





The dotted lines give the Voronoi cells.

Remark The connection with the Voronoi decomposition of $\mathrm{Sym}_2^{\geq 0}(\mathbb{R})$ is the following. All matrices in the relative interior of a cone belonging to the second Voronoi decomposition of $\mathrm{Sym}_2^{\geq 0}(\mathbb{R})$ define the same Delaunay decomposition in $\mathbb{R}^g$.

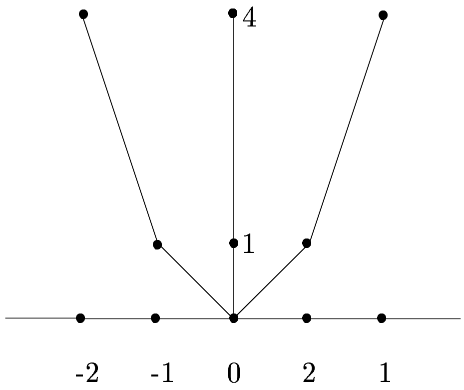
Another way of looking at Delaunay decompositions is as follows: Let $q = q_S$ by the quadratic form defined by S and consider

$$\{(q(x), x); x \in X\} \subset \mathbb{X}_{\mathbb{R}} := \mathbb{R} \oplus X_{\mathbb{R}}.$$

This is a multifaceted paraboloid and the projection of the facets to $X_{\mathbb{R}}$ gives precisely the Delaunay cells.

Example $X = \mathbb{Z}$; $q_S = x^2$

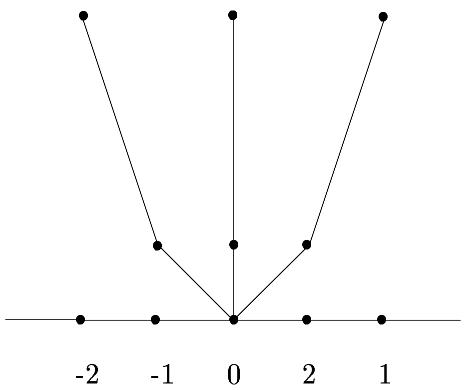
The following pictures shows how one obtains the unique Delaunay decomposition of $\mathbb{R}$.



Definition Let X be a lattice and $Y \subset X$ a sublattice of finite index. A cell decomposition Δ of $X_{\mathbb{R}}$ is called a *semi-Delaunay decomposition* with respect to (Y, X) if there exists a positive-definite symmetric bilinear function S on $X \times X$ and elements $r(\bar{x}) \in \mathbb{R}; \bar{x} \in X/Y$ such that Δ is the projection onto $X_{\mathbb{R}}$ of the lower envelope of the set

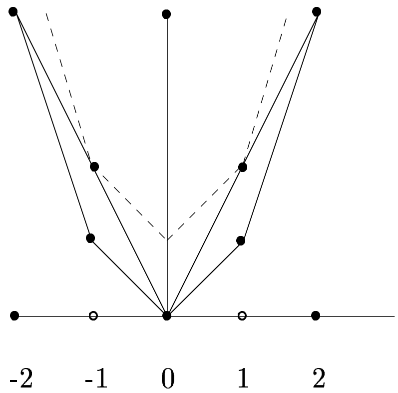
$$\{(q(x) + r(\bar{x}); x), x \in X\} \subset \mathbb{X}_{\mathbb{R}} = \mathbb{R} \oplus X_{\mathbb{R}}.$$

Example (1) $X = \mathbb{Z}, Y = 2\mathbb{Z};\ q(x) = x^2; r(\bar{0}) = r(\bar{1}) = 0.$



Here we obtain intervals of length 1.

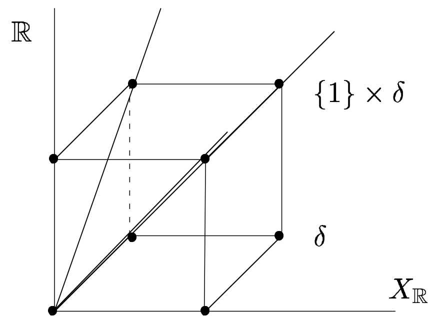
(2) $X = \mathbb{Z}; Y = 2\mathbb{Z}, q(x) = x^2; r(\bar{0}) = 0, r(\bar{1}) = 1$
In this case we obtain intervals of length 2.



More remarks on toric geometry

Let $\delta \subset X_{\mathbb{R}}$ be a cell. Then we consider the shift

$$\{1\} \times \delta \subset \mathbb{X}_{\mathbb{R}} = \mathbb{R} \oplus X_{\mathbb{R}}.$$



We take the cone over this set with vertex 0. In this way we obtain the semi-group

$$S_\delta := \mathbb{Z} \oplus X \cap \underbrace{(\text{Cone over}\{1\} \times \delta)}_{\substack{\| \\ \text{Cone } \delta}}$$

The elements of S_δ are of the form $\chi = (d, x); d \in \mathbb{Z}_{\geq 0}, x \in X$. The ring

$$R_\delta := \mathbb{C}[S_\delta] = \mathbb{C}[\zeta_x; x \in S_\delta]$$

is a $\mathbb{C}$-algebra. We can consider R_δ as a *graded*†$\mathbb{C}$-algebra via

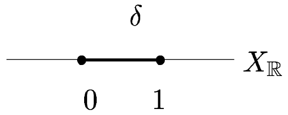
$$\deg(\chi) := \deg((d, x)) := d.$$

Let

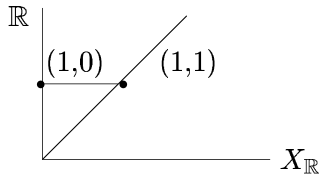
$$P_\delta := \text{Proj } R_\delta.$$

This is a projective variety and as such it carries a natural polarization $\mathcal{O}_{P_\delta}(1)$.

