Birational Geometry

Stefan Schreieder

SoSe 2020

Contents

1 Introduction

For us, a variety is an integral separated scheme of finite type over a field k. In this course, the field k will be algebraically closed (and it will be an advantage to assume that k is of characteristic zero, e.g. $k = \mathbb{C}$). Recall that two varieties X and Y over a field k are birational, if and only if there are non-empty open subsets $U \subset X$ and $V \subset Y$ that are isomorphic to each other. Equivalently, the function fields of X and Y are isomorphic to each other. The purpose of this course is to study the birational geometry of varieties. That is, we study varieties up to birational equivalence. By the above remark, this is the same thing as studying finitely generated field extensions of k, but this will not be the point of view of this course (and in fact, the geometric point of view taken in this class turns out to be much more successful than a purely field theoretic approach).

Here are some of the fundamental questions in the subject:

Question 1.1. *Let X and Y be explicitly given varieties over a field k. Can we decide whether X and Y are birational to each other?*

One instance of the above question is to decide whether a given variety X is rational, i.e. birational to $\mathbb{P}^{\dim X}$. This question is subtle and open even in seemingly simple cases. For instance, it is unknown whether a very general cubic hypersurface $X \subset \mathbb{P}^{n+1}_{\mathbb{C}}$ of dimension $n \geq 4$ is rational.

A related question is as follows:

Question 1.2. *Let X be a variety over a field k. Can we find a particularly 'nice' representative in its birational equivalence class?*

It is natural to hope that an answer to Question 1.2 might lead to an answer of Question 1.1. Indeed, once Question 1.2 is answered positively, Question 1.1 boils down to the case of comparing the 'nice' representatives of X and Y and it is natural to hope that these representatives are unique or at least not too far off from being unique, so that deciding whether the representatives are birational to each other might be much more tractable than the general question. For instance, if 'nice' models as in Question 1.2 would exist in a unique way, than X and Y would be birational if and only if their 'nice' models are isomorphic.

Instead of concentrating on explicitly given varieties, it is also natural to take the following more global point of view.

Question 1.3. *Can we classify varieties up to birational equivalence to some extent? That is, can we divide all varieties of given dimension over a field k into several classes, which capture important features of their birational geometry?*

The minimal model program is a (very important) branch in birational geometry that tries to answer Questions 1.2 and 1.3 and the purpose of this course is to give an introduction to this theory. The theory is developed to a large extent, even though it is still conjectural at some important points. Even though it turns out that an answer to Question 1.2 cannot answer Question 1.1 in all cases (e.g. the case of cubics cannot be answered via the minimal model program), this is indeed true in 'most cases'.

The purpose of this lecture is to study Question 1.2. This also leads to an answer of Question 1.3, as This is a very fundamental geometric problem, which allows us to pick up several useful concepts on the way.

1.1 Basic approach

Let X be a variety over an algebraically closed field k. We aim to find a nice representative of X in its birational equivalence class. Replacing X by the projective closure of a non-empty open affine subset of X, we may assume that X is projective. Taking the normalization of X, we may also assume that X is normal. (This uses that if X is projective and $X' \to X$ is a finite morphism, than X' is also projective, because the pullback of an ample line bundle on X will be ample on X', e.g. by Kleiman's ampleness criterion that we will discuss later in class.) This simple procedure solves the problem in dimension one.

Proposition 1.4. *Let X and Y be normal projective varieties of dimension one. Then X and Y are birational if and only if they are isomorphic.*

Proof. Let $\phi : X \dashrightarrow Y$ be a birational map. Since X and Y are normal and projective, the next lemma shows that ϕ and its inverse ϕ^{-1} are both defined in codimension one. Since X and Y are of dimension one, ϕ and its inverse ϕ^{-1} are in fact morphisms that must be inverse to each other. Hence X and Y are isomorphic to each other. □

The above proposition used the following well-known lemma.

Lemma 1.5. *Let $\phi : X \dashrightarrow Y$ be a rational map between varieties over k. Assume that X is normal and Y is projective. Then ϕ is a morphism in codimension one. That is, there is a closed subset $Z \subset X$ of codimension at least two, so that the resitrction of ϕ to $X \setminus Z$ is a morphism.*

Proof. By assumption, there is an open subset $U \subset X$ such that ϕ restricts to a morphism $U \to Y$. Let $D \subset X \setminus U$ be a prime divisor in the complement of U. Since X is normal, the local ring $\mathcal{O}_{X,D}$ of X at the generic point of D is a regular local ring of height one, hence a discrete valuation ring. Since Y is projective, there is a closed embedding $Y \subset \mathbb{P}^N$ and it suffices to show that the rational map $X \dashrightarrow \mathbb{P}^N$ induced by ϕ is a morphism in codimension one. Hence, we may w.l.o.g. assume that $Y = \mathbb{P}^N$. Then $\phi = [\phi_0 : \cdots : \phi_N]$ for rational functions $\phi_i \in k(X)$. Let π be a uniformizer of the dvr $\mathcal{O}_{X,D}$ (i.e. a generator of the maximal ideal). Let ν be the valuation on $k(X)$, induced by the dvr $\mathcal{O}_{X,D}$. Up to multiplying all ϕ_i by the same power of π, we may assume that $\nu(\phi_i) \geq 0$ for all i and $\nu(\phi_i) = 0$ for at least one i. Hence, $\phi_i \in \mathcal{O}_{X,D}$ for all i and $\phi_i|D$ is not identically to zero for at least one i. This shows that the rational map ϕ extends over the generic point of D, as we want.

Since $X \setminus U$ contains only finitely many irreducible components, it contains only finitely many prime divisors. We may therefore repeat the process and assume that ϕ is defined in codimension one, as we want. □

So far our discussion works over any field. Of course the next desirable step would be to replace a given normal projective variety X be a smooth projective model. By Hironaka's resolution of singularities theorem, this is known to be possible in characteristic zero.

Theorem 1.6 (Hironaka)**.** *Let X be a complex variety. Then there is a projective birational morphism $\tau : X' \to X$, given as a composition of blow-ups along regular centers, such that X' is regular (i.e. smooth) and τ is an isomorphism above the regular locus of X.*

Moreover, resolution of singularities over a field of positive characteristic is known in dimension two and three, but it is an open problem whether in dimension at least four, resolution of singularities still exist in positive characteristic. From now on, we thus assume that char $k = 0$ and there is no harm in assuming $k = \mathbb{C}$. Then, by the aforementioned theorem of Hironaka, we may assume that X is smooth and projective. Note that this smooth projective variety will be far from unique. Indeed, in the above process, we started by choosing some affine open subset and took its closure in some projective space. While taking normalizations is canonical, taking resolutions is not canonical and we could always blow-up more smooth subvarieties to arrive at another smooth projective variety that is still birational to X.

Instead of trying to control the non-uniqueness in the above process, it is much ore practicable to start with any smooth projective model X and to try to perform controlled birational maps with the aim of arriving at a 'nice' birational model.

1.2 Morphisms of varieties

Let X be a variety over a field. One of the most basic questions one can ask about X is to describe all possible morphisms $f : X \to Y$ to another variety Y. As we are mostly interested in (quasi-)projective varieties, there is no harm in assuming that Y is quasi-projective. This seemingly mild assumption allows us to understand the situation completely. Indeed, if Y is quasi-projective, then it admits an embedding into some projective space $\mathbb{P}^N$ and so to describe all morphisms $f : X \to Y$ from X to some quasi-projective variety Y, it suffices to describe all morphisms from X to some projective space. That is, we may assume that $f : X \to \mathbb{P}^N$ is a morphism to projective space. Let $[x_0 : \cdots : x_N]$ be homogeneous coordinates on $\mathbb{P}^N$. That is, $x_0, \ldots, x_N \in H^0(\mathbb{P}^N, \mathcal{O}(1))$ form a basis. Let $f_i := f^*x_i$ be the pullback of x_i. Then f_i is a section of the line bundle $L := f^*\mathcal{O}(1)$ and f is described by $f = [f_0 : \cdots : f_N] : X \to \mathbb{P}^N$ Since the sections $x_0, \ldots, x_N$ of $\mathcal{O}(1)$ are base point free, i.e. have no common zero, the same holds for $f_0, \ldots, f_N \in H^0(X, L)$.

Converesely, if L is a line bundle on X and $f_0, \ldots, f_N \in H^0(X, L)$ is a base point free set of sections , i.e. for all $x \in X$ there is at least one i with $f_i(x) \neq 0$, then

$$f = [f_0 : \cdots : f_N] : X \longrightarrow \mathbb{P}^N$$

is a well-defined morphism. Locally at $x \in X$, this morphism is defined as follows. Let s be a local section of L which is nonzero locally at x. Then $f_i = g_i s$ for some regular function g_i locally at x and we define the above morphism f locally around x to be given by

$$[g_0 : \cdots : g_N] : X \longrightarrow \mathbb{P}^N.$$

This is well-defined, i.e. does not depend on the choice of the section s, because a different choice of s amounts to multiply each g_i with the same invertible function, which does not change the morphism to projective space (by definition of $\mathbb{P}^N$). The well-definedness also shows that these local definitions glue to give a global morphism $f : X \longrightarrow \mathbb{P}^N$ as above.

Hence, we see that there is a one to one correspondence between (isomorphism classes of) morphisms $f : X \to \mathbb{P}^N$ and (equivalence classes of) pairs of line bundles L on X with base point-free sets of sections $f_0 \ldots, f_N \in H^0(X, L)$. This explains the fundamental importance of line bundles (aka invertible sheaves) when studying algebraic varieties and there morphisms. We will therefore recall some of these fundamental concepts in the next section.

2 Weil divisors, Cartier divisors and invertible sheaves

A standard reference for this section is [2, II.6.].

2.1 Weil divisors on normal varieties

Let X be a normal variety over a field k. A (Weil) divisor is a formal linear combination $D = \sum_{i=1}^n a_i D_i$, where $a_i \in \mathbb{Z}$ are integers and D_i are irreducible closed subvarieties of X of codimension one, i.e. $\dim D_i = \dim X - 1$. A prime divisor on X is a divisor of the form $D = 1 \cdot D_1$ consisting of a single irreducible closed subvariety of codimension one (with coefficient one). Sometimes we refer to the term 'prime divisor' simply to mean a closed irreducible codimension one subvariety of X.

If $\phi \in k(X)$ is a nonzero rational function, then there is a well-defined divisor $\mathrm{div}(\phi)$ consisting of zeros and poles of ϕ, counted with multiplicities. Indeed, since X is normal, the local ring $\mathcal{O}_{X,D}$ at the generic point of any prime divisor $D \subset X$ is a discrete valuation ring. Let ν_D be the corresponding valuation. Then $\nu_D(\phi) \geq 0$ if and only if ϕ is regular in a neighbourhood of the generic point of D and $\nu_D(\phi)$ denotes the order of vanishing of ϕ at the generic point of D. Moreover, if $\nu_D(\phi) < 0$, then ϕ has a pole of order $-\nu_D(\phi)$ at the generic point of D. We then set

$$\mathrm{div}(\phi) := \sum_D \nu_D(\phi) \cdot D,$$

where D runs through all prime divisors of X. Since ϕ is regular at some non-empty open subset, $\nu_D(\phi) \geq 0$ for all but finitely many prime divisors on X. Since $\nu_D(\phi^{-1}) = -\nu_D(\phi)$, the same argument applied to ϕ^{-1} shows that $\nu_D(\phi) = 0$ for all but finitely many prime divisors D on X. Hence the above sum is finite and $\mathrm{div}(\phi)$ is indeed a divisor.

Definition 2.1. *Two divisors D and D' on a normal variety X are linearly equivalent, denoted by $D \sim D'$, if there is a nonzero rational function $\phi \in k(X)$ with $D - D' = \mathrm{div}(\phi)$.*

Linear equivalence of divisors is an equivalence relation. We denote the free abelian group generated by all prime divisors on X by $\mathrm{WDiv}(X)$. The quotient $\mathrm{Cl}(X) := \mathrm{WDiv}(X)/\sim$ of all divisors on X modulo linear equivalence is an abelian group, called the class group of X.

2.1.1 Sheaf of sections of a divisor

Definition 2.2. *A divisor D on a normal variety X is effective, denoted by $D \geq 0$, if $D = \sum a_i D_i$ with $a_i \geq 0$ for all i.*

Let X be a normal variety over a field. For a divisor D on X, we define the space of sections of D on X, denoted by $\Gamma(X, \mathcal{O}_X(D)) \subset k(X)$, as the union of zero with the set of rational functions $\phi \in k(X)^*$ with $\mathrm{div}(\phi) + D \geq 0$. As the notation suggests, this definition gives rise to a presheaf of abelian groups $\mathcal{O}_X(D)$ on X, whose sections over a non-empty open subset $U \subset X$ consists of the union of zero with the set of all sections $\phi \in k(X)^*$ with

$$(\mathrm{div}(\phi) + D)|_U \geq 0.$$

It is easy to check that the presheaf of abelian groups defined this way satisfies the sheaf axioms, hence is a sheaf.

Lemma 2.3. *Let X be a normal variety, $f \in k(X)$. Then f is regular on X if and only if $\mathrm{div}(f) \geq 0$.*

Proof. Since both assertions are local, we may wlog assume that X is affine. If f is regular, then $\mathrm{div}(f) \geq 0$ is clear as f does not have any poles. Conversely, suppose that $\mathrm{div}(f) \geq 0$. Then $\nu_D(f) \geq 0$ for all $D \subset X$. Hence, $f \in k(X)$ lies in the localization $k[X]_{I(D)}$ for all prime divisors $D \subset X$. But the prime divisors on X are via $D \mapsto I(D)$ in one to one correspondence to the prime ideals of height one in $k[X]$ and so

$$f \in \bigcap_{\mathfrak{p}} k[X]_{\mathfrak{p}},$$

where the intersection runs through all prime ideals of height one. It is a deep result from commutative algebra that the latter intersection coincides with $k[X]$, and so f is regular. □

Definition 2.4. *Let X be a normal variety. A divisor D on X is Cartier if the $\mathcal{O}_X$-module $\mathcal{O}_X(D)$ is locally free of rank one.*

Lemma 2.5. *Let X be a normal variety. A divisor D on X is Cartier if and only if for all $x \in X$, there is a neighbourhood $x \in U \subset X$ and a rational function $\phi \in k(X)^*$ with $\operatorname{div}(\phi)|_U = D|_U$. In particular, if D is effective, then it is Cartier if and only if it is locally given by the vanishing of a single regular function.*

Proof. The if direction is clear. For the only if direction, assume that D is Cartier and let $x \in X$. Then there is a neibourhood $x \in U \subset X$ and a section s of $\mathcal{O}_X(D)|_U$ which generates the $\mathcal{O}_X$-module $\mathcal{O}_X(D)|_U$ in every point. That is, any local section of $\mathcal{O}_X(D)|_U$ is given as the product of a local regular function with s.

The section s corresponds by definition of $\mathcal{O}_X(D)$ to a rational function $\phi \in k(X)^*$ with $\operatorname{div}(\phi)|_U + D|_U \geq 0$. We claim that in fact equality holds in this inequality, which proves our claim in the lemma. For a contradiction, assume that $\operatorname{div}(\phi)|_U + D|_U \geq D'|_U$ for some prime divisor D' on X whose generic point is contained in U. Since X is normal, the local ring of X at the generic point of D' is a discrete valuation ring and so there is a rational function $\phi' \in k(X)^*$ such that $\operatorname{div}(\phi') = D' + D''$ for some divisor D'' whose support does not contain D'. Let $V \subset U$ be the complement of the support of $D''|_U$. Then, $\operatorname{div}(\phi')|_V = D'_V$. But this implies that

$$\operatorname{div}(\phi/\phi')|_V + D = \operatorname{div}(\phi)|_V + D - \operatorname{div}(\phi')|_V \geq D'|_V - D'|_V = 0.$$

Hence, ϕ/ϕ' is a section of $\mathcal{O}_X(D)|_V$. Since $1/\phi'$ is not regular on $V \subset U$, this contradicts our assumption that ϕ generates the $\mathcal{O}_X$-module $\mathcal{O}_X(D)$ in any point of U. $\square$

Weil divisors that are not Cartier are thus codimension one subvarieties which cannot be described locally by one function. An example is given by

$$X := \{x_0x_1 - x_2x_3 = 0\} \subset \mathbb{A}^4 \quad \text{and} \quad D := \{x_0 = x_2 = 0\}.$$

Here X is a cone over a smooth quadric surface and D is the cone over a line on this quadric surface. Clearly, D is a prime divisor on X. However, locally at the origin, D corresponds to the ideal

$$I_D := (x_0, x_2) \subset k[x_0, x_1, x_2, x_3]/(x_0x_1 - x_2x_3) = k[x_0, x_1, x_2][x_0x_1/x_2].$$

One checks that even after localizing at the maximal ideal (x_0, x_1, x_2, x_3), the ideal I_D does not become principal and so D is not Cartier.

Proposition 2.6. *If X is locally factorial, i.e. all local rings $\mathcal{O}_{X,x}$ are factorial, (e.g. X smooth) and D is a divisor on X, then $\mathcal{O}_X(D)$ is locally free of rank one. That is, any divisor on X is Cartier.*

Proof. Let $x \in X$. We need to find a neighbourhood $x \in U \subset X$ of x, such that $\mathcal{O}_X(D)|_U \cong \mathcal{O}_U$. Since X is smooth, $\mathcal{O}_{X,x}$ is a regular local ring. Let $D = \sum_i a_i D_i$ be a decomposition into prime divisors $D_i \subset X$. Up to shrinking X, we may assume $x \in D_i$ for all i. Each prime divisor D_i thus corresponds to a prime ideal

$$\mathfrak{p}_i \subset \mathcal{O}_{X,x},$$

consisting of all functions defined in some neighbourhood of x, which vanish along D_i. Since $\mathcal{O}_{X,x}$ is a regular local ring, the height one prime ideal $\mathfrak{p}_i$ is principal:

$$\mathfrak{p}_i = (g_i)$$

for some $g_i \in \mathcal{O}_{X,x}$ (in fact, any $g_i \in \mathfrak{p}_i$ irreducible will do the job). Up to shrinking X, we may assume that X is affine and $g_i \in k[X]$ is regular on X for each i. Then

$$g := \prod_i g_i^{a_i} \in k(X)$$

satisfies $\operatorname{div}(g) = D$ and so $\mathcal{O}_X(D) \cong \mathcal{O}_X(0)$ by exercise 4 on sheet 01. By Lemma 2.3, $\mathcal{O}_X(0) \cong \mathcal{O}_X$ and so the proposition follows. $\square$

2.2 Cartier divisors on schemes

On arbitrary schemes, Weil divisors, i.e. linear combinations of codimension one closed subschemes, do not work very well. Instead, the notion of Cartier divisor generalizes nicely to this broader context.

Definition 2.7. *Let A be a ring. The total quotient ring of A is the localization of A at the multiplicative set of all non-zero divisors.*

Let X be a scheme. Let $\mathcal{K}$ be the sheaf of rings that is associated to the presheaf which to an open subset $U \subset X$ associates the total quotient ring of $\mathcal{O}_X(U)$. This sheaf of rings replaces the notion of function field for varieties. We denote by $\mathcal{K}^* \subset \mathcal{K}$ the (multiplicative) subsheaf of invertible elements in the sheaf of rings $\mathcal{K}$. Similarly, $\mathcal{O}_X^* \subset \mathcal{O}_X$ denotes the multiplicative subsheaf of the sheaf of rings $\mathcal{O}_X$ on X.

Recall that a divisor on a normal variety is a divisor which is locally principle, i.e. the divisor of zeros and poles of a rational function ϕ. Moreover, two rational functions ϕ and ϕ' have the same divisor of zeros and poles on some open subset U if and only if $\operatorname{div}(\phi/(\phi')^{-1})|_U = 0$, which means that $\phi/(\phi')^{-1}$ is regular on U, see Lemma 2.3. This explains that the following definition generalizes our earlier definition of Cartier divisor from varieties to arbitrary schemes.

Definition 2.8. *A Cartier divisor on a scheme X is a global section of the quotient sheaf $\mathcal{K}_X^*/\mathcal{O}_X^*$.*

By the definition of quotient sheaves, this means that a Cartier divisor on X corresponds to a collection of pairs (U_i, ϕ_i), where $X = \bigcup U_i$ is an open cover, $\phi_i \in \mathcal{K}^*(U_i)$ and $\phi_i/\phi_j \in \mathcal{O}_X(U_i \cap U_j)^*$ for all i, j.

Definition 2.9. *Two Cartier divisors on a scheme X are linearly equivalent, if their difference is principal, i.e. the image of global section of $\mathcal{K}_X^*$. The group of all Cartier divisors modulo linear equivalence on X is denoted by* $\operatorname{CaCl}(X)$.

Given the canonical short exact sequence

$$0 \to \mathcal{O}_X^* \to \mathcal{K}^* \to \mathcal{K}^*/\mathcal{O}_X^* \to 0,$$

we get an inclusion $\operatorname{CaCl}(X) \hookrightarrow H^1(X, \mathcal{O}_X^*)$, which is an isomorphism if $H^1(X, \mathcal{K}^*) = 0$ (the latter holds e.g. if $\mathcal{K}^*$ is a flasque sheaf).

If X is regular in codimension one, then we can associated a well-defined Weil divisor to any Cartier divisor by

$$(U_i, \phi_i) \mapsto \{\text{Divisor given by glueing the divisors } \operatorname{div}(\phi_i) \text{ on } U_i\},$$

giving rise to an injective map $\operatorname{CaDiv}(X) \hookrightarrow \operatorname{WDiv}(X)$.

Proposition 2.10. *Let X be a normal variety, or more generally, an integral separated noetherian scheme which is regular in codimension one, i.e. $\mathcal{O}_{X,x}$ is regular for any codimension one point.*

(a) Mapping a Cartier divisor to the associated Weil divisor, we get an injective map

$$\mathrm{CaDiv}(X) \hookrightarrow \mathrm{WDiv}(X) \tag{1}$$

whose image consists of all Weil divisors that are locally principal, i.e. locally given by the divisor of a rational function. Moreover, this injection respects linear equivalence on both sides in the sense that two Cartier divisors are linearly equivalent if and only if the corresponding Weil divisors are linearly equivalent. In particular, the above injection induces an injection $\mathrm{CaCl}(X) \hookrightarrow \mathrm{Cl}(X)$ of the class group of Cartier divisors to the class group of Weil divisors.

(b) If X is locally factorial, i.e. $\mathcal{O}_{X,x}$ is factorial for any $x \in X$ (e.g. X is regular), then this injection is surjective.

Proof. See [2, II.6.11 and II.6.11.2].

To give some details, note that since X is integral, $\mathcal{K}$ is the constant sheaf with stalk K, the function field of X. Mapping a Cartier divisor $\{(U_i, \phi_i)\}$ to the divisor D given by glueing the divisors $\mathrm{div}(\phi_i) \in \mathrm{WDiv}(U_i)$ on U_i, we get a group homomorphism

$$\mathrm{CaDiv}(X) \longrightarrow \mathrm{WDiv}(X).$$

This is injective, because $\mathrm{div}(\phi_i) = 0 \in \mathrm{WDiv}(U_i)$ means that $\phi_i \in \mathcal{O}^*_{U_i}$ by Lemma 2.3. Clearly, any Weil divisor in the image of this injection is locally principal. Conversely, if D is a Weil divisor that is locally of the form $D|_{U_i} = \mathrm{div}(\phi_i)$, then $\mathrm{div}(\phi_i/\phi_j)|_{U_i \cap U_j} = 0$ and so ϕ_i/ϕ_j is an invertible regular function on $U_i \cap U_j$, see Lemma 2.3. Hence, $\{(U_i, \phi_i)\}$ is a Cartier divisor which associated Weil divisor D.

Also, if a given Cartier divisor is principal, then we can represent it as $\{(X, \phi)\}$ for a rational function $\phi \in K^*$. The corresponding Weil divisor $\mathrm{div}(\phi)$ is then principal as well. Conversely, if a Cartier divisor $\{(U_i, \phi_i)\}$ corresponds to a the Wil divisor D with $D = \mathrm{div}(\phi)$ for some rational function ϕ, then ϕ_i/ϕ is a section of $\mathcal{O}^*_{U_i}$ for all i and so the given Cartier divisor can be represented as $\{(X, \phi)\}$.

This concludes part (a). Part (b) follows by the same argument as in Proposition 2.6. □

If X is a normal variety, then the above proposition shows that the above definition of Cartier divisors is equivalent to the one given before in the context of varieties.

2.3 Invertible sheaves

Definition 2.11. *Let $(X, \mathcal{O}_X)$ be a ringed space. An $\mathcal{O}_X$-module on X is an invertible sheaf, or a line bundle, if it is locally free of rank one.*

The tensor product $L \otimes_{\mathcal{O}_X} L'$ of two invertible sheaves is again an invertibel sheaf. If L is an invertible sheaf, then the dual $L^\vee := \mathrm{Hom}(L, \mathcal{O}_X)$ is also an invertible sheaf and there is a canonical isomorphism

$$L \otimes_{\mathcal{O}_X} L^\vee \cong \mathcal{O}_X.$$

Altogether, we find that the set of isomorphism classes of invertible sheaves on X form a group, called the Picard group of X and denoted by $\mathrm{Pic}(X)$.

Remark 2.12. *Let X be a variety. Mapping an algebraic vector bundle on X to its sheaf of sections defines an equivalence between vector bundles of rank r on X and locally free $\mathcal{O}_X$-modules of rank r. For this reason, locally free sheaves of rank one can be thought of as line bundles.*

Definition 2.13. *Let D be a Cartier divisor on a scheme X, represented by $\{(U_i, \phi_i)\}$ as above. Then the associated sheaf of sections $\mathcal{O}_X(D)$ is the $\mathcal{O}_X$-submodule of $\mathcal{K}$, generated by ϕ_i^{-1} on U_i.*

Note that $\mathcal{O}_X(D)$ is well-defined, because ϕ_i/ϕ_j is invertible on $U_i \cap U_j$ and so ϕ_i and ϕ_j span the same $\mathcal{O}_X$-submodule over $U_i \cap U_j$.

Note also that the above definition is compatible with our earlier definition, because a rational function ϕ satisfies $\operatorname{div}(\phi)|_U + D|_U \geq 0$, where $D|_U = \operatorname{div}(\phi_i)$, if and only if $\phi \cdot \phi_i$ is regular, which means that ϕ is the product of ϕ_i^{-1} with a regular function.

Proposition 2.14. *Let X be a scheme and let D, D_1 and D_2 be Cartier divisors on X. Then*

(a) $\mathcal{O}_X(D)$ is an invertible sheaf;

(b) $\mathcal{O}_X(D_1 - D_2) \cong \mathcal{O}_X(D_1) \otimes \mathcal{O}_X(D_2)^{-1}$;

(c) $D_1 \sim D_2$ if and only if $\mathcal{O}_X(D_1) \cong \mathcal{O}_X(D_2)$ (as abstract $\mathcal{O}_X$-modules).

Proof. See [2, II.6.13].

To give some details, let $D = \{(U_i, \phi_i)\}$. Then $\mathcal{O}_X(D) \subset \mathcal{K}_X$ is generated as an $\mathcal{O}_X$-module by ϕ_i^{-1}. Since

$$\mathcal{O}_{U_i} \longrightarrow \mathcal{K}|_{U_i}, \quad f \mapsto f \cdot \phi_i^{-1}$$

is injective, the claim in (a) follows.

If $D_1 = \{(U_i, \phi_i)\}$ and $D_2 = \{(U_i, \psi_i)\}$, then $\mathcal{O}_X(D_1 - D_2)$ is locally generated by $\phi_i^{-1}\psi_i$ and this subsheaf of $\mathcal{K}$ is isomorphic to $\mathcal{O}_X(D_1) \otimes \mathcal{O}_X(D_2)^{-1}$, because $\mathcal{O}_X(D_1)$ is generated by ϕ_i and $\mathcal{O}_X(D_2)$ is locally generated by ψ_i. This proves (b).

By (b) it suffices to show that $D \sim 0$ if and only if $\mathcal{O}_X(D) \cong \mathcal{O}_X$. Clearly, if $D \sim 0$, then there is a global section ϕ of $\mathcal{K}^*$ such that $\mathcal{O}_X(D) = \phi \cdot \mathcal{O}_X \subset \mathcal{K}$ and so $\mathcal{O}_X(D) \cong \mathcal{O}_X$. Conversely, if $\mathcal{O}_X(D) \cong \mathcal{O}_X$, then the unit section $1 \in \Gamma(X, \mathcal{O}_X)$ corresponds to a global section ϕ of $\mathcal{O}_X(D)$ and so D is principal. This proves (c). □

By the above proposition, we get an injection $\operatorname{CaCl}(X) \hookrightarrow \operatorname{Pic}(X)$. This is surjective in most situations (i.e. e.g. if $\mathcal{K}$ is acyclic so that $H^1(X, \mathcal{K}) = 0$), but not always, as there might be situations where not every invertible sheaf on X is a subsheaf of $\mathcal{K}$.

Proposition 2.15. *If X is integral, then* $\operatorname{CaCl}(X) \hookrightarrow \operatorname{Pic}(X)$ *is surjective.*

Proof. Since X is integral, $\mathcal{K}$ is the constant sheaf with stalk K, the quotient field of X. This sheaf as well as its subsheaf $\mathcal{K}^* \subset \mathcal{K}$ is flasque and so $H^1(X, \mathcal{K}^*) = 0$. The short exact sequence $0 \to \mathcal{O}_X^* \to \mathcal{K}^* \to \mathcal{K}^*/\mathcal{O}_X^* \to 0$ thus yields the result, because

$$\operatorname{Pic}(X) \cong H^1(X, \mathcal{O}_X^*)$$

as one sees by representing line bundles by Cech one-cycles.

For a more elementary proof, we again use that $\mathcal{K}$ is constant and note that for any invertible sheaf L on X, the tensor product $L \otimes_{\mathcal{O}_X} \mathcal{K}$ is isomorphic to $\mathcal{K}$ and so we obtain a subsheaf

$$L \hookrightarrow L \otimes_{\mathcal{O}_X} \mathcal{K} \cong \mathcal{K}.$$

This subsheaf is an $\mathcal{O}_X$-submodule of rank one and so it is locally generated by a single nonzero section of $\mathcal{K}$, as we want. $\square$

Definition 2.16. *A Cartier divisor D is effective if $H^0(X, \mathcal{O}_X(D)) \neq 0$. This is equivalent to saying that D can be expressed as $\{(U_i, \phi_i)\}$ with $\phi_i \in H^0(U_i, \mathcal{O}_X)$.*

3 Morphisms to projective space and ample line bundles

The reference for this section is [2, II.7].

3.1 Morphisms to projective space

Definition 3.1. *Let X be a scheme. An $\mathcal{O}_X$-module $\mathcal{F}$ on X is globally generated, if for each $x \in X$, the images of the global sections $s \in H^0(X, \mathcal{F})$ in the stalk $\mathcal{F}_x$ of $\mathcal{F}$ at x generate $\mathcal{F}_x$ as an $\mathcal{O}_{X,x}$-module.*

Let A be a ring and let $\mathbb{P}^n_A := \operatorname{Proj} A[x_0, \dots, x_n]$. Then $\mathcal{O}(1)$ is globally generated by the global sections $x_0, \dots, x_n \in H^0(\mathbb{P}^n_A, \mathcal{O}(1))$.

Theorem 3.2. *Let A be a ring and let X be a scheme over A.*

(a) If $f : X \to \mathbb{P}^n_A$ is a morphism over A, then $L := f^*\mathcal{O}(1)$ is an invertible sheaf on X which is generated by the global sections f^*x_i, $i = 0, \dots, n$.

(b) Conversely, if L is an invertibal sheaf on X which is generated by global sections $f_0, \dots, f_n$, then there is a unique morphism

$$f : X \to \mathbb{P}^n_A$$

with $f^*\mathcal{O}(1) = L$ and $f_i = f^*x_i$.

Proof. See [2, II.7.1].

To see part (a), note that $L_x = \mathcal{O}(1)_{f(x)} \otimes_{\mathcal{O}_{Y,f(x)}} \mathcal{O}_{X,x}$. Hence, the statement follows from the fact that the global sections $x_0, \dots, x_n \in H^0(\mathbb{P}^n_A, \mathcal{O}(1))$ generate $\mathcal{O}(1)$.

To see part (b), we intuitively want to define f as $[f_0 : \dots : f_n]$. To make this precise in the current setting, let

$$V_i := X \setminus \{f_i = 0\} = \{x \in X \mid \overline{f}_i \neq 0 \in L \otimes_{\mathcal{O}_X} \kappa(x)\}.$$

Since L is globally generated by $f_0, \dots, f_n$, we see that $X = \bigcup V_i$ is an open cover of X. Let $U_i = \mathbb{P}^n_A \setminus \{x_i = 0\}$ be the standard open cover of $\mathbb{P}^n_A$. We then aim to produce morphisms $V_i \to U_i$ that glue on overlaps. For this, note that

$$U_i = \operatorname{Spec} A[y_0, \dots, \hat{y}_i, \dots y_n]$$

with $y_j = x_j/x_i$. The morphism $V_i \to U_i$ is then defined by

$$A[y_0, \dots, \hat{y}_i, \dots y_n] \longrightarrow \mathcal{O}_X(V_i), \quad y_j \mapsto f_j/f_i.$$

This makes sense, because f_j/f_i is a regular section of $L \otimes L^{-1} \cong \mathcal{O}_X$ over U_i, hence a regular function. One easily checks that this definition glues on overlaps, giving rise to the desired morphism $f : X \to \mathbb{P}^n_A$. It is also easy to check that $L = f^*\mathcal{O}(1)$ with $f_i = f^*x_i$. $\square$

Proposition 3.3. *Let k be an algebraically closed field and let X be a projective scheme over k. Let $f : X \to \mathbb{P}^n_k$ be a morphism over k, $L = f^*\mathcal{O}(1)$ and $V := f^*H^0(\mathbb{P}^n, \mathcal{O}(1)) \subset H^0(X, L)$. Then f is a closed immersion, if and only if:*

(a) elements of V separate points, i.e. for any distinct closed points $x, y \in X$, there is a section $s \in V$ with $s(x) = 0$ and $s(y) \neq 0$, i.e. $s_x \in \mathfrak{m}_x \cdot L_x$ and $s_y \notin \mathfrak{m}_y \cdot L_y$.

