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Master's thesis

Density of Noether–Lefschetz loci and rationality of quadric surface bundles

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Contents

1 Introduction

A classical problem in algebraic geometry asks whether a given variety X is *rational*, that is, whether it admits a birational map $X \dashrightarrow \mathbb{P}^n$ for some $n \geq 0$. Commonly, one also introduces the weaker notion of X being *stably rational*, which means that $X \times \mathbb{P}^r$ is rational for some $r \geq 0$. In general, very little is known about the class of rational or stably rational varieties.

In this thesis, we will consider *quadric surface bundles over* $\mathbb{P}^2$, i. e. projective varieties together with a flat morphism to $\mathbb{P}^2$ such that the generic fibre is a smooth quadric surface. Unless otherwise stated, we always work over the field of complex numbers. For these varieties, the rationality problem is a lot easier, since quadric surfaces are rational if and only if they have a rational point.

The behaviour of rationality in families turns out to be quite interesting in this case: In many natural families of quadric surface bundles, as we will prove, the locus of rational members is dense, while on the other hand it is known for these families that the locus of stably irrational members is dense as well.

A simple example for a quadric surface bundle is a smooth hypersurface in $\mathbb{P}^2 \times \mathbb{P}^3$ defined by a homogeneous polynomial of bidegree $(d, 2)$ for some integer $d \geq 0$. Projection to the first factor gives it the quadric bundle structure over $\mathbb{P}^2$. Apart from that, many interesting fourfolds arising in algebraic geometry are birational to quadric surface bundles over $\mathbb{P}^2$, for example

(i) a cubic fourfold containing a plane,

(ii) more generally, a hypersurface in $\mathbb{P}^5$ of degree $d + 2$ with multiplicity d along a plane for some integer $d \geq 1$ (see e. g. [Sch18a, Lemma 23]),

(iii) a double cover of $\mathbb{P}^4$ ramified in a quartic threefold singular along a line,

(iv) more generally, a double cover of $\mathbb{P}^4$ branched along a hypersurface in $\mathbb{P}^4$ of degree $d + 2$ with multiplicity d along a line for some even integer $d \geq 2$ (see e. g. [Sch18a, Lemma 24]),

(v) a smooth complete intersection of three quadrics in $\mathbb{P}^7$ (see e. g. [Bea77, Exemple 1.4.4]).

Recently, a lot of progress was made in the rationality problem for fourfolds by showing that in all examples from above except (i), a *very general* member is not stably rational. By this, one means the following: One parametrizes all fourfolds of a certain type by a smooth variety B and obtains a universal family $f\colon \mathcal{X} \to B$. Usually, B is a Zariski open subset in a high-dimensional projective space consisting of the possible defining equations for the examined varieties. We denote the fibre $f^{-1}(\{b\})$ at a point $b \in B$ by $\mathcal{X}_b$. Now saying that a very general member of the family $\mathcal{X} \to B$ is not stably rational means that the set

$$\{b \in B \mid \mathcal{X}_b \text{ is stably rational}\}$$

is contained in a countable union of proper closed subvarieties of B.

This result was proven for smooth hypersurfaces in $\mathbb{P}^2 \times \mathbb{P}^3$ of bidegree $(2, 2)$ by Hassett, Pirutka, and Tschinkel [HPT18a]. They also settled the case of quartic double fourfolds [HPT18b] and of complete intersections of quadrics [HPT17]. In order to prove stable irrationality of a very general member, they used the so called *specialization method* of Voisin [Voi15b] and Colliot-Thélène–Pirutka [CTP16], which also allowed to disprove rationality in several other families, see e. g. [Pir18] or [Voi18] for an overview.

Using his improvement [Sch18a] of the specialization method, Schreieder proved in [Sch18b] that a very general quadric surface bundle is not stably rational for a large class of families of quadric surface bundles over $\mathbb{P}^2$. His result includes all the remaining examples from above except (i) as special cases.

To describe the families studied in [Sch18b], we introduce the notion of *standard quadric surface bundles*. These underlie a general construction to obtain quadric surface bundles and arise in a natural way, albeit not all quadric surface bundles over $\mathbb{P}^2$ are standard ones (from the examples listed above, only item (v) is not). A standard quadric surface bundle over $\mathbb{P}^2$ is given by an equation of the form

$$\sum_{0 \le i,j \le 3} a_{ij} y_i y_j = 0 \,, \tag{1.1}$$

where $a_{ij} = a_{ji}$ is a homogeneous polynomial of degree $\frac{1}{2}(d_i + d_j)$ in the three coordinates of $\mathbb{P}^2$ for integers $d_0, d_1, d_2, d_3 \ge 0$ of the same parity. Here, y_0, y_1, y_2, y_3 denote local trivializations of a certain vector bundle $\mathcal{E}$ on $\mathbb{P}^2$ of rank 4, the details of which will be explained later in Section 4.1. We then say that the quadric surface bundle $X \subset \mathbb{P}(\mathcal{E})$ defined by equation (1.1) is of type (d_0, d_1, d_2, d_3).

With this notion, a hypersurface in $\mathbb{P}^2 \times \mathbb{P}^3$ of bidegree $(d, 2)$ for $d \ge 0$ is a standard quadric surface bundle of type (d, d, d, d). Further, the examples (ii) and (iv) from above are birational to quadric surface bundles of type $(d, d, d, d + 2)$ and $(0, d, d, d + 2)$, respectively.

Now, Schreieder has shown in [Sch18b] that a very general quadric surface bundle of type (d_0, d_1, d_2, d_3) is not stably rational except for the two cases $(1, 1, 1, 3)$ and $(0, 2, 2, 2)$ (up to reordering) which remain open and for trivial cases where the quadric surface bundle always has a rational section and is hence rational. Thus, the irrationality results of [HPT18a] and [HPT18b] can be generalized to almost any standard quadric surface bundle.

Hassett, Pirutka, and Tschinkel also proved in [HPT18a] for their family $\mathcal{X} \to B$ of hypersurfaces in $\mathbb{P}^2 \times \mathbb{P}^3$ of bidegree $(2,2)$ that the locus

$$\{b \in B \mid \mathcal{X}_b \text{ is rational}\} ,$$

while being contained in a countable union of proper closed subvarieties of B, is at the same time dense in B for the Euclidean topology. This is a striking result since it shows that rationality of the fibres is in general not a closed property on the base. In particular, rationality is not deformation invariant among smooth families, which was an open question before.

Subsequently, further examples of smooth families containing both rational and stably irrational fibres were identified, for example in [HPT18b], [HPT17], [Sch18a], [Sch18b], [ABP18], and [HKT18]. Typically, it is easy to provide certain rational members in the studied families. However, this does not exclude that the locus of rational fibres is contained in a proper closed subset of the base. In only a few cases, it was shown that the locus of rational fibres is dense in the moduli space.

The aim of this thesis is to prove the density assertion for any standard quadric surface bundle over $\mathbb{P}^2$, thus showing that also in this large class of families the locus of rational fibres is never contained in a proper closed subset of the moduli space. Concretely, we want to give a detailed proof of the following new result, which also appeared in [Pau18]:

Theorem 1.1. *Let $d_0, d_1, d_2, d_3 \geq 0$ be integers of the same parity and let $\mathcal{X} \to B$ be the universal family of smooth quadric surface bundles over $\mathbb{P}^2$ of type (d_0, d_1, d_2, d_3). Then the set*

$$\{b \in B \mid \mathcal{X}_b \textit{ is rational}\}$$

is dense in B for the Euclidean topology.

The first case where such a density result for quadric surface bundles was proven was for type $(0,2,2,4)$ and is due to Voisin [Voi15a, Section 2], see also [Sch18a, Proposition 25]. As mentioned, the case of type $(2,2,2,2)$ was shown in [HPT18a]. In particular, Theorem 1.1 generalizes their density result to hypersurfaces in $\mathbb{P}^2 \times \mathbb{P}^3$ of bidegree $(d,2)$ for arbitrary $d \geq 0$. It also provides a unified proof for the density of rational members in the aforementioned families of fourfolds that are birational to standard quadric surface bundles.

Further, Theorem 1.1 gives an affirmative answer to a question raised in [Sch18a, Remark 49]. Namely, Schreieder proved in [Sch18a, Theorem 47] that for all integers n, r, d such that $n \geq 2$, $2^{n-1} - 1 \leq r \leq 2^n - 2$, $d \geq 2(n+r)(r+1)$, and $d(r+1)$ is even, there exists a family $\mathcal{X} \to B$ of r-fold quadric bundles over $\mathbb{P}^n$ whose degeneration loci are of degree d such that $\mathcal{X}_b$ is stably irrational for a very general $b \in B$, but some fibres are rational for $r \geq 2$. Using Voisin's density result for standard quadric surface bundles of type $(0,2,2,4)$, he concluded that the locus of rational fibres is even dense in B (for the Euclidean topology) if $r \geq 3$ and d is even. As a consequence of Theorem 1.1, this actually holds for all $r \geq 2$ and without the restriction on the parity of d.

In order to prove Theorem 1.1, we follow Voisin's approach from [Voi15a, Section 2] that has later been used in [HPT18a] and [HPT17]. Using a theorem of Springer [Spr52] and the fact that the integral Hodge conjecture is known in codimension two for quadric bundles over surfaces [CTV12], we obtain a Hodge theoretic property guaranteeing the rationality of quadric surface bundles over $\mathbb{P}^2$:

Proposition 1.2. *Let $\pi\colon X \to \mathbb{P}^2$ be a quadric surface bundle. Then X is rational if there exists an integral Hodge class in $H^{2,2}(X, \mathbb{Z})$ meeting the generic fibre of π in odd degree.*

This leads to the study of the locus

$$\left\{ b \in B \;\middle|\; H^{2,2}_{\mathrm{odd}}(\mathcal{X}_b, \mathbb{Z}) \neq 0 \right\} . \tag{1.2}$$

Here, $H^{2,2}_{\mathrm{odd}}$ denotes the quotient of $H^{2,2}$ by the subgroup of classes having even intersection number with the generic fibre.

Similar loci already appeared more than 30 years earlier in the context of the Noether–Lefschetz theorem, which was conjectured by Noether and first proven by Lefschetz [Lef24] in 1924. This theorem states that for all $d \geq 4$, the Picard group of a very general smooth surface in $\mathbb{P}^3$ of degree d has only rank 1, i. e. is generated by the restriction of the line bundle $\mathcal{O}_{\mathbb{P}^3}(1)$. In analogy to the issue of rationality in families discussed above, it turns out that in the universal family $\mathcal{X} \to B$ of smooth surfaces in $\mathbb{P}^3$ of degree d, the so called *Noether–Lefschetz locus*

$$\{b \in B \mid \mathrm{Pic}(\mathcal{X}_b) \ncong \mathbb{Z} \cdot \mathcal{O}_{\mathbb{P}^3}(1)|_{\mathcal{X}_b}\}$$

is dense in the moduli space B for the Euclidean topology. This was first shown in [CHM88, Section 5] using an idea of Green and later by a different argument in [Kim91, Section 3]. The Noether–Lefschetz locus can be rephrased in terms of integral Hodge classes as the set

$$\left\{ b \in B \;\middle|\; H^{1,1}_{\mathrm{pr}}(\mathcal{X}_b, \mathbb{Z}) \neq 0 \right\} .$$

Here, the primitive cohomology $H^{1,1}_{\mathrm{pr}}$ may be regarded as the quotient of $H^{1,1}$ by the subgroup generated by the Kähler class, which is dual to the intersection with a plane in $\mathbb{P}^3$.

Both loci are defined via the existence of a non-zero integral Hodge class in (a quotient of) the middle cohomology groups $H^{1,1}$ and $H^{2,2}$, respectively. Since we have $H^{4,0} = H^{0,4} = 0$ for quadric surface bundles over $\mathbb{P}^2$, their Hodge structure on H^4 is only of weight 2. Hence, we will call the subset (1.2) a *Noether–Lefschetz locus* as well.

In [Voi03b, Proposition 5.20], Voisin stated an infinitesimal criterion for the density of such Noether–Lefschetz loci, based on Green's idea in [CHM88, Section 5]. Roughly speaking, it suffices to check that for some $b \in B$ there exists a class $\overline{\lambda} \in H^{1,1}(\mathcal{X}_b)$ such that the infinitesimal period map evaluated at $\overline{\lambda}$

$$\overline{\nabla}_b(\overline{\lambda})\colon T_{B,b} \to H^{0,2}(\mathcal{X}_b)$$

is surjective (one has to replace $H^{1,1}$ by $H^{2,2}$ and $H^{0,2}$ by $H^{1,3}$ in the case of a Hodge structure of weight 2 on H^4). This criterion was also used in [Kim91] for reproving the density result in the Noether–Lefschetz theorem, and in [Voi15a] and [HPT18a] for proving rationality of quadric surface bundles over a dense set of the moduli space.

In all of these applications, one can explicitly describe the infinitesimal period map $\overline{\nabla}_b(\overline{\lambda})$ as a multiplication map in a certain quotient of a polynomial ring. Therefore, the verification of the density criterion of Green and Voisin reduces to an elementary statement involving the multiplication of polynomials. This problem was solved in [HPT18a] with an explicit computation done in `Macaulay2` on a randomly chosen example. This was possible because they only dealt with quadric surface bundles of the fixed type $(2, 2, 2, 2)$. In [Voi15a, Section 2], Voisin proved the surjectivity of the infinitesimal period map for quadric surface bundles of type $(0, 2, 2, 4)$ via a more general argument. However, her approach seems to work only in low dimensions of $H^{1,3}$ and $H^{2,2}$. The argument of Kim in [Kim91, Section 3] for the density of the original Noether–Lefschetz locus is more sophisticated, since it involves the unknown degree d of the surface. We will face a similar challenge when handling quadric surface bundles of arbitrary type (d_0, d_1, d_2, d_3).

In order to prove Theorem 1.1, we use a result about the strong Lefschetz property of certain complete intersections which was proven in [HW03, Proposition 30]. With our approach involving the theory of Lefschetz properties, we can also simplify the argument of [Kim91], see Section 4.4.

The thesis is structured as follows. In Chapter 2, we prove Proposition 1.2 and see how the rationality of quadric surface bundles relates to the cohomology group $H^{2,2}$ arising in Hodge theory. In Chapter 3, we study Noether–Lefschetz loci in their generality and prove a slightly generalized version of Voisin's infinitesimal criterion for their density. In Chapter 4, we first compute the cohomology of standard quadric surface bundles to give an explicit description of the infinitesimal period map. Then we apply the preparations of the previous two chapters in order to reduce Theorem 1.1 to a statement about polynomials. Finally, we solve this problem in the last section.

Conventions

A variety is defined to be an integral separated scheme of finite type over a field. If not stated otherwise, varieties are always understood to be over the field of complex numbers. All Kähler manifolds are assumed to be compact and connected.

A quadric surface bundle over $\mathbb{P}^2$ is a complex projective variety X together with a flat morphism $\pi\colon X \to \mathbb{P}^2$ such that the generic fibre X_η over the generic point $\eta \in \mathbb{P}^2$ is a smooth quadric surface over the function field $\mathbb{C}(\mathbb{P}^2)$.

Preliminaries from Hodge theory

Hodge theory is a powerful tool in complex algebraic geometry. We will use it extensively in Chapter 3. We now recall some important results in order to introduce the notation used throughout this thesis.

A smooth projective variety X of dimension n is a Kähler manifold and hence admits a *Hodge decomposition*

$$H^k(X,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X)\ , \quad \overline{H^{p,q}(X)} = H^{q,p}(X)$$

for all $0 \leq k \leq 2n$ into the Hodge groups $H^{p,q}(X)$ generated by forms of type (p,q) in de Rham cohomology. The groups $H^{p,q}(X)$ are naturally isomorphic to the Dolbeault cohomology groups $H^q(X,\Omega_X^p)$. Conversely, it is a non-obvious result that a Kähler manifold embedded into projective space is algebraic and thus a smooth projective variety.