Examples (1) We start with the interval in $\mathbb{R}$ of length 1.

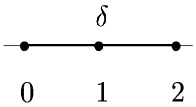


The cone over δ looks as shown below.



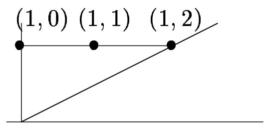
$$R_\delta = \mathbb{C}[x, y];\;\; x \hat{=} (1,0), y \hat{=} (0,1),$$
$$P_\delta = \text{Proj}(\mathbb{C}[x, y]) \cong \mathbb{P}^1,$$
$$\mathcal{O}_{P_\delta}(1) = \mathcal{O}_{\mathbb{P}^1}(1).$$

(2) Next we consider the interval of length 2.



R_δ has 3 generators, namely:
$\zeta_{(1,0)}, \zeta_{(1,1)}, \zeta_{(1,2)}$ $(\hat{=} x_0, x_1, x_2)$

The cone over δ is now given by

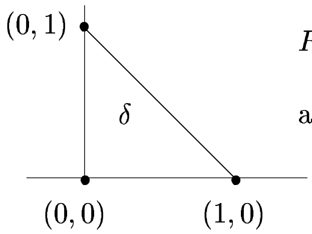


Since $(1,0) + (1,2) = 2(1,1)$
we have the relation $x_0 x_2 = x_1^2$.

In this case we find

$$P_\delta = \text{Proj } (\mathbb{C}[x_0, x_1, x_2]/(x_1^2 - x_0 x_2)) \cong (\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2)).$$

(3) The standard simplex in $\mathbb{R}^2$ gives the projective plane.



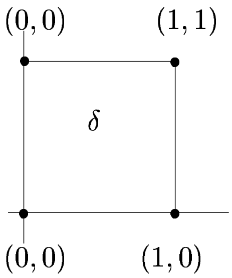
R_δ has 3 generators : $\zeta_{(1,(0,0))}, \quad \zeta_{(1,(1,0))}, \quad \zeta_{(1,(0,1))}$

$$\begin{array}{ccc} \zeta_{(1,(0,0))}, & \zeta_{(1,(1,0))}, & \zeta_{(1,(0,1))} \\ \| & \| & \|\dagger \\ x_0 & x_1 & x_2 \end{array}$$

and no relations.

$$P_\delta = \text{Proj } \mathbb{C}[x_0, x_1, x_2] \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)).$$

(4) The standard square in $\mathbb{R}^2$ gives the quadric.



R_δ has 4 generators:

$$\begin{array}{cccc} \zeta_{(1,(0,0))}, & \zeta_{(1,(1,0))}, & \zeta_{(1,(0,1))}, & \zeta_{(1,(1,1))} \\ \| & \| & \|\dagger & \| \\ x_0 & x_1 & x_2 & x_3 \end{array}$$

We have one relation:

$$(1, (0, 0)) + (1, (1, 1)) = (1, (0, 1)) + (1, (1, 0)).$$

Hence we obtain here a quadric:

$$P_\delta = \text{Proj } (\mathbb{C}[x_0, x_1, x_2, x_3]/(x_0x_3 - x_1x_2)) \cong (\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1)).$$

VII Alexeev's construction

In this section we describe a *set of degeneration data* from which one can construct the central fibre of a degeneration of abelian varieties:

(0) A *polarized abelian variety* $(A, \mathcal{L})$ where $\mathcal{L}$ represents a polarization $\lambda : A \to \widehat{A} = \text{Pic}^0(A)$ of type $\underline{d} = (d_1, \ldots, d_g)$.

(1) (i) A *semi-abelian variety*

$$1 \to T \to G \to A \to 1$$

given by a morphism $c : X \to \widehat{A}$ where $X \cong \mathbb{Z}^{g''}$ is the character group of the torus $T \cong (\mathbb{C}^*)^{g''}$.

(ii) A second *semi-abelian variety*

$$1 \to {}^tT \to {}^tG \to \widehat{A} \to 1$$

given by a homomorphism ${}^tc : Y \to A$ where Y is the character group of the torus ${}^tT$.

(2) In *inclusion* $\phi : Y \to X$ of lattices with finite cokernel such that $c \circ \phi = \lambda \circ {}^tc$.

(3) A *cell decomposition* Δ of $X_{\mathbb{R}} = X \otimes \mathbb{R}$ which is periodic with respect to Y and which has only finitely many cells modulo Y.

(4) Let $\mathcal{P}$ be the (birigidified) *PoincarÈ bundle.* Then we are given a *symmetric trivialization*

$$\tau_{Y \times X} : 1_{Y \times X} \longrightarrow ({}^tc \times c)^* \mathcal{P}^{-1}_{A \times \widehat{A}}$$

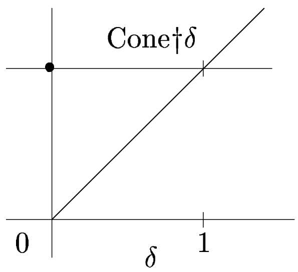
of $\mathbb{C}^*$-biextensions.

(Birigidified means that we have chosen isomorphisms $\mathcal{P}_{|A \times \{\mathcal{L}\}} \cong \mathcal{L}$ and $\mathcal{P}_{|\{0\}} \times \widehat{A} \cong \mathcal{O}_{\widehat{A}}$ and symmetric means that $\tau|_{Y \times Y}$ is symmetric.)