(b) elements of V separate tangent vectors, i.e. for any closed point $x \in X$, the set $\{s \in V \mid s_x \in \mathfrak{m}_x \cdot L_x\}$ spans the k-vector space

$$\mathfrak{m}_x L_x/\mathfrak{m}_x^2 L_x \cong (\mathfrak{m}_x/\mathfrak{m}_x^2) \otimes_{\mathcal{O}_{X,x}} L_x \cong m_x/m_x^2.$$

Proof. See [2, II.7.3]. □

3.2 Ample line bundles

Definition 3.4. *A line bundle L on a noetherian scheme X is ample, if for every coherent sheaf $\mathcal{F}$ on X, there is some integer $m_0 > 0$, such that $\mathcal{F} \otimes L^m$ is globally generated for all $m \geq m_0$, where $L^m = L^{\otimes m}$.*

Lemma 3.5 ([2, II.7.5])**.** *Let L be a line bundle on a noetherian scheme X. Then the following conditions are equivalent:*

(i) L is ample;

(ii) L^m is ample for all $m > 0$;

(iii) L^m is ample for some $m > 0$.

Proof. The implication $(i) \Rightarrow (ii) \Rightarrow (iii)$ is clear. We need to prove $(iii) \Rightarrow (i)$. So assume that L^m is ample for some $m > 0$ and let F be a coherent $\mathcal{O}_X$-module. We need to show that $F \otimes L^n$ is globally generated for all sufficiently large n, while we only know that this is true for all sufficiently large n that are divisible by m. However, since $F \otimes L^i$ is coherent for each i, the ampleness of L^m implies that

$$F \otimes L^i \otimes L^n \cong F \otimes L^{n+i}$$

is globally generated for each sufficiently large n that is divisible by n. Choosing n so large that this works for all $i = 0, \ldots, m-1$, we conclude that $F \otimes L^n$ is globally generated for all sufficiently large n (not necessarily divisible by m anylonger). This proves the lemma. □

Definition 3.6. *A line bundle L on a scheme X (over a base scheme S) is very ample, if $L = f^*\mathcal{O}(1)$ for some embedding $f : X \hookrightarrow \mathbb{P}^n_S$.*

The following two theorems of Serre are crucial:

Theorem 3.7. *Let X be a projective scheme over a noetherian ring A. Then any very ample line bundle on X over* $\operatorname{Spec} A$ *is ample.*

Proof. See [2, II.5.17].

We sketch a slightly different argument. For this, let $f : X \hookrightarrow \mathbb{P}^n_A$ be a closed embedding with $L = f^*\mathcal{O}(1)$. For a coherent $\mathcal{O}_X$-module F, we then have to show that $F \otimes L^m$ is globally generated for all sufficiently large m. Let

$$S := \bigoplus_{i \geq 0} H^0(X, L^i)$$

be the graded coordinate ring with respect to the embedding $f : X \hookrightarrow \mathbb{P}^n$. Then $X = \operatorname{Proj} S$ and any coherent $\mathcal{O}_X$-module F on X is of the form $F = \widetilde{M}$ for some finitely generated graded S-module M, see [2, II.5.15]. Since M is finitely generated, there is some m_0, such that M is generated as an S-module by its elements of degree at most m_0.

Since $L = f^*\mathcal{O}(1)$, we have $L^m = \widetilde{S(-m)}$, where $S(-m)$ is given by the ring S, where the grading is shifted, so that $S(-m)_i = S_{i-m}$. The tensor product $F \otimes L^m$ is thus associated to the graded S-module

$$M' := M \otimes_S S(-m).$$

The global sections of $F \otimes L^m$ correspond to the elements of degree zero in $M \otimes S(-m)$.

Let now $s \in M_{m'}$ be one of the (finitely many) generators of M as an S-module with $M' \leq m_0$. Then the elements

$$s \cdot x_i^{m-m'} \in M'_0$$

correspond to global sections of $F \otimes L^m$ for all $i = 0, \dots, n$. Since $x_0, \dots, x_n$ have no common zero on $\mathbb{P}^n_A$ (on the affine open subset $U_i = X \setminus \{x_i = 0\}$, x_i is nonzero by definition and X is covered by these open subsets), this proves that for $m \geq m_0$ the global sections of $F \otimes L^m$ of the form

$$s \cdot x_i^{m-m'} \in M'_0$$

with $s \in M_{m'}$, $m' \leq m_0$ and $i = 0, \dots, n$ generate the sheaf $F \otimes L^m$, as we want. (If you want to see more details, work out what this means on each open subset U_i from above.) This concludes the proof. □

Theorem 3.8 ([2, II.7.6])**.** *Let X be a scheme of finite type over a noetherian ring A, and let L be a line bundle on X. Then L is ample if and only if L^m is very ample over* Spec A *for some $m > 0$.*

In the proof of the above theorem, we need the following technical lemma.

Lemma 3.9. *Let X be a noetherian scheme and let L be an invertible sheaf on X. Let $s \in H^0(X, L)$ be a global section and let*

$$X_s := X \setminus \{s = 0\} := \{x \in X \mid s_x \notin \mathfrak{m}_x L_x\}.$$

If $\mathcal{F}$ is a quasi-coherent sheaf on X and $t \in H^0(X_s, \mathcal{F})$ is a section of $\mathcal{F}$ over X_s, then for some $n \gg 0$, the section $s^n t$ of $L^n \otimes \mathcal{F}$ extends from X_s to a global section of $L^n \otimes \mathcal{F}$ on X.

Proof. The assumption that X is noetherian implies that X as well as any open subset of X is quasi-compact. The statement thus follows from [2, II.5.14], see also [2, II.5.3]. □

Proof. Let us first assume that L^m is very ample over Spec A for some $m > 0$. The line bundle L^m induces by assumption an embedding $X \hookrightarrow \mathbb{P}^n_A$ over A and we let $\overline{X}$ be the closure of this embedding. Then $\overline{X}$ is projective over A. Since the pushforward of a coherent sheaf on X to $\overline{X}$ is again coherent, one easily reduces to the case where $X = \overline{X}$. (E.g. this would be automatic if we had assumed that X is proper to begin with.) Once we've reduced to the case where $X = \overline{X}$ is projective over A, the previous theorem and Lemma 3.5, imply that L is ample if a positive tensor power of L is very ample (over Spec A). This proves one direction in the theorem.

To prove the converse, assume that L is ample. We need to find a positive tensor power of L whose global sections embed X into some projective space. We proceed in several steps.

Step 1. Fix a point $x \in X$. Up to replacing L by a positive power L^m, there is a section $s \in H^0(X, L)$, such that

$$X_s := X \setminus \{s = 0\} := \{x \in X \mid s_x \notin \mathfrak{m}_x L_x\}$$

is affine and contains x; i.e. s does not vanish at x and the complement of the vanishing locus of s in X is affine.

To prove the claim in step 1, let $U \subset X$ be an affine open neighbourhood of x, such that $L|_U \cong \mathcal{O}_U$ is trivial. Let $Z := X \setminus U$ with the induced reduced scheme structure. Then the ideal sheaf $\mathcal{I}_Z$ is a coherent sheaf and so up to replacing L by a suitable power, $\mathcal{I}_Z \otimes L$ is generated by global sections. Since $\mathcal{I}_Z \subset \mathcal{O}_X$ is a subsheaf, we see that there is a global section $s \in H^0(X, L)$ which vanishes along Z and which does not vanish at x. Since $L|_U \cong \mathcal{O}_U$ is trivial, the restriction of s to U corresponds to a regular function f and so

$$X_s = U \setminus \{f = 0\} = \operatorname{Spec} A_{(f)}$$

is affine, where $U = \operatorname{Spec} A$. This proves step 1.

Step 2. Up to replacing L by a positive power L^m, we may assume that there are finitely many sections $s_i \in H^0(X, L)$ such that $X_i := X_{s_i}$ is affine for each i and $X = \bigcup X_i$ is an open cover.

Since X is noetherian, it can be covered by finitely many open subsets as in step 1. Step 2 follows immediately from this and the fact that $X_{s^m} = X_s$, so that passing to further powers is no problem.

Step 3. Since X is of finite type over A, $X_i = \operatorname{Spec} B_i$, where $B_i = H^0(X_i, \mathcal{O}_{X_i})$ is finitely generated as an A-algebra. Let $b_{ij} \in B_i$ be finitely many generators. Up to replacing s_i and L by suitable powers, the section $s_i b_{ij}$ of L over X_i extends to a global section $c_{ij} \in H^0(X, L)$.

This is a direct consequence of Lemma 3.9.

Step 4. The morphism $f : X \to \mathbb{P}^N_A$ that is induced by the global sections s_i and c_{ij} of L is an embedding/immersion (i.e. composition of a closed embedding and an open embedding).

Note first that X is covered by X_{s_i} and so the s_i have no common zeros, i.e. the sections S_i generate L and so f is indeed a morphism. Let x_i and x_{ij} be homogeneous coordinates on $\mathbb{P}^N$ that correspond to s_i and c_{ij}. Then $U_i := \mathbb{P}^N_A \setminus \{x_i = 0\}$ is an affine open subset of $\mathbb{P}^N_A$ with $f^{-1}(U_i) = X_i = \operatorname{Spec} B_i$. The morphism f restricts on X_i to a morphism

$$X_i = \operatorname{Spec} B_i \longrightarrow U_i = \operatorname{Spec} A_i.$$

Here A_i us a polynomial ring over A and the above map is induced by a homomorphism of A-algebras

$$A_i \longrightarrow B_i$$

which by construction has each b_{ij} in its image, hence is surjective. This proves that $X_i \to U_i$ is a closed embedding. Hence, $f : X \to \mathbb{P}^N_A$ is the composition of a closed embedding of X into $\bigcup U_i$, and the open embedding of $\bigcup U_i$ into $\mathbb{P}^N_A$. This concludes the proof. □

Theorem 3.10. *Let A be a noetherian ring and let X be a proper scheme over* $\operatorname{Spec} A$*. Let L be a line bundle on X. Then the following are equivalent:*

(a) L is ample;

(b) for each coherent sheaf $\mathcal{F}$ on X, there is an integer n_0 such that

$$H^i(X, \mathcal{F} \otimes L^n) = 0$$

for all $i > 0$ and $n \geq n_0$.

Proof. See [2, Proposition III.5.3]. □

4 Intersection numbers

4.1 Intersecting Cartier divisors with curves

Let X be a scheme of finite type over a field. Let D be a Cartier divisor on X and $C \subset X$ a proper curve, i.e. proper reduced subscheme of pure dimension one. Then we define

$$D \cdot C := \deg(\tau^*(\mathcal{O}_X(D)|_C))$$

where $\tau : C' \to C$ denotes the normalization of C. Here we use that

$$\tau^*(\mathcal{O}_X(D)|_C) \in \operatorname{Pic}(C') \cong \operatorname{CaCl}(C')$$

and taking the degree of a divisor on a smooth projective curve over a field respects linear equivalence of divisors.

Here we recall that on a smooth proper curve C' over a field k, the degree of a divisor $\sum a_i x_i$ is defined by $\sum a_i \deg(\kappa(x_i)/k)$, where $\deg(\kappa(x_i)/k)$ denotes the degree of the finite field extension $\kappa(x_i)/k$. If the ground field k is algebraically closed, then $\kappa(x_i) \cong k$ and so we arrive at $\deg \sum a_i x_i = \sum a_i$.

By definition, the intersection number of a Cartier divisor D on X with a proper curve depends only on the isomorphism class of the line bundle $\mathcal{O}_X(D))$ and hence only on the linear equivalence class of D.

The concept of intersecting Cartier divisors with (proper) curves is fundamental to birational geometry. To illustrate this, let X be a normal projective variety and let $f : X \to Y$ be a morphism to another projective variety Y. We know that f corresponds to a line bundle L on X together with a bunch of base-point-free sections $s_0, \dots, s_N \in H^0(X, L)$. Moreover, $L = f^*A$ for some ample line bundle A on Y.

Fundamental observation. Let $f : X \to Y$ be a proper morphism between quasi-projective varieties. Let $L := f^*A$ for some ample line bundle A on Y. A proper irreducible curve $C \subset X$ is contracted by f, i.e. $f(C)$ is a point, if and only if $L \cdot C = 0$.

Proof. If $f(C)$ is a point, then $f^*A|_C$ is the trivial line bundle, as it factors through the restriction of A to $f(C)$. Hence, $L \cdot C = 0$.

Conversely, let $C \subset X$ be a proper irreducible curve with $L \cdot C = 0$. Let $\tau : C' \to C$ be the normalization and let $L_{C'} := \tau^* L|_C$. We know that $f = [f_0 : \dots : f_N]$ for some base-point-free set of sections $f_i \in H^0(X, L)$. These sections f_i restrict to sections of $L|_C$ on C and so they pullback to sections $f_i' := \tau^* f_i$ of $L_{C'} = \tau^* L|_C$. The composition $C' \to C \subset X \to Y$ is given by

$$[f_0' : \dots : f_N'] : C' \to Y$$

and we need to show that this morphism is constant. But $L_{C'}$ is a line bundle of degree zero on C' that has nonzero sections (at least one of the f_i' does not vanish identically, because $f_0', \dots, f_N'$ is base-point-free, as $f_0, \dots, f_N$ is base-point-free.) Hence, $L_{C'} \cong \mathcal{O}_{C'}$ is trivial and so f_i' is constant for all i. This proves our claim. $\square$

4.2 Digression: morphisms with connected fibres

The above observation allows us to understand which curves on X are contracted by a morphism $f : X \to Y$. Of course, this does in general not describe the morphism, as f could for instance be finite and so it does not contract any curve, but still the morphism might be highly nontrivial.

A class of morphisms $f : X \to Y$ that are in some sense completely understood in terms of the curves they contract are given by morphisms with connected fibres:

Definition 4.1. *Let $f : X \to Y$ be a morphism between varieties. We say that f has connected fibres if $f_* \mathcal{O}_X = \mathcal{O}_Y$.*

Note as an example that if f is the normalization of a cuspidal curve, then f does not have connected fibres via the above definition, but still the set-theoretical fibres of f are connected.

The following important result shows conversely, that morphisms with conenected fibres indeed have connected fibres.

Theorem 4.2 ([2, III.11.3])**.** *Let $f : X \to Y$ be a projective morphism with connected fibres between noetherian schemes (e.g. varieties over a field k). Then $f^{-1}(y)$ is connected for each $y \in Y$.*

The following two results show why morphisms with connected fibres play an important role for us (which in turn explains why it is important for us to understand the intersection of line bundles with curves).

Theorem 4.3 (Zariski's Main Theorem [2, III.11.4])**.** *Let $f : X \to Y$ be a birational proper morphism between quasi-projective varieties. Then for any $y \in Y$, $f^{-1}(y)$ is connected.*

Theorem 4.4 (Stein factorization [2, III.11.5])**.** *Let $f : X \to Y$ be a projective morphism between noetherian schemes (e.g. varieties over a field). Then f factors into the composition of a morphism $f' : X \to Y'$ with connected fibres and a finite morphism $g : Y' \to Y$:*

$$f = g \circ f'.$$

Remark 4.5. *The three theorems above are all consequences of the theorem on formal functions [2, III.11.1], which asserts that if $f : X \to Y$ is a projective (or proper) morphism between noetherian schemes, and $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module, then for any $y \in Y$,*

$$R^i f_* \mathcal{F} \otimes \widehat{\mathcal{O}_{Y,y}} \cong \lim_{\leftarrow n} H^i((X_y)_n, \mathcal{F}|_{(X_y)_n}),$$

where $(X_y)_n = X \otimes \operatorname{Spec}(\mathcal{O}_{Y,y}/\mathfrak{m}_y^n)$ is an infinitesimal neighbourhood of the fibre X_y of f above y and

$$\widehat{\mathcal{O}_{Y,y}} = \lim_{\leftarrow n} (\mathcal{O}_{Y,y}/\mathfrak{m}_{Y,y}^n)$$

is the formal completion of the local ring $\mathcal{O}_{Y,y}$.

4.3 Intersecting Cartier divisors with each other

For most of this course (i.e. once Kleiman's ampleness criterion is proven), it will be enough to understand how to intersect a Cartier divisor with a curve. However, in the proof of this ampleness criterion, we will need to intersect Cartier divisors $D_1, \ldots, D_n$ on a projective scheme of dimension n (over an algebraically closed field) with each other (often the same Cartier divisor with itself). If the D_i meet properly, we would like to say that $D_1 \cdots D_n$ is the number of intersection points with suitable multiplicities. However, this definition is problematic if e.g. $D_i = D$ for all i and so the set-theoretic intersection is far from being zero-dimenisonal. One way around this is to show that we can replace the Cartier divisors D_i by lienarly equivalent Cartier divisors $D_i \sim D_i'$, such that the components of the D_i' meet properly and transversely in finitely many points. Then we define $D_1 \cdots D_n = D_1' \cdots D_n'$, but making this work one has to show that this definition is independent from choices. Instead of making this approach work, we sketch in this section another approach, taken in [1, Section 1.2]. This approach uses cohomology to define the required intersection numbers. Even though it is not very geometric, it is comparatively quick.

To motivate the discussion, note that if C is a smooth projective curve over an algebraically closed field k and L is a line bundle on C, then

$$\chi(C, L^m) = m \cdot \deg(L) - g(C) + 1$$

by the Riemann–Roch theorem. Hence, $\chi(C, L^m)$ is a linear polynomial in m whose leading coefficient is $\deg(L) = C \cdot L$. This point of view generalizes as follows.

Theorem 4.6. *Let $D_1, \ldots, D_n$ be Cartier divisors on a proper scheme X over a field k. Then the function*

$$(m_1, \ldots, m_n) \mapsto \chi(X, \mathcal{O}_X(m_1 D_1 + \cdots + m_n D_n))$$

is a polynomial function on $\mathbb{Z}^n$ with rational coefficients. The degree of this polynomial is at most equal to the dimension of X.

Proof. We sketch the proof in what follows, for more details, see [1, Theorem 1.5].

The main point is that if $0 \to A \to B \to C \to 0$ is a short exact sequence of coherent $\mathcal{O}_X$-modules, then the associated long exact sequence together with the basic fact that cohomology of coherent $\mathcal{O}_X$-modules vanishes in degree $i > \dim X$ shows that

$$\chi(X, B) = \chi(X, A) + \chi(X, C).$$

Using this and the fact that any coherent $\mathcal{O}_X$-module $\mathcal{F}$ admits a filtration by coherent submodules $\mathcal{F}_i \subset \mathcal{F}_{i+1} \subset \cdots \subset \mathcal{F}$ such that $\mathcal{F}_{i+1}/\mathcal{F}_i$ is torsion-free on an integral subscheme of X, we reduce to the case where X is integral and $\mathcal{F}$ is torsion-free. Moreover, any coherent $\mathcal{O}_X$-module is locally free on some non-empty open subset $U \subset X$: $\mathcal{F}|_U \cong \mathcal{O}_U^{\oplus r}$. This yields an embedding $\mathcal{F} \hookrightarrow \mathcal{K}^{\oplus r}$ and we let $\mathcal{G} := \mathcal{F} \cap \mathcal{O}_X^{\oplus r}$. Then we get two short exact sequences

$$0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{F} \longrightarrow \mathcal{G}_1 \longrightarrow 0$$

and

$$0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{O}_X^{\oplus r} \longrightarrow \mathcal{G}_2 \longrightarrow 0,$$

where $\mathcal{G}_1$ and $\mathcal{G}_2$ are supported on $Z := X \setminus U$, which has smaller dimension than X. By the additivity of Euler characteristics in short exact sequences, we reduce the problem by induction to the case where $\mathcal{F} = \mathcal{O}_X$ and X is integral.

Case 1. $n = 1$.

To prove the $n = 1$ case in the theorem, we need to show that on any proper scheme X over a field k and for any Cartier divisor D on X, $\chi(X, \mathcal{O}_X(mD))$ is a polynomial of degree at most $\dim X$ in m. Since D is Cartier, $\mathcal{O}_X(D) \subset \mathcal{K}$. Let

$$J' := \mathcal{O}_X(-D) \cap \mathcal{O}_X \quad \text{and} \quad J'' := \mathcal{O}_X(D) \cap \mathcal{O}_X.$$

These are $\mathcal{O}_X$-submodules of $\mathcal{O}_X$, hence they are ideal sheaves. Let $Y', Y'' \subset X$ be the closed subschemes associated to the ideal sheaves J' and J'', respectively. Concretely, these subschemes are given as follows: whenever D is locally given by a rational function f/g, where f and g are local regular functions on X that are not zero divisors, then $g \in J'$ and $f \in J''$. This implies

$$J'(D) = J'' \subset \mathcal{K}.$$

Hence there are exact sequences

$$0 \longrightarrow J'(mD) \longrightarrow \mathcal{O}_X(mD) \longrightarrow \mathcal{O}_{Y'}(mD) \longrightarrow 0$$

and

$$0 \longrightarrow J'(mD) = J''((m-1)D) \longrightarrow \mathcal{O}_X((m-1)D) \longrightarrow \mathcal{O}_{Y''}((m-1)D) \longrightarrow 0.$$

This implies

$$\chi(X, \mathcal{O}_X(mD)) - \chi(X, \mathcal{O}_X((m-1)D)) = \chi(X, \mathcal{O}_{Y'}(mD)) - \chi(X, \mathcal{O}_{Y''}((m-1)D)).$$

The right hand side is a polynomial of degree at most $\max(\dim Y', \dim Y'') \leq \dim X - 1$. This implies that $\chi(X, \mathcal{O}_X(mD))$ is a polynomial of degree at most $\dim X$, because if $f : \mathbb{Z} \to \mathbb{Z}$ is a function such that $f(m) - f(m-1)$ is a polynomial of degree $n-1$, then $f(m)$ is a polynomial of degree n, see [2, I.7.3(b)]. (Idea: Since $f(m) - f(m-1)$ is a polynomial of degree $n-1$, one easily cooks up a polynomial g of degree n with $f(m) - f(m-1) = g(m) - g(m-1)$ for all m. But then $f(m) - g(m)$ does not depend on m, hence is constant and so f and g differ by a constant. Hence, f is a polynomial of degree n.)

Case 2. $n > 1$.

By the same inductive argument used in case 1 (or in fact by the result of case 1, applied to a suitable sheaf $\mathcal{F}$), we see that for each fixed index $i \in \{1, \dots, n\}$, the function

$$f(m_1, \dots, m_n) := \chi(X, \mathcal{O}_X(m_1 D_1 + \dots + m_n D_n)),$$

thought of as a function in m_i is a polynomial of degree at most $\dim X$ in m_i. It is an elementary fact that this implies that the above function is a polynomial in $m_1, \dots, m_n$. (This is a slightly subtle point, as it was not true if we did not know that the degree of these polynomials is bounded from above by a constant – $\dim X$ – that does not depend on the fixed values of m_j for $j \neq i$. Indeed, using this boundedness result, we can write

$$f(m_1, \dots, m_n) := p(m_n) := \sum_{j=0}^{\dim X} a_j(m_1, \dots, m_{n-1}) m_n^j$$

and all we need to show is that $a_j(m_1, \dots, m_{n-1})$ is a polynomial. But by induction on the number of variables, it suffices to show that it is a polynomial function, and this in turn follows from the fact that we can express the coefficients a_j as suitable rational linear combinations of p evaluated at different values of m_n, e.g. at the values $m_n = 0, \dots, \dim(X)$, where we use that the Vandermonde matrix $(a_{ij})_{0 \leq i,j \leq \dim X}$ with $a_{ij} = i^j$ is invertible.)

Now that we know that f is a polynomial, we still need to see that its degree is only $\dim X$. Sloely the fact that the degree of f as a polynomial in each m_i is at most $\dim X$ does not guarantee this (think about the polynomial $\prod_i m_i^{\dim X}$ whose degree is $n \cdot \dim X$, even though as a function in each m_i its degree is $\dim X$). To show that in fact its degree is bounded from above by $\dim X$, one considers $m_i = m \cdot m'_i$ for suitable but fixed m'_i and applies case 1 to the divisor $D = \sum m'_i D_i$. (E.g. to rule out the polynomial $\prod_i m_i^{\dim X}$ mentioned above, use $m'_i = 1$ for all i; then we get the polynomial $m^{n \cdot \dim X}$ whose degree is too large if $n > 1$, as we want.) This proves the theorem. $\square$

Definition 4.7. *Let $D_1, \dots, D_n$ be Cartier divisors on a proper scheme X over a field k with $n = \dim(X)$. Then we define the intersection number*

$$D_1 \cdots D_n$$

to be the coefficient of $m_1 m_2 \cdots m_n$ in the polynomial $\chi(X, \mathcal{O}_X(m_1 D_1 + \dots + m_n D_n))$.

A priori, the intersection number defined above is only a rational number. But we will see in a second that it is in fact an integer.

Remark 4.8. *By definition, the intersection number $D_1 \cdots D_n$ depends only on the $\mathcal{O}_X$-modules $\mathcal{O}_X(D_i)$, and hence only on the linear equivalence class of the D_i.*

Definition 4.9. *Let X be a proper scheme over a field k. If $Y \subset X$ is a closed subscheme of dimension $\dim Y \leq s$, then for any Cartier divisors $D_1, \ldots, D_s$ on X, we define*

$$D_1 \cdots D_s \cdot Y := D_1|_Y \cdots D_s|_Y.$$

Here, $D_i|_Y$ denotes the pullback of D_i to Y. In general, this pullback is not well-defined on the level of Cartier divisors (e.g. when Y, or an associated point of Y, is contained in the support of D_i), but it is well-defined on the level of associated invertible sheaves, and that's what we mean by $D_i|_Y$ in the above formula. This works, because the intersection number defined above depends only on the linear equivalence class of divisors.

Example 4.10. *Let X be a proper scheme over a field k and let $C \subset X$ be a closed integral subscheme of dimension one. Let $\tau : C' \to C$ be the normalization of C. Then for any Cartier divisor D on X, we have*

$$D \cdot C = \deg(\tau^* \mathcal{O}_X(D)|_C).$$

Proof. To prove the claim made in the above example, note that we need to compute the leading coefficient of the linear polynomial $\chi(C, \mathcal{O}_X(mD))$ in m. Consider the normalization $\tau : C' \to C$. We get a short exact sequence

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \tau_* \mathcal{O}_{C'} \longrightarrow \delta \longrightarrow 0,$$

where δ is a sheaf on δ that is supported on the finitely many singular points of C. Twisting the above sequence with $\mathcal{O}_X(mD)$, we get

$$\chi(C, \tau_* \mathcal{O}_{C'}(mD)) = \chi(C, \mathcal{O}_C(mD)) + \chi(C, \delta(mD)).$$

Since τ is finite,

$$H^i(C', \mathcal{F}) \cong H^i(C, \tau_* \mathcal{F})$$

for any coherent $\mathcal{O}_{C'}$-module $\mathcal{F}$. (If you know about the Leray spectral sequence, this follows from $R^i \tau_* \mathcal{F} = 0$ for $i > 0$, which in turn follows from the fact that τ is affine and coherent modules on affine varieties have no higher cohomology.) Hence,

$$\chi(C', \mathcal{O}_{C'}(mD)) = \chi(C, \mathcal{O}_C(mD)) + \chi(C, \delta(mD)).$$

Since $\chi(C, \delta(mD))$ is constant (as the support of δ is zero dimensional), the claim follows from Riemann–Roch on C'. ☐

Proposition 4.11. *Let $D_1, \ldots, D_n$ be Cartier divisors on a proper scheme X of dimension n over a field k. Then,*

(a) the map

$$(D_1, \ldots, D_n) \mapsto D_1 \cdots D_n$$

is multilinear, symmetric and takes integral values;

(b) if D_n is effective with associated subscheme $Y \subset X$, then

$$D_1 \cdots D_n = D_1 \cdots D_{n-.1} \cdot Y.$$

Proof. See [1, Proposition 1.8]. We sketch the argument in what follows.

To prove (a), note that while symmetry of $D_1 \cdots D_n$ is clear by our definition, neither integrality nor multilinearity is clear. It turns out that both properties follow from formal properties of the definition as follows. The main point is the following. If $P(T_1, \ldots, T_n)$ is a polynomial of degree at most n, then the coefficient in front of $T_1 \cdots T_n$ is given by

$$\sum_{I \subset \{1,\ldots,n\}} (-1)^{|I|} P(-\epsilon_I),$$

where ϵ_I is the vector of length n whose i-th entry is 1 if $i \in I$ and 0 otherwise.

In the situation of the proposition, $n \geq \dim X$ and we know by the above theorem that $\chi(X, m_1 D_1 + \cdots + m_n D_n)$ is a polynomial of degree at most $\dim X$. Hence,

$$D_1 \cdots D_n = \sum_{I \subset \{1,\ldots,n\}} (-1)^{|I|} \chi(X, -\sum_{i \in I} D_i). \tag{2}$$

This implies that $D_1 \cdots D_n$ is indeed an integer (because the above right hand side obviously is).

For (a), it remains to prove that

$$(D_1 + D_1') \cdot D_2 \cdots D_n = D_1 D_2 \cdots D_n + D_1' D_2 \cdots D_n.$$

To see this, note that it is a formal consequence of (2) that

$$(D_1 + D_1') D_2 \cdots D_n - D_1 D_2 \cdots D_n - D_1' D_2 \cdots D_n = D_1' D_1 D_2 \cdots D_n,$$

where the right hand side is the coefficient of $m_1' m_1 \cdots m_n$ in the polynomial

$$\chi(X, m_1' D_1' + m_1 D_1 + \cdots + m_n D_n).$$

But by the previous theorem, this coefficient must be zero, because $\dim X = n$. This proves (a).

For (b), let $I \subset \{1, \ldots, n-1\}$ and consider the exact sequence

$$0 \longrightarrow \mathcal{O}_X(-D_n - \sum_{i \in I} D_i) \longrightarrow \mathcal{O}_X(-\sum_{i \in I} D_i) \longrightarrow \mathcal{O}_Y(-\sum_{i \in I} D_i) \longrightarrow 0.$$

This implies

$$\chi(X, \mathcal{O}_X(-\sum_{i \in I} D_i)) - \chi(X, \mathcal{O}_X(-D_n - \sum_{i \in I} D_i)) = \chi(Y, \mathcal{O}_Y(-\sum_{i \in I} D_i)).$$

The claim follows now easily from the formula (2). $\square$

4.4 Self-intersection of ample divisors and degree of projective schemes

In this section we collect four corollaries of the theory of intersection numbers (Theorem 4.6 and Proposition 4.11), developed in the previous section.

Corollary 4.12. *Let $X \subset \mathbb{P}^N_k$ be an integral projective scheme of dimension n over an algebraically closed field k and let $D := \mathcal{O}(1)|_X$. Then D^n is the degree of X, defined as the number of intersection points*

$$\deg X := \sharp(X \cap P)$$

of a general linear subspace $P \subset \mathbb{P}^N_k$ of codimension n with X. In particular, $D^n > 0$.

Proof. If $n = 0$, then nothing is to prove. If $n = 1$, then the statement follows directly from Example 4.10. If $n > 1$, then a general hyperplane section of X is integral by Bertini's theorem [2, II.8.18], and so the statement follows by induction from item (b) in Proposition 4.11. □

It is worth to try to understand to which extent the above corollary generalizes to projective schemes that are not necessarily integral.

Corollary 4.13. *Let $X \subset \mathbb{P}^N_k$ be a projective scheme of dimension n over an algebraically closed field k and let $D := \mathcal{O}(1)|_X$. Then D^n is the degree of X, defined as the number of intersection points of a general linear subspace $P \subset \mathbb{P}^N_k$ of codimension n with X counted with multiplicity. That is,*

$$\deg X := \sum_{x \in X \cap P} \operatorname{length}(\mathcal{O}_{X\cap P,x})$$

where $\operatorname{length}(\mathcal{O}_{X\cap P,x})$ denotes the length of the local ring $\mathcal{O}_{X\cap P,x}$ of the scheme-theoretic intersection $X \cap P$ at the point x (which in this case is an artinian k-algebra). (Note also that the above sum is finite, because $X \cap P$ is zero-dimensional.)

Proof. If $n = 0$, then X is a zero-dimensional scheme. Since $\mathcal{O}_X(m \cdot D)$ is a locally free $\mathcal{O}_X$-module for all m, it is in fact free, i.e. $\mathcal{O}_X(m \cdot D) \cong \mathcal{O}_X$. By definition, we get

$$D \cdot X = \chi(X, \mathcal{O}_X(m \cdot D)) = \chi(X, \mathcal{O}_X) = \dim_k(H^0(X, \mathcal{O}_X)),$$

because $H^i(X, \mathcal{O}_X) = 0$ for $i > 0$, since X has dimension zero. Again because X has dimension zero, we have

$$H^0(X, \mathcal{O}_X) = \bigoplus_{x\in X} \mathcal{O}_{X,x}$$

and the dimension of this k-vector space is exactly the sum of the length of the individual artinian k-akgebras $\mathcal{O}_{X,x}$. This proves the corollary in the case $n = 0$.