Especially when studying families of Kähler manifolds, it is often useful to consider the *Hodge filtration*

$$F^kH^k(X,\mathbb{C}) \subset \cdots \subset F^1H^k(X,\mathbb{C}) \subset F^0H^k(X,\mathbb{C}) = H^k(X,\mathbb{C})$$

where

$$F^pH^k(X,\mathbb{C}) = \bigoplus_{r\geq p} H^{r,k-r}(X)\ .$$

One can get back the Hodge decomposition from the Hodge filtration via the relation

$$H^{p,q}(X) = F^pH^k(X,\mathbb{C}) \cap \overline{F^qH^k(X,\mathbb{C})}$$

for $p+q=k$. The subspaces $F^pH^k(X,\mathbb{C})$ can also be seen as the hypercohomology groups $\mathbb{H}^k(X,\Omega_X^{\bullet\geq p})$ of the truncated holomorphic de Rham complex

$$0 \to \Omega_X^p \to \Omega_X^{p+1} \to \cdots\ .$$

For $p+q=k$, we write

$$H^{p,q}(X,\mathbb{Z}) = H^{p,q}(X) \cap H^k(X,\mathbb{Z}) \subset H^k(X,\mathbb{C})$$

for the Abelian group of integral classes of type (p,q), where $H^k(X,\mathbb{Z})$ is identified with its image in $H^k(X,\mathbb{C})$ under the inclusion of sheaves $\mathbb{Z} \subset \mathbb{C}$. If $p=q$, these classes are called *integral Hodge classes.* Since an integral class is real, we can express $H^{p,p}(X,\mathbb{Z})$ in terms of the Hodge filtration as

$$H^{p,p}(X,\mathbb{Z}) = H^{2p}(X,\mathbb{Z}) \cap F^pH^{2p}(X,\mathbb{C})\ .$$

This fact will be particularly useful in Chapter 3.

If X is a smooth projective variety and $Z \subset X$ is a subvariety of codimension k, we denote by

$$[Z] \in H^{2k}(X, \mathbb{Z})$$

the Poincaré dual of the homology class of Z. One can easily see that $[Z]$ is of type (k, k). Therefore, every *integral algebraic* $2k$*-cycle*, i. e. a finite formal sum $\sum n_i[Z_i]$ with $n_i \in \mathbb{Z}$ and subvarieties $Z_i \subset X$ of codimension k, gives an element of $H^{k,k}(X, \mathbb{Z})$. The *integral Hodge conjecture* asserts that all integral Hodge classes are of this form. For $k = 1$, this follows from the Lefschetz $(1, 1)$-theorem because the map $\operatorname{Pic} X \to H^{1,1}(X, \mathbb{Z})$ is surjective. For $k > 1$, however, the conjecture is false in general, as first shown by Atiyah and Hirzebruch [AH61]. Replacing $\mathbb{Z}$ by $\mathbb{Q}$ everywhere in the above discussion, we obtain the still unsolved *(rational) Hodge conjecture*, which is one of the seven Millennium Prize Problems.

2 Rationality of quadric surface bundles

The aim of this chapter is to prove Proposition 1.2, which gives a sufficient condition for the rationality of quadric surface bundles over $\mathbb{P}^2$ in terms of integral Hodge classes. Our treatment follows [Voi15a, Section 2] and [HPT18a, Section 3.1].

2.1 Quadric surfaces

Let us recall that a variety X of dimension n over a field k is called *rational* if it is birational to $\mathbb{P}^n_k$. This means there are non-empty Zariski open subsets $U \subset X$ and $V \subset \mathbb{P}^n_k$ which are isomorphic, or equivalently, the function field $k(X)$ is isomorphic to the purely transcendental extension $k(\mathbb{P}^n_k) = k(x_1, \ldots, x_n)$. While the problem of deciding whether a given variety is rational or not is very hard in general, it is easy to solve for quadric hypersurfaces. The following rationality criterion is well-known:

Lemma 2.1. *Let $Q \subset \mathbb{P}^{n+1}_k$ be a smooth quadric hypersurface over a field k. Then Q is rational if Q has a k-point.*

Proof. Suppose there exists a k-point $x \in Q$. Let $P \subset \mathbb{P}^{n+1}_k$ be the hyperplane tangent to Q at x. We claim that $Q \setminus P$ and $\mathbb{A}^n_k$ are isomorphic. Then it will follow that Q is rational because $Q \setminus P \subset Q$ and $\mathbb{A}^n_k \subset \mathbb{P}^n_k$ are non-empty Zariski open subsets (as Q is not contained in P). To see the claim, first note that $\mathbb{A}^n_k$ parametrizes all lines through x in $\mathbb{P}^{n+1}_k$ which are not contained in P. The intersection of a line $\ell \subset \mathbb{P}^{n+1}_k$ with $\ell \not\subset P$ through x with the quadric hypersurface Q is determined by a quadratic equation in t, where $t \in \mathbb{P}^1_k$ parametrizes the points on ℓ such that $t = \infty$ corresponds to $x \in \ell$. Since we already know that $x \in \ell \cap Q$, by Vieta's formula there is a unique second intersection point $y \in \ell \cap Q$, whose coordinates depend rationally on those of x and of the point in $\mathbb{A}^n_k$ describing ℓ. Further, y is different from x because ℓ is not contained in the tangent hyperplane of Q at x. Likewise, for any point $y \in Q \setminus P$ there is a unique line $\ell \subset \mathbb{P}^{n+1}_k$ with $\ell \not\subset P$ passing through x and y, whose associated point in $\mathbb{A}^n_k$ depends rationally on y. The two constructions are inverse to each other, so we have constructed an isomorphism between $Q \setminus P$ and $\mathbb{A}^n_k$. $\square$

In particular, over an algebraically closed field a smooth quadric hypersurface is always rational. The converse of Lemma 2.1 also holds by the Lang–Nishimura lemma. However, in this chapter we are mostly interested in sufficient conditions for rationality.

Now we consider a complex quadric surface bundle $\pi\colon X \to \mathbb{P}^2$. The generic fibre X_η is a quadric surface over the function field $k = \mathbb{C}(\mathbb{P}^2)$. Since $\mathbb{P}^2$ is rational, X is rational as soon as the generic fibre X_η is rational over k. Indeed, X is birational to X_η over k and thus its function field $\mathbb{C}(X)$ is a purely transcendental extension of k. Since $k = \mathbb{C}(x_1, x_2)$ itself is a purely transcendental extension of $\mathbb{C}$, the same holds for $\mathbb{C}(X)$ and it follows that X is rational (over $\mathbb{C}$).

In view of Lemma 2.1, we conclude:

Corollary 2.2. *Let $\pi\colon X \to \mathbb{P}^2$ be a quadric surface bundle. Then X is rational if the generic fibre X_η has a k-point where $k = \mathbb{C}(\mathbb{P}^2)$.*

2.2 Springer's theorem

The following theorem of Springer [Spr52] was originally a conjecture of Witt. It allows us to considerably weaken the requirement of X_η having a k-point.

Proposition 2.3 (Springer)**.** *Let $Q \subset \mathbb{P}^{n+1}_k$ be a quadric hypersurface over a field k and let K/k be a finite field extension of odd degree. If Q has a K-point, then Q has a k-point.*

The following proof only uses techniques from basic algebra.

Proof. The statement is obvious if $K = k$. Otherwise, let $\beta \in K \setminus k$. Then the degree $[K : k(\beta)]$ is odd and strictly less than $[K : k]$, so we may assume by induction on $[K : k]$ that $K = k(\beta)$. Let $p \in k[x]$ be the minimal polynomial of β over k and let $d = [K : k] = \deg p$. Let $f \in k[y_0, \dots, y_{n+1}]$ be the defining equation of Q, which is a homogeneous polynomial of degree 2. The assumed K-point of Q can be written as

$$[g_0(\beta) : \dots : g_{n+1}(\beta)] \in \mathbb{P}^{n+1}_K$$

with certain polynomials $g_0, \dots, g_{n+1} \in k[x]$ of degree less than d, not all being identically zero. Since the polynomial $f(g_0, \dots, g_{n+1}) \in k[x]$ has a zero at $\beta \in K$, it is divisible by the minimal polynomial p, i. e.

$$f(g_0, \dots, g_{n+1}) = p \cdot q$$

for some polynomial $q \in k[x]$. Let $m \geq 0$ be the maximal degree occuring among the polynomials $g_0, \dots, g_{n+1}$. By the choice of $g_0, \dots, g_{n+1}$, we have $m < d$. We may assume that $f(g_0, \dots, g_{n+1})$ has degree $2m$, since otherwise the coefficients of $g_0, \dots, g_{n+1}$ at degree m (which are not simultaneously zero) would already give a k-point of Q. Therefore, $\deg q = 2m - d$ is odd and less than d. Now let us take an irreducible factor of q having odd degree and consider one of its roots θ in an algebraic closure of k. It follows that

$$f(g_0(\theta), \dots, g_{n+1}(\theta)) = p(\theta) \cdot q(\theta) = 0\ ,$$

so Q has a $k(\theta)$-point. Since $k(\theta)/k$ is a finite field extension of odd degree $\leq \deg q < d$, Q has a k-point by induction. $\square$

Together with Corollary 2.2, Proposition 2.3 implies that in order to prove rationality of a quadric surface bundle $\pi\colon X \to \mathbb{P}^2$, it suffices to find a K-point on X_η for some field extension K/k of odd degree. This can be achieved through an odd degree multisection of π, i. e. a surface $Z \subset X$ such that

$$[Z] \cup [X_\eta] \in H^{4,4}(X, \mathbb{Z}) \cong \mathbb{Z}$$

is odd. Indeed, given a multisection Z of odd degree d, the projection $\pi|_Z\colon Z \to \mathbb{P}^2$ is a finite map of degree d, hence the extension $\mathbb{C}(Z)/\mathbb{C}(\mathbb{P}^2)$ of function fields is of finite degree d. Furthermore, Z intersects X_η in d points (counted with multiplicity), each one being of course rational over $\mathbb{C}(Z)$. Setting $K = \mathbb{C}(Z)$, we therefore found a K-point on X_η for a field extension K/k of odd degree.

To summarize:

Corollary 2.4. *Let $\pi\colon X \to \mathbb{P}^2$ be a quadric surface bundle. Then X is rational if π has a rational multisection of odd degree, that is, there exists a surface $Z \subset X$ such that $[Z] \cup [X_\eta]$ is odd.*

2.3 The integral Hodge conjecture

As mentioned in the introduction, the integral Hodge conjecture is false in general. However, it is true for certain special varieties. For $(2,2)$-classes on quadric bundles over surfaces, the integral Hodge conjecture was proven by Jean-Louis Colliot-Thélène and Claire Voisin [CTV12, Corollaire 8.2]. The following special case will be useful for us:

Proposition 2.5 (Colliot-Thélène–Voisin)**.** *Let $\pi\colon X \to \mathbb{P}^2$ be a smooth quadric surface bundle. Then the integral Hodge conjecture holds for $H^{2,2}(X, \mathbb{Z})$.*

This allows us to transform Corollary 2.4 into a Hodge theoretic condition. We are now ready to prove a reformulation of Proposition 1.2 from the introduction:

Corollary 2.6. *Let $\pi\colon X \to \mathbb{P}^2$ be a smooth quadric surface bundle. Then X is rational if there exists an integral Hodge class $\alpha \in H^{2,2}(X, \mathbb{Z})$ such that $\alpha \cup [X_\eta]$ is odd.*

Proof. By Proposition 2.5, there exist surfaces $Z_1, \ldots, Z_m \subset X$ such that

$$\alpha = \sum_{i=1}^{m} n_i [Z_i]\ , \quad n_i \in \mathbb{Z}\ .$$

Therefore, $[Z_i] \cup [X_\eta]$ is odd for at least one $i \in \{1, \ldots, m\}$. Hence, X is rational by Corollary 2.4. $\square$

It should be mentioned that the rationality condition in Corollary 2.6 is not necessary, because already the converse of Corollary 2.2 is not true in general. For example, the authors of [ABBVA14] construct smooth quadric surface bundles of type $(1,1,1,3)$ which are rational but do not have a rational section. Since our proof of Theorem 1.1 is based on Corollary 2.6, our proof shows the even stronger statement that the locus of quadric surface bundles of type (d_0, d_1, d_2, d_3) which admit a rational section is dense in the moduli space.

The question whether a very general quadric surface bundle of type $(1,1,1,3)$ is irrational is one of the two cases not handled by [Sch18b] and is still open. Actually, this is a longstanding and famous question, since any cubic fourfold containing a plane is birational to a quadric surface bundle of type $(1,1,1,3)$. It is conjectured that the answer is positive, though not even a single cubic fourfold is currently proven to be irrational. However, one can show that a very general quadric surface bundle of type $(1,1,1,3)$ does not have a rational section. Therefore, the actual locus we show to be dense is also for type $(1,1,1,3)$ a proper subset of the moduli space and, in particular, is not closed.

3 Density of Noether–Lefschetz loci

In the last chapter, we have seen how Hodge theory provides a sufficient condition for the rationality of quadric surface bundles over $\mathbb{P}^2$. In order to study the behaviour of rationality in families, we thus need to examine how the Hodge structure varies in families of Kähler manifolds. Therefore, in this chapter, we leave the algebraic world and study Noether–Lefschetz loci and their density in a general setting, detached from their applications to rationality. Our aim is to prove the infinitesimal criterion of Green and Voisin in variations of Hodge structure of weight 2. The content of this chapter is based on Voisin's books [Voi03a] and [Voi03b].

3.1 Variations of Hodge structure

Families of complex manifolds

A *family of complex manifolds* is a proper submersive holomorphic map $f\colon \mathcal{X} \to B$ between complex manifolds. It follows that all fibres $\mathcal{X}_b = f^{-1}(\{b\})$ are compact complex manifolds. Sometimes we fix a base point $0 \in B$. In this case, the manifolds $\mathcal{X}_b$ are called *deformations* of $\mathcal{X}_0$. We say that f is a *family of Kähler manifolds* if in addition all fibres are Kähler manifolds.

The following theorem from the theory of deformations is crucial:

Theorem 3.1 (Ehresmann's lemma)**.** *Let $f\colon \mathcal{X} \to B$ be a proper submersive smooth map between smooth manifolds. If B is contractible, there exists a diffeomorphism $\phi\colon \mathcal{X} \to B \times \mathcal{X}_0$ such that $f = \mathrm{pr}_1 \circ \phi$, where $0 \in B$ is an arbitrary base point.*

It is important to note that Ehresmann's lemma applied to f only gives an isomorphism between $\mathcal{X}$ and $B \times \mathcal{X}_0$ as smooth manifolds but not as complex ones. Therefore, by identifying different fibres $\mathcal{X}_b$ with $\mathcal{X}_0$ via the diffeomorphism ϕ, we get a varying complex structure on the differentiable manifold $\mathcal{X}_0$. In particular, if $f\colon \mathcal{X} \to B$ is a family of Kähler manifolds, the Hodge groups $H^{p,q}(\mathcal{X}_b)$ depend on the complex structure while the topological invariant $H^k(\mathcal{X}_b, \mathbb{Z}) \cong H^k(\mathcal{X}_0, \mathbb{Z})$ does not, so B parametrizes a varying Hodge decomposition on the fixed vector space $H^k(\mathcal{X}_0, \mathbb{C}) = H^k(\mathcal{X}_0, \mathbb{Z}) \otimes \mathbb{C}$.