The construction

Let δ be a cell of Δ. We consider the cone

$$\text{Cone } \delta \subset \mathbb{X}_{\mathbb{R}} = \mathbb{R} \oplus X_{\mathbb{R}}.$$



To each $\chi = (d, x) \in \mathbb{X} = \mathbb{Z} \oplus X$ we associate the line bundle

$$\mathcal{M}_\chi := \mathcal{L}^{\otimes d} \otimes c(x) = \mathcal{L}^{\otimes d} \otimes \mathcal{O}_x.$$

We shall assume that we have a rigidification of $\mathcal{L}$. Then $\mathcal{M}_\chi$ also has a natural rigidification at the origin and we have a canonical isomorphism of rigidified sheaves

$$\mathcal{M}_{\chi_1 + \chi_2} \cong \mathcal{M}_{\chi_1} \otimes \mathcal{M}_{\chi_2}.$$

All these sheaves are $\mathcal{O}_A$-algebras. Recall the semi-group

$$S_\delta = \text{Cone } \delta \cap (\mathbb{Z} \oplus X).$$

We then consider the scheme

$$\widetilde{V}_\delta = \text{Proj}_{\mathcal{O}_A} \mathbb{C}[\mathcal{M}_\chi; \chi \in S_\delta].$$

This is a scheme over A, i.e. it comes with a natural projection onto A. We now want to glue the varieties $\widetilde{V}_{\delta_1}$ and $\widetilde{V}_{\delta_2}$ along $\widetilde{V}_{\delta_1 \cap \delta_2}$. In order to do this, we introduce formal variables

$$\zeta_\chi = \mathcal{M}_\chi.$$

Let

$$\text{Cone } \Delta := \bigcup_{\delta \in \Delta} \text{Cone } \delta = \{(d, x) \in \mathbb{Z} \oplus X; d > 0\} \cup \{(0, 0)\}.$$

Then we define a semi-group algebra

$$R := \mathbb{C}[\zeta_\chi; \chi \in \text{ Cone } \Delta]$$

where the multiplication is defined as follows

$$\zeta_{\chi_1} \cdot \zeta_{\chi_2} := \begin{cases} \zeta_{\chi_1 + \chi_2} & \text{if } \chi_1, \chi_2 \text{ are cellmates (i.e. belong} \\ & \text{to a common cell)} \\ 0 & \text{otherwise.} \end{cases}$$

We define

$$(\widetilde{V}, \mathcal{O}_{\widetilde{V}}(1)) = (\text{Proj } R, \mathcal{O}_{\text{Proj}(R)}(1)).$$

We now want to define an action of Y on $(\widetilde{V}, \mathcal{O}_{\widetilde{V}}(1))$. We first notice that Y acts on A via ${}^t c$. For each $y \in Y$ we have to define an isomorphism

$$R \overset{\sim}{\to} T^*_{{}^t c(y)} R.$$

This is the same as giving a section

$$\varepsilon : Y \times \mathbb{Z} \times X \to \bigoplus_{d \in \mathbb{Z}} {}^t c^* \mathcal{L}^{-d} \otimes ({}^t c \times c)^* \mathcal{P}^{-1}_{A \times \hat{A}}.$$

This map ε can be described as follows:

(i) $\varepsilon|_{Y \times \{0\} \times X} := \tau_{Y \times X}$.

(ii) $\varepsilon|_{Y \times \{1\} \times \{0\}} := \psi : 1_Y \to {}^t c^* \mathcal{L}^{-1}$ where ψ is a cubical trivialization.

(iii) $\varepsilon(y, d, x) := \psi(y)^d \tau(y, x)$.

Here ψ can be chosen arbitrarily, the find result will not depend on the choice of ψ.

The group Y acts properly discontinuously on $\widetilde{V}$. We set

$$\boxed{(V, \mathcal{O}_V(1)) := (\widetilde{V}, \mathcal{O}_{\widetilde{V}}(1))/Y}$$

Properties of $(V, \mathcal{O}_V(1))$:

(1) V is projective, $\mathcal{O}_V(1)$ is an ample line bundle, $\dim V = g' + g''$.

(2) V is semi-normal.

(3) G acts on V with a finite number of orbits. The stabilizer G_x of every point $x \in V$ is reduced, connected and lies in the toric part of G.

Definition A variety $(V, \mathcal{O}_V(1))$ with properties (1)-(3) is called a *(polarized) stable semi-abelic variety* (PSSAV).

Remark

(i) $h^0(\mathcal{O}_V(1)) = d_1 \dots d_g \cdot |X/Y|$.

(ii) Let $x_0 \in V$ be a point in a maximal dimensional stratum of V. Then $G \to V, g \mapsto gx_0$ defines an immersion of G into V.

Theorem VII.0.3 (Alexeev) *The following holds*

(i) *The variety $(V, \mathcal{O}_V(1)))$ deforms to a smooth abelian variety if and only if the cell decomposition Δ of $X_{\mathbb{R}}$ is a semi-Delaunay decomposition with respect to (Y, X).*

(ii) *The type of the polarization is given by the elementary divisors of the group $H(\lambda) \times X/Y$ where $H(\lambda)$ is a totally isotropic subgroup of $K(\lambda)$. In particular $(V, \mathcal{O}_V(1))$ deforms to a p.p.a.v. if and only if $Y = X$ and Δ is a Delaunay decomposition.*

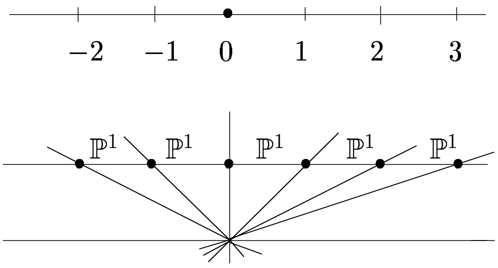
Remark Assume that Δ is a Delaunay decomposition with respect to a form S. Let S_0 be a positive definite symmetric bilinear form which defines the same Delaunay decomposition. Then $\tau(x, y)$ and

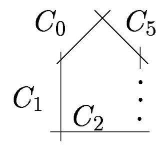
$$\tau_0(x, y) = \tau(x, y)c^{S_0(x,y)} \quad (c \in \mathbb{C}^*)$$

define the same variety V.