The general case follows by induction. For this we note that if we cut X with a general hyperplane section, than by Krull's Hauptideal Satz its dimension goes down by exactly one. The only subtle point in this inductive argument is the following: Since X might have embedded points, it is not true that a Cartier divisor D on X can as a divisor be restricted to any codimension one subscheme $Y \subset X$. It can only be restricted as a Cartier divisor up to linear equivalence, i.e. restricted as a line bundle. However, if the support of D meets the embedded points of Y as well as all components of Y properly, then the restriction of D to Y as a Cartier divisor makes sense and the corresponding divisor class is indeed the pullback of the divisor class of D.[1]

The genericity assumption on the linear subspace P implies that P is the intersection of n general hyperplanes in $\mathbb{P}^N$. Since k is algebraically close, it is infinite. On the other hand, X being projective over a field implies that it has only finitely many components and only finitely many embedded points. Hence, the genericity assumption on the aforementioned hyperplanes (i.e. Cartier divisors) ensures that we can restrict them as divisors, making the induction work. This concludes the proof. □

Corollary 4.14. *Let X be a projective scheme of dimension n over an algebraically closed field k and let D be an ample divisor on X. Then $D^n > 0$.*

[1] Recall from [2, p. 257] that an associated point of a scheme X is a point $x \in X$, such that every element of the maximal ideal $\mathfrak{m}_x$ is a zero divisor in the local ring $\mathcal{O}_{X,x}$. An associated point of X which is not a generic point of a component of X is called embedded point.

Proof. Up to replacing D by some multiple, we may assume that D is very ample. Then the corollary follows from Corollary 4.13, because $X \cap P$ is non-empty. To prove the latter by induction, we need to show that the intersection of X with a general hyperplane $H \subset \mathbb{P}^N_k$ has dimension $\dim X - 1$. Since k is algebraically closed, it is an infinite field and so H meets each component of X properly, by the genericity assumption on H. This implies that locally, the defining equation of H restricted to X is not a zero divisor. To prove that $\dim(X \cap H) = \dim X - 1$, it thus suffices by Krull's Hauptideal Satz (see e.g. [2, I.1.11A]) that the local equation of H is not a unit on any component of X. But if this was the case on one component X' of X, then $X' \subset \mathbb{P}^N_k \setminus H \cong \mathbb{A}^N_k$. Hence, X' is projective and affine at the same time, and this implies that X' is zero-dimensional. This concludes the proof. $\square$

Remark 4.15. *In the above arguments, we used repeatedly that our ground field k is infinite to ensure that for any given closed subscheme $X \subset \mathbb{P}^N$ we can find a hyperplane $H \subset \mathbb{P}^N$ which meets X properly in all its associated points. If X has only one associated point (i.e. it is integral), then this conclusion does not need k to be infinite, because if X was contained in $\{x_i = 0\}$ for all i, then $X = \emptyset$. However, if X has several components, than $|k| = \infty$ is necessary, as one can see by the simple example, where k is a finite field and X is the union of all closed points of $\mathbb{P}^N$ – in this case surely no hyperplane (defined over k) meets X properly.*

Corollary 4.16. *Let $X \subset \mathbb{P}^N_k$ be a projective scheme of dimension n over an algebraically closed field k and let $D := \mathcal{O}(1)|_X$. Then the Hilbert function*

$$m \mapsto h^0(X, \mathcal{O}_X(mD)) := \dim H^0(X, \mathcal{O}_X(mD))$$

coincides with $\chi(X, \mathcal{O}_X(mD))$ for large m and this is a polynomial of degree $\dim X$ *in m whose leading coefficient is given by*

$$\frac{D^n}{n!}.$$

Proof. The fact that

$$\chi(X, \mathcal{O}_X(mD)) = h^0(X, \mathcal{O}_X(mD))$$

for $m \gg 0$ is a direct consequence of Serre vanishing, see Theorem 3.10. By Theorem 4.6, it follows in particular, that the so called Hilbert function

$$m \mapsto h^0(X, \mathcal{O}_X(mD))$$

is a polynomial of degree at most $\dim X$ in m for $m \gg 0$. It is a formal consequence of this and the definition of D^n that for $m \gg 0$, we have

$$h^0(X, \mathcal{O}_X(mD)) = \frac{D^n}{n!} m^n + \text{"lower order terms"}.$$

As the left hand side is positive, it follows that $D^n \geq 0$. In fact, the previous corollary shows that $D^n > 0$, but there is also a slightly more elementary/classical way of seeing this.

The classical way of seeing this is as follows. We may up to replacing D by a positive multiple assume that D is very ample. That is, there is a closed embedding $f : X \hookrightarrow \mathbb{P}^N_k$ and $\mathcal{O}_X(D) \cong f^*\mathcal{O}(1)$. The embedding f yields a presentation $X = \operatorname{Proj} S$, where S is a graded ring, given as a quotient of $k[x_0, \dots, x_N]$ by the graded ideal given by the kernel of the natural map

$$k[x_0, \dots, x_N] = \bigoplus_{i \geq 0} H^0(\mathbb{P}^N_k, \mathcal{O}(i)) \longrightarrow \bigoplus_{i \geq 0} H^0(X, \mathcal{O}_X(iD)).$$

We use f to think about X as a closed subscheme of $\mathbb{P}^N_k$. We have a short exact sequence

$$0 \longrightarrow I_X \longrightarrow \mathcal{O}_{\mathbb{P}^N} \longrightarrow \mathcal{O}_X \longrightarrow 0,$$

where I_X is the ideal sheaf of X. Tensoring this with $\mathcal{O}(m)$, we find

$$0 \longrightarrow I_X(m) \longrightarrow \mathcal{O}_{\mathbb{P}^N}(m) \longrightarrow \mathcal{O}_X(mD) \longrightarrow 0.$$

By Serre's vanishing theorem (see Theorem 3.10), $H^1(\mathbb{P}^N, I_X(m)) = 0$ for $m \gg 0$. Hence,

$$H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(m)) \longrightarrow H^0(X, \mathcal{O}_X(mD))$$

is surjective for large $m \gg 0$. In other words, for large enough $m \gg 0$, we have

$$H^0(X, \mathcal{O}_X(mD)) \cong \dim_k(S_m),$$

where S_m denotes the degree m part of the graded ring S with $X = \operatorname{Proj} S$ from above. It is a basic theorem of Hilbert and Serre that the function

$$m \mapsto \dim_k(S_m)$$

is for $m \gg 0$ a polynomial of degree $\dim X$. Hence $D^n > 0$, as claimed. $\square$

5 Nakai–Moishezon Ampleness Criterion

Our final goal of this first part of this course is the Klaiman ampleness criterion, which shows that ampleness of a Cartier divisor only depends on its intersection behaviour with curves. A first step toward this direction is the Nakai–Moishezon ampleness criterion, which is the following theorem.

Theorem 5.1. *A Cartier divisor D on a proper scheme X over a field k is ample if and only if, for every integral subscheme Y of X, one has $D^{dim Y} \cdot Y > 0$.*

Remark 5.2. *The condition that Y is integral makes it easier to apply the theorem. But once D is ample, it follows that $D^{\dim Y} \cdot Y > 0$ for any closed subscheme $Y \subset X$ and so one could as well drop the integrality assumption on Y in the theorem.*

To prove the above theorem, we need the following consequence of Serre's cohomological characterization of ample line bundles in Theorem 3.10.

Corollary 5.3. *Let $f : X \to Y$ be a quasi-finite morphism between proper schemes over a field k. Then for any ample line bundle L on Y, f^*L is ample.*

Proof. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. By Theorem 3.10, we need to show that

$$H^i(X, \mathcal{F} \otimes f^*L^m) = 0$$

for all $i > 0$ and $m \gg 0$. Since f is quasi-finite, the theorem on formal functions (see Remark 4.5) shows that $R^i f_*\mathcal{F} = 0$ for all $i > 0$ and this implies by the Leray spectral sequence that

$$H^i(X, \mathcal{F} \otimes f^*L^m) = H^i(Y, f_*(\mathcal{F} \otimes f^*L^m))$$

By the projection formula, $f_*(\mathcal{F} \otimes f^*L^m) = f_*\mathcal{F} \otimes L^m$ and so

$$H^i(X, \mathcal{F} \otimes f^*L^m) = H^i(Y, f_*\mathcal{F} \otimes L^m)$$

which vanishes for $i > 0$ and $m \gg 0$ by the Serre vanishing theorem, see Theorem 3.10. This concludes the proof. $\square$

Proof of Theorem 5.1. One direction follows directly from Corollary 4.12. For the converse, assume that $D^{\dim Y} \cdot Y > 0$ for any closed subscheme.

Since the sheaf cohomology of a $\mathcal{O}_X$-module $\mathcal{F}$is the same as the sheaf cohomology of $\mathcal{F}$, viewed as a sheaf of k-algebras and not as a sheaf of algebras over $\mathcal{O}_X$, it follows that $H^i(X, \mathcal{F}) = H^i(X^{red}, \mathcal{F})$ for any proper scheme X over a field k. It thus follows from Serre's ampleness criterion 3.10, that D is ample on X if and only if its pullback to the reduction X^{red} is ample. Using Theorem 3.10 once again, one can also show that D is ample on X if and only if its pullback to each irreducible component of X^{red} is ample. This reduces the proof of the theorem to the case where X is integral. As in the proof of Theorem 4.6, consider

$$J' = \mathcal{O}_X(-D) \cap \mathcal{K} \quad \text{and} \quad J'' = \mathcal{O}_X(D) \cap \mathcal{K}$$

and let $Y', Y'' \subset X$ be the corresponding subschemes cut out by the ideal sheaves J' and J''. Recall that $J'(D) = J''$ and so we have two short exact sequences

$$0 \longrightarrow J'(mD) \longrightarrow \mathcal{O}_X(mD) \longrightarrow \mathcal{O}_{Y'}(mD) \longrightarrow 0$$

and

$$0 \longrightarrow J''((m-1)D) \longrightarrow \mathcal{O}_X((m-1)D) \longrightarrow \mathcal{O}_{Y''}((m-1)D) \longrightarrow 0.$$

By induction, the restriction of D to Y' and Y'' is ample. Hence, by Serre vanishing (see Theorem 3.10), we know that

$$h^i(\mathcal{O}_X((m-1)D)) = h^i(J''((m-1)D)) = h^i(J''(mD)) = h^i(\mathcal{O}_X(mD))$$

for all $i \geq 2$ and $m \gg 0$. Hence,

$$\chi(X, \mathcal{O}_X(mD)) = h^0(\mathcal{O}_X(mD)) - h^1(\mathcal{O}_X(mD)) + \text{'constant term'}$$

for all $m \gg 0$. Our assumption implies that $D^{\dim X} > 0$ and so the above expression is a polynomial of degree $\dim X$ which goes to $+\infty$ for $m \to \infty$. In particular, $h^0(\mathcal{O}_X(mD)) > 0$ for $m \gg 0$. Up to replacing D by a suitable positive multiple, we may thus assume that D is effective, i.e. $h^0(X, \mathcal{O}_X(D)) > 0$. In other words, $\mathcal{O}_X(-D) \subset \mathcal{O}_X$ is an ideal sheaf and by abuse of notation, we denote the subscheme $D \subset X$ cut out by this ideal sheaf by D. We then get a short exact sequence

$$0 \longrightarrow \mathcal{O}_X((m-1)D) \longrightarrow \mathcal{O}_X(mD) \longrightarrow \mathcal{O}_D(mD) \longrightarrow 0.$$

By induction, $\mathcal{O}_D(D)$ is ample. Hence, by Serre vanishing,

$$H^1(X, \mathcal{O}_{X(}(m-1)D)) \twoheadrightarrow H^1(X, \mathcal{O}_{X(}(m)D))$$

is surjective for all $m \gg 0$. Hence, $h^1(\mathcal{O}_X(mD))$ is constant for all $m \gg 0$. But then for $m \gg 0$, the above map is not only surjective but also injective, and so

$$H^0(X, \mathcal{O}_X(mD)) \twoheadrightarrow H^0(D, \mathcal{O}_D(mD))$$

is surjective for $m \gg 0$. Since $\mathcal{O}_D(D)$ is ample on D, $\mathcal{O}_D(mD)$ is globally generated for $m \gg 0$. The above surjection thus shows that mD has no base points on D, hence no base points at all, because mD has a section that vanishes exactly on D but nowhere else. Hence, for $m \gg 0$, $\mathcal{O}_X(mD)$ is base point free and so it gives a morphism

$$f : X \longrightarrow \mathbb{P}^N_k$$

with $f^*\mathcal{O}(1) = \mathcal{O}_X(mD)$ for some $m > 0$ and we may w.l.o.g assume $m = 1$. By assumptions, $\mathcal{O}_X(mD)$ has positive degree on each curve in X, hence f cannot contract any curve. That is, f is quasi-finite (it has finite fibres). Thus $\mathcal{O}_X(D) = f^*\mathcal{O}(1)$ is ample by Corollary 5.3. $\square$

Remark 5.4. *It is tempting to think that instead of asking $D^{\dim Y} \cdot Y > 0$ for all integral closed subschemes $Y \subset X$, it might be enough to ask this only for those that have dimension one, i.e. for curves. Somewhat surprisingly, this fails already for surfaces: there is a smooth projective surface S over $\mathbb{C}$, which is the projectivization of a rank two vector bundle over a curve C of genus at least two, such that there is a line bundle L on S that has positive degree on each curve, but L is not effective (the issue being that $L^2 = 0$), see [1, §1.35].*

6 Cone of curves

6.1 Basic definitions

Definition 6.1. *Let X be a proper scheme over a field k. Two Cartier divisors D_1 and D_2 on X are called numerically equivalent, denoted by $D_1 \equiv_{num} D_2$, if for any proper curve $C \subset X$,*

$$D_1 \cdot C = D_2 \cdot C.$$

The group of all Cartier divisors on X modulo numerical equivalence is denoted by

$$N^1(X) := \mathrm{CaDiv}(X)/\equiv_{num}$$

For convenience of notation, we sometimes denote numerical equivalence also by $\equiv$.

It is clear from the definition that $N^1(X)$ is an abelian group that must be torsion-free. It is an important fact that we will use without proof that this group is in fact finitely generated and hence free.

Theorem 6.2. *Let X be a proper scheme over a field k. Then $N^1(X)$ is finitely generated, hence a free abelian group $N^1(X) \cong \mathbb{Z}^{\rho(X)}$ whose rank is denoted by $\rho(X)$, called the Picard rank of X.*

By the above theorem,

$$N^1(X)_{\mathbb{Q}} := N^1(X) \otimes_{\mathbb{Z}} \mathbb{Q} \quad \text{and} \quad N^1(X)_{\mathbb{R}} := N^1(X) \otimes_{\mathbb{Z}} \mathbb{R}$$

are finite dimensional vector spaces over $\mathbb{Q}$ and $\mathbb{R}$ of dimension $\rho(X)$. An element in $N^1(X)_{\mathbb{Q}}$, resp. $N^1(X)_{\mathbb{R}}$ is represented by a $\mathbb{Q}$-linear (resp. $\mathbb{R}$-linear) combination of Cartier divisors on X. Numerical equivalence works well also for such linear combinations.

Definition 6.3. *A 1-cycle on a proper scheme X over a field k is a finite formal $\mathbb{Z}$-linear combination $\gamma = \sum_{i=1}^{r} n_i C_i$, where $n_i \in \mathbb{Z}$ and C_i is an integral proper curve on X. Two 1-cycles γ and γ' are numerically equivalent, denoted by $\gamma \sim \gamma'$ or $\gamma \equiv \gamma'$, if*

$$D \cdot \gamma = D \cdot \gamma'$$

for any Cartier divisor D on X, where the intersection number of a Cartier divisor and a 1-cycle are defined by linearity. The group of all 1-cycles on X modulo numerical equivalence is denoted by $N_1(X)$. Similarly, we put $N_1(X)_{\mathbb{Q}} := N_1(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ and $N_1(X)_{\mathbb{R}} := N_1(X) \otimes_{\mathbb{Z}} \mathbb{R}$.

By definition, the natural pairing

$$N^1(X) \times N_1(X) \longrightarrow \mathbb{Z}$$

is non-degenerate. In particular, $N_1(X)$ a free abelian group of rank $\rho(X)$ and $N_1(X)_{\mathbb{Q}}$ (resp. $N_1(X)_{\mathbb{R}}$) are dual to $N^1(X)_{\mathbb{Q}}$ (resp. $N^1(X)_{\mathbb{R}}$).

We endow the $\mathbb{R}$-vector spaces $N^1(X)_{\mathbb{R}}$ and $N_1(X)_{\mathbb{R}}$ with the natural euclidean topology, allowing us in particular to talk about closures of sets, etc.

Definition 6.4. *Let X be a proper scheme over a field k. Then the cone of curves on X is the subcone $\mathrm{NE}(X) \subset N_1(X)_{\mathbb{R}}$ generated by effective 1-cycles, i.e. 1-cycles of the form $\gamma = \sum n_i C_i$ with $n_i \geq 0$. The closed cone of curves, denoted by $\overline{\mathrm{NE}}(X) \subset N_1(X)_{\mathbb{R}}$ is the closure of $\mathrm{NE}(X)$.*

6.2 Rigidity Lemma

The rigidity lemma refers to the following statement.

Proposition 6.5. *Let $\pi : X \to Y$ and $\pi' : X \to Y'$ be proper morphisms between varieties over some field k. Assume that π has connected fibres.*

(a) If π' contracts one fibre $\pi^{-1}(y_0)$ of π to a point, there is a neighbourhood $U \subset Y$ of y_0, so that $\pi' : \pi^{-1}(U) \to Y'$ factors uniquely through π, i.e. there is a unique morphism $f : U \to Y'$, making the following diagram commutative

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\pi"] \arrow[d, "\pi'"'] & U \arrow[dl, "f"] \\ Y' & \end{tikzcd} .$$

(b) If π' contracts any fibre of π to a point, then π' factors uniquely through π, which means that we may take $U = Y$ in item (a).

Proof. Let Z be the image of

$$g := (\pi, \pi') : X \longrightarrow Y \times Y'$$

and let $p : Z \to Y$ and $p' : Z \to Y'$ be the two projections. Then $\pi = p \circ g$ and since this morphism has connected fibres, p must be surjective. Moreover, since π is proper, p is proper as well [2, Exercise II.4.4]. Since π' contracts $\pi^{-1}(y_0)$, one checks that $p^{-1}(y_0)$ is a single point. By the theorem on the dimension of fibres (see e.g.[6, Corollary 5.22]), p is quasi-finite above a neighbourhood $U \subset Y$ of y_0. Moreover, if we are in case (b), we can take $U = Y$. Up replacing Y by U and X by $X \times_Y U$, we may assume that p is proper and quasi-finite, hence finite (because quasi-finite proper morphisms are finite), and we aim to show that π' factors uniquely through π.

We first prove existence of such a factorization. Since π' clearly factors through g, it suffices for this to show that $p : Z \to Y$ is an isomorphism – the morphism $f : Y \to Y'$ is then given by the composition of p^{-1} with the second projection $Z \to Y'$ (i.e. Z is this way identified to the graph of f). On the other hand,

$$\mathcal{O}_Y \subset p_*\mathcal{O}_Z \subset p_*g_*\mathcal{O}_X = \pi_*\mathcal{O}_X = \mathcal{O}_Y,$$

where we used that π has connected fibres. Hence $p_*\mathcal{O}_Z = \mathcal{O}_Y$, i.e. p has connected fibres. But a finite morphism with connected fibres is an isomorphism, which shows that $p : Z \to Y$ is an isomorphism and so Z is the graph of a morphism $f : Y \to Y'$ with $\pi' = f \circ \pi$. It remains to proof uniqueness of f. For this, let $f' : Y \to Y'$ be another morphism with $\pi' = f' \circ \pi$. Then the graph of f' is a subvariety $\Gamma_{f'} \subset Y \times Y'$ with $Z \subset \Gamma_{f'}$. But since the first projection induces isomorphisms $Z \to Y$ and $/Gamma_{f'} \to Y$, we find that $Z = \Gamma_{f'}$ and so $f = f'$ because $Z = \Gamma_f$ by the construction of f above. This concludes the proof. $\square$

Let $\pi : X \to Y$ be a proper morphism. For any integral curve $C \subset X$, we define $\pi_*C = \deg(C \to \pi(C)) \cdot \pi(C)$ if $\pi(C)$ is a curve and $\pi_*C = 0$ otherwise.

Lemma 6.6. *The proper pushforward on 1-cycles defined above descends to 1-cycles modulo numerical equivalence, and hence yields a push-forward map $\pi_* : N_1(X) \to N_1(Y)$.*

Proof. By the projection formula,

$$C \cdot \pi^* L = \pi_* C \cdot L$$

for any integral curve C on X and line bundle L on Y. By linearity, this formula remains true in the case where C is a 1-cycle. The lemma follows immediately from this formula, as it says that if C and C' are numerically equivalent on X, then $\pi_* C$ and $\pi_* C'$ are numerically equivalent on Y. □

Definition 6.7. *Let $C \subset \mathbb{R}^n$ be a cone, i.e. a subset that is closed under addition and multiplication with non-negative real numbers. A subcone $C' \subset C$ is extremal if the following holds: whenever $a, b \in C$ with $a + b \in C'$, then $a, b \in C'$. Equivalently, there is a hyperplane $H \subset \mathbb{R}^n$ which contains C' and such that C is contained on one side of the hyperplane.*

Definition 6.8. *Let $\pi : X \to Y$ be a morphism of projective (or proper) varieties. We define the subcone $\mathrm{NE}(\pi) \subset \mathrm{NE}(X)$ to be spanned by all effective curves $C \subset X$ with $\pi_* C = 0$, i.e. which are contracted to points on Y.*

Theorem 6.9. *Let X, Y and Y' be projective varieties and let $\pi : X \to Y$ be a morphism.*

(a) The subcone $\mathrm{NE}(\pi)$ of $\mathrm{NE}(X)$ is extremal;

(b) Assume $\pi_* \mathcal{O}_X = \mathcal{O}_Y$ and let $\pi' : X \to Y'$ be a morphism to another projective variety Y'. If $\mathrm{NE}(\pi) \subset \mathrm{NE}(\pi')$, then there is a unique morphism $f : Y \to Y'$ with $\pi' = f \circ \pi$. In particular, π is uniquely determined by $\mathrm{NE}(\pi)$ up to unique isomorphism.

Proof. To prove the first statement, let $\gamma, \gamma' \in \mathrm{NE}(X)$ with $\gamma + \gamma' \in \mathrm{NE}(\pi)$. That is,

$$0 = \pi_* \gamma + \pi_* \gamma' \in N_1(Y).$$

Since γ and γ' are effective, the above class is represented by an effective curve. Since Y is projective, an effective curve is zero on $N_1(Y)$ if and only if the curve is zero. Hence, $0 = \pi_* \gamma$ and $0 = \pi_* \gamma'$. That is, $\gamma, \gamma' \in \mathrm{NE}(\pi)$. This proves (a).

To prove (b), assume that π has connected fibres and $\mathrm{NE}(\pi) \subset \mathrm{NE}(\pi')$. Since the targets of π and π' are projective, the morphisms π and π' correspond to line bundles L (resp. L') and an effective curve $C \subset X$ is contracted by π (resp. π') if and only if $L \cdot C = 0$ (resp. $L \cdot C' = 0$). This implies that the question whether an effective curve on X is contracted by a given morphism (with projective target) depends only on the class of C in $N_1(X)$. So the inclusion $\mathrm{NE}(\pi) \subset \mathrm{NE}(\pi')$ means that every irreducible curve that is contracted by π is also contracted by π'. The existence and uniqueness of $f : Y \to Y'$ follows therefore from the rigidity Lemma (Proposition 6.5). This concludes the proof. □

6.3 Examples

Most of the material discussed in this section can be found in [4, I.1.5].

6.3.1 $\rho = 1$

If $\rho(X) = 1$ (e.g. $X = \mathbb{P}^n$), then $N_1(X)_{\mathbb{R}}$ is the real line and $\mathrm{NE}(X) = \overline{\mathrm{NE}}(X)$ is the half-line, generated (over $\mathbb{R}$) by any effective curve on X.

6.3.2 Product of projective spaces

Let $X = \mathbb{P}^m \times \mathbb{P}^n$ with $m, n \geq 1$. Then $\mathrm{Pic}(X) \cong \mathbb{Z}^2$ and so $\rho(X) = 2$. One easily checks that $\mathrm{NE}(X) = \overline{\mathrm{NE}}(X)$ is generated by lines in the two factors

$$\ell_1 := \mathbb{P}^1 \times \{pt.\} \subset X \quad \text{and} \quad \ell_2 := \{pt.\} \times \mathbb{P}^1 \subset X.$$

In particular, $\mathrm{NE}(X) = \mathbb{R}^+\ell_1 + \mathbb{R}^+\ell_2$, where $\mathbb{R}^+ := \mathbb{R}_{\geq 0}$ denotes the set of non-negative real numbers, has two extremal subcones, corresponding to the projections to the first and second factor of $X = \mathbb{P}^m \times \mathbb{P}^n$, respectively.

6.3.3 Ruled surfaces

In this subsection, we follows [4, I.1.5.A].

Let C be a smooth projective curve an let E be a locally free sheaf of rank two on B, i.e. a rank two vector bundle on C. We define

$$\mathbb{P}(E) := \mathrm{Proj}_{\mathcal{O}_C} \bigoplus_{n \geq 0} (\mathrm{Sym}^n E)$$

to be the projective bundle of one-dimensional quotients of E. There is a projection $\pi : \mathbb{P}(E) \to C$ such that the fibre above $c \in C$ is the projectivization of the dual of the vector space $E \times \kappa(c)$.[2]

There is a natural line bundle $\mathcal{O}(1)$ on $\mathbb{P}(E)$, which arises as tautological quotient

$$\pi^* E \longrightarrow \mathcal{O}(1) \longrightarrow 0.$$

This line bundle satisfies

$$\pi_* \mathcal{O}(n) = \mathrm{Sym}^n E.$$

Let now for simplicity k be algebraically closed.

Lemma 6.10. *The Picard group* $\mathrm{Pic}(\mathbb{P}(E))$ *is freely generated by* $\pi^* \mathrm{Pic}(C)$ *and* $\mathcal{O}(1)$*. That is, there is a short exact sequence*

$$0 \longrightarrow \mathrm{Pic}(C) \xrightarrow{\pi^*} \mathrm{Pic}(\mathbb{P}(E)) \xrightarrow{p} \mathbb{Z} \longrightarrow 0$$

where p maps $\mathcal{O}(1)$ to a generator of $\mathbb{Z}$.

Proof. Since the vector bundle E is Zariski-locally trivial, $\pi : \mathbb{P}(E) \to C$ is a Zariski-locally trivial $\mathbb{P}^1$-bundle. This implies that π admits a section, i.e. there is a smooth projective curve $C' \subset \mathbb{P}(E)$ which maps isomorphically onto C via π. Using this we find that π^* is injective (because the composition of pullback via π and restriction to C' is an isomorphism). The cokernel of π^* surjects onto $\mathbb{Z}$, as one sees by the restriction map $\mathrm{Pic}(\mathbb{P}(E)) \to \mathrm{Pic}(F) \cong \mathbb{Z}$, where F denotes a fibre of π and where we note that $\mathcal{O}(1)$ restricts to a generator of F. It thus remains to proof that $\mathrm{Pic}(\mathbb{P}(E))$ is generated by $\mathcal{O}(1)$ and $\pi^* \mathrm{Pic}(C)$. To see this, note that there is a non-empty open subset $U \subset C$, such that $\pi^{-1}(U) \cong U \times \mathbb{P}^1$. Since the complement of U in C is a finite number of points, the complement of $\pi^{-1}(U)$ in $\mathbb{P}(E)$ is a finite union of prime divisors that are contained in $\pi^* \mathrm{Pic}(C)$.

Note that all our spaces are smooth, so that Picard groups coincide with class groups. By the localization sequence for class groups (see e.g. [2, II.6.5]), we thus conclude that it suffices to show that $\mathrm{Cl}(U \times \mathbb{P}^1)$ is generated by $\mathrm{Cl}(U)$ and $\mathcal{O}(1)$. To see this,, note that $U \times \mathbb{P}^1 \setminus U \times \{pt.\} \cong U \times \mathbb{A}^1$ has the same class group as U, and so the claim follows once again from the localization sequence for class groups, see e.g. [2, II.6.5]. □

[2]Caution: in the literature there is no common agreement whether $\mathbb{P}(E)$ denotes the projective bundle of one-dimensional subspaces or one-dimensional quotients. We follow here the convention used in [4].

By the above lemma, if f denotes a fibre F of π, then

$$N^1(\mathbb{P}(E)) = \xi \cdot \mathbb{Z} \oplus f \cdot \mathbb{Z}, \quad \text{where } \xi = [\mathcal{O}(1)].$$

The intersection form on $N^1(\mathbb{P}(E))_{\mathbb{R}}$ is determined by

$$\xi^2 = \deg E, \quad \xi \cdot f = 1 \quad \text{and} \quad f^2 = 0.$$

(Here $\deg E$ denotes the degree of the determinant line bundle $\det E$ of E and $\xi^2 = \deg E$ is a basic fact.)

Since $\mathbb{P}(E)$ is a surface, curves and divisors coincide and so $N_1(\mathbb{P}(E)) = N^1(\mathbb{P}(E))$.

The main observation in order to understand the cone $\mathrm{NE}(\mathbb{P}(E))$ of effective curves is the following lemma.

Lemma 6.11. *If a class $af + b\xi$ with $a, b \in \mathbb{Z}$ is represented by an effective curve $C' \subset \mathbb{P}(E)$ if and only if $b \geq 0$ and there is a line bundle A of degree a such that*

$$H^0(C, \operatorname{Sym}^b E \otimes A) \neq 0.$$

Proof. By the previous lemma, C' is lineary equivalent to

$$\mathcal{O}(n) \otimes \pi^* A$$

for some $n \in \mathbb{Z}$ and some line bundle A on C. Since $[C] = af + b\xi$, we finde $a = \deg A$ and $b = n$. Since C' is effective

$$H^0(\mathbb{P}(E), \mathcal{O}(n) \otimes \pi^* A) \neq 0$$

and conversely, any nonzero section of the above bundle yields an effective curve C' as above. By the projection formula,

$$H^0(\mathbb{P}(E), \mathcal{O}(n) \otimes \pi^* A) = H^0(C, \operatorname{Sym}^n E \otimes A)$$

and so it is clear that n must be non-negative and the above space is nonzero if and only if a curve C' as above exists. □

The projectivization $\mathbb{P}(E)$ does not change if we twist E with a line bundle. We may thus assume that $\deg E \in \{0, 1\}$ and we may from now on for simplicity assume that $\deg E = 0$. Hence,

$$\xi^2 = 0.$$

Case 1. E is unstable, i.e. there is a quotient line bundle $E \twoheadrightarrow A$, with $\deg A = a < 0 = \deg E$.

In this case, $C' := \mathbb{P}(A) \subset \mathbb{P}(E)$ is a section of π and so

$$[C'] = bf + c\xi.$$

for some $b, c \in \mathbb{Z}$. Intersecting with f shows $c = 1$. Intersecting with ξ yields

$$b = \deg(\mathcal{O}(1)|_{C'}).$$

Since $\mathcal{O}(1)$ is the universal quotient line bundle, we have $\mathcal{O}(1)|_{C'} = A$ and so $b = a$. Hence,

$$[C'] = af + \xi \quad \text{and so} \quad [C']^2 = 2a < 0.$$

Since C' is an effective curve with negative self-intersection on the surface $\mathbb{P}(E)$, it follows that $[C']$ spans an extremal ray of $\mathrm{NE}(\mathbb{P}(E))$. Indeed, any irreducible curve on $\mathbb{P}(E)$ which is not C' will meet C' in a non-negative number of points and so it intersects C' non-negatively, which is to say that $\mathrm{NE}(\mathbb{P}(E))$ is spanned by C' and by all effective curves contained in the half-space $N_1(\mathbb{P}(E))_{C'\geq 0}$ of curve classes that have non-negative intersection with C'.

Since f corresponds to the contraction π, it also spans an extremal ray of $\mathrm{NE}(\mathbb{P}(E))$. This analysis describes $\mathrm{NE}(\mathbb{P}(E))$ completely, showing that it is the closed cone generated by the effective curves F and C'.

Case 2. E is semistable, i.e. there is no quotient sheaf of negative degree.

We need to use the fact that as E is semistable, so is $\mathrm{Sym}^n E$ for all n. That is, $\mathrm{Sym}^n E$ does not admit quotient bundles of negative degree. This translates into:

Claim. $H^0(C, \mathrm{Sym}^n E \otimes A) = 0$ whenever A is a line bundle of negative degree on C.

Proof. Indeed, if $H^0(C, \mathrm{Sym}^n E \otimes A) \neq 0$, then there is an injection of $\mathcal{O}_C$-modules $\mathcal{O}_C \hookrightarrow \mathrm{Sym}^n E \otimes A$ and hence an injection

$$A^{-1} \hookrightarrow \mathrm{Sym}^n E$$

with quotient sheaf Q. Since E has degree zero, $\mathrm{Sym}^n E$ has degree zero as well and so Q has negative degree because A^{-1} has positive degree – a contradiction. (Strictly speaking, the above quotient sheaf Q might not be a vector bundle, as it might have torsion, but up to tensoring A^{-1} with a line bundle that corresponds to the torsion part of Q, we may indeed assume that Q is torsion-free and so it is locally free because C is a smooth curve and a finitely generated module over a principal ideal domain (here the local rings of C) is free if and only if it is torsion free.) □

By the above claim and Lemma 6.11, we see that $\mathrm{NE}(\mathbb{P}(E))$ is a subcone of the cone

$$\mathbb{R}_{\geq 0} \cdot \xi + \mathbb{R}_{\geq 0} \cdot f$$

generated by ξ and f.

