Birational Geometry

Sheet 06

Exercise 1. *Let R be a noetherian local ring with maximal ideal $\mathfrak{m}$ and let $I \subset \mathfrak{m}^2$ be an ideal in R. Assume that the canonical projection $\pi : R \to R/I$ has a section, i.e. a ring homomorphism $s : R/I \to R$ with $\pi \circ s = \mathrm{id}$. Show that $I = 0$.*

Exercise 2. *Let $X \subset \mathbb{P}^{n+1}_k$ be a smooth projective hypersurface of degree d and dimension n over a field k. Assume that k has characteristic zero and that $d \geq n+2$. Show that X is not uniruled, i.e. there is no dominant rational map $\mathbb{P}^1 \times Y \dashrightarrow X$ for some variety Y of dimension $n-1$.*

Exercise 3. *Let k be an algebraically closed field of characteristic zero and consider the Fermat surface*

$$X := \{x_0^d + x_1^d + x_2^d + x_3^d = 0\} \subset \mathbb{P}^3_k.$$

Let ζ and ζ' be primitive $2d$-th roots of unity in k and consider the line

$$\ell := \{x_0 = \zeta x_1,\ x_2 = \zeta' x_3\} \subset X.$$

Let $f : \mathbb{P}^1 \to X$ be a morphism of degree one with image ℓ.

(a) Show that if $d \geq 4$, then locally at $[f]$, $\mathrm{Mor}_1(\mathbb{P}^1, X)$ has dimension three.

*(**Hint:** Use Exercise 2 from sheet02 and Exercise 1 from this sheet.)*

(b) Let $\mathcal{N}_{\ell/X}$ be the normal bundle of ℓ in X. Show that there is an exact sequence

$$0 \longrightarrow \mathcal{N}_{\ell/X} \longrightarrow \mathcal{N}_{\ell/\mathbb{P}^3} \cong \mathcal{O}_\ell(1)^{\oplus 2} \longrightarrow \mathcal{N}_{X/\mathbb{P}^3} \longrightarrow 0.$$

Use this to show that $\mathcal{N}_{\ell/X} \cong \mathcal{O}_\ell(2-d)$.

(c) Use the exact sequence $0 \longrightarrow T_\ell \longrightarrow T_X|_\ell \longrightarrow \mathcal{N}_{\ell/X} \longrightarrow 0$ to compute

$$h^0(\mathbb{P}^1, f^*T_X) \quad \text{and} \quad h^1(\mathbb{P}^1, f^*T_X).$$

Conclude from this that $\mathrm{Mor}_1(\mathbb{P}^1, X)$ is singular at $[f]$ if $d \geq 4$.

(d) Show more generally that in fact $[f]$ lies on an irreducible but non-reduced three-dimensional component of $\mathrm{Mor}_1(\mathbb{P}^1, X)$ if $d \geq 4$.

Exercise 4. *Let k be an algebraically closed field of characteristic $p > 2$ and consider the smooth (!) Fermat surface*

$$X := \{x_0^d + x_1^d + x_2^d + x_3^d = 0\} \subset \mathbb{P}^3_k$$

of degree $d = p^r + 1$ for some positive integer r. The goal of this exercise is to show that X is unirational over k, i.e. it admits a dominant rational map $\mathbb{P}^2_k \dashrightarrow X$.

Let ζ be a primitive $2d$-th root of unity in k and consider the line

$$\ell := \{x_0 = \zeta x_1,\ x_2 = \zeta x_3\} \subset X.$$

Let $h_1 := \zeta x_1 - x_0$ and $h_2 := \zeta x_3 - x_2$ and consider the pencil of hyperplanes $h_1 - th_2 = 0$ in $\mathbb{P}^3$, whose restriction to X yields a dominant rational map

$$\pi := [h_1 : h_2] : X \dashrightarrow \mathbb{P}^1.$$

Let X_η be the generic fibre of $\mathbb{P}^1$ (by this we mean the generic fibre of some resolution $X' \to \mathbb{P}^1$ of the rational map π, but since X is a surface and π is dominant, this will not depend on the choice of X'). Identify $k(\mathbb{P}^1)$ with $k(t)$ and consider the field extension $k(t^{1/p^r})$ of $k(t)$. Show that over $k(t^{1/p^r})$, X_η becomes isomorphic to the rational plane curve with equation

$$y_2^{q-1} y_3 - y_1^q = 0.$$

Conclude from this that there is a dominant rational map $\mathbb{P}^2_k \dashrightarrow X$. Can such a map be separable if $d \geq 4$?

You can hand in your solutions via email to schreieder@math.uni-hannover.de before **Monday, June 15th, 10:00**. It is preferable if you submit solutions as a single (pdf) file, e.g. by using Latex or by converting pictures of your handwritten solutions into a single (pdf) file.