Example (1) We first study degenerations of elliptic curves.
(a) The standard degeneration of an elliptic curve with a degree n divisor is a fibre of type I_n. This arises as follows:

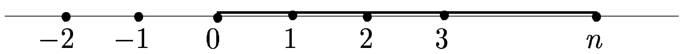
$$A = \{1\}, G = {}^tG = \mathbb{C}^*; \; X = \mathbb{Z}, \; Y = n\mathbb{Z} \subset \mathbb{Z}.$$

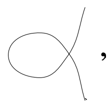


$V =$ 

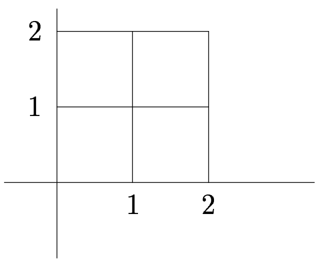
$\deg \mathcal{O}_V(1)|_{C_i} = 1$

(b) In order to obtain a nodal curve with a degree n polarization we take an interval of length n:



$V =$  , $\deg \mathcal{O}_V(1) = n.$

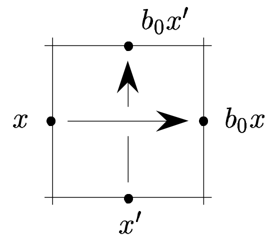
(2) Here we list all degenerations of principally polarized abelian surfaces with trivial abelian part.
† (a) $A = \{1\};\ G = {}^tG = (\mathbb{C}^*)^2;\ X = Y = \mathbb{Z}^2$
The standard form $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ defines the Delaunay decomposition of $\mathbb{R}^2$ into planes.



$$\tau = b : \mathbb{Z}^2 \times \mathbb{Z}^2 \to \mathbb{C}^*,$$
$$b_0 := b(e_1, e_2) \in \mathbb{C}^*.$$

The result is a $\mathbb{P}^1 \times \mathbb{P}$ with opposite sides glued

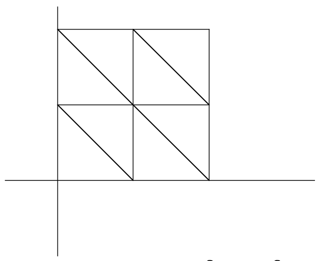
$$V = \mathbb{P}^1 \times \mathbb{P}^1/_\sim$$



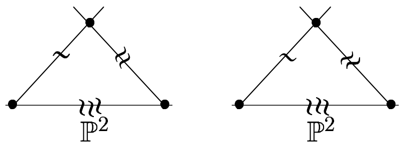
(b) Again we take

$$A = \{1\}; \quad G = {}^tG = (\mathbb{C}^*)^2; \quad X = Y = \mathbb{Z}^2; \quad \tau \text{ arbitrary.}$$

The form $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ defines the Delaunay decomposition of $\mathbb{R}^2$ into triangles. The result is a union of two projective planes glued as shown in the picture below. Here lines marked with the same symbol are identified as well as



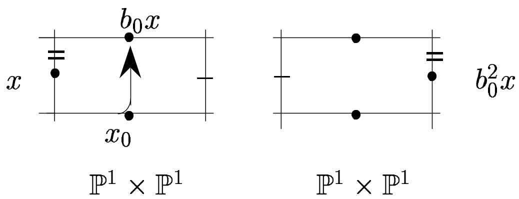
$$V = \mathbb{P}^2 \coprod \mathbb{P}^2 / \sim$$



all the points marked. The restriction of $\mathcal{O}_V(1)$ to each plane is $\mathcal{O}_{\mathbb{P}^2}(1)$.

(c) To obtain a degeneration with a non-principal polarization we can take a proper sublattice Y of X, e.g.

$$X = \mathbb{Z}^2; \; Y = \mathbb{Z}2e_1 + \mathbb{Z}e_2$$



We obtain two quadrics with identifications as indicated in the picture above. Here again $b_0 = b(e_1, e_2)$.

(3) Next we describe degenerations of principally polarized abelian surfaces with a 1-dimensional abelian part.

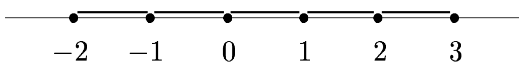
$$A = E_{[\tau_3]} = \mathbb{C}/(\mathbb{Z}\tau_3 + \mathbb{Z}); X = Y = \mathbb{Z};$$

$$\begin{array}{rccl} c = {}^t c: & \mathbb{Z} & \to & \operatorname{Pic}^0 E_{[\tau_3]} = E_{[\tau_3]} \\ & 1 & \mapsto & [\tau_2] =: e. \end{array}$$

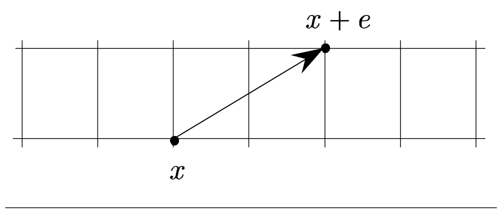
Then

$$\mathcal{O}(1) = \mathcal{O}_{E_{[\tau_2]}}([\tau_3] - 0) =: \mathcal{M}.$$

We choose the standard decomposition of $X_{\mathbb{R}} = X \otimes_{\mathbb{Z}} \mathbb{R}$, namely