While f is clearly contained in $\mathrm{NE}(\mathbb{P}(E))$, ξ is the class of an effective curve if and only if there is a degree zero line bundle A on C such that $H^0(C, \mathrm{Sym}^n E \otimes A) \neq 0$. This implies that E is semistable but not stable. A theorem of Narasimhan and Seshadri implies that such bundles exist if $g(C) \geq 2$ and $k = \mathbb{C}$. (This uses an equivalence between such bundles with certain representations of the fundamental group, which in turn explains the restriction on the genus, as the fundamental group needs to be sufficiently complicated for this to work.) So in this case, ξ is not the class of an effective curve.

On the other hand, ξ has positive intersection with any effective curve on $\mathbb{P}(E)$ – again because any such curve C' corresponds to a section of $\mathrm{Sym}^n E \otimes A$ for some line bundle A of positive degree (because E is stable) and unless C' is a fibre of π (in which case we have $\xi \cdot C' = \deg(\mathcal{O}(1)|_{C'}) = 1$), we have

$$C' \cdot \xi = \deg A > 0.$$

Finally, since ξ is positive on any curve, it follows from the Nakai–Moishezon criterion, that for any ample line bundle L on $\mathbb{P}(E)$, the line bundle $L \otimes \mathcal{O}(n)$ is ample or all $n \geq 0$ and so some high multiple has a section, which implies that the class of $L \otimes \mathcal{O}(n)$ is contained in $\mathrm{NE}(\mathbb{P}(E))$ for all $n \geq 0$. Letting $n \to \infty$, we find that $\xi \in \overline{\mathrm{NE}}(\mathbb{P}(E))$, and so $\mathrm{NE}(\mathbb{P}(E))$ is a cone with one "open side".

The above discussion verifies in particular the claim made in Remark 5.4.

6.3.4 Self-product of curves of genus ≥ 2

Let C be a smooth projective curve over $k = \mathbb{C}$, say. In general, the Picard rank of $X = C \times C$ depends on C. But if C is very general, then $\rho(X) = 3$ and $N_1(X)_{\mathbb{R}}$ is generated by the classes

$$f_1 := [C \times \{pt.\}], \quad f_2 := [\{pt.\} \times C] \text{ and } \delta := [\Delta_C],$$

where $\Delta_C \subset X = C \times C$ denotes the diagonal. Here we have $f_i^2 = 0$ and $f_1 f_2 = f_1\delta = f_2\delta = 1$. Moreover,

$$\delta^2 = \deg(\mathcal{N}_{\Delta_C/X}) = \deg(T_C) = -\deg(\omega_C) = 2 - 2g(C)$$

by Riemann–Roch, where we used the general fact that the normal bundle of the diagonal is the tangent bundle. In particular, Δ_C is an effective curve with negative self-intersection as long as $g(C) \geq 2$.

Somewhat surprisingly, the cones $\mathrm{NE}(X)$ or $\overline{\mathrm{NE}}(X)$ are in general unknown even if C is very general, see [4, I.1.5.B] for more details. (Even though it is reasonable to conjecture that these cones are both spanned by the three effective classes, mentioned above.) In fact, even the following conjecture of Kollár is open:

Conjecture 6.12. *Let C be a very general complex projective curve of genus at least two. Then the diagonal $\Delta_C \subset C \times C$ is the only integral curve on $C \times C$ with negative self-intersection.*

6.3.5 Self-product of elliptic curves

Let E and F be elliptic curves over $\mathbb{C}$ and let $X := E \times F$. Hodge theory shows that $\rho(X) \in \{2, 3, 4\}$. Moreover, if E and F are very general (and independent of each other), then $\rho = 2$ and if $E = F$ is very general, then $\rho = 3$.

Lemma 6.13. *Let X be an abelian surface (over $\mathbb{C}$, say), e.g. $X = E \times F$ as above. A class $\alpha \in N_1(X)$ lies in the closed cone of curves $\overline{\mathrm{NE}}(X)$ if and only if*

$$\alpha^2 \geq 0 \quad \textit{and} \quad \alpha \cdot h \geq 0$$

for some fixed ample class h on X.

Proof. Since X is a group, we can use the group structure to translate curves and this translation does not change the numerical equivalence class. (Roughly speaking because Euler characteristics are constant in flat families and flatness is more or less automatic for families over curves, i.e. for one-parameter families.) In particular, any class $\alpha \in \mathrm{NE}(X)$ satisfies

$$\alpha^2 \geq 0 \text{ and } \alpha \cdot h \geq 0$$

and so the same holds true by continuity for all $\alpha \in \overline{\mathrm{NE}}(X)$.

Conversely, assume that α satisfies the above inequalities. We then aim to show that $\alpha \in \overline{\mathrm{NE}}(X)$. By continuity, we may assume that $\alpha \in N_1(X)_{\mathbb{Q}}$ is a rational class and that in fact

$$\alpha^2 > 0 \text{ and } \alpha \cdot h > 0$$

holds. Multiplying α with some large integer, we may furthermore assume that $\alpha \in N_1(X)$ is an integral class. Since X is a smooth projective surface, there is a line bundle L on X with $[L] = \alpha \in N_1(X) = N^1(X)$ and we need to show that some multiple of L is effective. By Riemann–Roch:

$$\chi(X, L^m) = \chi(X, \mathcal{O}_X) + \frac{1}{2} L^m \cdot (L^m - K_X).$$

Since X is an abelian surface, $K_X = \mathcal{O}_X$ is trivial and $\chi(X, \mathcal{O}_X) = 0$. Hence,

$$\chi(X, L^m) = \frac{1}{2} m^2 L^2.$$

Since $L^2 = \alpha^2 > 0$, we find that

$$\chi(X, L^m) = h^0(L^m) + h^2(L^m) - h^1(L^m)$$

goes to infinity for large m. But by Serre duality, $h^2(L^m) = h^0((L^*)^m)$ and so either a positive multiple of L or L^* has a section. But $\alpha \cdot h > 0$ implies that no positive multiple of L^* can be effective, and so some positive multiple of L must be effective. This proves the lemma. $\square$

Remark 6.14. *In the situation of the lemma, where X is an abelian surface, the closed cone of effective curves is round, given by the conditions*

$$\alpha^2 \geq 0 \quad \textit{and} \quad \alpha \cdot h \geq 0$$

For instance, if $X = E \times E$ for a very general elliptic curve E, then $\rho = 3$ and $N_1(X)$ is generated by the classes of the two factors f_1, f_2 and the class δ of the diagonal. Here we have

$$f_1 f_2 = f_1 \delta = f_2 \delta = 1 \quad \textit{and} \quad f_1^2 = f_2^2 = \delta^2 = 0.$$

If we choose this as a basis, then an arbitrary class if of the form

$$\alpha = x f_1 + y f_2 + z \delta$$

and we find

$$\alpha^2 = xy + xz + yz.$$

By the Nakai-Moishezon cirterion, an ample class is for instance given by $h = f_1 + f_2 + \delta$ and so $\alpha \cdot h \geq 0$ turns into the condition

$$x + y + z \geq 0.$$

Hence, $\alpha^2 \geq 0$ and $\alpha \cdot h \geq 0$ define a round cone $\overline{\mathrm{NE}}(X)$, given by

$$xy + xz + yz \geq 0 \quad \textit{and} \quad x + y + z \geq 0.$$

6.4 Blow-up of $\mathbb{P}^2$

Let X be the blow-up of $\mathbb{P}^2$ in the nine intersection points of two general cubic curves. You prove on the Exercise sheet 4 that X contains an infinite sequence $C_1, C_2, \ldots$ of smooth rational curves with $C_i^2 = -1$ and $C_i \neq C_j$ for all $i \neq j$. By the adjunction formula,

$$K_{C_i} = K_X C_i + C_i^2.$$

Since $C_j \cong \mathbb{P}^1$, we find

$$-2 = \deg K_{C_i} = K_X C_i - 1 \ \text{ and hence } \ K_X C_i = -1.$$

The classes $\alpha_i = [C_i] \in N_1(X)$ are all integral and so their distance to the origin goes to infinity for $i \to \infty$. The fact that $K_X \alpha_i = -1$ remains constant means that the classes α_i accumulate towards the hyperplane $\{K_X = -1\} \subset N_1(X)_{\mathbb{R}}$. This is a very instructive example for the cone theorem that we will prove later: it turns out to be a general fact that the K_X-negative part of $\overline{NE}(X)$ of a smooth complex projective variety X is locally polyhedrial, i.e. locally generated by finitely many extremal rays, but there may be accumulation points towards the hyperplane of K_X-trivial curves.

Remark 6.15. *The cone of curves (and in particular its K_X-positive part) of the blow-up of $\mathbb{P}^2$ in more than nine points is in general unknown, see e.g. [4, I.1.5.D].*

Remark 6.16. *The bounded negativity conjecture says that on any smooth complex projective surface X, there is a constant $b(X) \in \mathbb{Z}$ such that for any integral curve $C \subset X$, we have $C^2 \geq b(X)$. This conjecture is wide open already in the example of blow-ups of $\mathbb{P}^2$ in sufficiently many points.*

7 Nef divisors and Kleiman's criterion

7.1 Nef divisors

Definition 7.1. *A Cartier divisor D on a proper scheme X over a field k is nef if for every closed integral subscheme $Y \subset X$, we have*

$$D^{\dim Y} \cdot Y \geq 0.$$

Remark 7.2. *By definition, the restriction of a nef divisor to any subscheme is again nef.*

Remark 7.3. *If D is nef then for any closed subscheme $Y \subset X$,*

$$D^{\dim Y} \cdot Y \geq 0.$$

To prove this, we need to show that the leading coefficient of $\chi(Y, \mathcal{O}_Y(mD))$ in front of $m^{\dim Y}$ is non-negative. More generally, one can show that for any coherent sheaf $\mathcal{F}$ on X whose support has dimension n, the coefficient in front of m^n of $\chi(X, \mathcal{F} \otimes \mathcal{O}_X(mD))$ is non-negative. This statement can be proven by induction on the dimension of the support of $\mathcal{F}$. Filtering $\mathcal{F}$ as in the proof of Theorem 4.6, the claim reduces to the case where $\mathcal{F}$ is a torsion-free sheaf on an integral subscheme Y of X. Using the fact that $\mathcal{F}$ is generically locally free, the same argument as at the beginning of the proof of Theorem 5.1 then reduces to the case where $\mathcal{F} = \mathcal{O}_Y$ and $Y \subset X$ is integral, which finally follows from our definition of nefness of D.

Lemma 7.4. *Let X be a proper scheme of dimension n over a field and let H be an ample divisor on X. Let D be a Cartier divisor on X such that $D^r \cdot Y \geq 0$ for any subscheme $Y \subset X$ of dimension r. Then $D^r \cdot H^{n-r} \geq 0$.*

Proof. We then prove the assertion by induction on n and we note that nothing has to be proven if $r = n$, so that we may assume $r < n$. The same reduction step as outlined in the above remark allows us to reduce to the case where X is integral. Up to replacing H by some high multiple, we may assume that H is very ample and so it defines an embedding $X \hookrightarrow \mathbb{P}^N$ for some $N \gg 0$. Let $W \subset X$ be a general hyperplane section. Then by Proposition 4.11,

$$D^r \cdot H^{n-r} = D|_W^r \cdot H^{n-r-1}|_W$$

and the latter is non-negative by induction, because $\dim W = n - 1$. □

Corollary 7.5. *Let X be a proper scheme over a field. The sum $D + H$ of a nef divisor D and an ample divisor H is ample on X.*

Proof. By the Nakai–Moishezon criterion, we need to prove

$$(D + H)^{\dim Y} \cdot Y \geq 0$$

for any integral subscheme $Y \subset X$. Since the restriction of D to Y is still nef and the restriction of H to Y is ample, we may replace Y by X and reduce to the case where $X = Y$, so that the result follows from the above Lemma, which implies

$$(D+H)^n = H^n + \sum_{i \geq 1} \binom{n}{i} D^i H^{n-i} \geq H^n > 0.$$

□

Corollary 7.6. *Let X be a projective scheme over a field. The sum $D + D'$ of two nef divisors D and D' is again nef.*

Proof. Let H be an ample divisor on X. By the previous corollary, $D' + tH$ is an ample $\mathbb{Q}$-divisor for all $t \in \mathbb{Q}_{>0}$ (i.e. some multiple is an honest ample Cartier divisor). Hence, by the lemma

$$D^{\dim Y - s}(D' + tH)^s \cdot Y \geq 0$$

for all closed subschemes $Y \subset X$. Taking the limit $t \to 0$, we find

$$D^{\dim Y - s}(D')^s \cdot Y \geq 0$$

and so

$$(D + D')^{\dim Y} \cdot Y \geq 0$$

by writing out the above power via the binomial formula. This concludes the corollary. □

7.2 Nefness is a numerical condition

Theorem 7.7. *Let X be a proper scheme over a field. A Cartier divisor D on X is nef if and only if for any integral curve $C \subset X$,*

$$D \cdot C \geq 0.$$

This theorem has for instance the following important consequences.

Corollary 7.8. *Let D be a Cartier divisor on a proper scheme X over a field.*

(a) The question whether D is nef depends only on the class of D in $N^1(X)$.

(b) If $f : X' \to X$ is a proper morphism and D is nef, then so is f^*D.

(c) If $f : X' \to X$ is a proper morphism and X' is projective, then D is nef if f^*D is nef.

Proof. Item (a) is an obvious consequence of the theorem and item (b) is a direct consequence of the projection formula and the theorem. Finally, item (c) follows from the projection formula and the fact that if X' is projective, then for any integral curve $C \subset X$, we can cut $f^{-1}(C)$ with general hyperplanes, giving rise to a curve $C' \subset X'$ with $f_*C' = \lambda C$ for some $\lambda \geq 1$. □

Corollary 7.9. *Let X be a proper scheme over a field. Let* $\mathrm{Nef}(X) \subset N^1(X)$ *be the cone generated by nef divisors on X. Then* $\mathrm{Nef}(X)$ *is a closed cone and any integral class in* $\mathrm{Nef}(X)$ *is represented by a nef divisor.*

Proof. By the theorem

$$\mathrm{Nef}(X) = \{D \in N^1(X)_{\mathbb{R}} \mid D \cdot \gamma \geq 0 \mid \gamma \in \mathrm{NE}(X)\}.$$

This is a closed set, because equality in the inequality is allowed. Hence,

$$\mathrm{Nef}(X) = \{D \in N^1(X)_{\mathbb{R}} \mid D \cdot \gamma \geq 0 \mid \gamma \in \overline{\mathrm{NE}}(X)\}.$$

This shows that $\mathrm{Nef}(X)$ is the dual cone of the closed cone of effective curves on X. Moreover, the theorem implies that the sum of two nef divisors is again nef. More generally, any positive linear combination of nef divisors is nef. Hence, any integral class in $\mathrm{Nef}(X)$ is nef. This concludes the proof. □

Proof of Theorem 7.7. One direction being trivial, it suffices to assume that $D \cdot C \geq 0$ for any integral curve C on X and we need to show that D is nef. Hence for an integral subscheme $Y \subset X$ we need to show

$$D^{\dim Y} \cdot Y \geq 0.$$

Replacing X by Y, we reduce to the case where X is integral and we need to show $D^n \geq 0$, where $\dim X = n$. By Chows Lemma, there is a projective variety X' and a proper birational morphism $\pi : X' \to X$. By the projection formula, π^*D has non-negative intersection with any effective curve on X'. Moreover, $D^n = (\pi^*D)^n$. Hence, up to replacing X by X' we may assume that X is projective.

Let H be an ample divisor on X and put $D_t := D + tH$ for $t \in \mathbb{R}$. Consider the real polynomial

$$p(t) := D_t^n = D^n + \sum_{i=1}^{n-1} \binom{n}{i} (H^i \cdot D^{n-i}) \cdot t^i + H^n t^n.$$

We need to show that $p(0) \geq 0$. Since the leading coefficient $H^n > 0$ is positive, we know $p(t) \to \infty$ for $t \to \infty$. If $p(t)$ has no real root, then we are done. We may thus assume that $p(t)$ has at least one real root and we let t_0 be the largest real root of $p(t)$. Hence, $p(t_0) = 0$ and $p(t) \geq 0$ for all $t \geq t_0$. If $t_0 \leq 0$, we are done and so we may assume that $t_0 > 0$.

To proceed, we claim that D_t is ample for all rational $t > t_0$. By the Nakai–Moshezon criterion, we need to compute the intersection

$$D_t^{\dim Y} \cdot Y$$

for a closed integral subscheme $Y \subset X$. Let $r = \dim Y$, then the above intersection number computes as follows:

$$D_t^{\dim Y} \cdot Y = D^r \cdot Y + \sum_{i=1}^{r-1} \binom{r}{i} (H^i \cdot D^{r-i} \cdot Y) \cdot t^i + (H^r \cdot Y) \cdot t^r.$$

By Proposition 4.11,

$$D_t^{\dim Y} \cdot Y = D_t|_Y^{\dim Y} = D|_Y^r + \sum_{i=1}^{r-1} \binom{r}{i} (H|_Y^i \cdot D|_Y^{r-i}) \cdot t^i + (H|_Y^r) \cdot t^r.$$

By Lemma 7.4, we have for $t > 0$ the inequality

$$D_t^{\dim Y} \cdot Y \geq (H|_Y^r) \cdot t^r > 0.$$

Moreover,

$$D_t^{\dim X} = p(t) > 0 \quad \text{for } t \geq t_0.$$

Hence, by the Nakai–Moishezon criterion, we find that D_t is ample for all rational $t > t_0$, meaning that some positive multiple of D_t is an ample Cartier divisor.

We can write
$$p(t) = q(t) + r(t)$$
where
$$q(t) = D_t^{n-1}D \quad \text{and} \quad r(t) = tD_t^{n-1} \cdot H = tD_t^{n-1}|_W$$
and where we assume wlog that H is very ample and $W \subset X$ is a general section of $\mathcal{O}_X(H)$. Since $t_0 > 0$, we know by induction that $D|_W$ is nef and so $D_t|_W$ is ample by Corollary 7.5. Hence, $r(t_0) > 0$. On the other hand, since D_t is ample for rational $t > t_0$ and D has positive degree on each curve, we find
$$q(t) \geq 0$$
for all $t \geq t_0$. Hence,
$$0 = p(t_0) = q(t_0) + r(t_0) \geq r(t_0) > 0.$$
This contradiction concludes the proof of the theorem. □

7.3 Kleiman's criterion

Lemma 7.10. *Let X be a quasi-projective scheme and let H be an ample divisor on X. For any Cartier divisor D on X, $nH + D$ is ample for $n \gg 0$.*

Proof. Since H is ample, there is some integer n_0 so that $\mathcal{O}_X(nH + D)$ is globally generated for all $n \geq n_0$. We claim that this implies that $nH + D$ is ample for $n \geq n_0 + 1$. In other words, replacing D by $n_0H + D$, we may assume that D is globally generated and we claim that this implies that $H + D$ is ample. One can deduce this directly from the definition of ampleness. Indeed, let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then
$$\mathcal{F} \otimes \mathcal{O}_X(nH + nD) = \mathcal{F} \otimes \mathcal{O}(nH) \otimes \mathcal{O}_X(nD).$$
for $n \gg 0$, $\mathcal{F} \otimes \mathcal{O}(nH)$ is globally generated and so the above tensor product is globally generated, because the tensor product of globally generated $\mathcal{O}_X$-modules is globally generated.

If X is projective, one can argue alternatively that any globally generated line bundle is nef (e.g. by Theorem 7.7) and 'ample+nef=ample' by Corollary 7.5. □

Theorem 7.11. *Let X be a projective variety over a field.*

(a) A Cartier divisor D on X is ample if and only if $D \cdot \gamma > 0$ for all nonzero $\gamma \in \overline{\mathrm{NE}}(X)$;

(b) For any ample divisor H on X and any integer N, the set
$$\{\gamma \in \overline{\mathrm{NE}}(X) \mid H \cdot \gamma \leq N\}$$
is compact, hence contains only finitely many classes of irreducible curves.

Proof. To prove (a), let us first assume that D is ample and let $\gamma \in \overline{\mathrm{NE}}(X)$ be nonzeri. Then there is a sequence of effective rational 1-cycles γ_i with $\gamma_i \to \gamma$ for $i \to \infty$. Hence, any ample divisor A on X satisfies
$$A \cdot \gamma = \lim_i A \cdot \gamma_i \geq 0.$$
In particular, $D\gamma \geq 0$ and we need to rule out equality in this inequality.

On the other hand, since the intersection pairing is non-degenerate and $\gamma \neq 0$, there is a divisor E on X with $E \cdot \gamma < 0$. Then $(D + tE) \cdot \gamma < 0$ for all $t \geq 0$ above observation then shows that $D + tE$ cannot be ample for any $t > 0$. In other words, $nD + E$ is not ample for any $n \gg 0$. But this contradicts Lemma 7.10. Hence,

$$D \cdot \gamma > 0.$$

Conversely, let us assume that $D \cdot \gamma > 0$ for all nonzero $\gamma \in \overline{\mathrm{NE}}(X)$. Choose a norm $||\cdot||$ on $N_1(X)$ and consider the compact set

$$K := \{\gamma \in \overline{\mathrm{NE}}(X) \mid ||\gamma|| = 1\}.$$

The divisor D viewed as a functional on $N_1(X)_{\mathbb{R}}$ is positive on K, hence bounded from below by a rational number $\epsilon > 0$. For an ample divisor H on X, the corresponding functional on K will also be bunded, hence be bounded from above by an integer $N \gg 0$. But then the $\mathbb{Q}$-divisor

$$D - \frac{\epsilon}{N} H$$

is non-negative on K, hence nef by Theorem 7.7. It follows that

$$D = \frac{\epsilon}{N} H + (D - \frac{\epsilon}{N} H)$$

is ample, because 'nef+ample=ample' by Corollary 7.5. This proves (a).

To prove (b), let $D_1, \ldots, D_\rho$ be Cartier divisors on X that form a basis of $N^1(X)_{\mathbb{Q}}$. By Lemma 7.10, there is an integer $m \gg 0$, so that $mH \pm D_i$ is ample for all i. Let $\gamma_1, \ldots, \gamma_\rho \in N_1(X)$ be the dual basis of $D_1, \ldots, D_\rho$. Then there is a norm $|\cdot|$ on $N_1(X)$, given by

$$|\sum a_i \gamma_i| = \sum |a_i|.$$

On the other hand, if $\gamma = \sum a_i \gamma_i \in \overline{\mathrm{NE}}(X)$, then

$$0 < \gamma \cdot (mH \pm D_i) = m\gamma \cdot H \pm a_i$$

for all i and so

$$m \cdot \rho \cdot (H \cdot \gamma) > |\gamma|.$$

But this implies that the closed set

$$\{\gamma \in \overline{\mathrm{NE}}(X) \mid H \cdot \gamma \leq N\}$$

is bounded, hence compact, as it is contained in the intersection of $\overline{\mathrm{NE}}(X)$ with a closed ball with respect to $|\cdot|$. This proves part (b), because a compact subset of $N_1(X)_{\mathbb{R}}$ contains only finitely many integral points, as those are discrete in $N_1(X)_{\mathbb{R}}$. This concludes the proof of the theorem. $\square$

Remark 7.12. *By Theorems 7.7 and 7.11, a Cartier divisor is nef (resp. ample) if and onyl if it is non-negative, (resp. positive) on $\overline{\mathrm{NE}}(X) \setminus \{0\}$. For this reason, it makes sense to talk about nefness and ampleness of $\mathbb{R}$-linear combinations of Cartier divisors, by asking the same condition on the image of such a linear combination in $N^1(X)_{\mathbb{R}}$. The set of all nef, resp. ample, classes generate cones*

$$\mathrm{Nef}(X) \subset N^1(X) \quad \textit{and} \quad \mathrm{Amp}(X) \subset N^1(X).$$

By Theorems 7.7 and 7.11, Nef(X) *is the dual of the closed cone of effective curves* $\overline{\mathrm{NE}}(X)$ *and* Amp(X) *is the interior of* Nef(X)*. In particular,* Nef(X) *is a closed cone and* Amp(X) *is an open cone.*

8 Morphism spaces

8.1 Parametrizing rational curves

Let $X \subset \mathbb{P}^N_k$ be a projective variety over a field k. We aim to construct a space $\mathrm{Mor}(\mathbb{P}^1, X)$ which parametrizes all morphisms $f : \mathbb{P}^1 \to X$. To begin with, we study the case $X = \mathbb{P}^N_k$ with $N \geq 1$. A morphism $f : \mathbb{P}^1 \to \mathbb{P}^N_k$ is given by a set of homogeneous polynomials $f_0, \dots, f_N$ of the same degree d with

$$f = [f_0 : f_1 : \cdots : f_N]$$

and where we may assume that $f_0, \dots, f_N$ have no common zero (in any algebraic closure of k). Here, two sets of polynomials $f_0, \dots, f_N$ and $f'_0, \dots, f'_N$ without common zero define the same morphism if and only if they differ by a constant factor. Since each degree d polynomial f_i has $d+1$-coefficients, we see that f is uniquely determined by a point

$$[f] \in \mathbb{P}^{(N+1)(d+1)-1}_k.$$

Moreover,

$$\mathrm{Mor}_d(\mathbb{P}^1_k, \mathbb{P}^N_k) \subset \mathbb{P}^{(d+1)(N+1)-1}_k$$

consists of all ordered tuples $(f_0, \dots, f_N)$ of polynomials of degree d that have no common zero in some algebraic closure of k.

Lemma 8.1. *The subset*

$$\mathrm{Mor}_d(\mathbb{P}^1_k, \mathbb{P}^N_k) \subset \mathbb{P}^{(d+1)(N+1)-1}_k$$

is Zariski-open and so it carries a natural structure of a quasi-projective variety. Moreover, this Zariski open subset can be defined over $\mathbb{Z}$, meaning that its complement can be defined by polynomials with integral coefficients and so there is a morphism space

$$\mathrm{Mor}_d(\mathbb{P}^1, \mathbb{P}^N) \subset \mathbb{P}^{(d+1)(N+1)-1}_{\mathbb{Z}}$$

over $\mathbb{Z}$ with

$$\mathrm{Mor}_d(\mathbb{P}^1_k, \mathbb{P}^N_k) = \mathrm{Mor}_d(\mathbb{P}^1, \mathbb{P}^N) \times_{\mathrm{Spec}\,\mathbb{Z}} \mathrm{Spec}\, k.$$

Proof. By the Nullstellensatz the f_i have a common zero in some algebraic closure $\overline{k}$ of k if and only if

$$(u, v)^r \subset I$$

for some $r \geq 1$. This means that the map

$$(\overline{k}[u, v]_{r-d})^{N+1} \longrightarrow \overline{k}[u, v]_r, \quad (g_0, \dots, g_N) \mapsto \sum g_i f_i$$

is surjective, i.e. has rank $r + 1$. This map is linear and defined over k, hence corresponds to a matrix A_r whose coefficients are linear combinations of the coefficients of the f_i. We thus conclude that $f_0, \dots, f_N$ have a common zero in $\overline{k}$ if and only if for all r, all $(r+1)$-minors of A_r vanish. This is a Zariski closed condition and so

$$\mathrm{Mor}_d(\mathbb{P}^1_k, \mathbb{P}^N_k) \subset \mathbb{P}^{(d+1)(N+1)-1}_k$$

is Zariski-open, as claimed. Since U_k is the complement of the vanishing locus of universal polynomials in the coefficients of f_i, it follows that

$$\mathrm{Mor}_d(\mathbb{P}^1_k, \mathbb{P}^N_k) \subset \mathbb{P}^{(d+1)(N+1)-1}_k$$

is in fact defined over $\mathbb{Z}$, as claimed in the lemma. $\square$

By the above lemma,

$$\mathrm{Mor}(\mathbb{P}^1_k, \mathbb{P}^N_k) = \bigcup_d \mathrm{Mor}_d(\mathbb{P}^1_k, \mathbb{P}^N_k)$$

is an infinite union of quasi-projective varieties, labeled by the degree of the morphism. Here,

$$\mathrm{Mor}_d(\mathbb{P}^1_k, \mathbb{P}^N_k) \subset \mathbb{P}_k^{(d+1)(N+1)-1}$$

is a Zariski open subset which can in fact be defined over the integers. Note also that there is a universal evaluation morphism

$$ev : \mathbb{P}^1 \times \mathrm{Mor}_d(\mathbb{P}^1, \mathbb{P}^N) \longrightarrow \mathbb{P}^N, \quad ([u:v],[f_0:\cdots:f_N]) \mapsto [f_0(u:v):\ldots f_N(u:v)].$$

If $X \subset \mathbb{P}^N_k$ is a closed subscheme, cut out by $h_1,\ldots,h_m$, then

$$\mathrm{Mor}_d(\mathbb{P}^1_k, X) \subset \mathrm{Mor}_d(\mathbb{P}^1_k, \mathbb{P}^N_k)$$

is cut out by the equations

$$h_j(f_0,\ldots,f_N) = 0 \ \text{ for all } \ j = 1,\ldots,m.$$

Hence,

$$\mathrm{Mor}(\mathbb{P}^1_k, X) = \bigcup_d \mathrm{Mor}_d(\mathbb{P}^1_k, X)$$

is also an infinite union of quasi-projective varieties, labeled by the degree of the morphism. Since $\mathrm{Mor}_d(\mathbb{P}^1_k, \mathbb{P}^N_k)$ can be defined over $\mathbb{Z}$, the morphism spaces $\mathrm{Mor}_d(\mathbb{P}^1_k, X)$ have the following property. If X is defined by equations with coefficients in a subring R of X, then these equations define a subscheme $\mathcal{X} \subset \mathbb{P}^N_R$ over R with $\mathcal{X} \times_R \operatorname{Spec} k = X$. By the above construction, $\mathrm{Mor}_d(\mathbb{P}^1_k, X)$ will also be defined over R, meaning that there is a scheme

$$\mathrm{Mor}_d(\mathbb{P}^1, \mathcal{X}) \longrightarrow \operatorname{Spec} R$$

over R so that for any morphism $\operatorname{Spec}\kappa \to \operatorname{Spec} R$, where κ denotes a field, we have

$$\mathrm{Mor}_d(\mathbb{P}^1, X) \times_R \operatorname{Spec}\kappa \cong \mathrm{Mor}_d(\mathbb{P}^1_\kappa, \mathcal{X} \times_R \operatorname{Spec}\kappa).$$

8.2 The general case: Grothendieck's theorem

Theorem 8.2 (Grothendieck)**.** *Let X and Y be varieties over a field k. If X is quasi-projective and Y is projective, then there is a locally noetherian scheme*

$$\mathrm{Mor}(Y, X)$$

over k which parametrizes morphisms $Y \to X$ in the following sense:

(a) there is a universal evaluation morphism

$$ev : Y \times \mathrm{Mor}(Y, X) \longrightarrow X.$$

(b) For any k-scheme T and any morphism of k-schemes

$$F : T \times Y \to X,$$

(which we think of as a family of morphisms $F_t : \{t\} \times Y \to X$ parametrized by $t \in T$), there is a unique k-morphism $\phi : T \to \mathrm{Mor}(Y, X)$ such that the following diagram commutes

$$\begin{tikzcd} Y \times T \arrow[d, "\mathrm{id}\times\phi"'] \arrow[dr, "F"] & \\ Y \times \mathrm{Mor}(Y,X) \arrow[r, "ev"'] & X. \end{tikzcd}$$

Moreover, even though $\mathrm{Mor}(Y,X)$ *is in general not of finite type over* k*, the subscheme*

$$\mathrm{Mor}_P(X,Y) \subset \mathrm{Mor}(X,Y)$$

which parametrizes those morphisms $f : X \to Y$ *whose Hilbert polynomial* $\chi(X, f^*\mathcal{O}_Y(m))$ *coincides with a given polynomial* $P(m)$ *is quasi-projective over* k *(and hence of finite type over* k*) and we have* $\mathrm{Mor}(Y,X) = \bigcup_P \mathrm{Mor}_P(Y,X)$ *where the union runs through all possible Hilbert polynomials.*

In the language of category theory, the above theorem says that the functor

$$\mathcal{M}or(Y,X) : \{\text{schemes}\} \longrightarrow \{\text{sets}\}$$

which maps a k-scheme T to the set $\mathrm{Hom}_k(T \times_k Y, X)$ of morphisms of k-schemes $T \times Y \to X$, is representable by a locally noetherian k-scheme $\mathrm{Mor}(Y,X)$, meaning that the above functor is naturally isomorphic to the functor of points $\mathrm{Hom}(-, \mathrm{Mor}(Y,X))$ of the scheme $\mathrm{Mor}(Y,X)$, given by

$$T \mapsto \mathrm{Hom}(T, \mathrm{Mor}(Y,X)).$$

This means that the above moduli problem has a fine moduli space, which roughly speaking means that the present situation is as nice as one could possibly hope for.

