Birational Geometry

Sheet 07

Exercise 1. *Let R be a finitely generated $\mathbb{Z}$-algebra. Show that*

(a) for any maximal ideal $\mathfrak{m} \subset R$, $R/\mathfrak{m}$ is a finite field;

*(**Hint:** You may use the following fact: if K/k is a field extension so that K is a finitely generated k-algebra, then K/k is algebraic and hence a finite extension. Use this to reduce the problem to showing that the preimage of $\mathfrak{m}$ via $\mathbb{Z} \to R$ is a maximal ideal of $\mathbb{Z}$.)*

(b) the closed points of $\operatorname{Spec} R$ *are dense.*

Exercise 2. *Let X and Y be projective varieties, let $\pi : X \to Y$ be a morphism, let $f : C \to X$ be a morphism from a smooth projective curve C and let $c_0 \in C$. Let T be an affine curve and let $F : T \times C \longrightarrow X$ be a morphism with*

(a) $F(t, c_0) = f(c_0)$ for all $t \in T$;

(b) there is some $t_0 \in T$ such that $F(t_0, c) = f(c)$ for all $c \in C$;

(c) for general $t \in T$ and general $c \in C$, $\pi(F(t, c)) \neq \pi(f(c))$.

Show that there is a morphism $f' : C \to X$ and an effective 1-cycle of rational curves R on X with $f(c_0) \in R$ and $\pi_* R \neq 0$ such that

$$f_*C \sim_{num} f'_*C + R.$$

Exercise 3 (Lueroth's theorem)**.** *Let $f : \mathbb{P}^1 \to C$ be a dominant morphism between smooth projective curves over an algebraically closed field k. The purpose of this exercise is to prove Lueroth's theorem, which says that in the above situation, $C \cong \mathbb{P}^1$. This implies in particular that the image of any non-constant morphism $f : \mathbb{P}^1 \to X$ is a rational curve, i.e. the image $f(\mathbb{P}^1)$ is birational to $\mathbb{P}^1$. (Note that the analogous statement fails for surface, see Exercise 4 on sheet 06.)*

(a) Prove Lueroth's theorem in the case where the differential $df : T_{\mathbb{P}^1} \to f^*T_C$ is not identically to zero.

(b) Prove that the df is not identically to zero if the field extension $k(C) \subset k(\mathbb{P}^1)$ is separable.

*(**Hint:** Use the theorem of the primitive element to find an explicit affine curve that is birational to C.)*

(c) Show that f factors as follows:

$$\mathbb{P}^1 \xrightarrow{h} C' \xrightarrow{g} C,$$

where g is separable and h is purely inseparable, i.e. $k(C) \subset k(C')$ is separable while $k(C') \subset k(\mathbb{P}^1)$ is purely inseparable. Conclude from (b) that this reduces Lueroth's theorem to the case where f is purely inseparable.

*(**Hint:** You may use without proof that a field existence $k(C) \subset k(C')$ with the required properties exists.)*

(d) Prove Lueroth's theorem in the case where f is purely inseparable, by showing that f factors through $F^{\circ r} : \mathbb{P}^1 \to \mathbb{P}^1$, where $F([x : y]) = [x^p : y^p]$ denotes the Frobenius endomorphism and where $\deg(f) = p^r$ for some positive integer r. Conclude from this that $f = F^{\circ r}$ and $C \cong \mathbb{P}^1$.

*(**Hint:** Note that it is essentially enough to prove the assertion on the level of function fields and recall from field theory that if L/K is a purely inseparable field extension of degree p^r, then $L^{p^r} \subset K$, or in other words, $L \subset K^{1/p^r}$.)*

You can hand in your solutions via email to schreieder@math.uni-hannover.de before **Monday, June 22th, 10:00**. It is preferable if you submit solutions as a single (pdf) file, e.g. by using Latex or by converting pictures of your handwritten solutions into a single (pdf) file.