The bilinear form τ can be chosen arbitrarily. In this case $G = {}^tG$ is a $\mathbb{C}^*$-bundle. It is the line bundle $\mathcal{M}$ with the 0-section removed.



$$G = {}^tG \subset \mathbb{P}(\mathcal{O} \oplus \mathcal{M}).$$

E

Adding the 0-section and the section at infinity one obtains the $\mathbb{P}^1$-bundle $\mathbb{P}(\mathcal{O} \oplus \mathcal{M})$. Then

$$V = \mathbb{P}(\mathcal{O} \oplus \mathcal{M})/_{\sim}$$

where $x \sim x + [\tau_2]$. Note that the shift e equals $\mathcal{M}$ under the natural identification $E_{[\tau_3]} \cong \operatorname{Pic}^0 E_{[\tau_3]}$. This degeneration corresponds to $\begin{pmatrix} \to i\infty & \tau_2 \\ \tau_2 & \tau_3 \end{pmatrix}$.

(4) Finally we consider non-principal degenerations. Take $A = E_{[\tau_3]}$ as above. On A we consider the polarization $\lambda : E_{[\tau_3]} \to \hat{E}_{[\tau_3]}, x \mapsto 2x$. We set $Y = 2\mathbb{Z} \subset X = \mathbb{Z}$. Then we have

$$\begin{array}{ccccccc} 2 & \in & 2\mathbb{Z} & \subset & \mathbb{Z} & \ni & 1 \\ \downarrow & & \downarrow {\scriptstyle {}^tc} & & \downarrow {\scriptstyle c} & & \downarrow \\ p_0 & \in & E_{[\tau_3]} & \xrightarrow{\cdot 2} & E_{[\tau_3]} & \ni & q_0. \end{array}$$

Note that

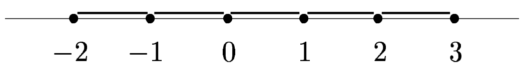
$$2p_0 = 2q_0.$$

Let

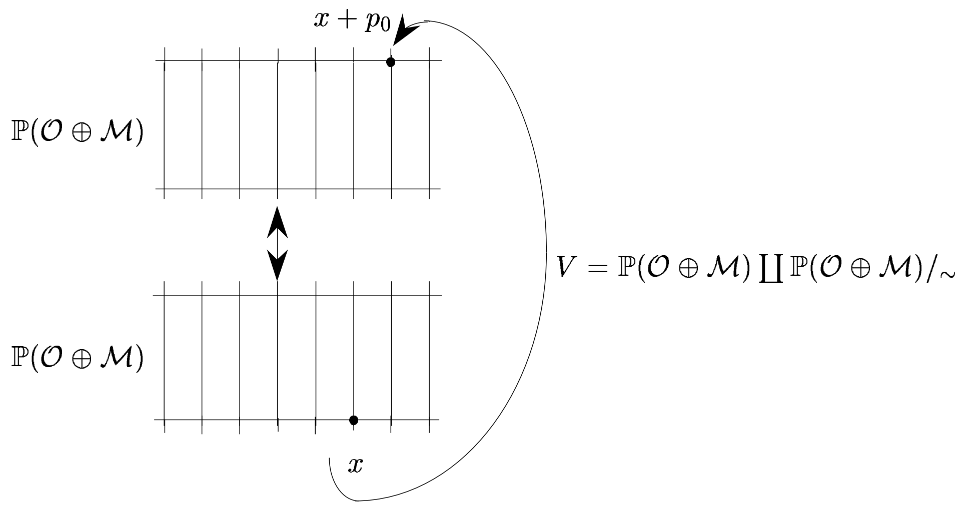
$$\mathcal{M} = \mathcal{O}_{E_{[\tau 3]}}(p_0 - 0).$$

Then G is the $\mathbb{C}^*$-bundle given by removing the 0-section from $\mathcal{M}$.

(a) We first choose Δ to be the decomposition of the line $\mathbb{R}$ into intervals of length 1:

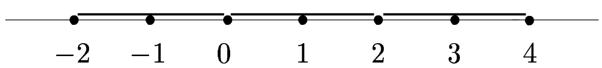


Then we obtain



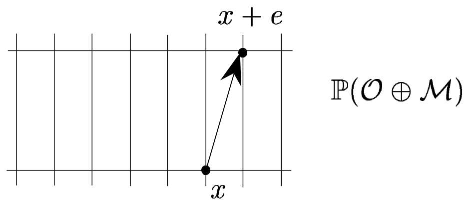
The polarization has degree 1 on the fibres of the ruled surfaces and degree 2 on the sections.