Example 8.3. *If* $Y = \operatorname{Spec} k$*, then* $\mathrm{Mor}(Y,X) \cong X$ *and the evaluation morphism* $Y \times \mathrm{Mor}(Y,X) \to X$ *is the identity.*

Example 8.4. *If* $Y = \operatorname{Spec} k[x]/x^2$*, then a morphism* $Y \to X$ *is uniquely determined by a point plus a tangent direction and so at least if* X *is smooth,* $\mathrm{Mor}(Y,X)$ *is the total space of the tangent bundle of* X*.*

Example 8.5. *Even if* X *and* Y *are smooth, the space* $\mathrm{Mor}(Y,X)$ *might be singular and in fact non-reduced. For instance, if* $X \subset \mathbb{P}^4_k$ *is the Fermat hypersurface* $\sum x_i^d = 0$ *of degree* d *with* $k = \overline{k}$ *of characteristic zero or* $p > d$*, then the morphism space* $\mathrm{Mor}_1(\mathbb{P}^1, X)$ *that parametrizes morphisms* $\mathbb{P}^1 \to X$ *of degree one (so the image is a line) is nowhere reduced, see [1, Section 2.16].*

8.3 The tangent space to $\mathrm{Mor}(Y,X)$

Note that Theorem 8.2 implies in particular that there is a bijection between the k-rational points of $\mathrm{Mor}(Y,X)$ and the k-morphisms $Y \to X$. In particular, for any morphisms $f : Y \to X$ of k-varieties, we may try to describe the tangent space of $\mathrm{Mor}(X,Y)$ at the k-rational point $[f] \in \mathrm{Mor}(Y,X)$. By definition

$$T_{\mathrm{Mor}(Y,X),[f]} \subset \mathrm{Hom}(\operatorname{Spec} k[x]/x^2, \mathrm{Mor}(Y,X))$$

is the k-vector space of homomorphisms $\mathrm{Spec}(k[\epsilon]/\epsilon^2) \to \mathrm{Mor}(Y,X)$ that maps the closed point to $[f]$. By Theorem 8.2 the space of such morphisms is in bijection to the set of morphisms

$$Y \times \mathrm{Spec}(k[\epsilon]/\epsilon^2) \longrightarrow X,$$

that restrict to f on Y.

Proposition 8.6. *Let* X *and* Y *be varieties over a field* k*, with* X *quasi-projective and* Y *projective, and let* $f : Y \to X$ *be a morphism over* k*. Then there is a natural isomorphism*

$$T_{\mathrm{Mor}(Y,X),[f]} \cong H^0(Y, \mathcal{H}om(f^*\Omega^1_X, \mathcal{O}_Y)).$$

Proof. Assume first that $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$ are both affine and let $f^\sharp : A \to B$ be the ring homomorphism that corresponds to $f : X \to Y$. By what we have said above, we are looking for morphisms

$$f_\epsilon^\sharp : A \longrightarrow B[\epsilon]/\epsilon^2$$

of k-algebras that restrict to $f^\sharp$ modulo ϵ. Hence, for $a \in A$ we get

$$f_\epsilon^\sharp(a) = f^\sharp(a) + \epsilon \cdot g(a)$$

and this determines $f_\epsilon^\sharp$ uniquely. The equality

$$f_\epsilon^\sharp(aa') = f_\epsilon^\sharp(a) f_\epsilon^\sharp(a')$$

is equivalent to

$$g(aa') = f^\sharp(a)g(a') + f^\sharp(a')g(a).$$

This is saying that $g : A \to B$ is a k-derivative of the A-module $f^\sharp : A \to B$. Hence, g factors uniquely through Ω_A:

$$g : A \longrightarrow \Omega_A \longrightarrow B.$$

Altogether, this analysis shows that there is a natural isomorphism

$$T_{\operatorname{Mor}(\operatorname{Spec} B, \operatorname{Spec} A),[f]} \cong \operatorname{Hom}_A(\Omega_A, B) \cong \operatorname{Hom}_B(\Omega_A \otimes_A B, B).$$

In general, cover X by affine open subsets $U_i = \operatorname{Spec} A_i$ and Y by affine open subsets $V_i = \operatorname{Spec} B_i$ with $f(V_i) \subset U_i$. As explained above, the tangent space $T_{\operatorname{Mor}(Y,X),[f]}$ is naturally isomorphic to extensions of f to morphisms

$$f_\epsilon : Y \times \operatorname{Spec}(k[\epsilon]/\epsilon^2) \longrightarrow X.$$

Such a morphism is determined by its restrictions

$$f_{\epsilon,i} : V_i \times \operatorname{Spec} k[\epsilon]/\epsilon^2 \longrightarrow U_i.$$

As shown above, each $f_{\epsilon,i}$ corresponds to a section of

$$\operatorname{Hom}_{B_i}(\Omega_{A_i} \otimes_{A_i} B_i, B_i).$$

These sections need to be compatible on overlaps, and so we conclude that there is a natural isomorphism

$$T_{\operatorname{Mor}(Y,X),[f]} \cong H^0(Y, \mathcal{H}om(f^*\Omega_X^1, \mathcal{O}_Y))$$

as claimed. This proves the proposition. □

As an immediate consequence, we find the following.

Corollary 8.7. *When X is smooth along the image $f(Y)$, then*

$$T_{\operatorname{Mor}(Y,X),[f]} \cong H^0(Y, f^*T_X).$$

Proof. Since X is smooth along $f(Y)$, we may up to shrinking X assume that X is smooth. But then

$$\mathcal{H}om(f^*\Omega_X^1, \mathcal{O}_Y)$$

is isomorphic to the pullback of the tangent bundle of X to Y, and so the corollary follows from Proposition 8.6. □

The above corollary say as a special case that for a smooth projective variety X, the tangent space to an automorphism $f \in \operatorname{Aut}(X)$ is given by $H^0(X, f^*T_X)$. In particular, if the latter group vanishes (i.e. if there are no global vector fields on X), then X has discrete automorphism group.

8.4 Local structure of $\mathrm{Mor}(Y,X)$

We prove now the main result of this section, which allows us to bound the local dimension of $\mathrm{Mor}(Y,X)$ in certain situations.

Theorem 8.8. *Let X and Y be projective varieties and let $f : Y \to X$ be a morphism so that X is smooth along $f(Y)$. Then locally around $[f] \in \mathrm{Mor}(Y,X)$, $\mathrm{Mor}(Y,X)$ can be defined by $h^1(X, f^*T_X)$ equations in a non-singular variety of dimension $h^0(Y, f^*T_X)$.*

The theorem has the following immediate consequence.

Corollary 8.9. *In the above situation, any irreducible component of $\mathrm{Mor}(Y,X)$ that passes through the point $[f]$ has dimension at least $h^0(Y, f^*T_X) - h^1(Y, f^*T_X)$.*

Before we can prove the above theorem, we need two technical lemmas.

Lemma 8.10. *Let k be a field and let R be a finitely generated local k-algebra with maximal ideal $\mathfrak{m}$ and residue field k. Let $I \subset \mathfrak{m}$ be an ideal with $\mathfrak{m}I = 0$. Let $f : Y \to X$ be a morphism so that X is smooth along $f(Y)$ and let*

$$f_{R/I} : Y \times \mathrm{Spec}(R/I) \longrightarrow X$$

be an extension of f. Then the obstruction to extending $f_{R/I}$ to a morphism

$$f_R : Y \times \mathrm{Spec}\, R \longrightarrow X$$

lies in $H^1(Y, f^*T_X) \otimes_k I$.

Remark 8.11. *An important special case of the above lemma is the case where $R = k[\epsilon]/\epsilon^m$ for some $m \geq 2$, $I = (\epsilon^{m-1})$ and $\mathfrak{m} = (\epsilon)$. In this case the lemma says that the obstructions from extending a deformation of f from $\mathrm{Spec}\, k[\epsilon]/\epsilon^{m-1}$ to $\mathrm{Spec}\, k[\epsilon]/\epsilon^m$ lies in $H^1(Y, f^*T_X)$.*

Proof of Lemma 8.10. We only sketch the argument, see [1, Lemma 2.7] for more details.
Assume first that $X = \mathrm{Spec}\, A$ and $Y = \mathrm{Spec}\, B$ are affine. Then $f_{R/I}$ is determined by a k-algebra homomorphism

$$f^\sharp_{R/I} : A \longrightarrow B \otimes_k R/I$$

and the extension f_R corresponds to a lift of $f^\sharp_{R/I}$ to a k-algebra homomorphism

$$f^\sharp_R : A \longrightarrow B \otimes_k R.$$

Because X is smooth along f and $I^2 = 0$ (because $\mathfrak{m}I = 0$), such a lifting exists by the infinitesimal lifting property, see [2, Exercise II.8.6]. Moreover, by a similar argument as in the previous proposition, one sees that any two such liftings differ by a k-derivation

$$g : A \longrightarrow B \otimes_k I,$$

hence by an element of $\mathrm{Hom}_A(\Omega_{A/k}, B \otimes_k I)$. This simplifies to

$$\begin{aligned} \mathrm{Hom}_A(\Omega_{A/k}, B \otimes_k I) &\cong \mathrm{Hom}_B(\Omega_{A/k} \otimes_A B, B \otimes_k I) \\ &\cong H^0(Y, \mathcal{H}om(f^*\Omega^1_X, \mathcal{O}_Y)) \otimes_k I \\ &\cong H^0(Y, f^*T_X) \otimes_k I. \end{aligned}$$

If X and Y are not affine, then we cover them by affine opens $U_i \subset X$ and $V_i \subset Y$ with $f(V_i) \subset U_i$. A global extension of f will be given by a collection of local extensions that agree on overlaps. Two different extensions on $V_{ij} = V_i \cap V_j$ differ by the above argument by an element of

$$H^0(V_{ij}, f^*T_X|_{V_{ij}}) \otimes_k I.$$

From this it follows that a given collection of local extensions gives rise to a Cech 1-cycle and hence to an element of

$$H^1(Y, f^*T_X) \otimes_k I$$

which vanishes if and only if the local choices can be made in a compatible way. This proves the lemma. $\square$

Remark 8.12. *It seems to me that one does not need all assumptions of the lemma in the proof. For instance, instead of $I\mathfrak{m} = 0$, $I^2 = 0$ seems enough.*

Lemma 8.13. *Let R be a noetherian local ring with maximal ideal $\mathfrak{m}$ and let $I \subset \mathfrak{m}^2$ be an ideal in R. If the canonical projection $\pi : R \to R/I$ has a section, i.e. a ring homomorphism $s : R/I \to R$ with $\pi \circ s = \mathrm{id}$, then $I = 0$.*

Proof. If $a, b \in R$, then $\pi(s(\pi(a))) = \pi(a)$ and $\pi(s(\pi(b))) = \pi(b)$. This implies

$$s(\pi(a)) = a + a' \quad \text{and} \quad s(\pi(b)) = b + b'$$

for some $a', b' \in I$. Let us from now on assume that $a, b \in \mathfrak{m}$. Then

$$s(\pi(ab)) = s(\pi(a)) \cdot s(\pi(b)) = ab + a'b + ab' + a'b' \in ab + \mathfrak{m}I + I^2.$$

Since $I \subset \mathfrak{m}^2$, this implies that we have for any $x \in I$,

$$0 = s(\pi(x)) \in x + \mathfrak{m}I.$$

Hence, $x \in \mathfrak{m}I$ and so $I \subset \mathfrak{m}I$, which implies $I = 0$ by Nakayama's lemma. This concludes the proof. $\square$

Proof of Theorem 8.8. Locally around $[f]$, $\mathrm{Mor}(Y, X)$ is defined by finitely many polynomials $g_1, \ldots, g_r$ in some affine space $\mathbb{A}_k^N$ over k, so that $[f]$ correponds to the origin in $\mathbb{A}_k^N$. The tangent space of $\mathrm{Mor}(Y, X)$ at $[f]$ is then cut out by the partial derivatives $\frac{\partial g_i}{\partial x_j}$. By Theorem 8.2, this tangent space has dimension $h^0(Y, f^*T_X)$. This implies that $r \geq r' := N - h^0(Y, f^*T_X)$ and up to renumeration, we may assume that the Jacobi matrix of $g_1, \ldots, g_{r'}$ has full rank. Hence, $\mathrm{Mor}(Y, X)$ is locally cut out from $V := \{g_1 = \cdots = g_{r'}\}$, which is locally around the origin smooth of dimension $h^0(Y, f^*T_X)$ by $r - r'$ polynomials. Up to shrinking V, we may assume that V is smooth of dimension $h^0(Y, f^*T_X)$ and we know that $\mathrm{Mor}(Y, X)$ is locally around $[f]$ a subscheme of V. We need to show that this subscheme is cut out by at most $h^1(X, f^*T_X)$ equations. For this it is enough to show that in the regular local

$$R := \mathcal{O}_{V,[f]},$$

the ideal I that locally defines $\mathrm{Mor}(Y, X)$ can be generated by $h^1(X, f^*T_X)$ elements. Let $\mathfrak{m} \subset R$ be the maximal ideal. Since the Zariski tangent spaces of V and $\mathrm{Mor}(Y, X)$ at $[f]$ coincide, we have

$$\mathfrak{m}/\mathfrak{m}^2 = \mathfrak{m}/\mathfrak{m}^2 \otimes R/I$$

and so

$$I \subset \mathfrak{m}^2.$$

Moreover, by Nakayama's lemma, it is enough to show that the k-vector space

$$I/\mathfrak{m}I$$

has dimension at most $h^1(X, f^*T_X)$.

The canonical morphism

$$\mathrm{Spec}(R/I) \longrightarrow \mathrm{Mor}(Y, X)$$

corresponds to a morphism

$$f_{R/I} : Y \times \mathrm{Spec}(R/I) \longrightarrow X$$

that extends $f : Y \to X$.

Since $I \subset \mathfrak{m}^2$, the obstruction to extend $f_{R/I}$ to a morphism

$$f_{R/\mathfrak{m}I} : Y \times \mathrm{Spec}(R/\mathfrak{m}I) \longrightarrow X$$

is by Lemma 8.10 given by an element in

$$H^1(Y, f^*T_X) \otimes_k (I/\mathfrak{m}I),$$

and so we can write this element as

$$\sum_{i=1}^{h^1} a_i \otimes \bar{b}_i$$

where $a_1, \ldots, a_{h^1}$ form a basis of $H^1(Y, f^*T_X)$ and $b_1, \ldots, b_{h^1}$ are elements in I. Hence, up to moding out the elements $b_1, \ldots, b_{h^1}$, the above extension problem is solvable and so

$$\mathrm{Spec}(R/I) \longrightarrow \mathrm{Mor}(Y, X)$$

lifts to a morphism

$$\mathrm{Spec}(R/(\mathfrak{m}I + (b_1, \ldots, b_{h^1}))) \longrightarrow \mathrm{Mor}(Y, X).$$

Since $\mathrm{Spec}(R/I) \longrightarrow \mathrm{Mor}(Y, X)$ is a local isomorphism, we find that the identity $R/I \to R/I$ factors as

$$R/I \longrightarrow R/(\mathfrak{m}I + (b_1, \ldots, b_{h^1})) \xrightarrow{\pi} R/I,$$

where π is the canonical projection. By Lemma 8.13,

$$I = \mathfrak{m}I + (b_1, \ldots, b_{h^1})$$

and so $I/\mathfrak{m}I$ can be generated by $b_1, \ldots, b_{h^1}$, as we want. This concludes the proof. $\square$

8.5 Morphisms with fixed points

Fix a subscheme B of Y and a morphism $g : B \longrightarrow X$. To study those morphisms $Y \to X$ that agree with g on B, we need to consider the restriction morphism

$$\rho : \mathrm{Mor}(Y, X) \longrightarrow \mathrm{Mor}(B, X)$$

and we denote by

$$\mathrm{Mor}(Y, X; g)$$

the fibre of this morphism above the point $[g]$. If $f : Y \to X$ is such that X is smooth along $f(Y)$, the tangent map of ρ is given by

$$d\rho : H^0(Y, f^*T_X) \longrightarrow H^0(B, g^*T_X)$$

and so the tangent space of $\mathrm{Mor}(Y, X; g)$ at $[f]$ is the kernel of the above map, which is nothing but

$$H^0(Y, f^*T_X \otimes I_B).$$

More generally, we have the following extension of Theorem 8.8.

Theorem 8.14. *Let X and Y be projective varieties, let $B \subset Y$ be a closed subscheme and let $g : B \to X$ be a morphism. Let further $f : Y \to X$ be a morphism with $f|_B = g$ and so that X is smooth along $f(Y)$. Then locally around $[f] \in \mathrm{Mor}(Y, X; g)$, $\mathrm{Mor}(Y, X; g)$ can be defined by $h^1(X, f^*T_X \otimes I_B)$ equations in a non-singular variety of dimension $h^0(Y, f^*T_X \otimes I_B)$.*

8.6 Morphisms from a curve

If $Y = C$ is a smooth projective curve and B is finite, then for any $f : C \to X$, we have by Riemann–Roch:

$$\dim_{[f]} \operatorname{Mor}(C, X) \geq \chi(C, f^*T_X) = -K_X \cdot f_*C + (1 - g(C)) \dim X$$

and

$$\begin{aligned}\dim_{[f]} \operatorname{Mor}(C, X; f|_B) &\geq \chi(C, f^*T_X) - \operatorname{length}(B) \dim X \\ &\geq -K_X \cdot C + (1 - g(C) - \operatorname{length}(B)) \cdot \dim X.\end{aligned}$$

If $B = \{c\}$ is a single reduced k-rational point of C, then this simplifies to:

$$\dim_{[f]} \operatorname{Mor}(C, X; f|_{\{c\}}) \geq -K_X \cdot C - g(C) \cdot \dim X. \tag{3}$$

9 Bend and break lemmas

From now on, the ground field k will always be algebraically closed and our main reference for the rest of the class will be [3].

Lemma 9.1 (Bend & Break I)**.** *Let X be a projective variety, C a smooth projective curve, $c_0 \in C$ a point and $f : C \to X$ a morphism. Assume that there is an affine curve T, $t_0 \in T$ a point and a morphism*

$$F : C \times T \longrightarrow X$$

with

- *$F(\{c_0\} \times T)$ is a point on X;*
- *$F|_{C\times\{t_0\}} = f$;*
- *for $t \in T$ general, $F|_{C\times\{t\}}$ is different from f.*

Then there is a morphism $f' : C \to X$ and an effective nontrivial cycle $Z = \sum a_i R_i$ of rational curves R_i on X with $f(c) \in Z$ and such that

$$f_*C \sim_{num} f'_*C + Z.$$

In particular, X contains a rational curve through the point $f(c) \in X$.

Remark 9.2. *The assumptions of the lemma are satisfied if*

$$\dim_{[f]} \operatorname{Mor}(C, X; f|_{\{c\}}) \geq 1.$$

If X is smooth along $f(C)$, then, by (3), this holds if

$$-K_X \cdot f_*C > g(C) \cdot \dim X.$$

Proof of Lemma 9.1. Replacing T by its normalization, we may w.l.o.g. assume that T is smooth. Let T' be a smooth projective compactification of T. Then the morphism F induces a rational map

$$F' : C \times T' \dashrightarrow X.$$

Note that $S := C \times T'$ is a smooth surface and so F' is defined away from possibly finitely many points.

We claim that F' is undefined somewhere along $\{c\} \times T'$. If not, then there is a neighbourhood $U \subset C$ of c such that F' is defined on $U \times T'$. By assumption, F' contracts one fibre of

$$\mathrm{pr}_1 : U \times T' \to U$$

and so it contracts any fibre of pr_1 by the rigidity lemma. Hence, for all $c \in U$:

$$F'(c, t) = F'(c, t_0) = f(c),$$

which contradicts the assumption that for $t \in T$ general, $F|_{C \times \{t\}}$ is different from f.

We have thus shown that if $\tau : S' \to S = C \times T'$ is a sequence of blow-ups along points so that F induces a morphism

$$F' : S' \to X,$$

then τ is not an isomorphism locally around $\{c\} \times \{t\}$ for at least one $t \in T'$. Hence the fibre of $\pi : S' \to T'$ above t is given by the union of C and a cycle of rational curves R.

Let $f' : C \to X$ be the morphism induced by the restriction of F' to $C \times \{t\}$. Since algebraically equivalent 1-cycles are numerically equivalent (or more precisely because Euler characteristics are constant in flat families and $S' \to T'$ is automatically flat),

$$f_* C \sim_{num} f'_* C + F'_* R$$

where $Z = F'_* R$ is a cycle of rational curves on X which passes through $f(c)$ (because if it would not pass through $f(c)$, then by the rigidity lemma, F' would be defined locally around $\{c\} \times \{t\}$). This proves the lemma. $\square$

The above bend and break Lemma allows us to produce rational curves, but a priori the degree of these curves cannot be controlled. The following lemma allows to break up also rational curves into cycles of rational curves with lower degree.

Lemma 9.3 (Bend & Break II)**.** *Let X be a projective variety and let $f : \mathbb{P}^1 \to X$ be a nonconstant morphism. Assume that there is an affine curve T, a point $t_0 \in T$ and a morphism $F : \mathbb{P}^1 \times T \longrightarrow X$ such that*

- *F contracts $\{0\} \times T$ and $\{\infty\} \times T$ to points on X;*
- *$F|_{\mathbb{P}^1 \times \{t_0\}} = f$;*
- *F is generically finite, i.e. $F(\mathbb{P}^1 \times T)$ is a surface.*

Then the 1-cycle $f_* R$ is algebraically (hence numerically) equivalent to a 1-cycle Z that passes through $f(0)$ and $f(\infty)$ and such that either Z is a reducible 1-cycle of rational curves or a multiple rational curve (i.e. one of the form aZ' with $a \geq 2$).

Remark 9.4. *Since the automorphism group of $\mathbb{P}^1$ that fixes two points is one-dimensional, the assumptions of the lemma are satisfied if*

$$\dim_{[f]}(\mathrm{Mor}(\mathbb{P}^1, X; f|_{\{0,\infty\}})) \geq 2.$$

If X is smooth along $f(C)$, then by Section 8.6 the assumptions of the lemma are satisfied if

$$-K_X \cdot f_* \mathbb{P}^1 \geq \dim X + 2.$$

Proof of Lemma 9.3. Replacing T by its normalization, we may w.l.o.g. assume that T is smooth. Let T' be a smooth projective compactification of T. Let $S \to T'$ be a $\mathbb{P}^1$-bundle which compactifies $\mathbb{P}^1 \times T$ and consider the rational map

$$F' : S \dashrightarrow X$$

that is induced by F. Let $\tau : S' \to S$ be a sequence of blow-ups along points so that F' induces a morphism

$$F'' : S' \longrightarrow X.$$

We may assume that the compactification S is chosen in such a way that the number r of blow-ups of τ is minimal with the property that F'' is a morphism.

The fibres of the morphism

$$\pi : S' \longrightarrow T'$$

induced by $\mathrm{pr}_2 : S \to T'$ are trees of rational curves and so there pushforwards to X are algebraically (hence numerically) equivalent 1-cycles whose components are rational curves. Note that π admits two sections T_0' and T_∞', given by the closure of $\{0\} \times T$ and $\{\infty\} \times T$, respectively. These sections are contracted via F' to $f(0)$ and $f(\infty)$, respectively. Hence, we conclude that

$$F'_*\pi^{-1}(t)$$

is a 1-cycle on X that passes through $f(0)$ and $f(\infty)$ and such that each of its components is rational. To prove the lemma, we need to show that for some $t \in T'$, $F'_*\pi^{-1}(t)$ is not integral.

For a contradiction, we assume that this is false, i.e. $F'_*\pi^{-1}(t)$ is integral for all $t \in T'$. This implies that for any $t \in T'$, exactly one component R of $\pi^{-1}(t)$ is not contracted by F''. Moreover, this component R must be a reduced component of $\pi^{-1}(t)$. We aim to arrive at a contradiction from this assumption. We will do so by induction on the number r of blow-ups in τ needed to resolve F'.

If $r = 0$, then $F' : S \to X$ is a morphism which contracts two different sections $T_0', T_\infty' \subset S$ of the $\mathbb{P}^1$-bundle $S \to T'$, given as extensions of $\{0\} \times T$ and $\{\infty\} \times T$, respectively. This implies that the pushforward map

$$F'_* : N_1(S) \longrightarrow N_1(X)$$

contracts the two sections $T_0', T_\infty' \subset S$. Since $\rho(S') = 2$ and F' is generically finite, it follows that T_0' and T_∞' must be numerically proportional to each other:

$$T_0' \sim_{num} \lambda T_\infty'$$

for some $\lambda \geq 0$ (λ cannot be negative, because S is ample and we can intersect with an ample divisor). Hence,

$$T_0' \cdot T_0' = \lambda T_0' \cdot T_\infty' \geq 0.$$

On the other hand, since $F' : S \to X$ is generically finite onto its image, the Stein factorization induces a birational map that contracts C_0 and so one of our exercises implies that $C_0^2 < 0$, which is a contradiction, as we want.

Let now $r > 0$ and let

$$S' = S_r \to S_{r-1} \to \cdots \to S_1 \to S_0 = S$$

be the sequence of blow-ups given by τ. Then S_1 is the blow-up of $S = \mathbb{P}^1 times T'$ in a single point (c, t). This implies that the fibre of $\pi_1 : S_1 \to T'$ above t has two components

$$\pi_1^{-1}(t) = R_0 \cup R_1$$

where $R_i \cong \mathbb{P}^1$ has self-intersection -1 for each $i = 0, 1$ and where R_0 and R_1 meet in a single point Q. Here R_0 denotes the proper transform of $\mathbb{P}^1 \times \{t\} \subset S$ and R_1 denotes the exceptional divisor of the blow-up $S_1 \to S$.

Consider the natural map $\tau_1 : S' \to S_1$. Since the number of blow-ups r is minimal, the rigidity lemma implies that at least one component of $\tau_1^{-1}(R_1)$ is not contracted by F'. Since F' contracts all but one components of $\pi^{-1}(t) = \tau_1^{-1}(R_0 \cup R_1)$, it follows by the same argument that the rational map $S_1 \dashrightarrow X$ is defined along $R_0 \setminus \{Q\}$ and so τ_1 is an isomorphism above $R_0 \setminus \{Q\}$. We claim that $S_1 \dashrightarrow X$ is also defined at Q. If not, then $\tau : S \to S_1$ blows up Q and the unique component of $\pi^{-1}(t) = \tau_1^{-1}(R_0 \cup R_1)$ that is not contracted by F'' lies in $\tau^{-1}(Q)$. But Q is a point of multiplicity two in $R_0 \cup R_1$ and so any component of $\tau^{-1}(Q)$ has multiplicity at least two, which contradicts our assumptions.

Hence, $S_1 \dashrightarrow X$ is defined locally at the -1-curve R_0. Since $r \geq 1$ and r is minimal, at least one component of $\tau_1^{-1}(R_1)$ is not contracted by F'' and so R_0 must be contracted by F''. Since $S_1 \dashrightarrow X$ is defined locally at the -1-curve R_0, $\tau_1 : S \to S_1$ is an isomorphism locally around R_0 and it follows that F'' descends to a morphism on the blow-down of the (-1)-curve $R_0 \subset S$. This concludes the proof by the number of blow-ups r. □

10 Fano varieties are uniruled

Definition 10.1. *A smooth projective variety X is Fano if $-K_X$ is ample.*

Example 10.2. *By the adjunction formula, a smooth projective hypersurface $X_d \subset \mathbb{P}^{n+1}$ of degree d satisfies*

$$K_{X_d} = \mathcal{O}_{\mathbb{P}^{n+1}}(-n - 2 + d)|_X.$$

Hence, X_d is Fano if $d \leq n + 1$.

Theorem 10.3. *Let X be a smooth projective variety that is Fano, i.e. $-K_X$ is ample. Then X contains a rational curve. In fact, through every point $x \in X$ there is a rational curve $C \subset X$ with*

$$0 < C \cdot (-K_X) \leq \dim X + 1.$$

Remark 10.4. *In the example of $X = \mathbb{P}^n$, the above theorem produces a line through each point, because $-K_X = \mathcal{O}(n+1)$. This shows in particular that the upper bound on $C \cdot (-K_X)$ in the theorem is sharp.*

Remark 10.5. *Th above theorem can be sharpened significantly, showing that in fact through any two general points $x, y \in X$, there is a chain of rational curves that joins x with y.*

The space of morphisms from $\mathbb{P}^1$ to X of bounded degree is quasi-projective. The above theorem therefore shows that there must be a morphism $f : \mathbb{P}^1 \to X$ whose deformations sweep out an open subset of X. In other words, there is a quasi-projective variety T of dimension $\dim X - 1$ and a dominant rational map $\mathbb{P}^1 \times T \dashrightarrow X$. This proves:

Corollary 10.6. *Smooth porjective Fano varieties are uniruled.*

Before we prove the theorem, we need the following auxiliary result.

Lemma 10.7. *Let R be a finitely generated integral $\mathbb{Z}$-algebra. Then*

(a) for any maximal ideal $\mathfrak{m} \subset R$, $R/\mathfrak{m}$ is a finite field;

(b) the closed points of $\operatorname{Spec} R$ *are dense.*

Proof. For (a), note that $\mathbb{Z} \cap \mathfrak{m}$ must be a prime ideal of $\mathbb{Z}$. We claim that it must in fact be maximal. To see this, assume for a contradiction that $\mathbb{Z} \cap \mathfrak{m} = 0$. Then the field $R/\mathfrak{m}$ is a finitely generated $\mathbb{Q}$-algebra, hence a finite field extension of $\mathbb{Q}$. Let $e_1, \ldots, e_n$ be a $\mathbb{Q}$-basis of $R/\mathfrak{m}$ and let $x_1, \ldots, x_m \in R/\mathfrak{m}$ be elements that generate $R/\mathfrak{m}$ as a $\mathbb{Z}$-algebra. Then there is an integer $N \gg 0$ such that $Nx_j \in \bigoplus \mathbb{Z} \cdot e_i$. Hence,

$$\bigoplus \mathbb{Q} \cdot e_i = R/\mathfrak{m} \subset \bigoplus \mathbb{Z}[1/N] \cdot e_i,$$

which is absurd. Hence, $R/\mathfrak{m}$ is a field extension of the finite field $\mathbb{F} = \mathbb{Z}/(\mathfrak{m} \cap \mathbb{Z})$. Since $R/\mathfrak{m}$ is finitely generated over $\mathbb{Z}$, this implies that it must be a finite extension of $\mathbb{F}$, hence a finite field. (This follows from the general fact that if $k \subset K$ is a field extension such that K is a finitely generated k-algebra, then K/k is algebraic, hence a finite extension.) This proves (a).

To prove (b), we need to show that for any nonzero element $a \in R$ the open subset $\operatorname{Spec} R_a \subset \operatorname{Spec} R$ contains a closed point of $\operatorname{Spec} R$. In other words, there is a maximal ideal of R that does not contain a. To see this, let $\mathfrak{n}$ be a maximal ideal of the localization R_a. Since R_a is still integral and finitely generated over R, we know by part (a) that $R_a/\mathfrak{n}$ is finite. Consider the natural ring map $\phi : R \to R_a$ and consider the prime ideal

$$\mathfrak{m} := \phi^{-1}(\mathfrak{n}) \subset R.$$

Then $R/\mathfrak{m}$ is a subring of $R_a/\mathfrak{n}$. Hence, $R/\mathfrak{m}$ is finite and so it must be a field. That is, $\mathfrak{m} \subset R$ is a maximal ideal that does not contain a, as we want. This proves the lemma. □

Proof of Theorem 10.3. Let $x \in X$. We aim to find a rational curve on X through x. The idea is to start with a smooth projective curve C of possible large genus and with a morphism $f : C \to X$ with $x \in f(C)$. By the first bend and break lemma, in order to find a rational curve through x we need to ensure that

$$-K_X \cdot f_*C > g(C) \cdot \dim X.$$

Our assumption implies that $-K_X$ is ample and so the rleft hand side will be positive, but in general it seems hard to make that intersection number large compared to the genus of C. For instance, if we let C be a general complete intersection curve, then $-K_X \cdot f_*C$ becomes bigger if we intersect more positive hyperplanes, but this will also make the genus of the resulting curve much larger, and so this strategy does not work.

Main Idea. If C admits an endomorphism $\phi : C \to C$ of degree at least two, then replacing f by $f \circ \phi^m$ the right hand side of the above inequality is independent of m, while the left hand side satisfies

$$-K_X \cdot (f \circ \phi^m)_*C = -K_X \cdot (\deg(\phi^m)) \cdot f_*C = (\deg \phi)^m \cdot (-K_X) \cdot f_*C.$$

Since $(-K_X) \cdot f_*C$ is positive, as X is Fano, the above quantity becomes arbitrary positive for $m \gg 0$, as long as $\deg \phi \geq 2$.

The above approach works for instance if C is an elliptic curve. But if $g(C) \geq 2$ and k is of characteristic zero, then the Hurwitz formula shows that C cannot admit any endomorphism of degree at least two.

However, there is a situation where such endomorphisms always exist: over (algebraic closures of) finite fields.

Case 1. $k = \overline{\mathbb{F}}_p$ is the algebraic closure of $\mathbb{F}_p$ for some prime p.

In this case $f : C \to X$ is defined over a finite field with $q = p^m$ elements. Explicitly, this means that $C \subset \mathbb{P}^N_k$, $X \subset \mathbb{P}^{N'}_k$ and f are all defined over $\mathbb{F}_q$. Since the Frobenius morphism

$$k \longrightarrow k, \quad x \mapsto x^q$$

fixes the subfield $\mathbb{F}_q \subset k$, it thus defines an endomorphism

$$\phi : C \longrightarrow C$$

of degree q, which is given by restriction of the endomorphism of $\mathbb{P}^N$ given by

$$[x_0 : x_1 : \cdots : x_N] \mapsto [x_0^q : x_1^q : \cdots : x_N^q].$$

(Note that this is an endomorphism of $\mathbb{P}^N$ over $\mathbb{F}_q$ and not over k, as it does not commute with the identity on $\operatorname{Spec} k$.) Replacing $f : C \to X$ by $f \circ \phi^m$ for some $m \gg 0$, we may assume that

$$-K_X \cdot f_*C > g(C) \cdot \dim X$$

and so there is a rational curve on X through x by Lemma 9.1 and Remark 9.2.