Local systems and the Gauß–Manin connection

Let $f\colon \mathcal{X} \to B$ be a family of complex manifolds. Let us suppose that the base B is connected but not necessarily contractible. In this case, different fibres $\mathcal{X}_b$ and $\mathcal{X}_0$ are still diffeomorphic, but we cannot identify them globally for all $b \in B$ in a consistent way. However, we can do so locally since every $b \in B$ has a contractible neighbourhood. This can be utilized in cohomology when considering for $k \geq 0$ the higher direct image sheaf $R^k f_* \mathbb{Z}$ on B with stalk $H^k(\mathcal{X}_b, \mathbb{Z})$ at $b \in B$. Restricted to a contractible neighbourhood of a point $b \in B$, this sheaf is isomorphic to the constant sheaf with values in $H^k(\mathcal{X}_b, \mathbb{Z}) \cong H^k(\mathcal{X}_0, \mathbb{Z})$ (the last isomorphism is not canonical). Sheaves which satisfy this property of being locally isomorphic to a constant sheaf with a fixed stalk are called *local systems*. Actually, it turns out that a local system is already trivial on every simply connected open subset.

As it may be easier to work with vector bundles, we consider the locally free $\mathcal{O}_B$-module

$$\mathcal{H}^k = R^k f_* \mathbb{Z} \otimes \mathcal{O}_B \ .$$

Then $\mathcal{H}^k$ is the sheaf of sections of a holomorphic vector bundle on B which we also denote by $\mathcal{H}^k$. Its fibre $\mathcal{H}^k_b$ at $b \in B$ (not to be confused with the stalk of the sheaf $\mathcal{H}^k$) is the cohomology group $H^k(\mathcal{X}_b, \mathbb{C})$. Furthermore,

$$\mathcal{H}^k_{\mathbb{R}} = R^k f_* \mathbb{Z} \otimes \underline{\mathcal{C}}^\infty_B$$

is the sheaf of smooth sections of a real subbundle $\mathcal{H}^k_{\mathbb{R}} \subset \mathcal{H}^k$ on B such that $\mathcal{H}^k_b = \mathcal{H}^k_{\mathbb{R},b} \otimes_{\mathbb{R}} \mathbb{C}$ for all $b \in B$.

Since $R^k f_* \mathbb{Z}$ is a local system, the vector bundle $\mathcal{H}^k$ comes with a flat connection

$$\nabla \colon \mathcal{H}^k \to \mathcal{H}^k \otimes \Omega_B \ .$$

On a local section $\sum \sigma_i \otimes f_i$ of $\mathcal{H}^k$, it is given by $\nabla \left(\sum \sigma_i \otimes f_i\right) = \sum \sigma_i \otimes df_i$. This is well-defined because the sections σ_i of $R^k f_* \mathbb{Z}$ are locally constant. The connection is *flat* since

$$\nabla \circ \nabla \left(\sum \sigma_i \otimes f_i\right) = \nabla \left(\sum \sigma_i \otimes df_i\right) = \sum \left(\sigma_i \otimes df_i \wedge df_i + \sigma_i \otimes ddf_i\right) = 0$$

where ∇ extends to a map

$$\nabla \colon \mathcal{H}^k \otimes \Omega_B \to \mathcal{H}^k \otimes \Omega^2_B$$

via the rule $\nabla(\sigma \otimes \alpha) = \sigma \otimes d\alpha + \nabla\sigma \wedge \alpha$. We call ∇ the *Gauß–Manin connection* on $\mathcal{H}^k$. It naturally descends to a flat connection on $\mathcal{H}^k_{\mathbb{R}}$.

One can regard $R^k f_* \mathbb{C} = R^k f_* \mathbb{Z} \otimes \mathbb{C}$ as a subsheaf of $\mathcal{H}^k$ consisting exactly of the *flat* sections of the vector bundle $\mathcal{H}^k$, i. e. those sections annihilated by ∇. Similarly, the local system $R^k f_* \mathbb{R} = R^k f_* \mathbb{Z} \otimes \mathbb{R}$ is the sheaf of ∇-flat sections of the real vector bundle $\mathcal{H}^k_{\mathbb{R}}$.

The Kodaira–Spencer map

For $b \in B$, we have a short exact sequence of holomorphic vector bundles on $X = \mathcal{X}_b$:

$$0 \to T_X \to T_{\mathcal{X}}|_X \to f^*T_B|_X \to 0\ .$$

Note that the vector bundle $f^*T_B|_X \cong X \times T_{B,b}$ is trivial. The associated long exact sequence in cohomology yields a map

$$\rho\colon T_{B,b} = H^0(X, f^*T_B|_X) \to H^1(X, T_X)$$

which is called the *Kodaira–Spencer map.*

Hodge bundles

Now suppose $f\colon \mathcal{X} \to B$ is a family of Kähler manifolds, so we have a Hodge decomposition

$$\mathcal{H}^k_b = H^k(\mathcal{X}_b, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(\mathcal{X}_b)$$

on each fibre of the vector bundle $\mathcal{H}^k$. Our aim is to study how this structure varies along B. One approach to do so locally is by considering the differential of the period map, which has values in a product of Grassmannians. Here, we will give a different formulation only using the language of subbundles and the Gauß–Manin connection ∇.

Note that there cannot exist holomorphic vector subbundles $\mathcal{H}^{p,q} \subset \mathcal{H}^k$ with fibre $\mathcal{H}^{p,q}_b = H^{p,q}(\mathcal{X}_b)$ at $b \in B$ for all $p + q = k$ because if $H^{p,q}(\mathcal{X}_b) \subset H^k(\mathcal{X}_b, \mathbb{C})$ would vary holomorphically, its conjugate $H^{q,p}(\mathcal{X}_b) = \overline{H^{p,q}(\mathcal{X}_b)} \subset H^k(\mathcal{X}_b, \mathbb{C})$ would vary antiholomorphically. This deficiency is resolved by considering the Hodge filtration

$$F^kH^k(\mathcal{X}_b, \mathbb{C}) \subset \cdots \subset F^1H^k(\mathcal{X}_b, \mathbb{C}) \subset F^0H^k(\mathcal{X}_b, \mathbb{C}) = H^k(\mathcal{X}_b, \mathbb{C})$$

instead.

In order to define the holomorphic vector bundles induced by the Hodge filtration, we first give another description of the sheaf $\mathcal{H}^k$. For this, let us consider the sheaf of relative holomorphic 1-forms $\Omega_{\mathcal{X}/B} = \Omega_{\mathcal{X}}/f^*\Omega_B$ on $\mathcal{X}$. We define $\Omega^p_{\mathcal{X}/B} = \bigwedge^p \Omega_{\mathcal{X}/B}$ for $p \geq 1$. By the relative holomorphic Poincaré lemma, the relative holomorphic de Rham complex

$$0 \to \Omega_{\mathcal{X}/B} \to \Omega^2_{\mathcal{X}/B} \to \cdots$$

is a resolution of the sheaf $f^*\mathcal{O}_B$ on $\mathcal{X}$. Therefore, we have

$$\mathcal{H}^k = R^kf_*\mathbb{Z} \otimes \mathcal{O}_B = R^kf_*\Omega^\bullet_{\mathcal{X}/B}\ .$$

For $p \geq 0$, the truncated relative holomorphic de Rham complex

$$0 \to \Omega^p_{\mathcal{X}/B} \to \Omega^{p+1}_{\mathcal{X}/B} \to \cdots$$

denoted by $\Omega^{\bullet\geq p}_{\mathcal{X}/B}$ gives rise to a sheaf

$$F^p\mathcal{H}^k = R^k f_* \Omega^{\bullet\geq p}_{\mathcal{X}/B}$$

on B. The degeneracy at E_1 of the Frölicher spectral sequence on the fibres $\mathcal{X}_b$ implies that this induces a decreasing filtration

$$F^k\mathcal{H}^k \subset \cdots \subset F^1\mathcal{H}^k \subset F^0\mathcal{H}^k = \mathcal{H}^k$$

on $\mathcal{H}^k$ by holomorphic vector subbundles with the expected fibres. These vector subbundles are called *Hodge bundles*.

For $p + q = k$, we define the quotients

$$\mathcal{H}^{p,q} = \frac{F^p\mathcal{H}^k}{F^{p+1}\mathcal{H}^k}$$

which are holomorphic vector bundles on B. The fibre

$$\mathcal{H}^{p,q}_b = \frac{F^p\mathcal{H}^k_b}{F^{p+1}\mathcal{H}^k_b} = \frac{F^p H^k(\mathcal{X}_b, \mathbb{C})}{F^{p+1}H^k(\mathcal{X}_b, \mathbb{C})}$$

can be identified with $H^{p,q}(\mathcal{X}_b)$ for $b \in B$. However, the structure of $H^{p,q}(\mathcal{X}_b)$ as a $\mathbb{C}$-linear subspace of $H^k(\mathcal{X}_b, \mathbb{C})$ cannot be realized by some structure of $\mathcal{H}^{p,q}$ as a holomorphic subbundle of $\mathcal{H}^k$, as explained earlier.

Griffiths transversality and the infinitesimal period map

Since the local trivializations $\mathcal{X}|_U \cong U \times \mathcal{X}_b$ of $f: \mathcal{X} \to B$ over contractible neighbourhoods $U \subset B$ of points $b \in B$ do not respect the complex structure, the holomorphic Hodge bundles $F^p\mathcal{H}^k$ are usually not flat with respect to ∇. However, Griffiths [Gri68] showed the important transversality property

$$\nabla\left(F^p\mathcal{H}^k\right) \subset F^{p-1}\mathcal{H}^k \otimes \Omega_B \ .$$

This allows to construct for $p + q = k$ a map

$$\overline{\nabla}: \mathcal{H}^{p,q} \to \mathcal{H}^{p-1,q+1} \otimes \Omega_B$$

on the quotient $\mathcal{H}^{p,q}$, called the *infinitesimal period map*. In contrast to ∇, the morphism of sheaves $\overline{\nabla}$ is $\mathcal{O}_B$-linear, since Leibniz' rule implies

$$\nabla(f\lambda) = \lambda \otimes df + f \cdot \nabla(\lambda) \equiv f \cdot \nabla(\lambda) \pmod{F^p\mathcal{H}^k \otimes \Omega_B}$$

for a local section λ of $F^p\mathcal{H}^k$ and hence $\overline{\nabla}(f\overline{\lambda}) = f \cdot \overline{\nabla}(\overline{\lambda})$ for a local section $\overline{\lambda}$ of $\mathcal{H}^{p,q}$. Therefore, $\overline{\nabla}$ induces for all $b \in B$ a $\mathbb{C}$-linear map

$$\overline{\nabla}_b: \mathcal{H}^{p,q}_b \to \mathcal{H}^{p-1,q+1}_b \otimes \Omega_{B,b}$$

on the fibres of the corresponding vector bundles. Via adjunction we obtain a map

$$\overline{\nabla}_b \colon T_{B,b} \to \operatorname{Hom}\left(\mathcal{H}_b^{p,q}, \mathcal{H}_b^{p-1,q+1}\right)$$

for all $b \in B$. Denoting $X = \mathcal{X}_b$, Griffiths [Gri68] computed that this map is just the Kodaira–Spencer map $\rho \colon T_{B,b} \to H^1(X, T_X)$ followed by the map

$$H^1(X, T_X) \to \operatorname{Hom}\left(H^q\left(X, \Omega_X^p\right), H^{q+1}\left(X, \Omega_X^{p-1}\right)\right) = \operatorname{Hom}\left(H^{p,q}(X), H^{p-1,q+1}(X)\right)$$

given by cup product and contraction.

Abstract setting

The above results motivate an axiomatic approach for the study of variations of Hodge structure where the original fibration $f \colon \mathcal{X} \to B$ is not needed anymore. Concretely, we start with a local system H of finitely generated free $\mathbb{Z}$-modules on a connected complex manifold B and consider the holomorphic vector bundle $\mathcal{H}$ with sheaf of sections

$$\mathcal{H} = H \otimes \mathcal{O}_B$$

together with its flat Gauß–Manin connection ∇. Let $\mathcal{H}_\mathbb{R} \subset \mathcal{H}$ be the corresponding real vector subbundle. We denote by $H_\mathbb{R} = H \otimes \mathbb{R}$ and $H_\mathbb{C} = H \otimes \mathbb{C}$ the induced real and complex local systems, respectively. The inclusion of sheaves $H_\mathbb{C} \subset \mathcal{H}$ corresponds to the ∇-flat sections of the vector bundle $\mathcal{H}$. We define an *integral variation of Hodge structure of weight k* on H to be a filtration

$$F^k\mathcal{H} \subset \cdots \subset F^1\mathcal{H} \subset F^0\mathcal{H} = \mathcal{H}$$

by holomorphic vector subbundles on $\mathcal{H}$ such that Griffiths transversality

$$\nabla\left(F^p\mathcal{H}\right) \subset F^{p-1}\mathcal{H} \otimes \Omega_B$$

is satisfied for all p. The quotients $\mathcal{H}^{p,q}$ and the infinitesimal period map $\overline{\nabla}$ are then defined as above.

3.2 Hodge loci

Let $(H, F^\bullet\mathcal{H})$ be an integral variation of Hodge structure of weight k on a connected complex manifold B. Let λ be a local section of H, defined on a connected open subset $U \subset B$. We may regard λ as a submanifold $\lambda \subset \mathcal{H}$ of the total space of the vector bundle $\mathcal{H}$. For $p \geq 0$, the *Hodge locus* $U_\lambda^p \subset U$ is defined as the image of the restricted projection map $\lambda \cap F^p\mathcal{H} \subset \mathcal{H} \to B$. In other words,

$$U_\lambda^p = \{u \in U \mid \lambda(u) \in F^p\mathcal{H}_u\}\,.$$

Since U^p_λ is by definition the zero locus of the induced holomorphic section $\overline{\lambda}$ on the quotient bundle $\mathcal{H}/F^p\mathcal{H}$, the Hodge locus U^p_λ is a complex analytic subset of U. Clearly, we have $U^p_\lambda = U$ if $\lambda = 0$ or $p = 0$.

The case $k = 2p$ is of particular interest, since in the geometric case where our variation of Hodge structure originates from a family $f\colon \mathcal{X} \to B$ of Kähler manifolds, U^p_λ is precisely the locus of points $u \in U$ where $\lambda(u)$ is an integral Hodge class of type (p,p) for $\mathcal{X}_u$. It would be a consequence of the (rational) Hodge conjecture that the locus U^p_λ is then actually algebraic. Surprisingly, Cattani, Deligne, and Kaplan [CDK95] showed in 1995 that U^p_λ is indeed algebraic in this setting. This is a strong indication of the validity of the Hodge conjecture.

The above description of U^p_λ as the zero locus of the section $\overline{\lambda}$ on $\mathcal{H}/F^p\mathcal{H}$ shows that the codimension of U^p_λ is not larger than the dimension of the vector bundle $\mathcal{H}/F^p\mathcal{H}$. Actually, one can show using Griffiths transversality that the codimension of U^p_λ is at most

$$h^{p-1,k-p+1} = \dim \mathcal{H}^{p-1,k-p+1} ,$$

see [Voi03b, Proposition 5.14].

More generally, we may consider the Hodge locus U^p_λ for a section λ of $H_\mathbb{C}$ defined on U, where λ does not need to be integral anymore. Obviously, we have $U^p_\lambda = U^p_{z\lambda}$ for all $z \in \mathbb{C}^*$, so U_λ actually only depends on a section of the projectivization $\mathbb{P}(H_\mathbb{C})$. Still the codimension of U^p_λ is at most $h^{p-1,k-p+1}$.

3.3 Noether–Lefschetz loci

Let $(H, F^\bullet\mathcal{H})$ be an integral variation of Hodge structure of weight 2 such that, as in the geometric setting, we have

$$\mathcal{H}_b = F^2\mathcal{H}_b \oplus \overline{F^1\mathcal{H}_b} \tag{3.1}$$

for all $b \in B$. Let us consider the union of the Hodge loci U^1_λ over all local sections $\lambda \neq 0$ of H, i. e. the subset

$$\mathrm{NL} = \{b \in B \mid H_b \cap F^1\mathcal{H}_b \neq 0\} \subset B ,$$

which is the image of the projection $H \cap F^1\mathcal{H} \subset \mathcal{H} \to B$ where H is regarded as a submanifold of the total space $\mathcal{H}$. In the geometric case, this agrees with the locus of $b \in B$ where the fibre $\mathcal{X}_b$ admits a non-zero integral Hodge class of type $(1,1)$ and is thus of special interest. The set NL is called the *Noether–Lefschetz locus* of our variation of Hodge structure $(H, F^\bullet\mathcal{H})$. A priori, we only know from Section 3.2 that NL is a countable union of locally analytic subsets of B of codimension at most $h^{0,2}$, but for instance, it is unclear whether NL is closed or in some sense large.