(b) Now let Δ be given by intervals of length 2:

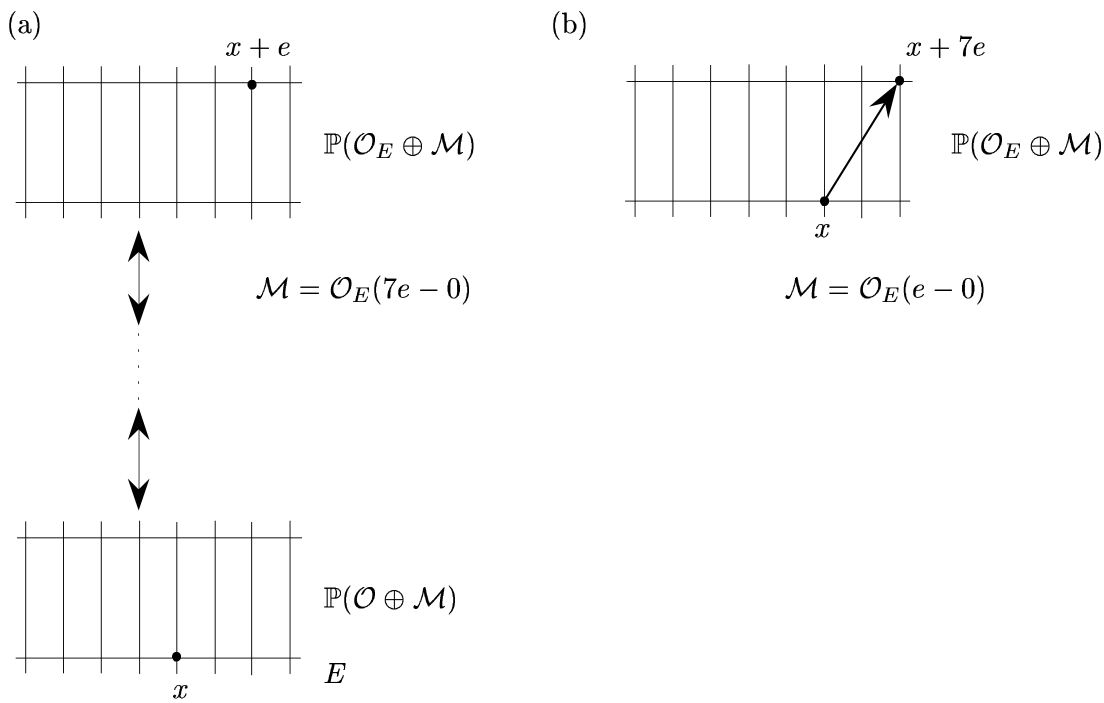


Then one obtains
We obtain

$$V = \mathbb{P}(\mathcal{O} \oplus \mathcal{M})/_\sim.$$

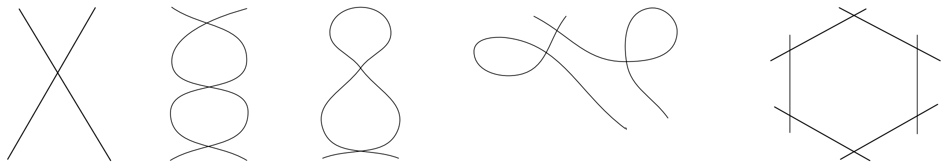


The degree of the polarization is 2, both on the fibres of the ruled surface and on the sections. In the case $p_0 = q_0$ this is the same variety as in (3) with the square of the polarization.
(5) In the $(1, 7)$-case we obtain the following possibilities



VIII The case of Jacobians

In this chapter I want to explain (following [A1]) how one can compactify the Jacobian $J(C)$ of a nodal curve C. Let C be a nodal curve, i.e. a connected projective curve with a most nodes as singularities and let $\nu : N \to C$ be the normalization of C.



The *Jacobian* of C is defined by

$$J(C) = \{\mathcal{L} \in \text{Pic } C;\ \mathcal{L} \text{ is algebraically equivalent to } \mathcal{O}_C\}.$$

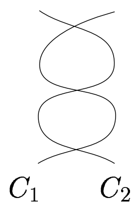
By pulling back to the normalization N we obtain an exact sequence

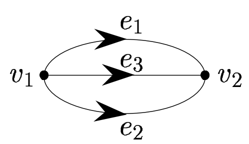
$$(*) \qquad 1 \to (\mathbb{C}^*)^r \to J(C) \to J(N) \to 1.$$

This shows that $J(C)$ is a semi-abelian variety. We first want to understand this extension.

Given a nodal curve C we associate to it its *dual graph* $\Gamma(C)$. The components of C correspond to the vertices of $\Gamma(C)$ and the nodes of C correspond to the edges of $\Gamma(C)$.

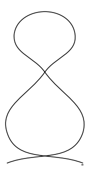
Examples (1)

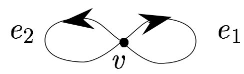




$\Gamma(C)$

(2)





$\Gamma(C)$

We set

$$C_0(\Gamma, \mathbb{Z}) = \bigoplus_{i \in I} \mathbb{Z} v_i \quad , \quad C_1(\Gamma, \mathbb{Z}) = \bigoplus_{j \in J} \mathbb{Z} e_j.$$

Here I parametrizes the components of C and J parametrizes the nodes of the curve C.

Now we choose an *orientation* on $\Gamma(C)$ (it will not matter which one). This allows us to define a *boundary operator*

$$\begin{array}{lcll} \partial : C_1(\Gamma,\mathbb{Z}) & \to & C_0(\Gamma,\mathbb{Z}) & \\ \partial e_j & = & v_i - v_i', & \text{if } e_j \text{ goes from } v_i \text{ to } v_i'. \end{array}$$

We define

$$H_1(\Gamma,\mathbb{Z}) := \ker \partial.$$

Examples In the above example we find

$$\begin{array}{lrcl} (1) & \Gamma(C) & = & \mathbb{Z}e_1 + \mathbb{Z}e_2 + \mathbb{Z}e_3, \\ & H_1(\Gamma,\mathbb{Z}) & = & \mathbb{Z}(e_1 - e_3) + \mathbb{Z}(e_2 - e_3). \end{array}$$

$$(2) \qquad \Gamma(C) = \mathbb{Z}e_1 + \mathbb{Z}e_2 = H_1(\Gamma,\mathbb{Z}).$$