To prove the theorem, we need to bound the degree of the rational curve. To this end, we may to begin with assume that $f : C = \mathbb{P}^1 \longrightarrow X$ is a non-constant morphism with $x \in f(C)$. Assume that $-K_X \cdot f_*C$ is minimal among all such morphisms. We need to show that

$$-K_X \cdot f_*C \leq \dim X + 1.$$

For a contradiction, assume that $-K_X \cdot f_*C \geq \dim X + 2$. Then Lemma 9.3 and Remark 9.4 show that f_*C can be replaced by a rational curve through x whose intersection number with the ample divisor $-K_X$ must be smaller than $-K_X \cdot f_*C$, which is a contradiction. This concludes the Theorem in the case where k is the algebraic closure of a finite field.

Case 2. k is an arbitrary algebraically closed field.

The general case of arbitrary ground field k (possibly of characteristic zero or not algebraic over its prime field) follows by a reduction to finite fields, as follows.

Since $-K_X$ is ample, there is some positive integer m such that $|-mK_X|$ is very ample and induces an embedding $X \subset \mathbb{P}^N_k$ There is a finitely generated $\mathbb{Z}$-algebra R such that the polynomials that cut out $X \subset \mathbb{P}^N_k$ as well as the k-point $x \in X$ are all defined over R. That is, there is a projective scheme $\mathcal{X} \subset \mathbb{P}^N_T$ over $T = \operatorname{Spec} R$ with structure morphism

$$\pi : \mathcal{X} \to T := \operatorname{Spec} R$$

such that the generic fibre $\mathcal{X} \times \operatorname{Frac} R$ becomes isomorphic to X after extension of scalars from $\operatorname{Frac} R$ to k. Up to shrinking the affine scheme $T = \operatorname{Spec} R$, we may assume that π is flat, and in fact smooth because X is smooth. Since ampleness is an open condition, we may also assume that

$$-K_{X_t} = -K_{\mathcal{X}}|_{X_t}$$

restricts to an ample line bundle on X_t for all $t \in T$. That is, all fibres of π are smooth projective Fano varieties.

For any maximal ideal $\mathfrak{m} \subset R$, the residue field $R/\mathfrak{m}$ is a finite field by Lemma 10.7. Hence, for any closed point $t \in T$, the fibre X_t is a smooth Fano variety with a distinguished point x_t over a finite field $\mathbb{F}$. By what we have proven in Case 1, X_t contains a raitonal curve through x_t whose degree with respect to $H = -mK_X$ is bounded from above by the constant

$$d := m(\dim X + 1)$$

which does not depend on the point $t \in T$. Let $\operatorname{Mor}_{\leq d}(\mathbb{P}^1, \mathcal{X})$ be the space of morphisms $\mathbb{P}^1_R \to \mathcal{X}$ over R of degree at most d. By construction of this space in Section 8.1, we know that $\operatorname{Mor}_{\leq d}(\mathbb{P}^1, \mathcal{X})$ is a quasi-projective scheme over $T = \operatorname{Spec} R$ and for any maximal ideal $\mathfrak{m}$,

$$\operatorname{Mor}_{\leq d}(\mathbb{P}^1, \mathcal{X}) \times \operatorname{Spec} R/\mathfrak{m} \cong \operatorname{Mor}_{\leq d}(\mathbb{P}^1, \mathcal{X} \times R/\mathfrak{m}).$$

The same compatibility holds for morphisms from $\mathbb{P}^1$ to X that pass through the given R-point x of $\mathcal{X}$:

$$\mathrm{Mor}_{\leq d}(\mathbb{P}^1, \mathcal{X}; 0 \mapsto x) \times \operatorname{Spec} R/\mathfrak{m} \cong \mathrm{Mor}_{\leq d}(\mathbb{P}^1, \mathcal{X} \times R/\mathfrak{m}; 0 \mapsto \overline{x}).$$

By Case 1, the image of the structure morphism

$$\pi : \mathrm{Mor}_{\leq d}(\mathbb{P}^1, \mathcal{X}; 0 \mapsto x) \longrightarrow T = \operatorname{Spec} R$$

contains all closed points. Since R is finitely generated over $\mathbb{Z}$, all closed points of the affine scheme R are dense in T, see Lemma 10.7. Since $\mathrm{Mor}_{\leq d}(\mathbb{P}^1, \mathcal{X}; 0 \mapsto x)$ is quasi-projective over $T = \operatorname{Spec} R$, the image of the structure morphism π is constructible (see [2, Exercise II.3.19]). As it contains all closed points, it is in fact dense and so it contains the generic poiont (see [2, Exercise II.3.18(b)]). This shows that X admits a rational curve $R \subset X$ through x of degree

$$R \cdot (-mK_X) \leq d = m(\dim X + 1).$$

Hence, $-R \cdot K_X \leq \dim X + 1$, as we want. This concludes the proof of the theorem. □

11 Rational curves on varieties whose canonical class is not nef

The goal of this section is to weaken the assumptions in Theorem 10.3, by asking only that $K_X \cdot f_*C$ is negative, and not that $-K_X$ is ample. The result (again due to Mori), is as follows.

Theorem 11.1. *Let X be a smooth projective variety and let H be an ample divisor on X. Assume that there is an irreducible curve $C' \subset X$ such that $-C' \cdot K_X > 0$. Then there is a rational curve $E \subset X$ such that*

$$\dim X + 1 \geq -(E \cdot K_X) > 0 \quad \text{and} \quad \frac{-E \cdot K_X}{E \cdot H} \geq \frac{-C' \cdot K_X}{C' \cdot H}.$$

Proof of Theorem 11.1.

Step 1. Reduction to the case where $k = \overline{\mathbb{F}}_p$ is the algebraic closure of $\mathbb{F}_p$.

Assume in this step that the theorem is proven over $\overline{\mathbb{F}}_p$ and let k be an arbitrary algebraically closed field. Let $R \subset k$ be a finitely generated $\mathbb{Z}$-algebra such that X and C admit models $\mathcal{X} \to T := \operatorname{Spec} R$ and $\mathcal{C}' \subset \mathcal{X}$ over R (that is, $X \cong \mathcal{X} \times_R k$ and $C' = \mathcal{C}' \times_R k$). Up to localizing R, we may assume that $\mathcal{X}$ is smooth over $T = \operatorname{Spec} R$, $\mathcal{C}'$ is flat over T. Up to shrinking T further, we may also assume that H extends to a Cartier divisor $\mathcal{H}$ on $\mathcal{X}$. Replacing H by some multiple, we may assume that H is very ample. Shrinking T if necessary, we may assume that a basis of $H^0(X, \mathcal{O}_X(H))$ extends to $H^0(\mathcal{X}, \mathcal{H})$. Since H is very ample, the full linear system $|H|$ yields an embedding $X \hookrightarrow \mathbb{P}^N_k$. Since the linear series $|H|$ extends over T, we get a rational map

$$\phi : \mathcal{X} \dashrightarrow \mathbb{P}^N_R$$

over $T = \operatorname{Spec} R$ which is a morphism and in fact an embedding on the generic fibre. The locus where ϕ is not defined is Zariski closed and so it maps to a proper closed subset of T. hence, up to shrinking T we may assume that ϕ is a morphism over R:

$$\phi : \mathcal{X} \longrightarrow \mathbb{P}^N_R.$$

Since ϕ is an embedding when restricted to the generic point of $T = \operatorname{Spec} R$, it must be an embedding when restricted to an open subset of T. Hence, up to shrinking T, we may assume that H_t is very ample for all $t \in T$.

Since intersection numbers are constant in flat families (because Euler characteristics are), we find that $K_{X_t} = \mathcal{K}_{\mathcal{X}}|_{X_t}$ satisfies

$$-K_{X_t} \cdot C'_t = -K_X \cdot C'$$

for all $t \in T$. For the same reason,

$$H_t \cdot C'_t = H \cdot C'$$

for all $t \in T$.

By Lemma 10.7, the residue field of any closed point of T is a finite field. By assumptions, we thus know that for each closed point $t \in T$, there is a rational curve $E_t \subset X_t$ with

$$E_t \cdot H_t \leq \frac{-(E_t \cdot K_{X_t}) \cdot (C'_t \cdot H_t)}{-C'_t \cdot K_{X_t}} = -(E_t \cdot K_{X_t}) \cdot \frac{C' \cdot H}{-C' \cdot K_X}$$

and

$$\dim X + 1 = \dim X_t + 1 \geq -(E_t \cdot K_{X_t}) > 0.$$

Hence, the degree of E_t with respect to the very ample divisor H_t:

$$E_t \cdot H_t \leq (\dim X + 1) \cdot \frac{C' \cdot H}{-C' \cdot K_X}$$

is bounded from above and this upper bound does not depend on t. As in the proof of Theorem 10.3, this implies that there must be a component of

$$\mathrm{Mor}_{\leq d}(\mathbb{P}^1_R, \mathcal{X})$$

such that the universal evaluation map

$$ev : \mathbb{P}^1 \times T \longrightarrow X$$

has the property that for infinitely many closed points $t \in T$, the restriction

$$ev_t : \mathbb{P}^1 \times \{t\} \longrightarrow X_t$$

of ev to the fibre above $t \in T$ is nothing but the normalization of the rational curve $E_t \subset X_t$.

Let

$$E \subset X$$

be the image of the base change of ev to $\operatorname{Spec} k$:

$$ev \times \operatorname{Spec} k : \mathbb{P}^1 \times \operatorname{Spec} k \longrightarrow X.$$

Then

$$E \cdot H = E_t \cdot H_t \quad \text{and} \quad E \cdot K_X = E_t \cdot K_{X_t}$$

for a Zariski dense set of $t \in T$ and so

$$\dim X + 1 \geq -(E \cdot K_X) > 0 \quad \text{and} \quad \frac{-E \cdot K_X}{E \cdot H} \geq \frac{-C' \cdot K_X}{C' \cdot H},$$

because the assumption in Step 1 implies that the corresponding result holds for the fibres above a dense set of closed points $t \in T$. This concludes step 1.

Step 2. The case where $k = \overline{\mathbb{F}}_p$.

Let $f : C \to X$ be the composition of the normalization $C \to C'$ and the inclusion $C' \hookrightarrow X$. Since $k = \overline{\mathbb{F}}_p$, C is defined over some finite field $\mathbb{F}_q$ with $q = p^s$. That is, there is a smooth projective curve $C_{\mathbb{F}_q}$ over $\mathbb{F}_q$ with $C_{\mathbb{F}_q} \times_{\mathbb{F}_q} k = C$. Let $F : C_{\mathbb{F}_q} \to C_{\mathbb{F}_q}$ be the Frobenius morphism, which is an endomorphism of degree q over $\mathbb{F}_q$ and let

$$\phi := F \times \mathrm{id} : C = C_{\mathbb{F}_q} \times_{\mathbb{F}_q} k \longrightarrow C = C_{\mathbb{F}_q} \times_{\mathbb{F}_q} k$$

be the base change of F, which is an endomorphism over k of degree q. Let further

$$f_m^0 := f \circ \phi^m : C \longrightarrow X.$$

Then

$$(f_m^0)_* C = q^m \cdot f_* C$$

and so

$$-(f_m^0)_* C \cdot K_X - g(C) \cdot \dim X > 0$$

for large m. By Bend and Break I, we conclude that

$$q^m \cdot f_* C = (f_m^0)_* C \sim_{num} (f_m^1)_* C + Z_m^1,$$

where Z_m^1 is a nontrivial sum of rational curves and $f_m^1 : C \to X$ is some morphism. If

$$-(f_m^1)_* C \cdot K_X - g(C) \cdot \dim X > 0$$

then we can apply Bend and Break I again (without composing with the Frobenius) and decompose $(f_m^1)_* C$ further. We do this and repeat the process. In each step, the intersection number $(f_m^1)_* C \cdot H$ goes down and so the process must terminate. This implies that in the above decomposition, we may assume that

$$-(f_m^1)_* C \cdot K_X \leq g(C) \cdot \dim X.$$

Applying Bend and Break II to each irreducible component of Z_m^1, we may also assume that each irreducible component E of Z_m^1 satisfies

$$-E \cdot K_X \leq \dim X + 1. \tag{4}$$

For simplicity of notation, let

$$a := -(f_m^1)_* C \cdot K_X, \quad b := -Z_m^1 \cdot K_X, \quad c := (f_m^1)_* C \cdot H \quad \text{and} \quad d := Z_m^1 \cdot H.$$

We also write

$$M := \frac{-C' \cdot K_X}{C' \cdot H}$$

and get

$$M = \frac{-(f_m^0)_* C \cdot K_X}{(f_m^0)_* C \cdot H} = \frac{a+b}{c+d},$$

because the factor q^m cancels in the fraction.

For large m, $a + b = -q^m \cdot f_* C \cdot K_X$ and $c + d = q^m \cdot f_* C \cdot H$ are large. On the other hand, $a \leq g(C) \cdot \dim X$ is bounded, so that b must be large.

We will use the following lemma.

Lemma 11.2. *Let a, b, c and d be integers such that $c, d > 0$. Then*

$$\frac{a+b}{c+d} \leq \max\left(\frac{a}{b}, \frac{c}{d}\right).$$

Proof. Exercise. □

With the above lemma, we can prove the following.

Claim. For any $\epsilon > 0$, if $m \gg 0$, then there is an irreducible component E of Z^1_m such that

$$\frac{-E \cdot K_X}{E \cdot H} > M - \epsilon.$$

Proof. If $a/c < M$, then $b/d \geq M$ by Lemma 11.2 and so E exists by another application of Lemma 11.2. For large m, $a \leq g(C) \cdot \dim X$ is bounded, hence if c gets large, we are done by the above case. We may thus assume that c remains bounded and we deduce that d gets large because $c + d = q^m \cdot f_* C \cdot H$ is large for large m. Hence, for large m, a and c remain bounded, while b and d get large. For $m \gg 0$, this implies that

$$\frac{b}{d} \approx \frac{b}{d+d} \approx \frac{a+b}{c+d} = M.$$

Hence, for large m,

$$\frac{b}{d} > \frac{a+b}{c+d} - \epsilon = M - \epsilon.$$

The claim then follows from Lemma 11.2. □

For sufficiently small ϵ, let E be the rational curve on X from the above claim. By (4), we know that $-E \cdot K_X \leq \dim X + 1$. On the other hand, M is positive and so for small ϵ, $M - \epsilon$ is also positive. This implies that $-E \cdot K_X$ is positive and so

$$0 < -E \cdot K_X \leq \dim X + 1.$$

Since

$$\frac{-E \cdot K_X}{E \cdot H} > M - \epsilon,$$

while $-E \cdot K_X$ is bounded, $E \cdot H$ must also be bounded. But then for sufficiently small ϵ, it follows that we can actually omit ϵ in the above estimate and deduce

$$\frac{-E \cdot K_X}{E \cdot H} \geq M = \frac{-C' \cdot K_X}{C' \cdot H}.$$

This concludes the proof of Theorem 11.1. □

12 Mori's cone theorem

Theorem 12.1. *Let X be a smooth projective variety over an algebraically closed field k. Then:*

(a) There are countably many rational curves $C_i \subset X$, $i \in I$ such that $0 < -K_X \cdot C_i \leq \dim X + 1$ and

$$\overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}(X)_{K_X \geq 0} + \sum_{i \in I} [C_i] \cdot \mathbb{R}_{\geq 0}.$$

(b) For any $\epsilon > 0$ and any ample divisor H,

$$\overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}(X)_{K_X + \epsilon H \geq 0} + \sum_{i \in I'} [C_i] \cdot \mathbb{R}_{\geq 0},$$

where the sum runs over a finite subset $I' \subset I$.

Proof. Let us first prove part (a). For this, note that $\mathrm{Mor}(\mathbb{P}^1, X)$ contains only countably many irreducible components and raitonal curves parametrized by the same component are numerically equivalent. Hence, X contains only finitely many rational curves up to numerical equivalence. In each numerical equivalence class $\gamma \in N_1(X)$ with $0 < -K_X\alpha \leq \dim X + 1$ we then pick (if possible) a rational curve C_i. This leads to a countable index set $i \in I$ and we define $W \subset N_1(X)_{\mathbb{R}}$ to be the closure of

$$\overline{\mathrm{NE}}(X)_{K_X \geq 0} + \sum_{i \in I} [C_i]\mathbb{R}_{\geq 0}.$$

Step 1. Here we prove that $W = \overline{\mathrm{NE}}(X)$.

For the contrary, assume that the natural inclusion $W \subset \overline{\mathrm{NE}}(X)$ is strict. Then there is a divisor D on X whose associated linear funcitonal on $N_1(X)_{\mathbb{R}}$ is positive in $W \setminus \{0\}$ but negative somewhere on $\overline{\mathrm{NE}}(X)$. Let H be an ample divisor on X and let $\mu > 0$ be the largest (real) number such that $H + \mu D$ is nef. Choose a nonzero element $z \in \overline{\mathrm{NE}}(X)$ with $(H + \mu D) \cdot z = 0$. Then $D \cdot z < 0$, because $H \cdot z > 0$ by Kleiman's criterion. Moreover, $K_X \cdot z < 0$ because $\overline{\mathrm{NE}}(X)_{K_X \geq 0} \subset W$.

By definition, there is a sequence of effective 1-cycles $Z_k = \sum a_{kj} Z_{kj}$ with

$$Z_k \overset{k \to \infty}{\longrightarrow} z,$$

where the limit happens inside $N_1(X)_{\mathbb{R}}$.

For any rational number $\mu' < \mu$, the $\mathbb{Q}$-divisor $H + \mu' D$ is ample on X. By Lemma 11.2, we may up to relabelling assume that

$$\frac{-K_X \cdot Z_{k0}}{(H + \mu' D) \cdot Z_{k0}} \geq \frac{-K_X \cdot Z_k}{(H + \mu' D) \cdot Z_k}.$$

Applying Theorem 11.1 to the curve Z_{k0} and the ample divisor $H + \mu' D$, we find that there is a rational curve $E_k \subset X$ with

$$0 < -K_X \cdot E_k \leq \dim X + 1$$

and

$$\frac{-K_X \cdot E_k}{(H + \mu' D) \cdot E_k} \geq \frac{-K_X \cdot Z_{k0}}{(H + \mu' D) \cdot Z_{k0}} \geq \frac{-K_X \cdot Z_k}{(H + \mu' D) \cdot Z_k}.$$

Since E_k is a rational curve with $0 < -K_X \cdot E_k \leq \dim X + 1$, it must be numerically equivalent to one of the C_i's and so $D \cdot E_k > 0$. Hence, for all $0 < \mu' < \mu$, we have

$$\frac{-K_X \cdot E_k}{H \cdot E_k} \geq \frac{-K_X \cdot E_k}{(H + \mu' D) \cdot E_k} \geq \frac{-K_X \cdot Z_k}{(H + \mu' D) \cdot Z_k}.$$

For $k \to \infty$, we thus find

$$\liminf_{k \to \infty} \frac{-K_X \cdot E_k}{H \cdot E_k} \geq \frac{-K_X \cdot z}{(H + \mu' D) \cdot z}$$

Note that

$$-K_X \cdot Z_k \longrightarrow -K_X \cdot z > 0$$

and

$$(H + \mu D) \cdot Z_k \longrightarrow (H + \mu D) \cdot z = 0.$$

Hence, for μ' sufficiently close to μ, the fraction

$$\frac{-K_X \cdot z}{(H + \mu' D) \cdot z}$$

becomes arbitrary large and so the limit

$$\liminf_{k \to \infty} \frac{-K_X \cdot E_k}{H \cdot E_k}$$

becomes arbitrarily large. On the other hand, there is a positive constant $c \gg 0$ such that $K_X + cH$ is ample on X. Then

$$(K_X + cH) \cdot E_k > 0$$

and so

$$c > \frac{-K_X \cdot E_k}{H \cdot E_k}$$

for all k, which contradicts the above observation that for μ' very close to μ, the above right hand side gets arbitrarily large for $k \to \infty$. This concludes Step 1.

Step 2. Part (b) of the theorem holds.

Let $\epsilon > 0$. If $(K_X + \epsilon H)C_i < 0$, then

$$H \cdot C_i \leq \frac{-K_X \cdot C_i}{\epsilon} \leq \frac{\dim X + 1}{\epsilon}.$$

By Theorem 7.11, there are only finitely many such curves C_i up to numerical equivalence, and so part (b) of the theorem follows from Step 1.

Step 3. Part (a) of the theorem holds.

The only missing ingredient is that

$$\overline{\mathrm{NE}}(X)_{K_X \geq 0} + \sum_{i \in I} [C_i]\mathbb{R}_{\geq 0}.$$

is closed. This is clear if I is finite (e.g. it was clear in the proof of part (b) above), but in general one has to argue a bit. However, by (b) we know that the accumulation points of the rays $[C_i]\mathbb{R}_{\geq 0}$ with $i \in I$ are only at $K_X = 0$ and it is a formal consequence of this that the above expression is indeed closed. □

Let $C \subset \mathbb{R}^n$ be a closed cone. Recall from Definition 6.7 that an extremal face $F \subset C$ is a subcone of C that lies in the boundary of C and that is obtained from C by intersecting it with a hyperplane in $\mathbb{R}^n$. An extremal face is called a ray if it is one-dimensional.

The cone theorem gives us control on the K_X-negative part of $\overline{\mathrm{NE}}(X)$. In particular, it shows that any K_X-negative ray $R \subset \overline{\mathrm{NE}}(X)$ is generated by an actual curve, and in fact by a rational curve. The key observation will be that these rays correspond to contractions of X in the following sense.

Definition 12.2. *Let X be a projective variety and let $F \subset \overline{\mathrm{NE}}(X)$ be an extremal face. A morphism $\mathrm{cont}_F : X \to Z$ to a projective variety Z is called the contraction of F if the following holds:*

- cont_F *has connected fibres, i.e.* $(\mathrm{cont}_F)_* \mathcal{O}_X = \mathcal{O}_Z$*;*
- *a curve $C \subset X$ is contracted by* cont_F*, i.e.* $(\mathrm{cont}_F)_* C = 0$*, if and only if* $[C] \in F$*.*

Recall from Theorem 6.9 that the contraction of F is unique up to isomorphism if it exists. Moreover, any morphism with connected fibres $\pi : X \to Z$ is the contraction of some extremal face, denoted by $\mathrm{NE}(\pi)$ in Theorem 6.9.

Lemma 12.3. *Let X be a normal projective variety and let $R \subset \overline{\mathrm{NE}}(X)$ be an extremal ray. Assume that X is $\mathbb{Q}$-factorial, i.e. some positive multiple of any Weil divisor is Cartier. Assume that the contraction of R exists: $f := \mathrm{cont}_R : X \to Z$. Then one of the following holds:*

- *f is birational and $\mathrm{Exc}(f)$ is an irreducible divisor (we say that f is divisorial contraction);*
- *f is birational and $\mathrm{Exc}(f)$ is of codimension at least two (we say that f is a small contraction);*
- *$\dim Z < \dim X$ (we say that f is of fibre type).*

Proof. The only assertion that has to be proven is the claim that $\mathrm{Exc}(f)$ is irreducible in the case where it contains a divisor. Assume that $\mathrm{Exc}(f)$ contains a divisor $E \subset \mathrm{Exc}(f)$. Let $H \subset Y$ be a general hyperplane section that contains $\pi(E)$. Since $\pi(E) \subset Y$ is irreducible of codimension at least two, the hyperplane H exists and is irreducible. Let $H' \subset X$ be the proper transform of H. Then H' meets the fibre $\pi^{-1}(y)$ over any point $y \in \pi(E)$ nontrivially. On the other hand, H' cannot contain the fibre $\pi^{-1}(y)$ completely for all $y \in \pi(E)$ because this would imply $E \subset \pi^{-1}\pi(E) \subset H'$ which is impossible because E is a divisor and H' is irreducible and different from E (as it maps to H via π). We have thus seen that for $y \in \pi(E)$ general, the fibre $\pi^{-1}(y)$ meets H' but is not contained in H'. Since $\pi^{-1}(y)$ is connected and X is projective, we can find a curve $C \subset \pi^{-1}(y)$ that meets H' nontrivially but is not contained completely in H'. Note that C is contracted via π to the point y and so $R = [C]\mathbb{R}_{\geq 0}$. Since H is a hyperplane section of Y, it is Cartier and so we may consider the pullback

$$\pi^* H = H' + \sum a_i E_i$$

where $E_i \subset \mathrm{Exc}(\pi)$ are some divisors on X that are contracted by π and where the a_i are non-negative. By assumption, each E_i is $\mathbb{Q}$-Cartier. Moreover, $\pi^* H \cdot C = 0$ by the projection formula and so

$$-H' \cdot C = \sum a_i E_i \cdot C.$$

Since $a_i \geq 0$ and $H' \cdot C > 0$ (because H meets C in finitely many points), we conclude that there must be a divisor $E' \subset \mathrm{Exc}(f)$ with $E' \cdot C < 0$.

Since f has connected fibres, through any point $x \in \mathrm{Exc}(f)$ there is a curve $C' \subset X$ with $f_* C' = 0$. Since f is the contraction of an extremal ray, C' is numerically proportional to C and so $E'C' < 0$. This implies $C' \subset E'$ and so $x \in E'$. Hence, $E' = \mathrm{Exc}(f)$, as we want. $\square$

We will see later that any K_X-negative face can be contracted. It is however important to keep in mind that not every extremal face can be contracted, as we see by the following example.

Example 12.4. *Let $\tau : X \to \mathbb{P}^2$ be the blow-up of 12 points $p_1, \ldots, p_{12}$ on a cubic curve $D \subset \mathbb{P}^2$. Let $C \subset X$ be the proper transform of D. Then*

$$K_X \cdot C = \tau^* \mathcal{O}(-3) C + \sum_{i=1}^{12} E_i C = -9 + 12 = 3$$

and

$$C^2 = (\tau^*\mathcal{O}(3) + \sum_{i=1}^{12} E_i) = 9 - 12 = -3.$$

Since $C^2 < 0$, the curve C spans an extremal ray R of $\overline{\mathrm{NE}}(X)$ (as we have seen on one of the exercise sheets). On the other hand, we claim that R can in general not be contracted. Indeed, assume that the contraction of R exists: $\mathrm{cont}_R : X \to Z$. Then X must carry a non-trivial line bundle

$$L = \tau^*\mathcal{O}(m) + \sum_{i=1}^{12} a_i E_i$$

whose restriction to C is trivial. But this implies that the line bundle

$$\mathcal{O}(m)|_D + \sum_{i=1}^{12} a_i p_i$$

is trivial on the elliptic curve C, which is impossible for general choices of $p_1, \ldots, p_{12}$ (unless $m = a_1 = \cdots = a_{12} = 0$). Hence, cont_F does not exist, as claimed.

13 Introduction to the minimal model program

We start this section by illustrating Mori's cone theorem in the case of surfaces. For this we will use Castelnuovo's contraction theorem.

Theorem 13.1 (Castelnuovo's contraction theorem)**.** *Let X be a smooth projective surface over an algebraically closed field k with a (-1)-curve $E \subset X$, i.e. a smooth rational curve with $E^2 = -1$. Then there is a smooth projective surface Y and a point $y \in Y$ so that $X \cong Bl_y Y$ and E corresponds to the exceptional divisor of the blow-up $Bl_y Y$. In other words, there is a blow-down map $\tau : X \to Y$ which contracts exactly E and such that Y is smooth.*

With this at hand, we obtain the following illustration of Mori's cone theorem.

Theorem 13.2. *Let X be a smooth projective surface over an algebraically closed field and let $R \subset \overline{\mathrm{NE}}(X)$ be an extremal ray with $R \cdot K_X < 0$ (i.e. R is generated by a K_X-negative curve). Then the contraction $\mathrm{cont}_R : X \to Z$ of R exists and one of the following holds:*

(a) Z is a smooth projective surface and X is obtained from Z by blowing up a point;

(b) Z is a smooth curve and X is a minimal ruled surface over Z, i.e. all fibres of cont_R are smooth $\mathbb{P}^1$'s.

(c) Z is a point and X is Fano, i.e. $-K_X$ is ample. (In fact, $X \cong \mathbb{P}^2$ but this is harder to prove.)

Proof. By Mori's cone theorem, there is a rational curve $C \in R$ with $0 < -K_X C \leq \dim X + 1 = 3$. Hence, $K_X C \in \{-1, -2, -3\}$. Recall the arithmetic genus

$$p_a(C) := 1 - \chi(C, \mathcal{O}_C) = 1 - h^0(C, \mathcal{O}_C) + h^1(C, \mathcal{O}_C) = h^1(C, \mathcal{O}_C),$$

where we used that C is integral and so $h^0(C, \mathcal{O}_C) = 1$. As a consequence of the Riemann-Roch formula for surfaces, there is the following formula (see [2, Exercise V.1.3]):

$$2p_a(C) - 2 = (K_X + C) \cdot C = K_X C + C^2.$$

Since $p_a(C) \geq 0$, this implies

$$C^2 \geq -2 - K_X \cdot C.$$

Since $K_X C \in \{-1, -2, -3\}$, we find that $C^2 \geq -1$ with equality if and only if $K_X C = -1$, in which case $p_a(C) = 0$ and this implies $C \cong \mathbb{P}^1$ (by a standard argument involving the normalization $\tau : C' \to C$ and considering the short exact sequence $0 \to \mathcal{O}_C \to \tau_* \mathcal{O}_{C'} \longrightarrow \delta \to 0$, where δ is a skyscraper sheaf that is supported on the singular points of C and which vanishes if and only if C is smooth). That is, C is a (-1)-curve and we are in case (a) by Castelnuovo's theorem. It remains to deal with the cases $C^2 = 0$ and $C^2 > 0$. In the latter case, a small perturbation of the class $[C] \in N_1(X)$ has still positive self-intersection and so some multiple will be effective by the Riemann–Roch theorem for surfaces. This implies that $[C] \in \overline{\mathrm{NE}}(X)$ is an interior point. Since $[C]$ also spans an extremal ray by assumptions, we find that $\rho(X) = 1$. Since $K_X \cdot C < 0$, we conclude that $-K_X$ is ample by Kleiman's criterion (bec. $\overline{\mathrm{NE}}(X) = [C] \cdot \mathbb{R}$) and so we are in case (c).

It remains to deal with the case $C^2 = 0$, in which case

$$2p_a(C) - 2 = K_X \cdot C \in \{-1, -2, -3\}$$

and so $p_a(C) = 0$ and $K_X \cdot C = -2$, which implies as above that C is smooth and in fact $C \cong \mathbb{P}^1$. We aim to prove that $|mC|$ gives the contraction morphism cont_R for $m \gg 0$. Since C is effective,

$$H^0(X, \mathcal{O}_X(mC)) \cong H^0(X, \mathcal{O}_X(K_X - mC))^\vee = 0$$

for $m \gg 0$. Hence, for large m, we find that

$$h^0(X, \mathcal{O}_X(mC)) \geq \chi(X, \mathcal{O}_X(mC)) = \frac{-K_X \cdot C}{2} m + \chi(X, \mathcal{O}_X)$$

which grows linearly, because $-K_X \cdot C$ is positive. In particular, for large m, we find that $h^0(X, \mathcal{O}_X(mC)) \geq 2$ and so there is a curve $D \subset X$ with $D \sim mC$ with $D \neq mC$. Up to cancelling common components, we may assume that D meets C in finitely many points. Since $C^2 = 0$, it follows that D is in fact disjoint from C. Hence, $|\mathcal{O}(mC)|$ is base point free for $m \gg 0$ and so it defines a morphism $f : X \to Z$ with $f^*H \sim \mathcal{O}_X(mC)$ for some ample divisor H on Z. Up to replacing f with its Stein factorization, we may assume that f has connected fibres. Note that f contracts C, as it contracts a curve if and only if the curve has trivial intersection with $f^*H \sim \mathcal{O}_X(mC)$. It remains to show that any fibre of f is isomorphic to $\mathbb{P}^1$. For this, let $F = \sum a_i C_i$ be a fibre of f. Then $\sum a_i C_i$ is numerically equivalent to any other fibre and hence to (a multiple of) C. Hence, $\sum a_i C_i \in R$. Since R is an extremal ray, $C_i \in R$ for all i. Since $C^2 = 0$, we find that $C_i^2 = 0$. Since in addition $(\sum a_i C_i)^2 = 0$, $a_i > 0$, we conclude that F is irreducible (as we know that it is connected). Hence, $F = aC'$ for some integral curve C' with $(C')^2 = 0$ and $K_X C' < 0$. This implies

$$2p_a(C') - 2 = K_X C' < 0$$

and so $p_a(C') = 0$, because $p_a(C') \geq 0$, since C' is integral. As before, we conclude $C' \cong \mathbb{P}^1$. This shows that all fibres of f are irreducible and the reduction of any fibre is $\mathbb{P}^1$. Since X is smooth, there can only be finitely many multiple fibres. So if $F = aC'$ is such a multiple fibre and F' is a general (hence non-multiple) fibre, then F and F' are numerically equivalent and so

$$K_X F' = a K_X C'$$

is a multiple of a. On the other hand, the above computations showed that F' and C' are smooth rational curves with trivial self-intersection and with $K_X F' = -2$ and $K_X C' = -2$. Hence, $a = 1$, as we want. This shows that $f : X \to Z$ is a ruled surface. Note also that Z is smooth, because it is normal (as it comes from the Stein factorization) of dimension one. $\square$

Note that in the above theorem if we are in case (a), then the Picard number of Z satisfies $\rho(Z) = \rho(X) - 1$. Since the Picard number of a projective variety is always positive, it follows that there cannot be an infinite sequence of such blow-downs. Hence, the above theorem shows that if we start with any smooth projective surface X, there is a finite sequence of birational morphisms

$$X = X_r \to X_{r-1} \to \cdots \to X_1 \to X_0 = X'$$

such that $X_i \to X_{i-1}$ is the blow-up of a point and X_i is smooth for all i and such that X' is one of the following:

(a) $K_{X'}$ is nef;

(b) X' is a ruled surface over a smooth curve;

(c) X' is Fano of Picard rank one.