In [Voi03b, Proposition 5.20], Voisin stated the following infinitesimal criterion for the density of NL, based on Green's idea for the proof in [CHM88, Section 5] of the density of the Noether–Lefschetz locus for the family of surfaces of degree $d \geq 4$ in $\mathbb{P}^3$:

Proposition 3.2 (Green–Voisin)**.** *Suppose there exists a point $b \in B$ and a class $\overline{\lambda} \in \mathcal{H}_b^{1,1}$ such that the infinitesimal period map evaluated at $\overline{\lambda}$*

$$\overline{\nabla}_b(\overline{\lambda})\colon T_{B,b} \to \mathcal{H}_b^{0,2}$$

is surjective. Then the Noether–Lefschetz locus NL *is dense in B for the Euclidean topology.*

Intuitively speaking, the surjectivity of $\overline{\nabla}_b(\overline{\lambda})$ means that the Hodge decompositions of local sections of $H_{\mathbb{C}}$ which are close to $\overline{\lambda}$ at b vary as much as possible, and hence the locus where a rational (and hence integral) class of pure type $(1,1)$ exists is dense in a neighbourhood of b. Since the stated property of $\overline{\lambda}$ is a Zariski open condition on $\mathcal{H}^{1,1}$, the density of NL follows.

In this section, we aim to formulate and prove a slightly more general version of this density criterion, which will be useful in our later application to the rationality of quadric surface bundles. For this, we generalize the notion of the Noether–Lefschetz locus naturally. Instead of taking the union of the Hodge loci for non-zero sections of H, that is, of locally constant non-zero integral classes, we may consider an arbitrary local subsystem of sets $D \subset H_{\mathbb{C}}$ and define its associated *Noether–Lefschetz locus* as

$$\mathrm{NL}_D = \{b \in B \mid D_b \cap F^1\mathcal{H}_b \neq \emptyset\} \subset B\,.$$

The local sections of D are certain designated ∇-flat sections of the holomorphic vector bundle $\mathcal{H}$, so we may regard D as a subset of the total space $\mathcal{H}$ and NL_D is the image of the restricted projection map $D \cap F^1\mathcal{H} \subset \mathcal{H} \to B$. If $D_b \subset H_{\mathbb{C},b}$ is discrete for one (and hence for all) $b \in B$, $D \subset \mathcal{H}$ is again a submanifold. As for NL, we have

$$\mathrm{NL}_D = \bigcup U_\lambda^1$$

where the union is taken over all connected open subsets $U \subset B$ and all sections $\lambda \in D(U)$. The Hodge loci U_λ^1 are called *local components* of NL_D. We can get back the classical Noether–Lefschetz locus NL by setting $D = H \setminus \{0\}$ (this is the sheaf of sets which assigns $H(U) \setminus \{0\}$ to a connected open subset $U \subset B$ and is clearly a local subsystem of $H_{\mathbb{C}}$).

The above definition is motivated by the results of Chapter 2, which suggest the study of the locus

$$\{b \in B \mid \exists \alpha \in H^{2,2}(\mathcal{X}_b, \mathbb{Z})\colon \alpha \cup [(\mathcal{X}_b)_\eta] \equiv 1 \pmod 2\}$$

for a family $\mathcal{X} \to B$ of quadric surface bundles over $\mathbb{P}^2$. This locus is precisely the generalized Noether–Lefschetz locus NL_D of a local system D with stalk

$$D_b = \{\alpha \in H^4(\mathcal{X}_b, \mathbb{Z}) \mid \alpha \cup [(\mathcal{X}_b)_\eta] \equiv 1 \pmod 2\}$$

at $b \in B$.

If one takes a close look at Voisin's proof of Proposition 3.2, one sees that the density comes into play when arguing that the integral classes H_b at a point $b \in B$ are multiples of the rational classes $H_{\mathbb{Q},b} = H_b \otimes \mathbb{Q}$, which in turn are dense in the real classes $H_{\mathbb{R},b} = H_b \otimes \mathbb{R}$. In place of this intermediate step involving $H_{\mathbb{Q}}$, one could instead directly use the fact that $\mathbb{R}^* H_b \subset H_{\mathbb{R},b}$ is dense. Therefore, let us consider any local subsystem of sets $D \subset H_{\mathbb{R}}$ such that $\mathbb{R}^* D_b \subset H_{\mathbb{R},b}$ is dense for one (and hence for all) $b \in B$. We then want to prove the following criterion for the analytical density of the generalized Noether–Lefschetz locus NL_D:

Proposition 3.3. *Let $D \subset H_{\mathbb{R}}$ be a local subsystem of sets such that $\mathbb{R}^* D_b \subset H_{\mathbb{R},b}$ is dense for at least one $b \in B$. Suppose there exists a point $b \in B$ and a class $\overline{\lambda} \in \mathcal{H}_b^{1,1}$ such that the infinitesimal period map evaluated at $\overline{\lambda}$*

$$\overline{\nabla}_b(\overline{\lambda}) \colon T_{B,b} \to \mathcal{H}_b^{0,2}$$

is surjective. Then the Noether–Lefschetz locus NL_D is dense in B for the Euclidean topology.

Proof. Let us consider the real vector subbundle

$$\mathcal{H}_{\mathbb{R}}^{1,1} = \mathcal{H}_{\mathbb{R}} \cap F^1 \mathcal{H} \subset \mathcal{H}_{\mathbb{R}} .$$

Condition (3.1) implies that $\mathcal{H}_{\mathbb{R},b}^{1,1} \otimes_{\mathbb{R}} \mathbb{C}$ may be identified with $\mathcal{H}_b^{1,1}$ for all $b \in B$ via the restricted projection

$$p \colon \mathcal{H}_{\mathbb{R}}^{1,1} \subset F^1\mathcal{H} \to F^1\mathcal{H}/F^2\mathcal{H} = \mathcal{H}^{1,1} .$$

By definition, NL_D is the image of the projection map $D \cap \mathcal{H}_{\mathbb{R}}^{1,1} \subset \mathcal{H}_{\mathbb{R}} \to B$. Since $\mathcal{H}_{\mathbb{R}}^{1,1}$ is a real vector bundle, the projections of $D \cap \mathcal{H}_{\mathbb{R}}^{1,1}$ and $(\mathbb{R}^* D) \cap \mathcal{H}_{\mathbb{R}}^{1,1}$ agree. By replacing D with $\mathbb{R}^* D$, we may thus assume that $D_b \subset H_{\mathbb{R},b}$ is dense for one and hence for all $b \in B$. It suffices to show that $D \cap \mathcal{H}_{\mathbb{R}}^{1,1}$ is dense in $\mathcal{H}_{\mathbb{R}}^{1,1}$ because the projection map $\mathcal{H}_{\mathbb{R}}^{1,1} \to B$ is trivially surjective.

For this, we first observe that the surjectivity of $\overline{\nabla}_b(\overline{\lambda})$ only fails on a locally analytic subset of the total space $\mathcal{H}^{1,1}$. Since $\mathcal{H}^{1,1} = \mathcal{H}_{\mathbb{R}}^{1,1} \otimes_{\mathbb{R}} \mathbb{C}$, the condition is hence fulfilled on a dense open subset of the real classes $p(\mathcal{H}_{\mathbb{R}}^{1,1}) \subset \mathcal{H}^{1,1}$. Therefore, it suffices to show the statement locally in $\mathcal{H}_{\mathbb{R}}^{1,1}$ around some $\lambda \in \mathcal{H}_{\mathbb{R},b}^{1,1}$ such that $\overline{\lambda} = p(\lambda)$ satisfies the hypothesis. By shrinking B, we may assume that the local system H and hence the vector bundles $\mathcal{H}_{\mathbb{R}}$ and $\mathcal{H}$ are trivial over B. We claim that the composed map

$$\phi \colon \mathcal{H}_{\mathbb{R}}^{1,1} \hookrightarrow \mathcal{H}_{\mathbb{R}} \xrightarrow{\cong} B \times H_{\mathbb{R},b} \to H_{\mathbb{R},b} ,$$

obtained via inclusion, isomorphism, and projection, is a submersion at $\lambda \in \mathcal{H}_{\mathbb{R}}^{1,1}$. Since D_b is dense in $\mathcal{H}_{\mathbb{R},b}$, it would then follow that the preimage $\phi^{-1}(D_b) = D \cap \mathcal{H}_{\mathbb{R}}^{1,1}$ is dense in $\mathcal{H}_{\mathbb{R}}^{1,1}$ around λ, which is what we wanted to show. It therefore remains to prove the claim that ϕ is a submersion at $\lambda \in \mathcal{H}_{\mathbb{R}}^{1,1}$. This is precisely the statement of [Voi03b, Lemma 5.21]. We will present Voisin's argument below. $\square$

Proof of the claim. Let us consider the analogous composed map

$$\psi\colon F^1\mathcal{H} \hookrightarrow \mathcal{H} \xrightarrow{\cong} B \times H_{\mathbb{C},b} \to H_{\mathbb{C},b}$$

in the complex case. Then ϕ is just the restriction of ψ to $\mathcal{H}^{1,1}_{\mathbb{R}} = F^1\mathcal{H} \cap \mathcal{H}_{\mathbb{R}} = \psi^{-1}(H_{\mathbb{R},b})$. Therefore, it suffices to show that ψ is a submersion at $\lambda \in F^1\mathcal{H}$. This will follow from the assumption that

$$\overline{\nabla}_b(\overline{\lambda})\colon T_{B,b} \to \mathcal{H}^{0,2}_b$$

is surjective for $\overline{\lambda} = p(\lambda) \in \mathcal{H}^{1,1}_b$. Indeed, for a local section α around b of the vector bundle $F^1\mathcal{H}$, $\nabla(\alpha)$ is a local section of $\mathcal{H} \otimes \Omega_B$ such that the induced map

$$\nabla_b(\alpha)\colon T_{B,b} \to \mathcal{H}_b = H_{\mathbb{C},b}$$

is exactly the differential of $\psi \circ \alpha$ at b. Therefore, $\overline{\nabla}_b(\overline{\lambda})$ can be computed by choosing a local section α of $F^1\mathcal{H}$ such that $\alpha(b) = \lambda$ and composing $d(\psi \circ \alpha)$ with the projection $H_{\mathbb{C},b} \to H_{\mathbb{C},b}/F^1\mathcal{H}_b = \mathcal{H}^{0,2}_b$ (we have shown in Section 3.1 using Griffiths transversality and Leibniz' rule that $\overline{\nabla}_b(\overline{\lambda})$ is well-defined). In other words, the following diagram commutes:

$$\begin{tikzcd}
T_{F^1\mathcal{H},\lambda} \arrow[r, "d\psi_\lambda"] & H_{\mathbb{C},b} \arrow[d] \\
T_{B,b} \arrow[u, "d\alpha_b"] \arrow[r, "\overline{\nabla}_b(\overline{\lambda})"] & H_{\mathbb{C},b}/F^1\mathcal{H}_b
\end{tikzcd}$$

To conclude that $d\psi_\lambda$ is surjective, it thus remains to prove that $F^1\mathcal{H}_b$ is contained in the image of $d\psi_\lambda$. This is obvious because the restriction of ψ to $F^1\mathcal{H}_b \subset F^1\mathcal{H}$ is simply the inclusion $F^1\mathcal{H}_b \subset H_{\mathbb{C},b}$ and hence $d\psi_\lambda$ is the identity on $T_{F^1\mathcal{H}_b,\lambda} = F^1\mathcal{H}_b$. □

Conversely, the commutative diagram from above also shows that $\overline{\nabla}_b(\overline{\lambda})$ is surjective if $d\psi_\lambda$ is surjective, because any tangent vector in $T_{F^1\mathcal{H},\lambda}$ is modulo $T_{F^1\mathcal{H}_b,\lambda} = F^1\mathcal{H}_b$ in the image of $d\alpha_b$ for a suitable section α of $F^1\mathcal{H}$ with $\alpha(b) = \lambda$.

Proposition 3.3 has numerous applications. For instance, Voisin used this infinitesimal condition in [Voi06] when proving the integral Hodge conjecture for $(2,2)$-classes on uniruled or Calabi–Yau threefolds. More recently, a real analogue of the criterion was applied in [Ben18] to prove that sums of three squares are dense among bivariate positive semidefinite real polynomials.

It is often not easy to verify the condition that $\overline{\nabla}_b(\overline{\lambda})$ is surjective somewhere and different strategies have been developed to accomplish this task. While [Ben18] follows the approach of [CL91] by constructing components of the Noether–Lefschetz locus of maximal codimension, Kim gave in [Kim91, Theorem 2] a new proof of the density theorem from [CHM88, Section 5] by proving a statement about the Jacobian rings appearing in the description of $\overline{\nabla}_b(\overline{\lambda})$. The most general arguments are due to Voisin, for example in [Voi00] and [Voi06].

In the next chapter, we present two applications of Proposition 3.3 where we use Kim's method of computing the infinitesimal period map explicitly, most prominently we give a proof of Theorem 1.1. However, we solve the underlying algebraic problem in a different manner than in [Kim91, Section 3]. Our approach involving the strong Lefschetz property, the use of which seems to be new in this area, also allows to give a short proof for the density of the original Noether–Lefschetz locus for surfaces in $\mathbb{P}^3$.

Note that the density criterion is not directly transferable to variations of Hodge structure of weight $k = 2p > 2$, because by Griffiths transversality, $\overline{\nabla}_b(\overline{\lambda})\colon T_{B,b} \to \mathcal{H}_b/F^p\mathcal{H}_b$ cannot be surjective unless $F^{p+1}\mathcal{H} = \mathcal{H}$.

4 Applications

In this chapter, we come back to an algebraic setting. We want to prove Theorem 1.1 by putting together the results of Chapters 2 and 3. For this, we need to show that the infinitesimal period map is surjective somewhere. To tackle this, we regard standard quadric surface bundles as toric hypersurfaces and explicitly describe their middle cohomology groups and the map $\overline{\nabla}$ by polynomials. We then arrive at the question whether, under some genericity conditions, certain polynomials in a bigraded polynomial ring generate all polynomials of a specific bidegree. The theory of Lefschetz properties will help us to solve this problem with an involved but quite elementary approach.

As another application of Proposition 3.3, we reprove the density result for the original Noether–Lefschetz locus. It turns out that the surjectivity of the respective multiplication map is a direct consequence of a classical result of Stanley [Sta80] and Watanabe [Wat87] concerning the strong Lefschetz property.

4.1 Standard quadric surface bundles

The previous two chapters were motivated by our aim to prove Theorem 1.1, which states that the locus

$$\{b \in B \mid \mathcal{X}_b \text{ is rational}\} \tag{4.1}$$

is analytically dense in the moduli space B for the universal family $\mathcal{X} \to B$ of smooth quadric surface bundles of type (d_0, d_1, d_2, d_3). We yet have to explain how this family is defined.

