Birational Geometry

Sheet 03

Exercise 1. *Let X be a proper scheme of dimension n over a field k and let D be a Cartier divisor on X. Show that the leading coefficient of $\chi(X, \mathcal{O}_X(mD))$ is given by $D^n/n!$. Use this to give an example where $\chi(X, \mathcal{O}_X(mD))$ is not an integral polynomial in m.*

Exercise 2. *Let (for simplicity) k be an algebraically closed field.*

(a) *Let X be a smooth projective surface over k and let $\tau : X' = Bl_x X \to X$ be the blow-up of X in a point $x \in X$. Let $E \subset X'$ be the exceptional divisor of X'. Show that $E^2 = -1$.*

*(**Hint:** Let $D \subset X$ be a smooth hyperplane section which passes through x. Compute the intersection number $\tau^*\mathcal{O}_X(D) \cdot E$ and use it to deduce the result by computing the pullback of D to X' as a Cartier divisor.)*

(b) *Let X be a smooth projective variety over k and let $f : X \to Y$ be a birational morphism to a normal variety Y. Let $E \subset X$ be a prime divisor that is contracted, i.e. $f(E)$ is of codimension at least two in Y. Show that there is a proper curve $C \subset X$ with $C \cdot E < 0$.*

*(**Hint:** Generalize the approach to part (a).)*

Exercise 3. *Let $f : X \to Y$ be a birational morphism between normal projective varieties over an algebraically closed field. Let $\mathrm{Exc}(f)$ be the exceptional locus of f, i.e. the locus of all points in X where f is not a local isomorphism.*

(a) *Show that $\mathrm{Exc}(f) = f^{-1}(Z)$, where $Z \subset Y$ is the locus of points $y \in Y$ such that the fibre $X_y = f^{-1}(y)$ is positive-dimensional.*

*(**Hint:** Use Zariski's main theorem.)*

(b) *Deduce fro (a) that $\mathrm{Exc}(f)$ is closed, $f(\mathrm{Exc}(f))$ has codimension at least two in Y and that set-theoretically, we have $\mathrm{Exc}(f) = f^{-1}(f(\mathrm{Exc}(f)))$.*

*(**Hint:** You may use without proof that the subset $Z \subset Y$ in part (a) is closed.)*

(c) *Assume that $\mathrm{Exc}(f)$ is non-empty and of codimension at least two in X. Show that then Y must be singular, and in fact non-$\mathbb{Q}$-factorial, which means that there is a Weil divisor D on Y such that mD is not Cartier for any nonzero integer $m \in \mathbb{Z} \setminus \{0\}$.*

Exercise 4. *(Projection formula)*
Let $f : X \to Y$ be a surjective morphism between proper varieties over an algebraically closed field k. Let C be a proper integral curve on X and let D be a Cartier divisor on Y. Show that

$$f^*D \cdot C = D \cdot f_*C,$$

where f_*C is defined as follows: it is zero if f contracts C to a point and it is $d \cdot f(C)$, where d denotes the degree of the finite morphism $C \to f(C)$, if C is not contracted by f.

You can hand in your solutions via email to schreieder@math.uni-hannover.de before **Monday, May 11th, 10:00**. It is preferable if you submit solutions as a single (pdf) file, e.g. by using Latex or by converting pictures of your handwritten solutions into a single (pdf) file.