This is the minimal model program for surfaces!

Remark 13.3. *Classically, a smooth projective surface X is called minimal if it does not contain any (-1)-curve, which by Castelnuovo's theorem is equivalent to saying that X is not the blow-up of another smooth projective surface in a point. In view of the above theorem, such a surface has either nef canonical class or it is Fano or it is a fibration into Fano varieties of lower dimensinos (here $\mathbb{P}^1$'s – the unique Fano variety of dimension one). From a modern point of view, it is thus more natural to distinguish among these cases in our terminology by reserving the term 'minimal model' to the case where K_X is nef (in which case it is automatic that there is no (-1)-curve, because such curves are K_X-negative). The other two cases that appear above are called Mori fibre spaces. The general definition is that it is a projective contraction $\pi : X \to Z$ with connected fibres and of relative Picard number one (i.e. $\rho(X) = \rho(Z) + 1$), such that the general fibre of π is Fano. (Here also the trivial fibre space $X \to \operatorname{Spec} k$ is allowed but it can only appear if $\rho(X) = 1$.)*

It is natural to try to generalize the above approach to higher dimensions. In view of Mori's cone theorem, what we have to understand is the following:

- Can we contract a K_X-negative extremal ray R on a threefold?
- How does the contraction $\operatorname{cont}_R : X \to Z$ of R look like?

Mori solved the second problem above in the case of smooth projective threefolds in characteristic zero. The result is as follows:

Theorem 13.4 (Mori)**.** *Let X be a smooth projective threefold over an algebraically closed field k of characteristic zero. Let R be a K_X-negative extremal ray of $\overline{\mathrm{NE}}(X)$. Assume that the contraction $f := \operatorname{cont}_R : X \to Z$ of R exists. The following is a list of all possible cases.*

(a) f is birational and contracts an irreducible surface to a point or a curve. More precisely, the following cases may occur:

(i) f is the blow-up of a smooth curve in a smooth threefold Z;

(ii) f is the blow-up of a smooth point of a smooth threefold Z;

(iii) f is the blow-up of an ordinary double point of Z, given locally analytically by the equation $\sum x_i^2 = 0$;

(iv) f is the blow-up of a singular point that is locally analytically of the form $x_1^2 + x_2^2 + x_3^2 + x_4^3 = 0$;

(v) f contracts a $\mathbb{P}^2$ with normal bundle $\mathcal{O}(-2)$ to a point, which locally analytically is isomorphic to the quotient of $\mathbb{A}^3$ by the involution $x \mapsto -x$.

(b) $\dim Z = 2$ *and $f : X \to Z$ is a conic bundle (in particular, general fibres are isomorphic to $\mathbb{P}^1$);*

(c) $\dim Z = 1$ *and the general fibre of $f : X \to Z$ is a Fano surface;*

(d) $\dim Z = 0$ *and X is Fano of Picard rank one.*

The above theorem gives a clear hint that the structure that we have seen for surfaces remains true in dimension three. However, the theorem is not enough to prove the minimal model program for threefolds for at least three reasons:

(1) We are missing a statement that each extremal ray can be contracted;

(2) In cases (aiii)-(av), the base of the divisorial contraction $f : X \to Z$ is no longer smooth and so we cannot repeat the process, as we have only dealt with smooth projective varieties so far.

(3) If X is a threefold that is produced from a smooth projective threefold by a contraction as in cases (aiii)-(av), then it may happen that X has a K_X-negative ray R whose contraction exists and is small: $\pi = \mathrm{cont}_R : X \to Z$. In this case K_Z cannot be $\mathbb{Q}$-Cartier, as it would imply that $\pi^* K_Z = K_X$ has trivial intersection with R, which is false. Hence, our program rapidly stops on Z (because we cannot talk about K_Z-negative rays anymore).

The solution to problems (1) and (2) will be to generalize Mori's cone theorem to an appropriate singular setting and to prove a contraction theorem for K_X-negative rays in this context. The solution to problem (3) is in some sense more subtle. It relies on the observation that if $\pi : \mathrm{cont}_R : X \to Z$ is a small contraction of a K_X-negative ray, then one can in practise often find another variety X^+ with a small contraction $\pi^+ : X^+ \to Z$ so that X and X^+ are birational over Z and $\pi^+ : X^+ \to Z$ is a small contraction of a K_{X^+}-positive ray. In other words, X^+ is obtained from X by somehow replacing the K_X-negative curves contracted by π by K-positive curves. This process is called a flip and we can think about it as a sort of surgery operation in codimension two. Note however that at this point it is completely unclear that such a 'surgery operation' should exist.

Definition 13.5. *Let X be a normal projective variety with mild singularities*[3] *(in particular, K_X should be $\mathbb{Q}$-Cartier). Let $\pi : X \to Z$ be the contraction of a K_X-negative extremal ray R of X and assume that this is a small contraction, i.e.* $\mathrm{codim}(\mathrm{Exc}(\pi)) \geq 2$. *A projective variety X^+ together with a birational morphism $\pi^+ : X^+ \to Z$ is called flip of π if*

(a) X^+ has mild singularities (in particular, K_{X^+} is $\mathbb{Q}$-Cartier);

(b) K_{X^+} is π^+-ample, i.e. the curves contracted by π^+ have positive intersection with K_{X^+};

(c) the exceptional set $\mathrm{Exc}(\pi^+)$ *has codimension at least two in X^+.*

By slight abuse of notation, the rational map $X \dashrightarrow X^+$ is also called flip.

With this definition at hand, we can formulate the following central conjecture, which generalizes the picture we have seen for surfaces to arbitrary dimensions.

[3] We will specify later what that should mean precisely.

Conjecture 13.6. *Let X be a smooth projective variety over an algebraically closed field of characteristic zero. Then there is a sequence of birational maps*

$$X = X_r \dashrightarrow X_{r-1} \dashrightarrow X_{r-2} \dashrightarrow \cdots \dashrightarrow X_1 \dashrightarrow X_0 := X',$$

where $X_i \dashrightarrow X_{i-1}$ is either a divisorial contraction of a K_{X_i}-negative extremal ray of X_i or it is the flip of K_{X_i}-negative extremal ray of X_i, and such that X' is one of the following:

(a) $K_{X'}$ is nef, in which case X' is called minimal model;

(b) X' admits a $K_{X'}$-negative extremal ray R whose contraction $\pi = \mathrm{cont}_R : X \to Z$ is of fibre type; in particular, the general fibre of π is a Fano variety. In this case X' is called a Mori fibre space.

The main problems we face to solve the above conjecture and to run the outlined program are as follows:

(1) identify the sort of singularities we should allow in the process (i.e. answer the question what 'mild singularities' should be);

(2) prove a cone and contraction theorem for this class of singularities;

(3) prove the existence of flips;

(4) prove that any sequence of divisorial contractions and flips has to stop at some point. Since the Picard rank drops by one in each divisorial contraction, this boils down to proving that there are no infinite sequences of flips.

The above conjecture is not fully known yet, even though it is known in dimension three and for large classes of varieties of arbitrary dimensions (e.g. those of general type). While problems (1)–(3) are completely solved by now (due to work of a lot of people over the last 40 years), termination of flips is still open in general.

Finally, it is natural to wonder why the condition that $K_{X'}$ is nef is a natural and desirable condition. To see this, note that a line bundle that is base point free is automatically nef, and so nefness is a necessary condition for being base point free. The example of torsion bundles (e.g. the canonical bundle of an Enriques surface) shows that nefness of a line bundle can at most imply that some multiple is base point free. THis is wrong for arbitrary line bundles, but it is conjectured to be true for the canonical bundle.

Conjecture 13.7 (Abundance conjecture)**.** *Let X be a projective variety with mild singularities (e.g. smooth). If K_X is nef, then some multiple mK_X for $m \gg 0$ is base point free.*

Assume that the abundance conjecture holds for X and let $f : X \to Z$ be the morphism induced by the sections of mK_X. Then $mK_X = f^*L$ for some ample line bundle L on Z. Taking the Stein factorization, we may assume that f has connected fibres (this will be automatic if $m \gg 0$ but we don't need this here). Let F be a general fibre of f. Then F has trivial canonical bundle and so

$$K_F = K_X|_F$$

by the adjunction formula. This implies

$$mK_F = mK_X|_F = f^*L|_F = \mathcal{O}_F$$

and so the canonical bundle of F is torsion. In this case F is called a Calabi-Yau variety (in a weak sense). For instance, F must be an elliptic curve if it is a curve and there are finitely many deformation types if F is a surface. It is conjectured (but unknown) that the deformation types of such varieties (with mild singularities) are finite in any dimension. In any case, the above discussion shows that the morphism $f : X \to Z$ realizes X either as the total space of a fibration with Calabi-Yau varieties as general fibres or f is birational, in which case L must be mK_Z and so Z has ample canonical class and is birational to X. This shows that the minimal model conjecture (Conjecture 13.6) together with the Abundance conjecture would imply that up to birational equivalence, every variety is made up from the following three building blocks:

- varieties with ample canonical class;
- varieties whose canonical class is torsion (Calabi-Yau varieties);
- varieties whose canonical class is antiample (Fano varieties).

14 Singularities in the minimal model program

14.1 Motivation and provisional definition of terminal singularities

From now on it will be important to be able to talk about the canonical divisor K_X of any normal (projective) variety X. If X is smooth, then K_X is a divisor such that $\mathcal{O}_X(K_X) \cong \Lambda^{\dim X}\Omega^1_X$. If X is not smooth, we use the following definition.

Definition 14.1. *Let X be a normal variety. Then K_X is the divisor given by taking the closure of a canonical divisor on the smooth locus of X. Since $X^{\mathrm{sing}} \subset X$ has codimension at least two, the divisor K_X is unique up to linear equivalence on X.*

To get a feeling which kind of singularities we need to allow to be able to run the minimal model program outlined in Conjecture 13.6, we consider the following toy example. Let X be a normal projective variety that is $\mathbb{Q}$-factorial, i.e. any Weil divisor has a positive multiple that is Cartier. Assume that X has a unique singular point $x \in X$ such that

$$X' = Bl_x X$$

is smooth and such that the exceptional divisor $E \subset X'$ is irreducible. This implies that for any divisor D on X', $\tau^*\tau_*D - D$ is a (rational) multiple of E, where we use that X is $\mathbb{Q}$-factorial and where $\tau : X' \to X$ denotes the blow-down map. We have thus seen that

$$\rho(X') = \rho(X) + 1.$$

This implies that the curves contracted by $\tau : X' \to X$ are all numerically proportional and so τ is the contraction of an extremal ray R of $\overline{\mathrm{NE}}(X')$. Moreover,

$$K_{X'} \sim \tau^*K_X + aE$$

for some $a \in \mathbb{Q}$, because K_X is $\mathbb{Q}$-Cartier.[4] If C is a curve that is contracted by τ, then $E \cdot C < 0$ (e.g. by Exercise 2b on sheet 3) and so

$$K_{X'} \cdot C = (\tau^*K_X + aE) \cdot C = aE \cdot C$$

is negative as long as $a > 0$. In other words, if $a > 0$, then X' is a smooth projective variety with an extremal $K_{X'}$-negative ray R whose contraction $\mathrm{cont}_R : X' \to X$ is divisorial. But

[4] More precisely, there is some integer $m > 0$ such that mK_X is Cartier and so τ^*mK_X is a Cartier divisor on X' that is linearly equivalent to $mK_{X'}$ outside of E and so $mK_{X'} = \tau^*mK_X + bE$ for some integer b. In the above formula, we divided by slight abuse of notation by m and got $a = b/m \in \mathbb{Q}$.

then our program outlined in Conjecture 13.6 forces us to replace X' by X and so we need to allow the singularities of X. However, the only thing we know is that X has a resolution X' with

$$K_{X'} = \tau^* K_X + aE$$

where $a > 0$. This leads to the following definition.

Definition 14.2. *Let X be a normal projective variety such that mK_X is Cartier for some positive integer m. Let $f : Y \to X$ be a proper birational morphism from a normal variety Y and let $E_i \subset Y$ denote the prime divisors contracted by f. Assume that mK_Y is Cartier (e.g. Y is smooth). Then*

$$mK_Y \sim f^* mK_X + \sum (m \cdot a_i(E_i, X)) E_i$$

for some rational numbers $a_i(E_i, X)$ that are called the discrepancies of E_i with respect to X.

Definition 14.3 (Preliminary Definition (works if characteristic is zero))**.** *Let X be a normal projective variety such that K_X is $\mathbb{Q}$-Cartier and such that there is a resolution $\tau : Y \to X$ of singularities (e.g. $\mathrm{char}(k) = 0$). We say that X has terminal (resp. canonical) singularities if the discrepancy $a_i(E_i, X)$ of any τ-exceptional divisor E_i is positive (resp. non-negative). In other words,*

$$K_Y = \tau^* K_X + \sum a_i E_i$$

for some rational numbers $a_i > 0$ (resp. $a_i \geq 0$) where the $E_i \subset Y$ run through all τ-exceptional divisors.

Remark 14.4. *We will see later that if one resolution τ as in the above definition exists, then any resolution has that property. Moreover, in positive characteristic where resolutions of singularities are unknown, we may ask that the discrepancies are positive for all exceptional divisors of any proper birational morphisms $Y \to X$ from a normal variety Y.*

14.2 Divisors over X and their discrepancies

To formalize the above discussion and to generalize the above definition to the case where resolutions of singularities might not exist, it is useful to use the following terminology/definition.

Definition 14.5. *Let X be a quasi-projective variety over a field k. A divisor over X is a divisor E on a normal variety Y that admits a birational morphism $f : Y \to X$ (not necessarily proper). The closure of $f(E)$ is called the center of E over X, denoted by $\mathrm{center}_X E$. A divisor over X is exceptional if $\mathrm{center}_X E$ has codimension at least two on X.*

Lemma 14.6. *Any divisor E over X induces a unique valuation $\nu(E)$ on the function field $k(X)$ that is given by measuring poles and zeros of rational functions on X (hence on Y) along E. In other words, $\nu(E)$ is the unique valuation on $k(X)$ that corresponds to the discrete valuation ring*

$$\mathcal{O}_{Y,E} \subset k(Y) = k(X).$$

Proof. Clear. □

Lemma 14.7. *Let X be a quasi-projective variety over an algebraically closed field. Two divisors E and E' over X, that lie on normal birational models $Y \to X$ and $Y' \to X$, induce the same valuations on $k(X)$ if and only if the composition $Y \to X \dashrightarrow Y'$ induces an isomorphism between the generic points of E and E', respectively.*

Proof. This follows directly from the fact that the valuation $\nu(E, X)$ determines uniquely (and is determined uniquely) by the local ring $\mathcal{O}_{X,E}$. In particular, $\nu(E) = \nu(E')$ implies

$$\mathcal{O}_{X,E} = \mathcal{O}_{X,E'} \subset k(X)$$

and this is equivalent to saying that the composition $Y \to X \dashrightarrow Y'$ induces an isomoprhism between a neighbourhood of the generic point of E with a neighbourhood of the generic point of E'. This proves the lemma. $\square$

Let X be a normal variety over a field and assume that mK_X is Cartier for some positive integer m. Let E be a divisor over X, i.e. $E \subset Y$ where $f : Y \to X$ is birational but necessarily proper and Y is normal. Let $e \in E$ be a general point. Since Y is normal, e is a smooth point of Y and we may pick a coordinate system $y_1, \ldots, y_n$ around e (i.e. the y_i are regular functions that vanish at e and generate the maximal ideal $\mathfrak{m}_e \subset \mathcal{O}_{X,e}$) such that $E = \{y_1 = 0\}$. Then the canonical bundle on Y is locally generated by $dy_1 \wedge \cdots \wedge dy_n$. If s is a local generator of $\mathcal{O}_X(mK_X)$ (recall that mK_X is Cartier), then

$$f^*s = y_1^{m \cdot a(E,X)} \cdot (\text{unit}) \cdot (dy_1 \wedge \cdots \wedge dy_n)^{\otimes m}$$

for some integer $a(E, X)$.

Definition 14.8. *Let E be a divisor over X and assume that K_X is $\mathbb{Q}$-Cartier . The integer $a(E, X)$ from above is called the discrepancy of E over X. This generalizes the definition of discrepancies to the case where f is not necessarily proper and E is not necessarily exceptional.*

Note that $a(E, X) = 0$ if E over X is not exceptional.

Lemma 14.9. *Let X be a variety so that K_X is $\mathbb{Q}$-Cartier. The discrepancy $a(E, X)$ of a divisor E over X depends only on the valuation $\nu(E)$ of E on the function field $k(X)$ and not on the explicit choice of f and Y.*

Proof. This follows directly from the definition and Lemma 14.7. $\square$

Definition 14.10. *Let X be a normal variety over a field k such that K_X is $\mathbb{Q}$-Cartier. The total discrepancy $discrep(X)$ of X is given by*

$$discrep(X) := \inf\{a(E, X) \mid E \text{ is an exceptional divisor over } X\}.$$

X has terminal (resp. canonical) singularities if $discrep(X) > 0$ (resp. $discrep(X) \geq 0$).

Remark 14.11. *Using the fact that in characteristic zero, any birational map between projective varieties can be resolved by a sequence of blow-ups along smooth centers, it is not hard to show that this new definition coincides with the old one if k has characteristic zero.*

14.3 Pairs and their singularities

Even though one might a priori mostly be interested in smooth varieties or in varieties X that have terminal or canonical singularities, it turns out that there is a major advantage if one allows pairs (X, Δ) with mild singularities.

Definition 14.12. *A pair (X, Δ) consists of the following data:*

- *a normal variety X;*

- *an effective $\mathbb{Q}$-divisor $\Delta = \sum a_i D_i$, where D_i are distinct prime divisors on X and the a_i are non-negative rational numbers.*[5]

The divisor Δ of a pair (X, Δ) is referred to as boundary divisor. The case $\Delta = 0$ corresponds to the case of a variety without boundary (that is, the notaiton of pairs is really more general then that of varieties). In the case of a pair (X, Δ), the $\mathbb{Q}$-divisor

$$K_X + \Delta$$

plays the role of the canonical class of a variety. Discrepancies of pairs (X, Δ) are defined as follows.

Definition 14.13. *Let (X, Δ) with $\Delta = \sum a_i D_i$ be a pair and assume that $m(K_X + \Delta)$ is Cartier for some $m \gg 0$. Let $f : Y \to X$ be a birational morphism (not necessarily proper). Let $f_*^{-1}\Delta = \sum a_i f_*^{-1} D_i$ be the birational transform of Δ (i.e. $f_*^{-1} D_i$ is zero if D_i lies outside the image of f and it is given by the closure of the preimage of the generic point of D_i otherwise). Then the Weil divisors*

$$m(K_Y + f_*^{-1}\Delta) \quad \textit{and} \quad f^* m(K_X + \Delta)$$

agree outside of the f-exceptional locus. Hence,

$$m(K_Y + f_*^{-1}\Delta) \sim f^* m(K_X + \Delta) + m \cdot \sum_i a(E_i, X, \Delta) E_i$$

where E_i runs through the f-exceptional divisors on Y and where $a_i(E_i, X)$ are rational numbers such that $m \cdot a(E_i, X, \Delta)$ are integers. This defines the discrepancy of any f-exceptional divisor E on Y. If $E \subset Y$ is a prime divisor that is not f-exceptional, then we put

$$a(E, X, \Delta) := \begin{cases} -a_i, & \text{if } E = f_*^{-1} D_i; \\ 0, & \text{if } E \neq f_*^{-1} D_i \text{ for all } i. \end{cases}$$

Dividing by m, we get the following $\mathbb{Q}$-linear equivalence:

$$K_Y + f_*^{-1}\Delta \sim_{\mathbb{Q}} f^*(K_X + \Delta) + \sum_i a(E_i, X, \Delta) E_i.$$

Moreover, the definition of the discrepancies of non-exceptional divisors is formed in such a way that we get a formula of the form

$$K_Y \sim_{\mathbb{Q}} f^*(K_X + \Delta) + \sum_E a(E, X, \Delta) E.$$

where E runs through all prime divisors on Y.

Remark 14.14. *Let E be a divisor over X and let (X, Δ) be a pair such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Then Lemma 14.7 implies as in the case where $\Delta = 0$ that the discrepancy $a(E, X, \Delta)$ of E with respect to (X, Δ) that is defined in the above definition depends only on the valuation $\nu(E)$ of $k(X)$ and not on the explicit model $f : Y \to X$.*

The discrepancy of a pair (X, Δ) is now defined in analogy with the case where $\Delta = 0$ as follows – note however that in the case $\Delta \neq 0$, there are two different such notions.

[5] A priori one may also allow the a_i to be arbitrary rational numbers. This is done in [3, Definition 2.25] but in practice the case where the a_i are non-negative is most important.

Definition 14.15. *Let* (X, Δ) *be a pair and assume that* $K_X + \Delta$ *is* $\mathbb{Q}$*-Cartier. Then the discrepancy of* (X, Δ) *is defined as*

$$\textit{discrep}(X, \Delta) := \inf\{a(E, X, \Delta) \mid E \textit{ is an exceptional divisor over } X\}.$$

Similarly, the total discrepancy is defined as

$$\textit{totaldiscrep}(X, \Delta) := \inf\{a(E, X, \Delta) \mid E \textit{ is a divisor over } X\}.$$

Clearly,

$$\textit{discrep}(X, \Delta) \geq \textit{totaldiscrep}(X, \Delta).$$

Moreover, if $\Delta = \sum a_i D_i$, then

$$-a_i \geq \textit{totaldiscrep}(X, \Delta)$$

for all i. In particular, $\textit{totaldiscrep}(X, \Delta)$ can never be positive if $\Delta \neq 0$.
The following proposition summarizes important properties of the above notions.

Proposition 14.16. *Let* (X, Δ) *be a pair with* $\Delta = \sum a_i D_i$ *and such that* $K_X + \Delta$ *is* $\mathbb{Q}$*-factorial.*

(a) Either $\textit{discrep}(X, \Delta) = -\infty$ *or*

$$-1 \leq \textit{totaldiscrep}(X, \Delta) \leq \textit{discrep}(X, \Delta) \leq 1.$$

(In particular, $\textit{discrep}(X, \Delta) = -\infty$ *as soon as* $a_i > 1$ *for some* i*.)*

(b) If X *is smooth then* $\textit{discrep}(X, 0) = 1$*.*

(c) Assume that X *is smooth,* $\sum_i D_i$ *is a simple normal crossing divisor and* $a_i \leq 1$ *for every* i*. Then*

$$\textit{discrep}(X, \Delta) = \min\left\{\min_{i \neq j, D_i \cap D_j \neq \emptyset}\{1 - a_i - a_j\}, \min_i\{1 - a_i\}, 1\right\}.$$

Remark 14.17. *Item (a) in the above proposition explains that one usually restricts the notion of pairs* (X, Δ) *to the case where the coefficients of* Δ *are bounded from above by* 1*.*

Definition 14.18. *Let* (X, Δ) *be pair with* $D = \sum a_i D_i$ *and such that* $K_X + \Delta$ *is* $\mathbb{Q}$*-factorial. Then we say that* (X, Δ) *is*

(1) terminal if $\textit{discrep}(X, \Delta) > 0$*;*

(2) canonical if $\textit{discrep}(X, \Delta) \geq 0$*;*

(3) Kawamata log terminal (klt) if $\textit{discrep}(X, \Delta) > -1$ *and* $\lfloor D \rfloor = 0$*, i.e.* $a_i < 1$ *for all* i*;*

(4) log canonical (lc) if $\textit{discrep}(X, \Delta) \geq -1$*.*

Remark 14.19. *Note that item (a) in Proposition 14.16 implies that* $a_i \in [0, 1]$ *if* (X, Δ) *is terminal, canonical, klt or lc.*

14.3.1 Why pairs?

Reasons for this are:

- one gains tremendous flexibility;
- proofs tend to work in this level of generality;
- in fact, certain proofs force one to allow this level of generality, because it is natural to use induction on the dimension but when passing from X to a subvariety $Y \subset X$ of lower dimensions, e.g. a divisor, then the canonical bundle of X does not transform to the canonical bundle of Y, but we have the slightly more complicated formula $(K_X+Y)|_Y = K_Y$, which explains that allowing boundary components can be very useful (and might in fact be necessary).

14.3.2 Why is the class of klt singularities natural in the context of pairs?

We've explained in Section 14.1 that the class of terminal varieties naturally shows up if we start with a smooth variety X and successively contract K_X-negative rays. Which class do we get if we start with a smooth variety X and a boundary divisor $\Delta = \sum a_i D_i$ with $a_i \in [0,1]$?[6] For this, assume that $K_X + \Delta$ is $\mathbb{Q}$-Cartier and let R be a $K_X + \Delta$ negative extremal ray of $\overline{\mathrm{NE}}(X)$. Assume that the contraction $f = \mathrm{cont}_R : X \to Z$ of R exists. Assume furthermore that f is divisorial and that $K_Z + \Delta'$ is $\mathbb{Q}$-factorial, where $\Delta' = f_*\Delta$. Then we have

$$K_X + \Delta = f^*(K_Y + \Delta') + aE$$

for some $a \in \mathbb{Q}$. In fact, if $C \in R$ is nonzero, then as in Section 14.1 we get that $E \cdot C < 0$ and so $a > 0$ because

$$0 > (K_X + \Delta) \cdot C = (f^*(K_Y + \Delta') + aE) \cdot C = aE \cdot C.$$

On a first superficial sight, one might get the impression that this suggests that the singularities of (Y, Δ') are terminal. However, this is false! Indeed, if we look carefully at the definition of discrepancies of pairs, then we note that we have to compare $K_Y + \Delta'$ with $K_X + f_*^{-1}\Delta'$ and not with $K_X + \Delta$. This makes a difference if E is a component of Δ! Indeed, in this case

$$\Delta = bE + f_*^{-1}\Delta'$$

for some $b \in [0,1]$ and we find that

$$K_X + f_*^{-1}\Delta' + bE = f^*(K_Y + \Delta') + aE$$

Hence,

$$a(E, Y, \Delta') = a - b.$$

Here $a > 0$ and $b \leq 1$ are rational numbers, and so $a - b$ will be larger than -1, but it could apriori be arbitrarily close to -1. This explains why klt singularities are the natural class of singularities that appear in the MMP for pairs (X, Δ), even if we start with the nicest possible situation, where X is smooth and $\Delta = \sum a_i D_i$ with $\sum D_i$ a simple normal crossing divisor.

[6] Here the assumption that $a_i \leq 1$ is very natural from the point of view of item (a) in Proposition 14.16, as otherwise we essentially allow all possible singularities and get no restriction on the type of singularities whatsoever.

14.3.3 Proof of Proposition 14.16

To prove Proposition 14.16, we need the following two lemmas.

Lemma 14.20. *Let X be a smooth variety and let $\Delta = \sum a_i D_i$ be a sum of distinct prime divisors. Let $Z \subset X$ be a closed subvariety of codimension k. Let $p : Bl_Z X \to X$ be the blow-up of Z and let $E \subset Bl_Z X$ be the irreducible component of the exceptional divisor which dominates Z. (If Z is smooth this is the only component.) Then*

$$a(E, X, \Delta) = k - 1 - \sum a_i \cdot mult_Z D_i$$

Proof. Replacing X by $X \setminus Z^{\mathrm{sing}}$, we may assume that Z is smooth. Let $X' = Bl_Z X$. Then we get

$$K_{X'} = \tau^* K_X + (k-1)E$$

and

$$\tau^* \Delta = \Delta' + \sum a_i \cdot mult_Z D_i \cdot E$$

where Δ' is the birational transform of Δ. Putting everything together, we find

$$K_{X'} + \Delta' = \tau^*(K_X + \Delta) + (k - 1 - \sum a_i \cdot mult_Z D_i) \cdot E$$

which proves the lemma. $\square$

Lemma 14.21. *Let $f : X' \to X$ be a proper birational morphism between normal varieties. Let $\Delta_{X'}$ resp. Δ_X be effective $\mathbb{Q}$ divisors on X' resp. X such that $K_X + \Delta_X$ is $\mathbb{Q}$-Cartier and*

$$K_{X'} + \Delta_{X'} = f^*(K_X + \Delta_X) \quad \textit{and} \quad f_* \Delta_{X'} = \Delta_X.$$

Then for any divisor E over X,

$$a(E, X', \Delta_{X'}) = a(E, X, \Delta_X).$$

Proof. Let $g : Y \to X$ be a birational morphism with $E \subset Y$. Up to replacing Y by a suitable blow-up, we may assume that the natural birational map $Y \dashrightarrow X'$ is a morphism $g' : Y \to X'$ with $g = f \circ g'$. Let Δ_Y (resp. Δ'_Y) be the birational transform of Δ_X (resp. of $\Delta_{X'}$) We then find

$$K_Y + \Delta_Y = g^*(K_X + \Delta_X) + \sum_{E_i} a(E_i, X, \Delta_X) E_i$$

where E_i runs through all g-exceptional divisors on Y. Similarly,

$$K_Y + \Delta'_Y = g'^*(K_{X'} + \Delta_{X'}) + \sum_{E'_i} a(E'_i, X', \Delta_{X'}) E'_i,$$

where E'_i runs through all g'-exceptional divisors on Y. Since $g = f \circ g'$ and $K_{X'} + \Delta_{X'} = f^*(K_X + \Delta_X)$, we find that

$$g^*(K_X + \Delta_X) = g'^*(K_{X'} + \Delta_{X'})$$

and so

$$g'^*(K_{X'} + \Delta_{X'}) + \sum_{E_i} a(E_i, X, \Delta_X) E_i + \Delta_Y - \Delta'_Y = g'^*(K_{X'} + \Delta_{X'}) + \sum_{E'_i} a(E'_i, X', \Delta_{X'}) E'_i.$$

Hence,

$$\sum_{E_i} a(E_i, X, \Delta_X) \cdot E_i = \sum_{E'_i} a(E'_i, X', \Delta_{X'}) \cdot E'_i + \Delta'_Y - \Delta_Y.$$

Here, $\Delta'_Y - \Delta_Y$ is effective and f-exceptional, because $f_* \Delta_{X'} = \Delta_X$. This shows $a(E, X', \Delta_{X'}) = a(E, X, \Delta_X)$ if E is g'-exceptional.

If E is not g'-exceptional but g-exceptional, then $g'_* E$ is f-exceptional and $a(E, X', \Delta_{X'}) = -a$ where a is the coefficient of E in $\Delta_{X'}$. The above equality thus shows $a(E, X', \Delta_{X'}) = a(E, X, \Delta_X)$ also in this case.

Finally, the case where E is not g-exceptional follows from the fact that $f_* \Delta_{X'} = \Delta_X$. This concludes the lemma. □

Proof of Proposition 14.16. Let us first prove (a). Blowing up a codimension two subvariety that meets the smooth locus of X we find that $discrep(X, \Delta) \leq 1$. It remains to show that $discrep(X, \Delta) = -\infty$ if $a(E, X, \Delta) < -1$ for some divisor E over X. To prove this assume that $E \subset Y$, where $f : Y \to X$ is birational and write

$$a(E, X, \Delta) = -1 - c$$

for some $c > 0$. Define the effective $\mathbb{Q}$-divisor Δ_Y on Y via

$$f^*(K_X + \Delta) = K_Y + \Delta_Y.$$

Let $Z_0 \subset Y$ be a closed subvariety of codimension two that is contained in E but in general position otherwise (more precisely: not in any other f-exceptional divisor and not in $f_*^{-1}\Delta$). Let $Y_1 \to Y$ be the blow-up along Z_0 and let $E_1 \subset Y_1$ be the unique component of the exceptional divisor that dominates Z_0. Then

$$a(E_1, X, \Delta) = a(E_1, Y, \Delta_Y) = -c$$

by Lemma 14.21. Blowing up the (unique) irreducible component Z_1 of the intersection $E \cap E_1$ that dominates Z_0 on Y_1, we get a birational model Y_2 and a divisor E_2 on Y_2 dominating Z_2 with

$$a(E_2, X, \Delta) = a(E_2, Y, \Delta_Y) = -2c.$$

Blowing up the component Z_3 of $E_2 \cap E$ that dominates Z_2 and repeating this process inductively, we find a sequence of divisors E_i over X with

$$a(E_i, X, \Delta) = a(E_i, Y, \Delta_Y) = -i \cdot c.$$

This proves $discrep(X, \Delta) = -\infty$, as we want.