In Chapter 1, we introduced standard quadric surface bundles over $\mathbb{P}^2$ as zero sets of equations of the form

$$\sum_{0 \le i,j \le 3} a_{ij} y_i y_j = 0\ . \tag{4.2}$$

We first give a more precise definition of standard quadric surface bundles, following [Sch18a, Section 3.5]. For integers $r_0, r_1, r_2, r_3 \ge 0$, let us consider the vector bundle

$$\mathcal{E} = \bigoplus_{j=0}^{3} \mathcal{O}_{\mathbb{P}^2}(-r_j)$$

on $\mathbb{P}^2$. For some integer $d \ge 0$, let $q \colon \mathcal{E} \to \mathcal{O}_{\mathbb{P}^2}(d)$ be a line bundle valued quadratic form on $\mathcal{E}$, i. e. a global section of $\operatorname{Sym}^2 \mathcal{E}^\vee \otimes \mathcal{O}_{\mathbb{P}^2}(d)$. We assume that the quadratic form q_η at the generic point $\eta \in \mathbb{P}^2$ is non-degenerate and that $q_s \neq 0$ for all $s \in \mathbb{P}^2$. Then the zero set $X = \{q = 0\} \subset \mathbb{P}(\mathcal{E})$ is a quadric surface bundle over $\mathbb{P}^2$. Indeed, the non-degeneracy of q_η implies that the generic fibre X_η is a smooth quadric surface, and the condition $q_s \neq 0$ for all $s \in \mathbb{P}^2$ implies that all fibres of the projection $\pi\colon X \subset \mathbb{P}(\mathcal{E}) \to \mathbb{P}^2$ have the same Hilbert polynomial. Since X and $\mathbb{P}^2$ are projective varieties, this is equivalent to the flatness of the morphism $\pi\colon X \to \mathbb{P}^2$.

The deformation type of X only depends on the integers $d_j = 2r_j + d$ for $j \in \{0,1,2,3\}$. We call X a *standard quadric surface bundle* over $\mathbb{P}^2$ of type (d_0, d_1, d_2, d_3). Conversely, quadric surface bundles of type (d_0, d_1, d_2, d_3) for given integers $d_j \geq 0$ exist whenever d_0, d_1, d_2, d_3 are of the same parity.

Let
$$V = H^0\left(\mathbb{P}^2, \operatorname{Sym}^2 \mathcal{E}^\vee \otimes \mathcal{O}_{\mathbb{P}^2}(d)\right) \cong \bigoplus_{0 \leq i \leq j \leq 3} H^0\left(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(r_i + r_j + d)\right) .$$
Under this isomorphism, a quadratic form $q \in V$ is indeed given by an equation of the form (4.2) with homogeneous polynomials a_{ij} of degree $r_i + r_j + d = \frac{1}{2}(d_i + d_j)$, where y_j is a local trivialization of $\mathcal{O}_{\mathbb{P}^2}(-r_j)$. Since the standard quadric surface bundle associated to $q \in V$ is just $X = \{q = 0\} \subset \mathbb{P}(\mathcal{E})$, we may parametrize the smooth quadric surface bundles of type (d_0, d_1, d_2, d_3) by a non-empty Zariski open subset $B \subset \mathbb{P}(V)$ in the projectivization of the finite dimensional vector space V. In this way, we obtain a family $\mathcal{X} \to B$ with fibre $\mathcal{X}_b = \{q = 0\}$ at $b = \mathbb{C} \cdot q \in B$. This is called the *universal family* of smooth quadric surface bundles of type (d_0, d_1, d_2, d_3), which we referred to in Theorem 1.1.

In order to prove this theorem, it is enough by Corollary 2.6 to show that the Noether–Lefschetz locus
$$\left\{ b \in B \;\middle|\; \exists \alpha \in H^{2,2}(\mathcal{X}_b, \mathbb{Z})\colon \alpha \cup [(\mathcal{X}_b)_\eta] \equiv 1 \pmod 2 \right\}$$
is dense in B for the Euclidean topology. Since it is easier to compute, we consider instead the *vanishing cohomology*
$$H^4_{\mathrm{van}}(\mathcal{X}_b, \mathbb{C}) = \{\alpha \in H^4(\mathcal{X}_b, \mathbb{C}) \mid \alpha \cup \iota^*\beta = 0 \ \forall \beta \in H^4(\mathbb{P}(\mathcal{E}), \mathbb{C})\}$$
where the map $\iota^*\colon H^4(\mathbb{P}(\mathcal{E}), \mathbb{C}) \hookrightarrow H^4(\mathcal{X}_b, \mathbb{C})$ is induced by inclusion and is injective by the Lefschetz hyperplane theorem. This construction is also applicable to the Hodge groups $H^{p,q}$ and gives a Hodge decomposition
$$H^4_{\mathrm{van}}(\mathcal{X}_b, \mathbb{C}) = \bigoplus_{p+q=4} H^{p,q}_{\mathrm{van}}(\mathcal{X}_b) .$$

Using Proposition 3.3, we then want to show that the possibly smaller locus
$$\left\{ b \in B \;\middle|\; \exists \alpha \in H^{2,2}_{\mathrm{van}}(\mathcal{X}_b, \mathbb{Z})\colon \alpha \cup [(\mathcal{X}_b)_\eta] \equiv 1 \pmod 2 \right\} \tag{4.3}$$
is dense in B for the Euclidean topology. We will now explain how our family $\mathcal{X} \to B$ fits to the setup required by Proposition 3.3.

As for the usual integral cohomology, there exists a local system H of finitely generated free $\mathbb{Z}$-modules on B with stalk $H_b = H^4_{\mathrm{van}}(\mathcal{X}_b, \mathbb{Z})$ at $b \in B$. This is because the definition of vanishing cohomology is independent of the complex structure on $\mathcal{X}_b$. Since $H^{4,0}_{\mathrm{van}}(\mathcal{X}_b) = H^{0,4}_{\mathrm{van}}(\mathcal{X}_b) = 0$ for all $b \in B$ (for example, this follows from the explicit computation of the middle cohomology groups of $\mathcal{X}_b$ in Section 4.2), the local system H is endowed with a variation of Hodge structure of weight 2

$$F^2\mathcal{H} \subset F^1\mathcal{H} \subset F^0\mathcal{H} = \mathcal{H}$$

corresponding to the Hodge filtrations on the fibres $\mathcal{H}_b = H^4_{\mathrm{van}}(\mathcal{X}_b, \mathbb{C})$ at $b \in B$. Note that we reuse the notation introduced in Chapter 3. In particular, the infinitesimal period map

$$\overline{\nabla} \colon \mathcal{H}^{1,1} \to \mathcal{H}^{0,2} \otimes \Omega_B$$

is defined. We may identify $\mathcal{H}^{p,q}_b$ with $H^{p+1,q+1}_{\mathrm{van}}(\mathcal{X}_b)$ for $p + q = 2$ and $b \in B$.

For all $b \in B$, let us consider the discrete subset

$$D_b = \{\alpha \in H^4_{\mathrm{van}}(\mathcal{X}_b, \mathbb{Z}) \mid \alpha \cup [(\mathcal{X}_b)_\eta] \equiv 1 \pmod 2\} \subset H_{\mathbb{R},b}\ .$$

Since the definition of D_b is purely topological and thus compatible with the local trivializations of $\mathcal{X} \to B$ from Ehresmann's lemma, we obtain a local subsystem of sets $D \subset H_{\mathbb{R}}$. Note that the locus (4.3) is precisely the Noether–Lefschetz locus NL_D from Section 3.3. We are now able to prove the following result:

Proposition 4.1. *Let $\mathcal{X} \to B$ be the universal family of smooth quadric surface bundles over $\mathbb{P}^2$ of type (d_0, d_1, d_2, d_3). Suppose there exists a point $b \in B$ and a class $\overline{\lambda} \in H^{2,2}_{van}(\mathcal{X}_b)$ such that the infinitesimal period map evaluated at $\overline{\lambda}$*

$$\overline{\nabla}_b(\overline{\lambda}) \colon T_{B,b} \to H^{1,3}_{van}(\mathcal{X}_b)$$

is surjective. Then the set

$$\{b \in B \mid \mathcal{X}_b \text{ is rational}\}$$

is dense in B for the Euclidean topology.

Proof. In view of Proposition 3.3, it remains to show that $\mathbb{R}^* D_b \subset H_{\mathbb{R},b}$ is dense for at least one $b \in B$. By [Sch18a, Lemma 20], there exists a smooth quadric surface bundle $\mathcal{X}_b$ of type (d_0, d_1, d_2, d_3) which admits a rational section. Therefore, $D_b \neq \emptyset$ for this $b \in B$. By definition, D_b is then a coset of a subgroup of H_b of index 2. Since the integral classes H_b form a lattice in $H_{\mathbb{R},b} = H_b \otimes \mathbb{R}$, it is easy to see that $\mathbb{R}^* D_b$ is dense in $H_{\mathbb{R},b}$. □

4.2 Computation of the cohomology

In order to prove Theorem 1.1, we need to understand the infinitesimal period map $\overline{\nabla}$ for our family $\mathcal{X} \to B$. This will be done by using Griffiths description of $\overline{\nabla}$ and an explicit computation of the middle cohomology groups of standard quadric surface bundles using results of [BC94] for toric hypersurfaces, which generalize earlier results of Griffiths [Gri68] for projective hypersurfaces. To this end, we aim to interpret equation (4.2) differently as a global equation inside the polynomial ring

$$S = \mathbb{C}[x_0, x_1, x_2; y_0, y_1, y_2, y_3] \ .$$

The equation will turn out to be homogeneous with respect to a non-standard bigrading.

By [CLS11, Example 7.3.5], the total space $\mathbb{P}(\mathcal{E})$ of a split vector bundle

$$\mathcal{E} = \bigoplus_{j=0}^{3} \mathcal{O}_{\mathbb{P}^2}(-r_j)$$

is a toric variety associated to a fan Σ in $\mathbb{R}^2 \times \mathbb{R}^3$ and has coordinate ring S. If u_1, u_2 and v_1, v_2, v_3 denote the standard basis vectors of $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively, then the seven 1-dimensional cones of Σ are generated by $u_0, u_1, u_2, v_0, v_1, v_2, v_3$ where

$$u_0 = -\sum_{i=1}^{2} u_i + \sum_{j=1}^{3} (r_j - r_0) v_j \quad \text{and} \quad v_0 = -\sum_{j=1}^{3} v_j \ .$$

Further, the maximal cones of Σ are given by

$$\langle u_0, \dots, \hat{u}_i, \dots, u_2, v_0, \dots, \hat{v}_j, \dots, v_3 \rangle \ , \quad i \in \{0,1,2\} \ , \quad j \in \{0,1,2,3\} \ .$$

By [BC94, Definition 1.7], we have $\mathrm{Cl}(\Sigma) \cong \mathbb{Z}^7 / \operatorname{Im} C$ where

$$C = \begin{pmatrix} -1 & -1 & r_1 - r_0 & r_2 - r_0 & r_3 - r_0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \in \operatorname{Hom}(\mathbb{Z}^5, \mathbb{Z}^7) \ .$$

It is easy to see that the surjection

$$\begin{aligned} \mathbb{Z}^7 &\to \mathbb{Z}^2 \\ (m_0, m_1, m_2, n_0, n_1, n_2, n_3) &\mapsto \left(\sum_{i=0}^{2} m_i - \sum_{j=0}^{3} r_j n_j, \sum_{j=0}^{3} n_j \right) \end{aligned}$$

has kernel $\operatorname{Im} C$. Hence, this map descends to an isomorphism $\mathrm{Cl}(\Sigma) \cong \mathbb{Z}^2$ and endowes the coordinate ring S with the non-standard bigrading

$$\deg x_i = (1, 0) \ , \quad \deg y_j = (-r_j, 1)$$

for $i \in \{0,1,2\}$ and $j \in \{0,1,2,3\}$.

For $m,n \in \mathbb{Z}$, we denote by $S(m,n)$ the subspace of homogeneous polynomials of bidegree (m,n) in S. This gives a decomposition

$$S = \bigoplus_{m,n\in\mathbb{Z}} S(m,n)$$

into finite dimensional complex vector spaces.

A quadratic form $q\colon \mathcal{E} \to \mathcal{O}_{\mathbb{P}^2}(d)$ corresponds to an element in $S(d,2)$. In this way, the local description (4.2) of the zero set of q can be seen globally as a defining equation for a toric hypersurface $X \subset \mathbb{P}(\mathcal{E})$.

This allows us to compute the middle cohomology groups of a smooth quadric surface bundle $\pi\colon X \to \mathbb{P}^2$ of type (d_0,d_1,d_2,d_3) defined by a polynomial $f \in S(d,2)$ via the method of [BC94]. According to [BC94, Theorem 10.13], we have

$$H^{1,3}_{\mathrm{van}}(X) = R(t,4) \quad \text{and} \quad H^{2,2}_{\mathrm{van}}(X) = R(t-d,2)$$

where

$$t = 4d - 3 + r_0 + r_1 + r_2 + r_3$$

and where R denotes the Jacobian ring of f, i. e. the quotient of S by all partial derivatives of f. We also see that $H^{4,0}_{\mathrm{van}}(X) = R(t-3d,-2) = 0$ and hence $H^{0,4}_{\mathrm{van}}(X) = 0$, as asserted in Section 4.1.

Now we return to the family $\mathcal{X} \to B$ of smooth quadric surface bundles of type (d_0,d_1,d_2,d_3). If we identify $T_{B,b} \cong (S/fS)(d,2)$ where $f \in S(d,2)$ is the defining equation of $\mathcal{X}_b$ for some $b \in B$, Griffiths has shown that the infinitesimal period map

$$\overline{\nabla}_b\colon T_{B,b} \otimes H^{2,2}_{\mathrm{van}}(\mathcal{X}_b) \to H^{1,3}_{\mathrm{van}}(\mathcal{X}_b)$$

is given, up to a sign, as the multiplication map

$$(S/fS)(d,2) \otimes R(t-d,2) \to R(t,4)\ .$$

In order to show that the assumption of Proposition 4.1 holds, it therefore suffices to provide polynomials $f \in S(d,2)$ and $g \in S(t-d,2)$ such that the quadric surface bundle $\{f = 0\} \subset \mathbb{P}(\mathcal{E})$ is smooth and the composed map $S(d,2) \to R(t,4)$ given by multiplication with g followed by projection is surjective. By Bertini's theorem, the hypersurface $\{f = 0\} \subset \mathbb{P}(\mathcal{E})$ is smooth for a general polynomial $f \in S(d,2)$. The surjectivity part is equivalent to claiming that the ideal generated by g and all partial derivatives of f contains all polynomials in $S(t,4)$. Consequently, we can reduce Theorem 1.1 to an elementary statement:

Proposition 4.2. *Let $\mathcal{X} \to B$ be the universal family of smooth quadric surface bundles over $\mathbb{P}^2$ of type (d_0, d_1, d_2, d_3). Let $r_0, r_1, r_2, r_3 \geq 0$ be integers such that $d_j = 2r_j + d$ for some integer $d \geq 0$. Let S and t be defined as above. Suppose that for general polynomials $f \in S(d, 2)$ and $g \in S(t-d, 2)$, the ideal in S generated by the polynomials*

$$\frac{\partial f}{\partial x_0}, \quad \frac{\partial f}{\partial x_1}, \quad \frac{\partial f}{\partial x_2}, \quad \frac{\partial f}{\partial y_0}, \quad \frac{\partial f}{\partial y_1}, \quad \frac{\partial f}{\partial y_2}, \quad \frac{\partial f}{\partial y_3}, \quad g$$

contains all polynomials in $S(t, 4)$. Then the locus

$$\{b \in B \mid \mathcal{X}_b \text{ is rational}\}$$

is dense in B for the Euclidean topology.

4.3 The strong Lefschetz property

Proposition 4.2 raises the following question about a bigraded polynomial ring: Under which conditions do sufficiently general polynomials of fixed bidegrees generate all polynomials of a certain bidegree? For the case of singly graded polynomial rings, this question was already studied extensively, though no definitive answer is yet known.

As a first and easy observation, we show that this condition is open in the Zariski topology. Of course, there is nothing special in the fact that the bigrading of our $\mathbb{C}$-algebra S has values in $\mathbb{Z}^2$, just any Abelian group works for this.