Every edge e_j determines two points P_j^+, P_j^- on the normalization N. If e_j is not a loop, then we choose P_j^+ on C_i and P_j^- and $C_{i'}$, if e_j goes from v_i to $v_{i'}$. Otherwise we choose P_j^+ and P_j^- arbitrarily. In this way we obtain a homomorphism

$$\begin{array}{rcl} c : C_1(\Gamma,\mathbb{Z}) & \to & \operatorname{Pic}(N) \\ e_j & \mapsto & \mathcal{O}_N(P_j^+ - P_j^-). \end{array}$$

This restricts to a homomorphism

$$c : \quad H_1(\Gamma,\mathbb{Z}) \to \operatorname{Pic}^0(N) = J(N) = \widehat{J(N)}.$$

By our general theory of semi-abelian varieties this defines an extension

$$1 \to H^1(\Gamma,\mathbb{Z}) \otimes \mathbb{C}^* \to G \to J(N) \to 1.$$

This extension can be identified with $(*)$.

Set of data for the compactified Jacobian

(J0) The underlieing abelian variety is

$$A = J(N); \ \lambda_A = c_1(\mathcal{O}_{J(N)}(\Theta))$$

where Θ is the theta divisor.

(J1) (i) $G = J(C)$, i.e.

$$1 \to H^1(\Gamma,\mathbb{Z}) \otimes \mathbb{C}^* \to G = J(C) \to J(N) = \widehat{J(N)} \to 1$$

is given by $c : X := H_1(\Gamma, \dagger\mathbb{Z}) \to \widehat{J(N)}$.

(ii) ${}^tG = G$.

(J2) $Y = X$; $\Phi =$ id : $Y = X \to X$.

(J3) We consider the standard form S on $C_1(\Gamma, \mathbb{R})$ given by

$$S(e_i, e_j) = \delta_{ij}.$$

The corresponding Delaunay decomposition Del of $C_1(\Gamma, \mathbb{R})$ is that into standard cubes. We intersect this with $H_1(\Gamma, \mathbb{R})$, i.e.

$$\Delta = \text{Del } \cap H_1(\Gamma, \mathbb{R}).$$

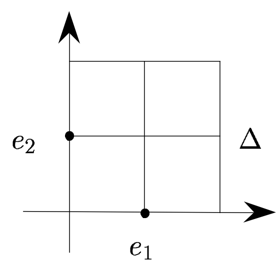
More precisely, we mean the following

$$\Delta = \{\delta \cap H_1(\Gamma, \mathbb{R}); \delta \in \Delta; \text{ rel. int. } \delta \cap H_1(\Gamma, \mathbb{R}) \neq \emptyset\}.$$

Examples In the examples given above we find

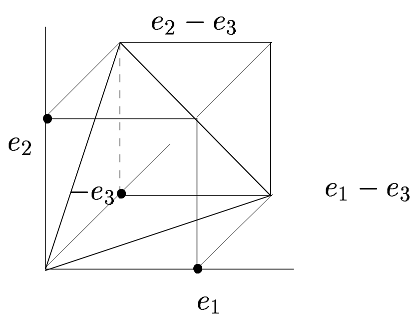
(2)

$$H_1(\Gamma, \mathbb{Z}) = C_1(\Gamma, \mathbb{Z})$$

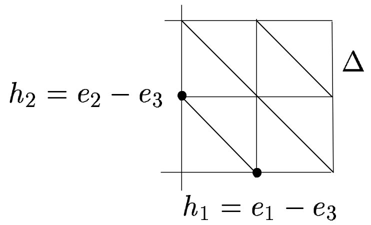


(1)

$$H_1(\Gamma, \mathbb{Z}) = \mathbb{Z}(e_1 - e_3) \oplus \mathbb{Z}(e_2 - e_3)$$



This gives the following picture



$$H_1(\Gamma, \mathbb{Z}) = \mathbb{Z}h_1 + \mathbb{Z}h_2.$$

Alternatively, we can restrict the form S from $C_1(\Gamma, \mathbb{Z})$ to $H_1(\Gamma, \mathbb{Z})$ and then take the associated Delaunay decomposition of $H_1(\Gamma, \mathbb{R})$. In particular, Δ is a Delaunay decomposition.

Before I can define the fourth datum $\tau_{X \times X}$ it is necessary to say something about the *Deligne pairing*. For this purpose let B be an arbitrary *smooth* curve (we shall later put $B = N$). Let f, g be two rational functions on B with $\mathrm{div}(f) \cap \mathrm{div}(g) = \emptyset$. One then defines

$$(f, g) = f(\mathrm{div}(g)) = \prod_{x \in B} f(x)^{v_x(g)}.$$

Here $v_x(g)$ is the valuation of g at x and one has $(f, g) = (g, f)$ by a result of Weil.

Example Let $B = \mathbb{P}^1$ be a rational curve,

$$f(z) = \tfrac{z-a}{z-b} \quad ; \quad g(z) = \tfrac{z-c}{z-d}.$$

Then

$$(f, g) = \frac{c-a}{c-b} \cdot \frac{d-b}{d-a} = \frac{a-c}{a-d} \cdot \frac{b-d}{b-c} = (g, f).$$

This is the classical *cross ratio.*

Now let $f \in H^0(L), g \in H^0(M)$ be two sections of line bundles L and M with $\mathrm{div}(f) \cap \mathrm{div}(g) = \emptyset$. Then we can still define

$$(f, g) := f(\mathrm{div}(g)) = \prod_{x \in B} f(x)^{v_x(g)} \in \bigotimes_{x \in B} L_x^{\otimes v_x(g)} =: (L, M)_{f,g}.$$

One can then show that

$$(f, g) = (-1)^{\deg L \deg M}(g, f).$$

One also shows that for other sections f', g' one has a natural identification

$$(L, M)_{f,g} = (L, M)_{f',g'} =: (L, M).$$

When L, M vary in $\mathrm{Pic}^0 B$ one obtains in this way a line bundle on $\mathrm{Pic}^0 B \times \mathrm{Pic}^0 B$ which is the inverse of the Poincaré-bundle.

