Birational Geometry

Sheet 05

Exercise 1. *(Ruled surface)*

Let E be a vector bundle of rank two on a smooth projective curve C. Let

$$X = \mathbb{P}(E) = \mathrm{Proj}_{\mathcal{O}_C} \bigoplus_{n \geq 0} \mathrm{Sym}^n E$$

be the projective bundle of one-dimensional quotients of E with projection $\pi : X \to C$. Let $\mathcal{O}(1)$ be the relative dualizing sheaf, which is given as tautological quotient bundle $\pi^* E \twoheadrightarrow \mathcal{O}(1)$ and which satisfies $\pi_* \mathcal{O}(n) = \mathrm{Sym}^n E$ for all n.

(a) *Show that there is a short exact sequence*

$$0 \longrightarrow \mathrm{Pic}\, C \xrightarrow{\pi^*} \mathrm{Pic}\, X \xrightarrow{\alpha} \mathrm{Pic}(F) \cong \mathbb{Z} \longrightarrow 0$$

where $F \cong \mathbb{P}^1$ denotes a smooth fibre of π and α denotes the restriction map which in particular maps $\mathcal{O}(1)$ to a generator of $\mathrm{Pic}\, F$.

*(**Hint:** Use that X is smooth, so that $\mathrm{Pic}\, X = \mathrm{Cl}\, X$ and use the localization sequence for class groups, see e.g. II.6.5 in Hartshorne's book. It will also be helpful to note that π admits a section, because $\mathbb{P}(E)$ is a Zariski locally trivial $\mathbb{P}^1$-bundle.)*

(b) *Show that $N_1(X) = N^1(X)$ is freely generated by $f = [F]$ and $\xi = [\mathcal{O}(1)]$. Compute f^2 and $f\xi$, and show that f spans an extremal ray of $\mathrm{NE}(X)$.*

(c) *Deduce from the fact that X contains some ample line bundle that $H = \pi^* A \otimes \mathcal{O}(1)$ is ample as long as $A \in \mathrm{Pic}\, C$ has sufficiently large degree.*

(d) *Use part (c) to show that $\xi \in \overline{\mathrm{NE}}(X)$ and $-f \notin \overline{\mathrm{NE}}(X)$.*

(e) *Show that $\xi^2 = \deg E$, where $\deg E = \deg(\Lambda^2 E)$ denotes the degree of the determinant line bundle $\Lambda^2 E$ associated to E.*

*(**Hint:** One way of solving this is as follows: By (c), $H = \pi^* A \otimes \mathcal{O}(1)$ is ample as long as $A \in \mathrm{Pic}\, C$ has sufficiently large degree. Since you computed f^2 and ξf in (b), the intersection number ξ^2 can be deduced from the self-intersection of H. Compute the latter by analysing the leading term of $\chi(X, \mathcal{O}_X(mH))$ – at this point you will need Serre vanishing on X and Riemann–Roch for vector bundles on C. Moreover, you will need to use Serre duality on C as well as the second assertion in part (d) applied to the projective bundle $\pi' : \mathbb{P}(E^*) \to C$ of one-dimensional quotients of E^*.)*

(f) *Assume now for simplicity that $\deg E = 0$. Show that* *(i) If there is a line bundle A on C of degree $a \leq 0$ such that*

$$H^0(C, A \otimes \operatorname{Sym}^n E) \neq 0,$$

then $\mathrm{NE}(X) = \overline{\mathrm{NE}}(X)$ is generated by $af + \xi$ and f.

(ii) If for any line bundle A on C of degree $a \leq 0$ we have

$$H^0(C, A \otimes \operatorname{Sym}^n E) = 0,$$

then $\xi \notin \mathrm{NE}(X)$ and so $\mathrm{NE}(X) \neq \overline{\mathrm{NE}}(X)$. Moreover, $\mathcal{O}(1)$ is in this case a line bundle on X of positive degree on each effective curve on X, yet $\mathcal{O}(1)$ is not ample in this case.

*(**Remark:** If C is a smooth complex projective curve of genus at least two, then a rank two vector bundle E of degree zero on C with the above property exists.)*

Exercise 2. *(Big divisors)*

Let X be a projective scheme of dimension n over a field. A Cartier divisor D on X is called big if

$$\limsup_{m \to \infty} \frac{h^0(X, \mathcal{O}_X(mD))}{m^n} > 0.$$

(a) Show that any ample divisor on X is big;

(b) Find an example of a big divisor that is not nef;

(c) Find an example of a big divisor that is nef but not ample.

You can hand in your solutions via email to schreieder@math.uni-hannover.de before **Monday, May 25th, 10:00**. It is preferable if you submit solutions as a single (pdf) file, e.g. by using Latex or by converting pictures of your handwritten solutions into a single (pdf) file.