Lemma 4.3. *Let G be an Abelian group and let A be a G-graded $\mathbb{C}$-algebra whose homogeneous components $A(m)$ are finite dimensional $\mathbb{C}$-vector spaces for all $m \in G$. Let $m_0, \ldots, m_k \in G$. Then the set*

$$\{(f_1, \ldots, f_k) \in A(m_1) \oplus \cdots \oplus A(m_k) \mid A(m_0) \subset f_1 A + \cdots + f_k A\}$$

is Zariski open.

Proof. The condition on $(f_1, \ldots, f_k)$ is equivalent to saying that the $\mathbb{C}$-linear map

$$\begin{aligned} A(m_0 - m_1) \oplus \cdots \oplus A(m_0 - m_k) &\to A(m_0) \\ (g_1, \ldots, g_k) &\mapsto f_1 g_1 + \cdots + f_k g_k \end{aligned}$$

is surjective. This map is represented by a matrix B with $r = \dim_{\mathbb{C}} A(m_0)$ rows, whose entries are linear polynomials in the coefficients of $f_1, \ldots, f_k$. The locus in $A(m_1) \oplus \cdots \oplus A(m_k)$ where this linear map is not surjective is precisely where the determinants of all $(r \times r)$-submatrices of B vanish (in particular, it is the whole affine space if B has less than r columns) and thus Zariski closed. Therefore, the set in question is open for the Zariski topology. ☐

Since taking partial derivatives is a linear and hence Zariski continuous map between the respective bigraded pieces of S, Lemma 4.3 already shows that the assumption of Proposition 4.2 is Zariski open on f and g.

Apart from S, we will often apply Lemma 4.3 to the polynomial ring $\mathbb{C}[x_0, x_1, x_2]$ together with its usual grading, since due to the chosen bigrading on S, $\mathbb{C}[x_0, x_1, x_2]$ corresponds to the homogeneous elements in S of bidegree $(m, 0)$ for some $m \geq 0$. For this situation, we can give sufficient criteria whether three or four polynomials satisfy the Zariski open condition in the lemma. More generally, for $n \geq 0$ we can give such criteria for $n+1$ and $n+2$ polynomials in the graded polynomial ring

$$P_n = \mathbb{C}[t_0, \dots, t_n] = \bigoplus_{m \geq 0} P_n(m) \ .$$

Lemma 4.4. *If $f_0, \dots, f_n \in P_n$ form a complete intersection, i. e. they have no common zero in $\mathbb{P}^n$, then*

$$P_n(m) \subset f_0 P_n + \cdots + f_n P_n$$

for all $m \geq m_0 + \cdots + m_n - n$ where $f_j \in P_n(m_j)$ for $j \in \{0, \dots, n\}$.

Proof. This immediately follows from Macaulay's Theorem (see for example [Voi03b, Section 6.2.2]) which tells us that the quotient of P_n by the ideal generated by $f_0, \dots, f_n$ is a graded Gorenstein ring with socle degree $\sum(m_j - 1)$, and hence its m-th graded piece is zero-dimensional for all $m \geq \sum m_j - n$. □

To state a sufficient criterion whether $n+2$ polynomials in P_n belong to the Zariski open set in Lemma 4.3, we use the so called *strong Lefschetz property*, see e. g. [Sta80]. A quotient Q of P_n by homogeneous polynomials $f_0, \dots, f_n \in P_n$ is said to have the strong Lefschetz property if there exists a linear homogeneous polynomial $\ell \in P_n(1)$ such that the map $Q(m) \to Q(m+i)$ given by multiplication with ℓ^i has maximal rank for all $m, i \geq 0$. The polynomial ℓ is then called a *strong Lefschetz element* for the system $f_0, \dots, f_n$.

Lemma 4.5. *If $f_0, \dots, f_n \in P_n$ form a complete intersection having the strong Lefschetz property and $f_{n+1} \in P_n$ is a power of a strong Lefschetz element for $f_0, \dots, f_n$, then*

$$P_n(m) \subset f_0 P_n + \cdots + f_{n+1} P_n$$

for all $m \geq \frac{1}{2}(m_0 + \cdots + m_{n+1} - n - 1)$ where $f_j \in P_n(m_j)$ for $j \in \{0, \dots, n+1\}$.

Proof. As in Lemma 4.4, the quotient Q of P_n by $f_0, \dots, f_n$ is a graded Gorenstein ring with socle degree $s = \sum(m_j - 1)$. Macaulay's Theorem also shows that $\dim_{\mathbb{C}} Q(i) = \dim_{\mathbb{C}} Q(s-i)$ for all $i \in \mathbb{Z}$. Because of the strong Lefschetz property, $\dim_{\mathbb{C}} Q(i)$ needs to be increasing for $i \leq \frac{s}{2}$ and decreasing for $i \geq \frac{s}{2}$. The claimed statement is equivalent to saying that the map $Q(m - m_{n+1}) \to Q(m)$ given by multiplication with f_{n+1} is surjective. Since f_{n+1} is a power of a strong Lefschetz element, it suffices to show

$\dim_{\mathbb{C}} Q(m - m_{n+1}) \geq \dim_{\mathbb{C}} Q(m)$. This is clear if $m - m_{n+1} \geq \frac{s}{2}$. For $m - m_{n+1} \leq \frac{s}{2}$, we have $\dim_{\mathbb{C}} Q(m) = \dim_{\mathbb{C}} Q(s-m) \leq \dim_{\mathbb{C}} Q(m - m_{n+1})$ because $s - m \leq m - m_{n+1}$ holds due to the given bound on m. $\square$

To make use of Lemma 4.5, it is convenient to have a rich source of complete intersections enjoying the strong Lefschetz property. The following important result, proved in 1980 by Stanley [Sta80] and independently in 1987 by Watanabe [Wat87], was the starting point for the theory of Lefschetz properties:

Proposition 4.6 (Stanley–Watanabe)**.** *A monomial complete intersection $x_0^{m_0}, \ldots, x_n^{m_n}$ in P_n with $m_0, \ldots, m_n \geq 0$ has the strong Lefschetz property for all $n \geq 0$.*

It is known for $n \leq 1$ and conjectured for $n \geq 2$ that actually all complete intersections in P_n have the strong Lefschetz property. For $n = 2$, the following partial result proven in [HW03, Proposition 30] satisfies our needs for the proof of Theorem 1.1:

Proposition 4.7 (Harima–Watanabe)**.** *If $f_0, f_1, f_2 \in P_2 = \mathbb{C}[x_0, x_1, x_2]$ form a complete intersection such that f_0 is a power of a linear polynomial, then f_0, f_1, f_2 has the strong Lefschetz property.*

4.4 Noether–Lefschetz loci for surfaces

Before we prove Theorem 1.1 via Proposition 4.2, we show how the Lemmas 4.4 and 4.5 from the previous section can also be applied to other contexts where the infinitesimal density criterion of Green and Voisin is used. Specifically, we want to simplify Kim's proof [Kim91, Section 3] for the density of the original Noether–Lefschetz locus for surfaces in $\mathbb{P}^3$. We will solve the elementary problem underlying [Kim91, Theorem 2] in a different manner which does not require the technical statement [Kim91, Proposition 3] anymore. For this, we do not need the more recent result from [HW03] stated in Proposition 4.7, but only Proposition 4.6. Since the setup here is a lot easier than in the case of standard quadric surface bundles, this will also be a good preparation for the more involved arguments in Section 4.5.

Let $X \subset \mathbb{P}^3$ be a surface of degree $d \geq 4$. A natural question about X is whether any curve on X is a complete intersection. In terms of the class group of X which is isomorphic to the Picard group $\operatorname{Pic} X$ of isomorphism classes of holomorphic line bundles on X, this question asks whether $\operatorname{Pic} X$ is generated by the class of the intersection of X with a plane in $\mathbb{P}^3$, i. e. by the restriction of the line bundle $\mathcal{O}_{\mathbb{P}^3}(1)$. By the Lefschetz $(1,1)$-theorem, the exponential exact sequence induces an exact sequence

$$0 \to H^1(X, \mathbb{Z}) \to H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X^*) \to H^{1,1}(X, \mathbb{Z}) \to 0$$

where

$$H^1(X, \mathcal{O}_X) \cong H^{0,1}(X) \subset H^1(X, \mathbb{C}) \quad \text{and} \quad H^1(X, \mathcal{O}_X^*) \cong \operatorname{Pic} X\ .$$

The image of an element of $\operatorname{Pic} X$ in $H^{1,1}(X,\mathbb{Z})$ is also called its *first Chern class*. By the Lefschetz hyperplane theorem, we have

$$H^1(X,\mathbb{C}) \cong H^1(\mathbb{P}^3,\mathbb{C}) = 0 ,$$

hence the first Chern class gives an isomorphism $\operatorname{Pic} X \cong H^{1,1}(X,\mathbb{Z})$ in our case. Choosing the Fubini–Study metric on $\mathbb{P}^3$, the class of the Kähler form is integral and coincides with the first Chern class of the line bundle $\mathcal{O}_{\mathbb{P}^3}(1)$. By restriction, we obtain an integral Kähler class on X, which is the Poincaré dual of the homology class of the intersection of X with a plane in $\mathbb{P}^3$. The question whether $\operatorname{Pic} X$ is generated by $\mathcal{O}_X(1) = \mathcal{O}_{\mathbb{P}^3}(1)|_X$ is therefore equivalent to asking whether the integral *primitive cohomology*

$$H^{1,1}_{\mathrm{pr}}(X,\mathbb{Z}) = \{\alpha \in H^{1,1}(X,\mathbb{Z}) \mid \alpha \cup [\omega_X] = 0\}$$

vanishes. Here, $\omega_X \in H^0(X,\Omega^2_X)$ denotes the restriction of the integral Kähler form $\omega_{\mathbb{P}^3}$ on $\mathbb{P}^3$ to X. Since $H^2(\mathbb{P}^3,\mathbb{Z})$ is generated by $[\omega_{\mathbb{P}^3}]$, $H^{1,1}_{\mathrm{pr}}(X,\mathbb{Z})$ actually agrees with the vanishing cohomology we considered in Section 4.1 for standard quadric surface bundles. In particular, $H^{1,1}_{\mathrm{pr}}(X,\mathbb{Z})$ is defined via a purely topological property.

Let us consider the family $\mathcal{X} \to B$ of all smooth surfaces in $\mathbb{P}^3$ having degree d, so $B \subset \mathbb{P}(W)$ is a non-empty Zariski open subset in the projectivization of the $\binom{d+3}{3}$-dimensional vector space

$$W = H^0\left(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(d)\right) = P_3(d) .$$

The Noether–Lefschetz theorem says that for a very general $b \in B$, any curve on $\mathcal{X}_b$ is a complete intersection. In other words, the *Noether–Lefschetz locus*

$$\mathrm{NL}_d = \left\{ b \in B \;\middle|\; H^{1,1}_{\mathrm{pr}}(\mathcal{X}_b,\mathbb{Z}) \neq 0 \right\}$$

is contained in a countable union of proper closed subvarieties of B.

In [CHM88, Section 5] and [Kim91, Section 3] it was shown that NL_d is analytically dense in B for all $d \geq 4$. Green's argument in [CHM88, Section 5] was the foundation for the infinitesimal density criterion developed in Chapter 3, but works a bit differently than most later uses of this criterion, since the proof goes by providing one component of NL_d having maximal codimension in B and then deriving from this the density of the union of all such components.

The proof in [Kim91, Section 3] is closer to the later approaches in [HPT18a] and [HPT17], and also to our method, as it tries to explicitly prove the surjectivity of $\overline{\nabla}_b(\overline{\lambda})$ for at least one b and $\overline{\lambda}$. However, Kim introduces an additional step in her proof by first dualizing the map $\overline{\nabla}_b(\overline{\lambda})$ and then using Macaulay's Theorem in order to show an injectivity result instead. With the help of Lemma 4.5, our proof for the density of the Noether–Lefschetz locus for surfaces is fairly straightforward.

Theorem 4.8. *For $d \geq 4$, let $f\colon \mathcal{X} \to B \subset \mathbb{P}(W)$ be the family of smooth surfaces in $\mathbb{P}^3$ of degree d as above. Then the Noether–Lefschetz locus*

$$\mathrm{NL}_d = \left\{ b \in B \;\middle|\; H^{1,1}_{\mathrm{pr}}(\mathcal{X}_b, \mathbb{Z}) \neq 0 \right\}$$

is dense in B for the Euclidean topology.

Proof. Let H be the local system on B with stalk $H_b = H^2_{\mathrm{pr}}(\mathcal{X}_b, \mathbb{Z})$ at $b \in B$ and let

$$F^2\mathcal{H} \subset F^1\mathcal{H} \subset F^0\mathcal{H} = \mathcal{H}$$

be the associated variation of Hodge structure of weight 2 on $\mathcal{H} = H \otimes \mathcal{O}_B$, as explained in Section 3.1. Note that NL_d is precisely the classical Noether–Lefschetz locus from Section 3.3. By Proposition 3.2, it suffices to show that there exists a point $b \in B$ and a class $\overline{\lambda} \in H^{1,1}_{\mathrm{pr}}(\mathcal{X}_b)$ such that the infinitesimal period map evaluated at $\overline{\lambda}$

$$\overline{\nabla}_b(\overline{\lambda})\colon T_{B,b} \to H^{0,2}_{\mathrm{pr}}(\mathcal{X}_b)$$

is surjective.

For a surface $X \subset \mathbb{P}^3$ defined by a polynomial $f \in P_3(d)$, by [BC94, Theorem 10.13] (or the earlier results of Griffiths) we have

$$H^{0,2}_{\mathrm{pr}}(X) = R(3d-4) \quad \text{and} \quad H^{1,1}_{\mathrm{pr}}(X) = R(2d-4)$$

where R denotes the Jacobian ring of f, i. e. the quotient of P_3 by the four partial derivatives of f. If we identify $T_{B,b} \cong (P_3/fP_3)(d)$ where $f \in P_3(d)$ is the defining equation of $\mathcal{X}_b$ for some $b \in B$, Griffiths has shown that the infinitesimal period map

$$\overline{\nabla}_b\colon T_{B,b} \otimes H^{1,1}_{\mathrm{pr}}(\mathcal{X}_b) \to H^{0,2}_{\mathrm{pr}}(\mathcal{X}_b)$$

is given, up to a sign, as the multiplication map

$$(P_3/fP_3)(d) \otimes R(2d-4) \to R(3d-4)\ .$$

Therefore, it suffices to find polynomials $f \in P_3(d)$ and $g \in P_3(2d-4)$ such that the surface $\{f = 0\} \subset \mathbb{P}^3$ is smooth and the ideal generated by g and the partial derivatives of f contains the whole of $P_3(3d-4)$.

One can achieve this with the Fermat surface defined by

$$f = x_0^d + x_1^d + x_2^d + x_3^d\ ,$$

which was also used in [Kim91, Section 3]. Clearly, the surface $\{f = 0\} \subset \mathbb{P}^3$ is smooth. Since the complete intersection $x_0^{d-1}, x_1^{d-1}, x_2^{d-1}, x_3^{d-1}$ consisting of the partial derivatives of f has the strong Lefschetz property by Proposition 4.6, we can take g to be a power of a corresponding strong Lefschetz element and obtain via Lemma 4.5

$$P_3(m) \subset x_0^{d-1}P_3 + x_1^{d-1}P_3 + x_2^{d-1}P_3 + x_3^{d-1}P_3 + gP_3$$

for all $m \geq \frac{1}{2}(4(d-1) + 2d - 4 - 4) = 3d - 6$. Since $3d - 4 \geq 3d - 6$, this finishes the proof. □

4.5 Proof of the main result

As we have seen in Proposition 4.2, Theorem 1.1 is proven once we can show the following:

Proposition 4.9. *For general polynomials $f \in S(d,2)$ and $g \in S(t-d,2)$, the ideal in S generated by the polynomials*

$$\frac{\partial f}{\partial x_0}\,,\quad \frac{\partial f}{\partial x_1}\,,\quad \frac{\partial f}{\partial x_2}\,,\quad \frac{\partial f}{\partial y_0}\,,\quad \frac{\partial f}{\partial y_1}\,,\quad \frac{\partial f}{\partial y_2}\,,\quad \frac{\partial f}{\partial y_3}\,,\quad g$$

contains all polynomials in $S(t,4)$.