Now we return to the case of Jacobians. Let $e_k, e_l \in C_1(\Gamma, \dagger\mathbb{Z})$. If $k \neq l$ then $P_k^+ - P_k^-$ and $P_l^+ - P_l^-$ have disjoint support and the Deligne pairing defines an element

$$(e_k, e_l) \in V_{kl} (= \text{ suitable vector space of dimension } 1).$$

If $k = l$, then we still have the vector space V_{kk}, but no elements (e_k, e_k). We choose$(e_k, e_k) \in V_{kk}$ arbitrarily. This defines by linearity a map

$$(\ ,\ ) : C_1(\Gamma, \mathbb{Z}) \times C_1(\Gamma, \mathbb{Z}) \to \bigoplus_{k,l} V_{kl}.$$

Restricting this to $H_1(\Gamma, \mathbb{Z})$ and identifying the spaces V_{kl} with the stalks of the inverse of the Poincaré-bundle we obtain a pairing

$$(\ ,\ )\ H_1(\Gamma, \mathbb{Z}) \times H_1(\Gamma, \mathbb{Z}) \to \bigoplus_{x,y \in X} \mathcal{P}^{-1}_{({}^t c(x), c(y))}$$

where $\mathcal{P}$ is the Poincaré-bundle on $J(N) \times \widehat{J(N)}$ and

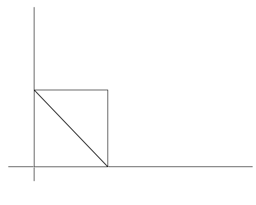
$${}^t c = c : X \to J(N) = \widehat{J(N)}.$$

(J4) The symmetric trivialization is given by

$$\begin{array}{rcl} \tau_{X \times X} :\ 1_{X \times X} \dagger & \dagger \to & ({}^t c \times c)^* \mathcal{P}^{-1} \\ \tau_{X \times X}(x, y) & = & (x, y). \end{array}$$

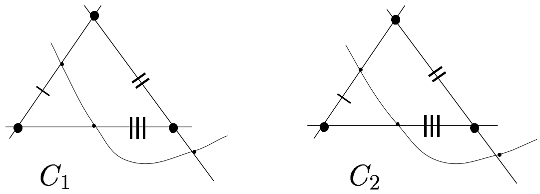
These data define a polarized semi-abelic variety $\overline{J(C)}$, the compactified Jacobian of C.

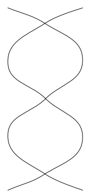
Examples (1) Here we have



and this gives

$$\overline{J(C)} = \mathbb{P}^2 \coprod \mathbb{P}^2 /_{\sim}.$$



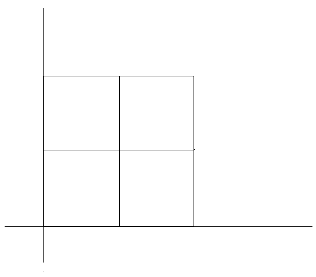


$$\subset \overline{J(C)}.$$

$$C = C_1 + C_2.$$

The form $\tau_{X \times X}$ plays no role here.

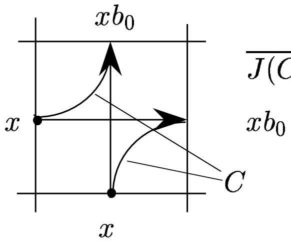
(2) Here we have



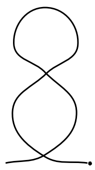
The form $\tau_{X \times X}$ is given by

$$\tau_{X \times X} = b : \mathbb{Z}^2 \times \mathbb{Z}^2 \to \mathbb{C}^*$$
$$b = \begin{pmatrix} 1 & b_0 \\ b_0 & 1 \end{pmatrix}.$$

(We can choose $(e_1, e_1), (e_2, e_2)$ arbitrarily). We obtain

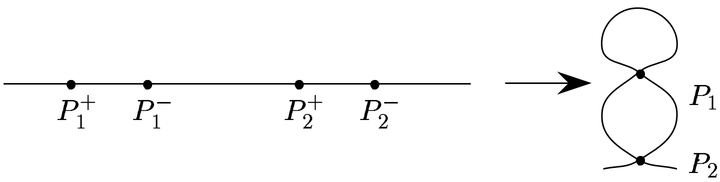


$$\overline{J(C)} = \mathbb{P}^1 \times \mathbb{P}^1/_\sim$$

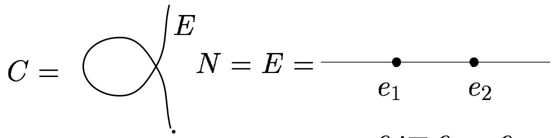


$$C \subset \overline{J(C)}.$$

The parameter b_0 is the cross ratio of the four points in $\mathbb{P}^1 = N$ lieing over the 2 nodes of C.

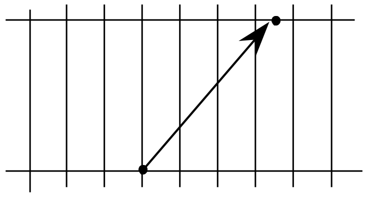


(3) Now let E be an elliptic curve.



$$e := e_1 - e_2$$

We then obtain



$$J(C) = X/_{\sim},\ X = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(e-0)).$$

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