Birational Geometry

Sheet 04

Exercise 1. *(Negative curves on surfaces span extremal rays of the cone of curves)*

Let X be a smooth projective surface over a field k.

(a) Let $C \subset X$ be an integral curve with $C^2 := C \cdot C < 0$. Show that $[C] \in N_1(X)$ spans an extremal ray of the cone $\overline{NE}(X)$, i.e. whenever $\gamma, \gamma' \in \overline{NE}(X)$ with $\gamma + \gamma' \in [C]\mathbb{R}_{\geq 0}$, then $\gamma, \gamma' \in [C]\mathbb{R}_{\geq 0}$.

(b) Let $C, C' \subset X$ be two different integral curves with negative self-intersections. Show that the extremal rays of $\overline{NE}(X)$ spanned by C and C' are different.

Exercise 2. *(An example of a surface with infinitely many negative curves)*

The purpose of this exercise is to construct an example of a smooth projective surface X over $k = \mathbb{C}$ (with $\rho(X) = 10$) which admits an infinite sequence of integral curves $C_i \subset X$ with negative self-intersections (in fact $C_i^2 = -1$ for all i). By Exercise 1 it follows in particular that $\overline{NE}(X)$ has infinitely many extremal rays.

Construction: *Let $F, G \in k[x_0, x_1, x_2]$ be general cubic forms and let $p_1, \ldots, p_9$ be the base locus of the pencil spanned by $f, g \in H^0(\mathbb{P}^2, \mathcal{O}(3))$, i.e. $\{f = g = 0\} = \{p_1, \ldots, p_9\}$. Let S be the blow-up of $\mathbb{P}^2$ along the nine points $\{p_1, \ldots, p_9\}$ with exceptional divisor $E_i \subset S$ above p_i. Consider the morphism $\pi : X \to \mathbb{P}^1$ induced by the rational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ given by $x \mapsto [g(x) : f(x)]$. Let $C_t := \{f - tg = 0\}$ be the fibre of π above $[(1 : t)]$, which is a smooth curve of genus one for all but finitely many values of t.*

(a) Show that for all but countably many $t \in \mathbb{C}$, the line bundle $\mathcal{O}_{C_t}(p_2 - p_1)$ yields a point in $\mathrm{Pic}(C_t)$ which is not torsion.

(b) Show that there is an automorphism $\phi : X \to X$ which commutes with the projection $\pi : X \to \mathbb{P}^1$ and which maps E_1 to E_2.

(c) Show that ϕ has infinite order and deduce that X admits infinitely many integral curves with negative self-intersections. In fact, the curves $C_n := \phi^n(E_1)$ form an infinite sequence of smooth rational curves on S with $C_n^2 = -1$.

*(**Hint:** each exceptional divisor E_i yields a section of π. Using one of these sections, say E_1, we get a canonical base point p_1 in each fibre C_t, so that the (k-rational points of the) smooth fibres C_t carry the structure of a group with neutral element p_1 via the injection $C_t(k) \to \mathrm{Pic}(C_t)$, $x \mapsto \mathcal{O}_E(x - p_1)$, whose image is given by all classes of line bundles of degree zero on C_t. You should think about ϕ as the isomorphism which is fibrewise given by translation by p_2 with respect to the group structure just mentioned.)*

Exercise 3. *Let X be an abelian variety over an algebraically closed field k. Let $D_1, \dots, D_n$ be effective Cartier divisors on X, where $n = \dim X$. Show that*

$$D_1 \cdots D_n \geq 0.$$

*(**Hint:** Skip this exercise if you don't know what an abelian variety is.)*

Exercise 4. *Let C be a smooth projective curve of genus g over an algebraically closed field k. Show that the smooth projective surface $X := C \times C$ carries an integral curve of negative self-intersection, if and only if $g \geq 2$.*

You can hand in your solutions via email to schreieder@math.uni-hannover.de before **Monday, May 18th, 10:00**. It is preferable if you submit solutions as a single (pdf) file, e.g. by using Latex or by converting pictures of your handwritten solutions into a single (pdf) file.