Let us recall that the non-standard bigrading on the polynomial ring

$$S = \mathbb{C}[x_0, x_1, x_2; y_0, y_1, y_2, y_3]$$

is given by

$$\deg x_i = (1,0)\ , \quad \deg y_j = (-r_j, 1)$$

for $i \in \{0,1,2\}$ and $j \in \{0,1,2,3\}$. Here, $r_0, r_1, r_2, r_3 \geq 0$ are integers such that $d_j = 2r_j + d$ for some integer $d \geq 0$. Without loss of generality, let $d_0 \leq d_1 \leq d_2 \leq d_3$ and thus $r_0 \leq r_1 \leq r_2 \leq r_3$. Let us further recall that

$$t = 4d - 3 + r_0 + r_1 + r_2 + r_3\ .$$

By Lemma 4.3, the property stated in Proposition 4.9 is Zariski open on f and g. Hence, it suffices to show the existence of polynomials $f \in S(d,2)$ and $g \in S(t-d,2)$ such that the homogeneous ideal $I \subset S$ generated by

$$\frac{\partial f}{\partial x_0}\,,\quad \frac{\partial f}{\partial x_1}\,,\quad \frac{\partial f}{\partial x_2}\,,\quad \frac{\partial f}{\partial y_0}\,,\quad \frac{\partial f}{\partial y_1}\,,\quad \frac{\partial f}{\partial y_2}\,,\quad \frac{\partial f}{\partial y_3}\,,\quad g$$

contains all polynomials in $S(t,4)$.

Let

$$f = f_0 y_0^2 + f_1 y_1^2 + f_2 y_2^2 + f_3 y_3^2 \in S(d,2)$$

where $f_j \in S(d_j, 0)$ are general for $j \in \{0,1,2,3\}$. Further let

$$g = g_{11} y_1^2 + g_{33} y_3^2 + \sum_{0 \leq i < j \leq 3} g_{ij} y_i y_j \in S(t-d, 2)$$

where $g_{ij} \in S(t-d+r_i+r_j, 0)$ are general for $i,j \in \{0,1,2,3\}$. Instead of proving directly that $S(t,4) \subset I$, we will consider the homogeneous ideal

$$J = \bigoplus_{m,n \in \mathbb{Z}} \{ r \in S(m,n) \mid rS \cap S(t,4) \subset I \}\ ,$$

and aim to show $J = S$. One can think of J as all relations which hold if a polynomial of bidegree $(t, 4)$ is considered modulo I. Since $I \subset J$, the following congruences hold:

$$f_j y_j \equiv 0 \pmod{J}\,, \quad j \in \{0,1,2,3\}\,, \tag{4.4}$$

$$\frac{\partial f_0}{\partial x_i} y_0^2 + \frac{\partial f_1}{\partial x_i} y_1^2 + \frac{\partial f_2}{\partial x_i} y_2^2 + \frac{\partial f_3}{\partial x_i} y_3^2 \equiv 0 \pmod{J}\,, \quad i \in \{0,1,2\}\,, \tag{4.5}$$

$$g_{11} y_1^2 + g_{33} y_3^2 + \sum_{0 \le i < j \le 3} g_{ij} y_i y_j \equiv 0 \pmod{J}\,. \tag{4.6}$$

It suffices to show $S(t, 4) \subset J$. For this it is enough to prove the following four claims for all permutations σ of $\{0, 1, 2, 3\}$:

(i) $y_{\sigma(0)} y_{\sigma(1)} y_{\sigma(2)} \in J$

(ii) $y^3_{\sigma(0)} y_{\sigma(1)} \in J$

(iii) $y^2_{\sigma(0)} y^2_{\sigma(1)} \in J$

(iv) $y^4_{\sigma(0)} \in J$

These claims are proved in the Lemmas 4.10, 4.11, 4.12, and 4.13 below. For each claim, it suffices to show that a given monomial of bidegree $(t, 4)$ containing the specified variables y_j can be reduced to 0 modulo J using the congruences (4.4), (4.5), (4.6), and the previous claims. Actually, the assertion $r_0 \le r_1 \le r_2 \le r_3$ and the congruence (4.6) will not be used in Lemmas 4.10 and 4.11, so we are allowed to restrict ourselves to the case $\sigma = \mathrm{id}$ in these two proofs.

Lemma 4.10. *We have $y_{\sigma(0)} y_{\sigma(1)} y_{\sigma(2)} \in J$ for all permutations σ of $\{0, 1, 2, 3\}$.*

Proof. Without loss of generality, let $\sigma = \mathrm{id}$. We first note that

$$S(d_0 + d_1 + d_2 - 2, 0) \subset f_0 S + f_1 S + f_2 S\,. \tag{4.7}$$

This follows from Lemmas 4.3 and 4.4 because there are complete intersections f_0, f_1, f_2 in $\mathbb{C}[x_0, x_1, x_2]$. Now let us take a monomial $h y_0 y_1 y_2 y_j \in S(t, 4)$ where $j \in \{0, 1, 2, 3\}$ and $h \in S(t + r_0 + r_1 + r_2 + r_j, 0)$. We may assume that $r_j > 0$ or $d > 0$, since for $d_j = 2r_j + d = 0$ we have $y_j \equiv 0 \pmod{J}$ by (4.4) and hence $h y_0 y_1 y_2 y_j \equiv 0 \pmod{J}$. In view of (4.4) and (4.7), it suffices to show that

$$t + r_0 + r_1 + r_2 + r_j \ge d_0 + d_1 + d_2 - 2\,.$$

This is equivalent to

$$2r_0 + 2r_1 + 2r_2 + r_3 + r_j + 4d - 3 \ge 2r_0 + 2r_1 + 2r_2 + 3d - 2$$

or just $r_3 + r_j + d \ge 1$, which is true because $r_j > 0$ or $d > 0$. ∎

Lemma 4.11. *We have $y^3_{\sigma(0)} y_{\sigma(1)} \in J$ for all permutations σ of $\{0, 1, 2, 3\}$.*

Proof. Without loss of generality, let $\sigma = \mathrm{id}$. Multiplying (4.5) with y_0y_1 and using Lemma 4.10 yields

$$\left(\frac{\partial f_0}{\partial x_i}y_0^2 + \frac{\partial f_1}{\partial x_i}y_1^2\right) y_0y_1 \equiv 0 \pmod J , \quad i \in \{0,1,2\} . \tag{4.8}$$

We introduce the new polynomial ring $T = \mathbb{C}[x_0, x_1, x_2; z_0, z_1]$ with the bigrading

$$\deg x_i = (1,0) , \quad \deg z_j = (-d_j, 1)$$

for $i \in \{0,1,2\}$ and $j \in \{0,1\}$.

Claim. We have

$$T(d_0+d_1-3,1) \subset f_0T + f_1T + \left(\frac{\partial f_0}{\partial x_0}z_0 + \frac{\partial f_1}{\partial x_0}z_1\right)T + \left(\frac{\partial f_0}{\partial x_1}z_0 + \frac{\partial f_1}{\partial x_1}z_1\right)T . \tag{4.9}$$

Proof of the claim. The claim is true if $d_0 = 0$ or $d_1 = 0$ because f_0 or f_1 is a unit then. If $d_0, d_1 > 0$, setting $f_0 = (x_0+x_1)^{d_0} + x_2^{d_0}$ and $f_1 = (x_0 - x_1)^{d_1} + x_2^{d_1}$ yields

$$\left(\frac{\partial f_0}{\partial x_0}z_0 + \frac{\partial f_1}{\partial x_0}z_1\right) + \left(\frac{\partial f_0}{\partial x_1}z_0 + \frac{\partial f_1}{\partial x_1}z_1\right) = 2d_0(x_0+x_1)^{d_0-1}z_0 .$$

Since $(x_0+x_1)^{d_0-1}, f_0, f_1$ form a complete intersection in $\mathbb{C}[x_0,x_1,x_2]$, Lemma 4.4 implies that (4.9) holds for all polynomials in $T(d_0+d_1-3,1)$ of type hz_0 where $h \in T(2d_0+d_1-3,0)$. Similarly,

$$\left(\frac{\partial f_0}{\partial x_0}z_0 + \frac{\partial f_1}{\partial x_0}z_1\right) - \left(\frac{\partial f_0}{\partial x_1}z_0 + \frac{\partial f_1}{\partial x_1}z_1\right) = 2d_1(x_0-x_1)^{d_1-1}z_1$$

and $(x_0-x_1)^{d_1-1}, f_0, f_1$ are again a complete intersection, so all polynomials in $T(d_0 + d_1 - 3, 1)$ divisible by z_1 fulfill (4.9) as well. Hence, the claim follows from Lemma 4.3 applied the polynomial ring T, since the coefficients of the four polynomials which are supposed to generate $T(d_0+d_1-3,1)$ depend linearly and thus Zariski continuously on those of the general polynomials f_0 and f_1. □

Now let us take a monomial $hy_0^3y_1 \in S(t,4)$ where $h \in S(t+3r_0+r_1, 0)$. We have

$$t + 3r_0 + r_1 = 4r_0 + 2r_1 + r_2 + r_3 + 4d - 3 \geq 4r_0 + 2r_1 + 3d - 3 = 2d_0 + d_1 - 3 .$$

Therefore, as a consequence of (4.9) we obtain

$$hz_0 = h_0f_0 + h_1f_1 + h_2\left(\frac{\partial f_0}{\partial x_0}z_0 + \frac{\partial f_1}{\partial x_0}z_1\right) + h_3\left(\frac{\partial f_0}{\partial x_1}z_0 + \frac{\partial f_1}{\partial x_1}z_1\right)$$

for certain polynomials $h_0, h_1, h_2, h_3 \in T$. Substituting z_j by y_j^2 for $j \in \{0,1\}$ and multiplying with y_0y_1, we get by (4.4) and (4.8)

$$\begin{aligned} hy_0^3y_1 &= \tilde{h}_0f_0y_0y_1 + \tilde{h}_1f_1y_0y_1 + h_2\left(\frac{\partial f_0}{\partial x_0}y_0^2 + \frac{\partial f_1}{\partial x_0}y_1^2\right)y_0y_1 + h_3\left(\frac{\partial f_0}{\partial x_0}y_0^2 + \frac{\partial f_1}{\partial x_0}y_1^2\right)y_0y_1 \\ &\equiv \tilde{h}_0y_1\cdot 0 + \tilde{h}_1y_0 \cdot 0 + h_2 \cdot 0 + h_3 \cdot 0 \equiv 0 \pmod J \end{aligned}$$

where $\tilde{h}_0$ and $\tilde{h}_1$ denote the results of the substitution inside h_0 and h_1. □

Lemma 4.12. *We have* $y^2_{\sigma(0)} y^2_{\sigma(1)} \in J$ *for all permutations* σ *of* $\{0,1,2,3\}$.

Proof. Multiplying (4.6) with $y_i y_j$ for $0 \le i < j \le 3$ and using Lemmas 4.10 and 4.11, we obtain

$$g_{ij} y_i^2 y_j^2 \equiv 0 \pmod{J} . \tag{4.10}$$

For $j \in \{0,1,2,3\}$, let $\hat{A}_j$ be the (3×3)-matrix where we leave out the j-th column (counted from 0) of the matrix

$$\left(\frac{\partial f_j}{\partial x_i}\right)_{\substack{i\in\{0,1,2\}\\ j\in\{0,1,2,3\}}} .$$

An easy calculation shows that (4.5) implies

$$\left(\det \hat{A}_j\right) y_i^2 \equiv \varepsilon_{ij} \left(\det \hat{A}_i\right) y_j^2 \pmod{J} , \quad i,j \in \{0,1,2,3\} \tag{4.11}$$

where $\det \hat{A}_j \in S(d_0+d_1+d_2+d_3-d_j-3,0)$ for $j \in \{0,1,2,3\}$ and $\varepsilon_{ij} \in \{\pm 1\}$ is a sign depending on $i,j \in \{0,1,2,3\}$. If $d_0 = 0$, we have $\frac{\partial f_0}{\partial x_i} = 0$ for $i \in \{0,1,2\}$ and hence $\det \hat{A}_i = \det \hat{A}_j = 0$ for $i,j \in \{1,2,3\}$ which makes (4.11) useless in these cases. However, if we define in the case $d_0 = 0$ the matrix $\hat{A}_j$ for $j \in \{1,2,3\}$ to be the (2×2)-matrix where one leaves out the j-th column (counted from 1) of the matrix

$$\left(\frac{\partial f_j}{\partial x_i}\right)_{\substack{i\in\{0,1\}\\ j\in\{1,2,3\}}}$$

we observe that because (4.4) implies $y_0 \equiv 0 \pmod{J}$ one can still conclude from (4.5) that

$$\left(\det \hat{A}_j\right) y_i^2 \equiv \varepsilon_{ij} \left(\det \hat{A}_i\right) y_j^2 \pmod{J} , \quad i,j \in \{1,2,3\} \tag{4.12}$$

where $\det \hat{A}_j \in S(d_1+d_2+d_3-d_j-2,0)$ for $j \in \{1,2,3\}$ and $\varepsilon_{ij} \in \{\pm 1\}$ may be different for $i,j \in \{1,2,3\}$.

Let us first suppose that $\{\sigma(0),\sigma(1)\} = \{1,2\}$. Multiplying (4.6) with y_2^2 and using Lemmas 4.10 and 4.11 yields

$$g_{11} y_1^2 y_2^2 + g_{33} y_2^2 y_3^2 \equiv 0 \pmod{J} . \tag{4.13}$$

Let us consider the polynomial ring $U = \mathbb{C}[x_0,x_1,x_2;z_1,z_3]$ with the bigrading

$$\deg x_i = (1,0) , \quad \deg z_j = (-d_j,1)$$

for $i \in \{0,1,2\}$ and $j \in \{1,3\}$. We claim that

$$U(t-d+2r_2,1) \subset K , \tag{4.14}$$

where K denotes the ideal in U generated by

$$f_1 z_1 , \quad f_2 , \quad f_3 z_3 , \quad g_{12} z_1 , \quad g_{23} z_3 , \quad g_{11} z_1 + g_{33} z_3 , \quad \left(\det \hat{A}_3\right) z_1 - \varepsilon_{13} \left(\det \hat{A}_1\right) z_3 .$$

Since the coefficients of these seven polynomials in U depend algebraically on those of $f_0, f_1, f_2, f_3, g_{11}, g_{12}, g_{23}, g_{33}$, Lemma 4.3 with $A = U$ shows that it is enough to provide a special choice for the general polynomials $f_j, g_{ij} \in \mathbb{C}[x_0,x_1,x_2]$ making (4.14) true.

