Birational Geometry

Sheet 02

Exercise 1. *Let X be a projective scheme over a field k with $\mathrm{Pic}(X) \cong \mathbb{Z}$ (e.g. $X = \mathbb{P}^N_k$). Let L be a very ample line bundle on X and let $f : X \hookrightarrow \mathbb{P}^N_k$ be the closed embedding induced by a basis of $H^0(X, L)$. Let $f : X \to X$ be an isomorphism over k. Show that*

(a) $f^*L \cong L$;

(b) the automorphism f is given by restriction of an automorphism of the ambient projective space $\mathbb{P}^N_k$.

Exercise 2. *Show that any automorphism of $\mathbb{P}^N_k$ is induced by an element of the projective general linear group over k:*

$$\mathrm{PGL}(N, k) = \mathrm{GL}(N + 1, k)/k^*.$$

(Hint: You may use $\mathrm{Pic}(\mathbb{P}^N_k) \cong \mathbb{Z} \cdot [\mathcal{O}(1)]$ without proof. Show then that any isomorphism $f : \mathbb{P}^N_k \to \mathbb{P}^N_k$ over k pulls back $\mathcal{O}(1)$ to itself and hence induces a linear automorphism of the k-vector space $H^0(\mathbb{P}^N_k, \mathcal{O}(1))$ of dimension $N + 1$.)

Exercise 3. *Consider the subscheme $X \subset \mathbb{P}^2_k = \mathrm{Proj}\, k[x_0, x_1, x_2]$, cut out by the ideal*

$$I_X = (x_2^2, x_1x_2).$$

Consider the Cartier divisors

$$D_1 := \{x_1 = 0\} \quad \textit{and} \quad D_2 := \{x_2 = 0\}$$

on $\mathbb{P}^2_k$.

(a) Compute the restrictions of D_i to X as divisors classes and compare the corresponding Cartier divisor classes with the scheme-theoretic intersection $D_i \cap X$.

(b) Is it possible to pullback D_1 or D_2 to X as a Cartier divisor by pulling back local equations for D_1 or D_2, respectively?

(c) Can you explain what is going wrong here?

Exercise 4. *Let $f : X \to Y$ be a morphism of schemes and let D be a Cartier divisor on Y. We say that the pullback f^*D of the Cartier divisor D by the morphism f is well-defined as a Cartier divisor (and not only as a divisor class), if locally at each point of x, D is given locally around $f(x)$ by a quotient $\phi = a/b$ of local regular functions on Y which are not zero-divisors, such that $f^*\phi = f^*a/f^*b$ is a quotient of local regular functions on X that are not zero divisors.*

(a) *Assume that f^*D is well-defined as a Cartier divisor. Show that in this case,*

$$f^*\mathcal{O}_Y(D) \cong \mathcal{O}_X(f^*D).$$

(b) *Let k be a field and consider the projective scheme $X \subset \mathbb{P}^2_k$, cut out by $x_0x_1^2 - x_2^3$. Let D be the Cartier divisor on $\mathbb{P}^2_k$, given by by $x_0 = 0$. Show that the restriction (i.e. pullback) of D to X is well-defined as a Cartier divisor. Compare this to what you have observed in the previous exercise for the Cartier divisor D_1.*

(c) *Let for simplicity $f : X \to Y$ be a morphism of schemes that are separated of finite type over an algebraically closed field k. Let D be a Cartier divisor on Y and assume for simplicity that D is effective, i.e. $\mathcal{O}_X(-D) \subset \mathcal{O}_X$, which means that D is locally given by $\phi = a/1$, where a is a local regular function that is not a zero divisor. Can you find conditions on f, X and D that ensure that f^*D is well-defined as a Cartier divisor (and not only as a divisor class)?*

You can hand in your solutions via email to schreieder@math.uni-hannover.de before **Monday, May 4th, 10:00**. It is preferable if you submit solutions as a single (pdf) file, e.g. by using Latex or by converting pictures of your handwritten solutions into a single (pdf) file.