Birational Geometry

Sheet 01

Unless specified otherwise, we will always work over an algebraically closed field k.

Exercise 1. *Let X be a projective variety with $\mathrm{Pic}(X) \cong \mathbb{Z}$. Let $f : X \to Y$ be a surjective morphism to a projective variety Y. Show that f contracts either X to a point $Y = \{pt.\}$, or $\dim X = \dim Y$ and f is quasi-finite (i.e. has finite fibres). (In the latter possibility f is finite, bnecause 'proper + quasi-finite = finite'.)*

Exercise 2. *Let $X := \mathbb{P}^1 \times \mathbb{P}^1$. Compute $\mathrm{Pic}(X)$ and characterize all line bundles on X that may arise as pullbacks of $\mathcal{O}(1)$ via morphisms from X to some projective space.*

Exercise 3. *Consider*

$$X := \{x_0x_1 - x_2x_3 = 0\} \subset \mathbb{A}^4 \quad \textit{and} \quad D := \{x_0 = x_2 = 0\}.$$

Show that D is a prime divisor on X which is not Cartier. Is it true that D is $\mathbb{Q}$-Cartier, i.e. is there a positive integer m such that mD is Cartier?

(Hint: You may use without proof that X is normal.)

Exercise 4. *Let X be a normal variety over an algebraically cosed field k and let $D \in \mathrm{WDiv}(X)$ be a divisor on X. Consider the presheaf $\mathcal{O}_X(D)$ on X which on a nonempty open subset $U \subset X$ is given by*

$$\mathcal{O}_X(D)(U) := \{0\} \cup \{f \in k(X)^* \mid \mathrm{div}(f)|_U + D|_U \geq 0\}.$$

In other words, the sections of $\mathcal{O}_X(D)$ over a nonempty open subset $U \subset X$ are given by all rational functions f on X such that the divisor $\mathrm{div}(f) + D$ is effective when restricted to U, i.e. $\mathrm{div}(f) + D = D' + D''$ for an effective divisor D' and a divisor D'' which is supported on $X \setminus U$.

(a) Show that $\mathcal{O}_X(D)$ is a sheaf of abelian groups. Show further that $\mathcal{O}_X(D)$ is an $\mathcal{O}_X$-module, i.e. for any nonempty open subset $U \subset X$, $\mathcal{O}_X(D)(U)$ is a module over the ring $\mathcal{O}_X(U)$ and this module structure is compatible with the restriction maps on both sides.

(b) Let D_1 and D_2 be two divisors on X with $D_1 \sim D_2$, i.e. $D_1 - D_2 = \mathrm{div}(f)$ for some $f \in k(X)^*$. Show that $\mathcal{O}_X(D_1) \cong \mathcal{O}_X(D_2)$.

(c) Let D_1 and D_2 be two divisors on X with an isomorphism $\varphi : \mathcal{O}_X(D_1) \xrightarrow{\sim} \mathcal{O}_X(D_2)$ of sheaves which is compatible with the natural $\mathcal{O}_X$-module structures on both sides. Prove that $D_1 \sim D_2$.

You can hand in your solutions via email to schreieder@math.uni-hannover.de before **Monday, April 27th, 10:00**. It is preferable if you submit solutions as a single (pdf) file, e.g. by using Latex or by converting pictures of your handwritten solutions into a single (pdf) file.