Claim. This can be achieved in the following way, where $\mu, \nu \in U(1,0)$ denote suitable strong Lefschetz elements of complete intersections that will be specified later:

$$\begin{aligned} f_0 &= x_0^{d_0} & g_{11} &= x_2^{t-d+2r_1} \\ f_1 &= x_0^{d_1} & g_{12} &= \nu^{t-d+r_1+r_2} \\ f_2 &= x_0^{d_2} + x_1^{d_2} & g_{23} &= \mu^{t-d+r_2+r_3} \\ f_3 &= x_0^{d_3} + x_2^{d_3} & g_{33} &= \mu^{t-d+2r_3} \end{aligned}$$

Proof of the claim. The claim is obvious for $d_2 = 0$, so we may assume $d_2 > 0$ in the following. As in the case of the ideal I, we consider instead the larger homogeneous ideal

$$L = \bigoplus_{m,n \in \mathbb{Z}} \{r \in U(m,n) \mid rU \cap U(t-d+2r_2, 1) \subset K\}$$

and we want to show that $U(t-d+2r_2,1) \subset L$ (or equivalently, $L = U$). This will be done by proving first $z_1 \in L$ and then $z_3 \in L$. Since $K \subset L$, we have

$$0 \equiv g_{11}z_1 + g_{33}z_3 = g_{11}z_1 + \mu^{r_3-r_2} g_{23}z_3 \equiv g_{11}z_1 \pmod{L} .$$

By Proposition 4.7, the complete intersection f_1, f_2, g_{11} in $\mathbb{C}[x_0, x_1, x_2]$ possesses the strong Lefschetz property. We may thus assume that ν is a strong Lefschetz element for f_1, f_2, g_{11}. Lemma 4.5 then implies

$$z_1 U(m,0) \subset f_1 z_1 U + f_2 z_1 U + g_{11} z_1 U + g_{12} z_1 U \subset L$$

for all $m \geq \frac{1}{2}(d_1 + d_2 + t - d + 2r_1 + t - d + r_1 + r_2 - 3)$. In order to show $z_1 \in L$, we thus need to check that

$$2(t-d+2r_2+d_1) \geq d_1 + d_2 + t - d + 2r_1 + t - d + r_1 + r_2 - 3 .$$

This is equivalent to

$$4r_2 + 2d_1 \geq d_1 + d_2 + 3r_1 + r_2 - 3 ,$$

which simplifies to $r_2 \geq r_1 - 3$. The last inequality is obviously true.

Next we show $z_3 \in L$. If $d_0 > 0$, we have

$$\det \hat{A}_1 = \det \begin{pmatrix} d_0 x_0^{d_0-1} & d_2 x_0^{d_2-1} & d_3 x_0^{d_3-1} \\ 0 & d_2 x_1^{d_2-1} & 0 \\ 0 & 0 & d_3 x_2^{d_3-1} \end{pmatrix} = d_0 d_2 d_3 x_0^{d_0-1} x_1^{d_2-1} x_2^{d_3-1} .$$

Together with $K \subset L$ and $z_1 \in L$, this implies

$$0 \equiv (d_0 d_2 d_3)^{-1} x_1 x_2 \left(\det \hat{A}_1\right) z_3 = x_0^{d_0-1} x_1^{d_2} x_2^{d_3} z_3 \equiv x_0^{d_0+d_2+d_3-1} z_3 \pmod{L} .$$

Similarly, for $d_0 = 0$ we have

$$\det \hat{A}_1 = \det \begin{pmatrix} d_2 x_0^{d_2-1} & d_3 x_0^{d_3-1} \\ d_2 x_1^{d_2-1} & 0 \end{pmatrix} = -d_2 d_3 x_0^{d_3-1} x_1^{d_2-1}$$

and thus

$$0 \equiv (d_2 d_3)^{-1} x_1 \left(\det \hat{A}_1\right) z_3 = -x_0^{d_3-1} x_1^{d_2} z_3 \equiv x_0^{d_0+d_2+d_3-1} z_3 \pmod{L}$$

as well. By Proposition 4.7, the complete intersection $x_0^{d_0+d_2+d_3-1}, f_2, f_3$ has the strong Lefschetz property. Hence, we may assume that μ is a strong Lefschetz element for $x_0^{d_0+d_2+d_3-1}, f_2, f_3$. Lemma 4.5 implies

$$z_3 U(m,0) \subset x_0^{d_0+d_2+d_3-1} z_3 U + f_2 z_3 U + f_3 z_3 U + g_{23} z_3 U \subset L$$

for all $m \geq \frac{1}{2}(d_0 + d_2 + d_3 - 1 + d_2 + d_3 + t - d + r_2 + r_3 - 3)$. It thus remains to check

$$2(t - d + 2r_2 + d_3) \geq d_0 + d_2 + d_3 - 1 + d_2 + d_3 + t - d + r_2 + r_3 - 3$$

or

$$2r_0 + 2r_1 + 6r_2 + 6r_3 + 8d - 6 \geq 3r_0 + r_1 + 6r_2 + 6r_3 + 8d - 7 \,.$$

This reduces to $r_1 \geq r_0 - 1$, which is clearly true. This finishes the proof of (4.14). □

Now let us take a monomial $h y_1^2 y_2^2 \in S(t,4)$ where $h \in S(t + 2r_1 + 2r_2, 0)$. We have $h z_1 \in U(t - d + 2r_2, 1)$ and thus

$$\begin{aligned} h z_1 &= h_1 f_1 z_1 + h_2 f_2 + h_3 f_3 z_3 + h_4 g_{12} z_1 + h_5 g_{23} z_3 \\ &\quad + h_6 (g_{11} z_1 + g_{33} z_3) + h_7 \left(\left(\det \hat{A}_3\right) z_1 - \varepsilon_{13} \left(\det \hat{A}_1\right) z_3 \right) \end{aligned}$$

for certain polynomials $h_1, \ldots, h_7 \in U$. Substituting z_j by y_j^2 for $j \in \{1,3\}$ and multiplying with y_2^2, we get

$$\begin{aligned} h y_1^2 y_2^2 &= h_1 f_1 y_1^2 y_2^2 + \tilde{h}_2 f_2 y_2^2 + h_3 f_3 y_2^2 y_3^2 + h_4 g_{12} y_1^2 y_2^2 + h_5 g_{23} y_2^2 y_3^2 \\ &\quad + h_6 \left(g_{11} y_1^2 y_2^2 + g_{33} y_2^2 y_3^2 \right) + h_7 \left(\left(\det \hat{A}_3\right) y_1^2 - \varepsilon_{13} \left(\det \hat{A}_1\right) y_3^2 \right) y_2^2 \\ &\equiv h_1 y_1 y_2^2 \cdot 0 + \tilde{h}_2 y_2 \cdot 0 + h_3 y_2^2 y_3 \cdot 0 + h_4 \cdot 0 + h_5 \cdot 0 + h_6 \cdot 0 + h_7 y_2^2 \cdot 0 \\ &\equiv 0 \pmod{J} \end{aligned}$$

where we used the congruences (4.4), (4.10), (4.11), (4.12), and (4.13), and where $\tilde{h}_2$ denotes the result of the substitution inside h_2. This concludes the proof of $y_1^2 y_2^2 \in J$.

At this point, we are ready to handle the general case of $\{\sigma(0), \sigma(1)\}$. For this, we show the following claim:

Claim. Any multiple of $y_{\tau(0)}^2 y_{\tau(1)}^2$ in $S(t,4)$ can be replaced modulo J by a multiple of $y_{\tau(0)}^2 y_{\tau(2)}^2$ in $S(t,4)$ where τ is a permutation of $\{0,1,2,3\}$ such that $\tau(3) < \tau(2)$.

Proof of the claim. In view of (4.4), (4.10), (4.11), and (4.12), it suffices to show that

$$S(t+2r_{\tau(0)}+2r_{\tau(1)},0) \subset f_{\tau(0)}S + f_{\tau(1)}S + g_{\tau(0)\tau(1)}S + \left(\det \hat{A}_{\tau(2)}\right) S \ .$$

This will follow from Lemma 4.3 once we provide a special choice for the general polynomials $f_{\tau(0)}, f_{\tau(1)}, f_{\tau(3)}, g_{\tau(0)\tau(1)}$ satisfying this property. Let $a = d_{\tau(0)}$, $b = d_{\tau(1)}$, and $c = d_{\tau(3)}$. We may assume $a, b > 0$ because otherwise we would already have $y_{\tau(0)}^2 y_{\tau(1)}^2 \equiv 0 \pmod J$ by (4.4). We take

$$f_{\tau(0)} = x_0^a + x_1^a \ , \quad f_{\tau(1)} = x_0^b + x_2^b \ , \quad f_{\tau(3)} = x_0^c \ .$$

If also $c > 0$, we have

$$\det \hat{A}_{\tau(2)} = \pm \det \begin{pmatrix} ax_0^{a-1} & bx_0^{b-1} & cx_0^{c-1} \\ ax_1^{a-1} & 0 & 0 \\ 0 & bx_2^{b-1} & 0 \end{pmatrix} = \pm abc x_0^{c-1} x_1^{a-1} x_2^{b-1} \ .$$

Therefore, we get

$$x_0^{a+b+c-1} \in f_{\tau(0)}S + f_{\tau(1)}S + \left(\det \hat{A}_{\tau(2)}\right) S \ .$$

If $c = 0$, it follows that $d_0 = 0$. Since $a, b > 0$ and $\tau(3) < \tau(2)$, only $\tau(3) = 0$ is possible. Then we have

$$\det \hat{A}_{\tau(2)} = \pm \det \begin{pmatrix} ax_0^{a-1} & bx_0^{b-1} \\ ax_1^{a-1} & 0 \end{pmatrix} = \mp ab x_0^{b-1} x_1^{a-1}$$

und thus again

$$x_0^{a+b+c-1} = x_0^{a+b-1} \in f_{\tau(0)}S + f_{\tau(1)}S + \left(\det \hat{A}_{\tau(2)}\right) S \ .$$

In either case, the complete intersection $x_0^{a+b+c-1}, f_{\tau(0)}, f_{\tau(1)}$ has the strong Lefschetz property by Proposition 4.7, so we may pick for $g_{\tau(0)\tau(1)}$ an adequate power of a strong Lefschetz element and obtain via Lemma 4.5

$$S(m,0) \subset f_{\tau(0)}S + f_{\tau(1)}S + g_{\tau(0)\tau(1)}S + \left(\det \hat{A}_{\tau(2)}\right) S$$

for all $m \geq \frac{1}{2}(a+b+c-1+a+b+t-d+r_{\tau(0)}+r_{\tau(1)}-3)$. Therefore, it remains to prove that

$$2(t+2r_{\tau(0)}+2r_{\tau(1)}) \geq a+b+c-1+a+b+t-d+r_{\tau(0)}+r_{\tau(1)}-3 \ .$$

This simplifies to

$$6r_{\tau(0)}+6r_{\tau(1)}+2r_{\tau(2)}+2r_{\tau(3)}+8d-6 \geq 6r_{\tau(0)}+6r_{\tau(1)}+r_{\tau(2)}+3r_{\tau(3)}+8d-6$$

or just $r_{\tau(2)} \geq r_{\tau(3)}$, which holds because $\tau(3) < \tau(2)$. □

With this result at hand, we proceed as follows: We start with a monomial of degree $(t, 4)$ divisible by $y^2_{\sigma(0)} y^2_{\sigma(1)}$ and repeatedly apply transitions of the form

$$y^2_{\tau(0)} y^2_{\tau(1)} \rightsquigarrow y^2_{\tau(0)} y^2_{\tau(2)}$$

with $\tau(3) < \tau(2)$ for a suitable permutation τ until we arrive at a polynomial divisible by $y_1^2 y_2^2$, for which we have already shown that it vanishes modulo J. The fact that such a sequence of transitions always exists can be most easily seen from the following diagram:

$$\begin{tikzcd}
\{0,1\} \arrow[r, "2<3"] \arrow[d, "2<3"] & \{1,3\} \arrow[d, "0<2"] \arrow[dr, "0<2"] & \\
\{0,3\} \arrow[r, "1<2"] & \{2,3\} \arrow[r, "0<1"] & \{1,2\} \\
\{0,2\} \arrow[u, "1<3"] \arrow[ur, "1<3"'] & &
\end{tikzcd}$$

The arrows are labeled with the inequalities $\tau(3) < \tau(2)$ which hold for the employed permutations τ. For every possible subset $\{\sigma(0), \sigma(1)\} \subset \{0, 1, 2, 3\}$, there exists at least one directed path ending in $\{1, 2\}$. This completes the proof of Lemma 4.12. $\square$

Lemma 4.13. *We have* $y_j^4 \in J$ *for all* $j \in \{0, 1, 2, 3\}$.

Proof. Let us take a monomial hy_j^4 where $h \in S(t + 4r_j, 0)$. If $d_j = 0$, we are done by (4.4). Otherwise, multiplying (4.5) with y_j^2 and using Lemma 4.12 produces

$$\frac{\partial f_j}{\partial x_i} y_j^4 \equiv 0 \pmod{J} , \quad i \in \{0, 1, 2\} .$$

First suppose $j < 3$. By Lemmas 4.3 and 4.4, we have

$$S(3d_j - 5, 0) \subset \frac{\partial f_j}{\partial x_0} S + \frac{\partial f_j}{\partial x_1} S + \frac{\partial f_j}{\partial x_2} S$$

since the partial derivatives of $f_j = x_0^{d_j} + x_1^{d_j} + x_2^{d_j}$ form a complete intersection. Therefore, it remains to show that $t + 4r_j \geq 3d_j - 5$. This is equivalent to

$$r_0 + r_1 + r_2 + r_3 + 4r_j + 4d - 3 \geq 6r_j + 3d - 5 ,$$

which in turn is equivalent to

$$r_0 + r_1 + r_2 + r_3 + d + 2 \geq 2r_j .$$

The last inequality is true because $j \leq 2$ implies $r_2 + r_3 \geq r_j + r_j$.

Now let $j = 3$. If we multiply (4.6) with y_3^2 and use Lemmas 4.10, 4.11, and 4.12, we obtain

$$g_{33} y_3^4 \equiv 0 \pmod{J} .$$

We claim that

$$S(t + 4r_3, 0) \subset \frac{\partial f_3}{\partial x_0} S + \frac{\partial f_3}{\partial x_1} S + \frac{\partial f_3}{\partial x_2} S + g_{33} S .$$

By Lemma 4.3, it is enough to give one working example for f_3 and g_{33}. If we take again $f_3 = x_0^{d_3} + x_1^{d_3} + x_2^{d_3}$, the complete intersection given by the partial derivatives of f_3 has the strong Lefschetz property by Proposition 4.7, so we may choose for g_{33} a power of a strong Lefschetz element and obtain via Lemma 4.5 that

$$S(m, 0) \subset \frac{\partial f_3}{\partial x_0} S + \frac{\partial f_3}{\partial x_1} S + \frac{\partial f_3}{\partial x_2} S + g_{33} S$$

for all $m \geq \frac{1}{2}(3d_3 - 3 + t - d + 2r_3 - 3)$. Therefore, we are finished if

$$2(t + 4r_3) \geq 3d_3 - 3 + t - d + 2r_3 - 3 .$$

This simplifies to

$$2r_0 + 2r_1 + 2r_2 + 10r_3 + 8d - 6 \geq r_0 + r_1 + r_2 + 9r_3 + 6d - 9 ,$$

or equivalently,

$$r_0 + r_1 + r_2 + r_3 + 2d + 3 \geq 0 .$$

The last statement is clearly true. □

Since every monomial in $S(t, 4)$ is divisible by an element handled in one of the four lemmas above, we obtain $S(t, 4) \subset J$ as desired. This finally proves Proposition 4.9 and thus Theorem 1.1.

Note that it was crucial in the choice of g to leave out the terms g_{00} and g_{22}, i. e. the ones belonging to the smallest and second-largest values among the degrees d_0, d_1, d_2, d_3. With any other two indices, the above proof would not work. Furthermore, if we would also set $g_{33} = 0$, the proof of Lemma 4.12 would be much simpler, but then Lemma 4.13 would work out only if $d_3 \leq d_0 + d_1 + d_2 + 4$. And if we would instead set $g_{11} = 0$, Lemma 4.13 could be left untouched, but Lemma 4.12, though its proof would be simpler, would turn out right only if $d_3 \leq d_2 + 6$. It is also worth to mention that the properties of J we are proving in each of the four claims are in general *not* open on the polynomials f_j and g_{ij}, thus an argument where one specializes to $g_{33} = 0$ in one claim but not in another one does not succeed.

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Selbstständigkeitserklärung

Hiermit versichere ich, die vorliegende Arbeit selbstständig und nur unter Verwendung der angegebenen Quellen und Hilfsmittel verfasst zu haben.

Miesbach, den 7. Dezember 2018