Item (b) is a special case of (c), which we aim to prove now. For this let

$$r(X, \Delta) := \min \left\{ \min_{i \neq j, D_i \cap D_j \neq \emptyset} \{1 - a_i - a_j\}, \min_i \{1 - a_i\}, 1 \right\}.$$

Blowing up a codimension two subvariety of X shows that

$$discrep(X, \Delta) \leq 1.$$

Blowing up a codimension two subvariety of X that lies on E_i but is in general position otherwise shows that

$$discrep(X, \Delta) \leq 1 - a_i.$$

Blowing up the intersection $D_i \cap D_j$ (if non-empty), shows that

$$discrep(X, \Delta) \leq 1 - a_i - a_j.$$

This proves $discrep(X, \Delta) \leq r(X, \Delta)$. For the converse inequality, let E be a divisor over X. By Lemma 14.22 below, there is a sequence of blow-ups

$$Y = X_n \to X_{n-1} \to \cdots \to X_1 \to X_0 = X$$

such that $E \subset Y$. We need to prove $a(E, X, \Delta) \geq r(X, \Delta)$. To this end we may shrink X around the point $f(y) \in X$, because $r(X, \Delta)$ increases if we shrink X. Hence, up to shrinking X we may assume that the center Z_i of the blow-up $X_i \to X_{i-1}$ is smooth. In particular, X_i is smooth for $i \geq 0$ (because X is smooth by assumption). The claimed inequality then follows by induction on the number of blow-ups n in the above sequence by an explicit computation, see the end of proof of [3, Corollary 2.31]. □

Lemma 14.22. *Let X be an algebraic variety over a field and let ν be a discrete valuation on $k(X)$ with valuation ring $(R, \mathfrak{m})$. Assume that*

$$\dim R/\mathfrak{m} = \dim X - 1.$$

Then $\nu = \nu(E)$ for an exceptional divisor E over X. Explicitly, E can be constructed as follows.

Let $Y = \operatorname{Spec} R$ and consider the natural morphism $f_0 : Y \to X_0 := X$. Let $y \in Y$ be the unique closed point and let $Z_0 \subset X$ be the closure of $f_0(y)$. We then put $X_1 = Bl_{Z_1} X$, $f_1 : Y \to X_1$ the induced morphism and $Z_1 \subset X_1$ the closure of $f_1(y)$. Inductively, if $f_i : Y \to X_i$ is defined and $Z_i \subset X_i$ is the closure of $x_i := f_i(y)$, then $X_{i+1} = Bl_{Z_i} X_i$. Then for some $n \gg 0$, $f_n : Y \to X_n$ induces an isomorphism $\mathcal{O}_{Y,y} = R \cong \mathcal{O}_{X_n, x_n}$. In particular, $E = Z_n = \overline{\{x_n\}}$ is a divisor on X_n with $\nu = \nu(E)$.

Proof. See [3, Lemma 2.45]. □

15 Nef and big divisors and vanishing theorems

15.1 Kodaira vanishing theorem

Recall the following result from Complex Geometry:

Theorem 15.1 (Kodaira vanishing)**.** *Let X be a smooth projective variety over an algebraically closed field of characteristic zero and let L be an ample line bundle on X. Then*

$$H^i(X, K_X \otimes L) = 0$$

for all $i \geq 1$.

After the Serre vanishing theorem, the Kodaira vanishing theorem is one of the most basic vanishing theorems. The usage of vanishing theorems like that lie in the fact that they allow one to lift sections. To explain this in a baby example, let X be a smooth complex projective variety and let $A \in |L|$ be a smooth ample divisor on X. Then there is a short exact sequence

$$0 \longrightarrow K_X + L \longrightarrow K_X + 2L \longrightarrow (K_X + 2L)|_A = K_A + L|_A \longrightarrow 0$$

giving rise to a short exact sequence

$$0 \to H^0(X, K_X + L) \to H^0(X, K_X + 2L) \to H^0(A, K_A + L|_A) \to H^1(X, K_X + L) = 0$$

where we used Kodaira vanishing to see that

$$H^0(X, K_X + 2L) \to H^0(A, K_A + L|_A)$$

is surjective. That is, any section of $K_A + L|_A$ lifts to a section of $K_X + 2L$ and such lifting results are very useful in proving base point freeness results, which is exactly what we need to prove if we want to prove a contraction theorem for extremal K_X-negative rays.

The above approach seems promising, but the assumption that L is ample is usually to strong for the applications we have in mind. In this section we thus study a class of line bundles that gets pretty close to being ample and such that a similar vanishing theorem such as Kodaira's vanishing theorem still holds for those bundles.

15.2 Nef and big divisors

Lemma 15.2. *Let X be a projective scheme of dimension n over a field k. Let D be a Cartier divisor on X. Then $h^0(X, \mathcal{O}_X(mD)) \leq C \cdot m^n$ for some positive constant C.*

Proof. Let A be an ample divisor on X. For $l \gg 0$, the divisor $-D + lA$ is globally generated and so it admits a nonzero section. Up to replacing A by lA, we may assume that $l = 1$. This shows that we can write

$$D \sim A - E$$

for some effective divisor E. Hence,

$$h^0(X, \mathcal{O}_X(mD)) \leq h^0(X, \mathcal{O}_X(mA)).$$

But for large m, Serre vanishing shows

$$h^0(X, \mathcal{O}_X(mA)) = \chi(X, \mathcal{O}_X(mA))$$

and the latter is a polynomial of degree n by the results in Section 4. This concludes the lemma. □

Definition 15.3. *Let X be a projective scheme of dimension n over a field k. A Cartier divisor D on X is called big if there is a positive constant c such that $h^0(X, \mathcal{O}_X(mD)) > c \cdot n^m$ for all $m \gg 0$.*

Note that D is big if it is ample, because in this case for $m \gg 0$ we have

$$h^0(X, \mathcal{O}_X(mD)) = \chi(X, \mathcal{O}_X(mD))$$

by Serre vanishing and the above left hand side is a polynomial of degree n with leading coefficient $\frac{D^n}{n!}$.

However, not every big divisor is ample.

For instance, if $f : Y \to X$ is a proper birational morphism, then

$$H^0(Y, f^*\mathcal{O}_X(D)) = H^0(X, f_*f^*\mathcal{O}_X(D)) = H^0(X, \mathcal{O}_X(D))$$

and so f^*D is big if and only if D is big. (On the other hand, f^*D ample is equivalent to the fact that f is an isomorphism and that D is ample.)

Lemma 15.4. *Let X be a projective variety of dimension n and let D be a Cartier divisor on X. Then the following are equivalent.*

(a) D is big;

(b) *for some* $m > 0$ *we have* $mD \sim A + E$*, where* A *is ample and* E *is effective;*

(c) *for some* $m > 0$ *the rational map* $\phi_{|mD|}$ *associated to* mD *is birational;*

(d) *for some* $m > 0$ *the rational map* $\phi_{|mD|}$ *associated to* mD *has* n*-dimensional image.*

Proof. Clearly (b) $\Rightarrow$ (c) $\Rightarrow$ (d). To prove (d) $\Rightarrow$ (a), let $Y \subset \mathbb{P}^N$ be the closure of the rational map $\phi_{|mD|} : X \dashrightarrow \mathbb{P}^N$ and let $\mathcal{O}_Y(1)$ be the restriction of $\mathcal{O}_{\mathbb{P}^N}(1)$ to Y. Then there is a natural inclusion

$$H^0(Y, \mathcal{O}_Y(i)) \hookrightarrow H^0(X, \mathcal{O}_X(imD))$$

for all $i \geq 0$. On the other hand,

$$h^0(Y, \mathcal{O}_Y(i)) = \chi(Y, \mathcal{O}_Y(i))$$

is a polynomial of degree i^n and so the result follows from Lemma 15.2.

It remains to prove (a) $\Rightarrow$ (b). For this we need to show that $mD - A$ admits a section for some $m \gg 0$. To this end, consider the short exact sequence

$$0 \longrightarrow \mathcal{O}_X(mD - A) \longrightarrow \mathcal{O}_X(mD) \longrightarrow \mathcal{O}_X(mD)|_A \longrightarrow 0.$$

By Lemma 15.2, $h^0(A, \mathcal{O}_X(mD)|_A)$ growth at most like m^{n-1}, while $h^0(X, \mathcal{O}_X(mD))$ growth like m^n. Hence, for $m \gg 0$, the restriction map

$$H^0(X, \mathcal{O}_X(mD)) \longrightarrow H^0(A, \mathcal{O}_X(mD)|_A)$$

must have a nontrivial kernel, and so $\mathcal{O}_X(mD - A)$ must have a nontrivial section, as we want. This concludes the lemma. $\square$

Definition 15.5. *Let* X *be a projective scheme over a field* k*. A Cartier divisor* D *on* X *is called nef and big if it is nef and big.*

Proposition 15.6. *Let* X *be a projective variety of dimension* n *over a field* k *of characteristic zero and let* D *be a Cartier divisor on* X*. Then the following are equivalent.*

(a) D *is nef and big;*

(b) D *is nef and* $D^n > 0$*;*

(c) *there is an effective divisor* E *such that* $A_m := D - \frac{1}{m}E$ *is an ample* $\mathbb{Q}$*-divisor for all* $m \gg 0$*;*

(d) *For any effective* $\mathbb{Q}$*-divisor* Δ*, there is a log resolution* $\tau : Y \to X$ *of the pair* (X, Δ) *and an effective divisor* E' *on* Y *with simple normal crossings such that* $A_m := f^*D - \frac{1}{m}E'$ *is an ample* $\mathbb{Q}$*-divisor for all* $m \gg 0$*.*

Proof. (a) $\Rightarrow$ (b): Assume that (a) holds. We prove (b) by induction on n. By Lemma 15.4, $mD \sim A + E$ for some $m \gg 0$, where A is ample and E is effective. We aim to show that $D^n > 0$ and for this we may repace D by mD. Hence we may assume $m = 1$ and $D \sim A + E$. Then

$$D^n = D^{n-1}A + D^{n-1}E = D|_A^{n-1} + D|_E^{n-1}.$$

By Lemma 15.4, $D|_A$ is big. Since the restriction of nef divisors is nef, we get by induction $D|_A^{n-1} > 0$. Moreover, $D|_E^{n-1} \geq 0$ because D is nef and E is effective. This shows $D^n > 0$, as we want.

(b) $\Rightarrow$ (a): Assume that D is nef and $D^n > 0$. We aim to show that D is big. It suffices to show this after pulling everything back via some birational morphism $Y \to X$ and so we may w.l.o.g. assume that X is smooth. Let then B be an effective ample divisor on X such that $B - K_X$ is also ample. Then $mD + B - K_X$ is ample by Kleiman's criterion and so

$$H^i(X, mD + B) = H^i(X, mD + B - K_X + K_X) = 0$$

for all $i \geq 1$ by the Kodaira vanishing theorem (where we use that k has characteristic zero). Hence,

$$h^0(X, mD + B) = \chi(X, mD + B).$$

From the short exact sequence

$$0 \longrightarrow \mathcal{O}_X(mD) \longrightarrow \mathcal{O}_X(mD + B) \longrightarrow \mathcal{O}_X(mD + B)|_B \longrightarrow 0$$

we find

$$\chi(X, mD + B) = \chi(X, mD) + \chi(B, \mathcal{O}_X(mD + B)|_B)$$

and so by induction on n we find that

$$\chi(X, mD + B) = m^n/n! \cdot D^n + O(m^{n-1}).$$

and

$$\chi(B, mD|_B + B|_B) = m^{n-1}/(n-1)! \cdot D^{n-1} + O(m^{n-2}).$$

Using the above sequence once again, we find that

$$\begin{aligned} h^0(X, mD) &\geq h^0(X, mD + B) - h^0(B, mD|_B + B|_B) \\ &= \frac{D^n}{n!} m^n + O(m^{n-1}) \end{aligned}$$

and so D is big, as we want.

(c) $\Rightarrow$ (a): Assume that $D - \frac{1}{m}E$ is ample for $m \gg 0$ and some effective divisor E. Then Lemma 15.4 implies that D is big, as it is linearly equivalent to an ample divisor plus some effective $\mathbb{Q}$-divisor. Moreover, $D - \frac{1}{m}E$ is ample for $m \gg 0$ and so

$$D = \lim_{m \to \infty} D - \frac{1}{m}E$$

is nef, which concludes the proof.

(d) $\Rightarrow$ (a): This follows by literally the same argument as in (c) $\Rightarrow$ (a) above.

(a), (b), (c) $\Rightarrow$ (d): It suffices to prove (d) after replacing X by some resolution of X. Hence, we may assume that X is smooth. By Lemma 15.4 we can write $D \sim A + E$ where A is ample and E is an effective $\mathbb{Q}$-divisor. Let $f : Y \to X$ be a log resolution of $(X, E + \Delta)$. Then $f^*D \sim f^*A + f^*E$ and f^*E is a snc divisor. The issue now is that f^*A will not be ample (unless f is an isomorphism). However, since X is smooth, there is an f-exceptional divisor F such that $-F$ is f-ample. (Indeed, start with a general effective ample divisor H' on Y. Then $f^*f_*H' = H' + F$ for some effective f-exceptional divisor F that must be f-anti-ample, as we want.) Then $A' = f^*A - \epsilon F$ is ample for $0 < \epsilon \ll 1$ (see e.g. [2, II.7.10]) and $E' = f^*E + \epsilon F$ is an effective snc divisor, as we want. We may then conclude by setting

$$A'_m := \frac{1}{m}(A' + (m-1)f^*D).$$

□

15.3 Kawamata–Vieweg vanishing

Historically, the Kodaira vanishing theorem was first proven via Hodge theory. Roughly speaking, one picks a (Kähler) metric on the complex manifold X. Cohomology classes of

$$H^i(X, K_X \otimes L) \cong H^{n,i}(X, L)$$

can be represented by harmonic (n,i)-forms with values in the line bundle L and it follows from a metric computation that such forms need to be zero if L has a positive metric, i.e. if L is ample.

The usage of metrics explains the restriction to the ground field $\mathbb{C}$. In fact, it is known that the result does not hold for ample line bundles on smooth projective varieties over fields of arbitrary characteristic. Nonetheless, there is a much more algebraic proof of the Kodaira vanishing theorem, which only uses the following consequence of Hodge theory: the natural map of sheaves $\underline{\mathbb{C}}_X \to \mathcal{O}_X$ induces a surjection

$$H^i(X, \mathbb{C}) \to H^i(X, \mathcal{O}_X)$$

for all $i \geq 0$. Using this fact it is possible to reduce Kodaira's vanishing theorem above to Serre vanishing. The main trick here is to use ramified coverings of X to pass from L to some high power of L. More precisely, one uses a branched covering $Y \to X$ of X to show that

$$H^i(X, L^{-N}) \to H^i(X, L^{-1})$$

is surjective for $i < \dim X$ and some large N, which allows to conclude what we want because

$$H^i(X, L^{-r}) \cong H^{\dim X - i}(X, L^r \otimes K_X)$$

by Serre duality and the above group vanishes for large r by Serre vanishing. For more details on this argument, see [3, Section 2.4] for details.

This algebraic proof of Kodaira's vanishing theorem has the advantage that it also works in the case where X is slightly singular, as long as $H^i(X, \mathbb{C}) \to H^i(X, \mathcal{O}_X)$ is still surjective. In fact, the strategy is very robust enough to allow not only to weaken the smoothness assumption on X but also the positivity assumption on L. This leads to the Kawamata–Vieweg vanishing theorem, which plays a central role in higher dimensional birational geometry.

Theorem 15.7 (Kawamata–Vieweg vanishing theorem)**.** *Let X be a smooth projective variety over an algebraically closed field of characteristic zero. Let $\Delta = \sum d_i D_i$ be a $\mathbb{Q}$-divisor and let L be a line bundle on X. Assume that $L + \Delta$ is nef and big and that $\sum D_i$ has only simple normal crossings. Then*

$$H^i(X, \mathcal{O}_X(K_X + L + \lceil \Delta \rceil)) = 0 \quad \textit{for} \quad i > 0.$$

Proof. See [3, Theorem 2.64 and 3.1]. □

Note that in the special case where $\Delta = 0$, the above theorem says that

$$H^i(X, \mathcal{O}_X(K_X + L)) = 0 \quad \text{for} \quad i > 0$$

whenever L is a big and nef line bundle on X.

Serre duality shows that

$$H^i(X, \mathcal{O}_X(-N)) \cong H^{\dim X - i}(X, \mathcal{O}_X(K_X + N)).$$

The following theorem thus generalizes the above version to the situation of klt pairs.

Theorem 15.8 (Logarithmic Kawamata–Vieweg vanishing theorem)**.** *Let (X, Δ) be a proper klt pair over an algebraically closed field of characteristic zero. Let N be a $\mathbb{Q}$-Cartier divisor on X such that $N \sim_{num} M + \Delta$, where M is a nef and big $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor . Then*

$$H^i(X, \mathcal{O}_X(-N)) = 0 \quad \textit{for} \quad i < \dim X.$$

16 Cone and contraction theorem for klt pairs

16.1 Statement of results

We start by formulating four central theorems culminating in the cone and contraction theorem for klt pairs. Throughout we work over an algebraically closed field of characteristic zero.

Theorem 16.1 (Non-vanishing Theorem)**.** *Let (X, Δ) be a proper klt pair over an algebraically closed field of characteristic zero. Let D be a nef Cartier divisor on X and suppose that for some $a > 0$ the divisor $aD - K_X - \Delta$ is $\mathbb{Q}$-Cartier, nef and big.*

Then, for all $m \gg 0$, $H^0(X, mD) \neq 0$.

Theorem 16.2 (Basepoint-free Theorem)**.** *Let (X, Δ) be a proper klt pair over an algebraically closed field of characteristic zero. Let D be a nef Cartier divisor such that $aD - K_X - \Delta$ is nef and big for some $a > 0$. Then the linear series $|bD|$ has no basepoints for all $b \gg 0$.*

Theorem 16.3 (Rationality Theorem)**.** *Let (X, Δ) be a proper klt pair over an algebraically closed field of characteristic zero such that $K_X + \Delta$ is not nef. Let $a(X) > 0$ be an integer such that $a(X) \cdot (K_X + \Delta)$ is Cartier. Let H be a nef and big Cartier divisor and define*

$$r = r(X) := \max\{t \in \mathbb{R} \mid H + t(K_X + \Delta) \ \ \textit{is nef}\}.$$

Then r is a rational number of the form $r = u/v$ where u, v are integers with

$$0 < v \leq a(X) \cdot (\dim X + 1).$$

Moreover, there is a $(K_X + \Delta)$-negative extremal ray R with $R \cdot (H + r(K_X + \Delta)) = 0$.

Theorem 16.4 (Cone Theorem)**.** *Let (X, Δ) be a projective klt pair over an algebraically closed field of characteristic zero. Then*

(1) There are countably many rational curves $C_j \subset X$ such that $0 < -(K_X+\Delta)\cdot C_j \leq 2\dim X$ and

$$\overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}(X)_{(K_X+\Delta)\geq 0} + \sum [C_j] \cdot \mathbb{R}_{\geq 0}.$$

(2) For any $\epsilon > 0$ and ample $\mathbb{Q}$-divisor H,

$$\overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}(X)_{(K_X+\Delta+\epsilon H)\geq 0} + \sum_{\textit{finite}} [C_j] \cdot \mathbb{R}_{\geq 0}$$

(3) Let $F \subset \overline{\mathrm{NE}}(X)$ be a $(K_X + \Delta)$-negative extremal face. Then the contraction of F, i.e. the unique morphism $cont_F : X \to Z$ with connected fibres to a projective variety Z such that a curve is contracted by $cont_F$ iff it lies in the face F, exists.

(4) Let F and $cont_F$ be as in (3) and let L be a line bundle on X with $L \cdot C = 0$ for every curve C with $[C] \in F$. Then there is a line bundle M on Z with $L = cont_F^* M$.

The proofs of the above theorems are pretty interwoven with each other. The logical order in which these theorems are proven is as follows:

$$\text{non-vanishing} \quad \Rightarrow \quad \text{basepoint free} \quad \Rightarrow \quad \text{rationality} \quad \Rightarrow \quad \text{cone.}$$

A complete proof of the non-vanishing theorem as well as of all implications above can be fuond e.g. in [3, Section 3]. To illustrate the mechanism of the proofs and because of lack of time, we will only explain one step, namely the first implication above.

16.2 The non-vanishing theorem implies the basepoint-free theorem

The purpose of this section is to prove that Theorem 16.1 implies Theorem 16.2.

Let (X, Δ) be a proper klt pair over an algebraically closed field of characteristic zero. Let D be a nef Cartier divisor on X and suppose that for some $a > 0$ the divisor $aD - K_X - \Delta$ is $\mathbb{Q}$-Cartier, nef and big. By the non-vanishing theorem, we may assume that for all $m \gg 0$, $H^0(X, \mathcal{O}_X(mD)) \neq 0$.

For any positive integer s, let $B(s)$ denote the reduced base locus of the linear series $|sD|$. Clearly $B(st) \subset B(s) \cup B(t)$ and so $B(s^a) \subset B(s^b)$ for any positive integers s, a and b with $a \geq b$. By Noetherian induction

$$B_s := \bigcap_{a \geq 1} B(s^a)$$

is either empty or it is algebraic and coincides with $B(s^a)$ for $a \gg 0$.

Step 1. It suffices to show that $B_s = \emptyset$ for all $s \geq 2$.

Proof. If $B_s = \emptyset$ for all $s \geq 2$, then in particular $B_s = \emptyset$ and $B_t = \emptyset$ for coprime integers $s, t|geq2$. This implies $B(s^a) = \emptyset$ and $B(t^b) = \emptyset$ for some $a, b \geq 1$. Replacing s and t by s^a and t^b, we may assume that $a = b = 1$. Since $|sD|$ and $|tD|$ are basepoint-free, the linear series $|n_1 sD|$ and $|n_2 tD|$ is basepoint-free for all $n_1, n_2 \geq 1$. Now, any sufficiently large integer $m \gg 0$ is a linear combination $m = n_1 s + n_2 t$ and so $|mD|$ is basepoint-free, as we want. This concludes step 1. □

From now on we assume for a contradiction that $B_s \neq \emptyset$ for some $s \geq 2$. In particular, there is some $m = s^a$ such that $B_s = Bs(mD)$ is non-empty.

Step 2. There is a resolution $f : Y \to X$ with the following properties:

(a) $f^*mD \sim L + \sum r_j F_j$ where L is basepoint-free, $r_j \geq 0$ are integers and $r_j > 0$ if and only if F_j is a divisorial component of the base locus of f^*mD.

(b) the divisor $f^*(aD - K_X - \Delta) - \sum p_j F_j$ is ample for suitable $0 < p_j \ll 1$, where the F_j need not be f-exceptional;

(c) $K_Y \sim_{\mathbb{Q}} f^*(K_X + \Delta) + \sum a_j F_j$, where $a_j > -1$ are rational numbers and and F_j are prime divisors on Y that are either f-exceptional or proper transforms of components of Δ; Moreover, $-1 < a_j < 0$ whenever F_j is not f-exceptional.

(d) the divisor $\bigcup F_j$ given by the union of all F_j that appear above is a snc divisor

Proof. Let $Z \subset X$ be the base locus of $|mD|$. By Hironaka's theorem, there is a log resolution $f : Y \to X$ such that $f^{-1}(Z \cup \operatorname{supp}(\Delta) \cup X^{\text{sing}})$ is a simple normal crossing divisor. Since X is normal and f is proper birational, it has connected fibres by Zariski's main theorem. Hence,

$$f_* f^* \mathcal{O}_X(mD) \cong f_* \mathcal{O}_X \otimes \mathcal{O}_X(mD) \cong \mathcal{O}_X(mD).$$

This implies that f^* induces an isomorphism $H^0(X, mD) \cong H^0(Y, f^*mD)$ and so

$$Bs(f^*mD) = f^{-1}Bs(mD) = f^{-1}(Z).$$

By construction, this is a divisor on Y with simple normal crossings. If F_j denote the components of this divisor, then we find

$$f^*mD \sim L + \sum r_j F_j$$

for some integers $r_j \geq 1$ and a basepoint-free Cartier divisor L. This proves item (a).

Note that item (a) is stable under replacing Y by a further log resolution $\tau : Y' \to Y$ of the pair $(X, \sum F_j)$, i.e. a proper birational morphism so that Y' is smooth and $\tau^{-1} \sum F_j$ is a snc divisor. Since $aD - K_X - \Delta$ is nef and big, it thus follows from item (d) in Proposition 15.6 that we may assume that item (b) in Step 2 holds true. Item (c) in Step 2 follows directly from the fact that (X, Δ) is klt. Finally, item (d) follows frmo the fact that $\cup F_j$ is supported on $f^{-1}(Z \cup \mathrm{supp}(\Delta) \cup X^{\mathrm{sing}})$, which is a snc divisor by construction. This concludes step 2. $\square$

Recall from the above proof that $Bs(f^*mD) = f^{-1}Bs(mD)$. Hence, in order to arrive at a contradiction to our assumption that $B_s = Bs(mD)$ is non-empty, it suffices to show that there is some component of the base locus of $f * mD$ that is not in the base locus of f^*bD for $b \gg 0$. In other words, we need to show that there is soe component F_j with $r_j > 0$ in item (a) of Step 2 such that F_j is not contained in $Bs(f^*bD)$ for $b \gg 0$.

Step 3. For an integer $b > 0$ and a rational number $c > 0$ such that $b > cm + a$, the $\mathbb{Q}$-Cartier divisor

$$N(b, c) := f^*bD - K_Y + \sum(-cr_j + a_j - p_j)F_j$$

is ample.

Proof. By step 2, we have $-K_Y + \sum a_j F_j \sim_{\mathbb{Q}} -f^*(K_X + \Delta)$. Using this we get

$$N(b, c) \sim_{\mathbb{Q}} (b - cm - a)f^*D + c(mf^*D - \sum r_j F_j) + f^*(aD - K_X - \Delta) - \sum p_j F_j.$$

Since $b > cm + a$, $(b - cm - a)f^*D$ is nef. By step 2,

$$mf^*D - \sum r_j F_j \sim L$$

is basepoint-free, hence nef, and

$$f^*(aD - K_X - \Delta) - \sum p_j F_j$$

is ample. Hence, $N(b, c)$ is linearly equivalent to the sum of a nef divisor and an ample divisor, and so it is ample by Kleiman's criterion. This concludes step 3. $\square$

By item (d) in step 2, the fractional part of the ample divisor $N(b, c)$ from step 3 has simple normal crossing support. The Kawamata-Vieweg vanishing theorem (Theorem 15.7) thus implies that

$$H^1(Y, K_Y + \lceil N(b, c) \rceil) = 0.$$

Since

$$\lceil N(b, c) \rceil = f^*bD - K_Y + \sum \lceil -cr_j + a_j - p_j \rceil F_j,$$

this is saying that

$$H^1(Y, f^*bD + \sum \lceil -cr_j + a_j - p_j \rceil F_j) = 0. \tag{5}$$

Step 4. Up to changing the p_j's from step 2 slightly, we may assume that there is a rational number $c > 0$ such that

$$\sum \lceil -cr_j + a_j - p_j \rceil F_j = A - F$$

where A is effective and f-exceptional and $F = F_j$ for some j with $r_j > 0$.

Proof. We choose $c > 0$ such that

$$\min_j(-cr_j + a_j - p_j) = -1$$

Up to wiggeling the p_j's from step 2 slightly, we may assume that this minimum is achieved for a unique index j_0, which then necessarily has to satisfy $r_j > 0$. With this choice for the rational number $c > 0$, we find that

$$\sum \lceil -cr_j + a_j - p_j \rceil F_j = A - F$$

where A is effective and $F = F_{j_0}$ with $r_{j_0} > 0$. It remains to show that A is f-exceptional. For this note that $r_j > 0$ for all j and so $-cr_j \leq 0$ for all j. Moreover, $-p_j \leq 0$ is negative. Since

$$\min_j(-cr_j + a_j - p_j) = -1,$$

and this minimum is achieved for the index $j = j_0$ only, we conclude that for $j \neq j_0$ we have $\lceil -cr_j + a_j - p_j \rceil \geq 0$ and strict inequality holds here only if $a_j > 0$. But $a_j > 0$ implies that F_j is f-exceptional by item (c) in step 2. This concludes step 4. $\square$

By (5) and step 4, we find that

$$H^1(Y, f^*bD + A - F) = 0$$

for all integers $b > cm + a$, where A is an effective f-exceptional divisor and $F = F_{j_0}$ with $r_{j_0} > 0$. This vanishing implies that the restriction map

$$H^0(Y, f^*bD + A) \longrightarrow H^0(F, (f^*bD + A)|_F)$$

is surjective. Since A is effective and f-exceptional, we find that $f_*\mathcal{O}_X(A) = 0$ and so the projection formula yields

$$H^0(Y, f^*bD + A) \cong H^0(X, bD).$$

Since F is in the base locus of f^*bD, we conclude that the above surjection is the zero map and so

$$H^0(F, (f^*bD + A)|_F) = 0.$$

To derive the desired contradiction, we aim to apply the non-vanishing theorem to $(f^*bD + A)|_F$ as follows.

Step 5. The non-vanishing theorem applies to the divisor $(f^*bD + A)|_F$ and this shows that

$$H^0(F, (f^*bD + A)|_F) \neq 0$$

for $b \gg 0$.

Proof. Let

$$A' := \sum_{j \neq j_0} (-cr_j + a_j - p_j)F_j.$$

Then $A = \lceil A' \rceil$. Moreover, $F \not\subset \operatorname{supp} A'$. Since $F = F_{j_0}$ and $\bigcup F_j$ is a snc divisor by step 2, we find that the support of the restriction $A'|_F$ is also a simple normal crossing divisor. In particular, the fractional part of the restriction $(f^*bD + A')|_F$ is a snc divisor. By step 3,

$$N(b, c) = f^*bD - K_Y - F + A'$$

is ample on Y and so this divisor restricts to an ample divisor on F. On the other hand, $K_F = (K_Y + F)|_F$. Hence, if we knew that the pair $(Y, (A - A')|_F)$ was klt, where $(A - A')|_F$ denotes the fractional part of the divisor $N(b, c)|_F$, then the non-vanishing theorem (Theorem 16.1) applied to F would show that

$$H^0(F, (f^*bD + A)|_F) \neq 0$$

as we want. Hence it remains to show that the pair (F, Δ_F) with $\Delta_F := (A - A')|_F$ is klt. Since A is the round up of A' and F is not contained in the support of A', we find that

$$\Delta_F = \sum_{j \neq j_0} e_j F_j|_F \quad \text{with} \quad e_j = \lceil -cr_j + a_j - p_j \rceil - (-cr_j + a_j - p_j).$$

Clearly, $0 \leq e_j < 1$ and so *discrep*$(F, \Delta_F) > -1$ by Proposition 14.16, which implies that (F, Δ_F) is klt, as we want. This concludes step 5 and hence the proof of the basepoint-free theorem. □

17 Canonical models

Definition 17.1. *Let (X, Δ) be a proper klt (or lc) pair. Then the canonical ring of (X, Δ) is defined as*

$$\mathcal{R}(X, \Delta) := \bigoplus_{m=0}^{\infty} H^0(X, \mathcal{O}_X(mK_X + \lfloor m\Delta \rfloor)).$$

Note that the canonical ring is indeed a ring, because

$$\lfloor m_1 \Delta \rfloor + \lfloor m_2 \Delta \rfloor \leq \lfloor (m_1 + m_2)\Delta \rfloor.$$

(This is the reason why we have to take round downs, as the inequality would go in the wrong direction if we were using round ups in the definition of canonical rings.)

Theorem 17.2. *Let (X, Δ) be a proper klt pair over an algebraically closed field k of characteristic zero. If $K_X + \Delta$ is nef and big, then the canonical ring $\mathcal{R}(X, \Delta)$ is finitely generated.*

Proof. Let $a \geq 2$ be a positive integer such that $D := r(K_X + \Delta)$ is Cartier. Since $K_X + \Delta$ is nef and big, the same holds true for $D - K_X - \Delta$. By the basepoint-free theorem (Theorem 16.2), aD is basepoint-free for all $a \gg 0$. Since D is nef and big, Proposition 15.6 implies that the associated morphism

$$\phi := \phi_{|aD|} : X \longrightarrow \mathbb{P}(H^0(X, aD)^\vee)$$

is birational onto its image for a sufficiently large and we denote by Y the image of $\phi_{|aD|}$. There is an ample line bundle L on Y such that

$$\phi^* L = aD = ar(K_X + \Delta).$$

Let $G_m := \phi_* \mathcal{O}_X(mK_X + \lfloor m\Delta \rfloor)$. Then

$$\mathcal{R}(X, \Delta) = \bigoplus_{m=0}^{\infty} H^0(Y, G_m).$$

Moreover,

$$G_{m+ar} \cong L \otimes G_m$$

by the projection formula, because $\phi^*L = ar(K_X + \Delta)$. Since $L = \mathcal{O}_Y(1)$, we have

$$Y = \operatorname{Proj} S, \quad \text{with} \quad S = \bigoplus_{i=0}^{\infty} H^0(Y, L^i).$$

In particular, S is finitely generated. On the other hand, $\mathcal{R}(X, \Delta)$ is generated as an S module by

$$\bigoplus_{m=0}^{ar-1} H^0(Y, G_m)$$

which is a finite-dimensional k-vector space. Since S is finitely generated over k, it follows that $\mathcal{R}(X, \Delta)$ is a finitely generated k-algebra, as we want. $\square$

References

[1] O. Debarre, *Higher-Dimensional Algebraic Geometry*, Springer, New York, 2001.

[2] R. Hartshorne, *Algebraic Geometry*, Springer, 1977.

[3] J. Kollár and S. Mori, *Birational Geometry of Algebraic Varieties*, Cambridge University Press, Cambridge, 1998.

[4] R. Lazarsfeld, *Positivity in algebraic geometry, I and II*, Springer, 2004.

[5] K. Matsuki, *Introduction to the Mori Program*, Springer, New York, 2002.

[6] S. Schreieder, *Algebraic Geometry I*, Lecture course at LMU Munich, WiSe 2018/19, https://www.iag.uni-hannover.de/fileadmin/institut/team/schreieder/AlgGeo1.pdf

[7] S. Schreieder, *Algebraic Geometry II*, Lecture course at LMU Munich, SoSe 2018/19, https://www.iag.uni-hannover.de/fileadmin/institut/team/schreieder/AlgGeo2.pdf