Birational Geometry

Sheet 08

Exercise 1. *Let X be a Fano variety over an algebraically closed field k. The index of X, denoted by ind(X), is the largest integer l such that $-K_X = l \cdot H$ for some ample divisor H on X. Show that ind$(X) \leq \dim X + 1$.*

Remark: *One can show that ind$(X) = \dim X + 1$ if and only if $X \cong \mathbb{P}^n$.*

Exercise 2. *Let X be a projective variety over an algebraically closed field k. Assume that k is uncountable. Show that X is uniruled if for any closed point $x \in X$ there is a rational curve $R \subset X$ with $x \in R$.*

Remark: *Let $X \subset \mathbb{P}^3$ be a smooth quartic over an algebraically closed field k of characteristic zero. By Sheet 6, we know that X is not uniruled. On the other hand, it is known that X contains infinitely (but countably) many rational curves. In particular, if $k = \overline{\mathbb{Q}}$, it is a priori possible that through any closed point of X there is a rational curve. Whether or not this happens is unknown for any smooth quartic $X \subset \mathbb{P}^3$ over $k = \overline{\mathbb{Q}}$.*

Exercise 3. *Let X be a projective variety over an algebraically closed field k. Let $R \subset k$ be a finitely generated $\mathbb{Z}$-algebra and let*

$$\pi : \mathcal{X} \longrightarrow T := \operatorname{Spec} R$$

be a projective flat R-scheme with $X \cong \mathcal{X} \times_R k$. Hence $\mathcal{X} \subset \mathbb{P}^N_T$ for some $N \gg 0$ and we let $H := \mathcal{O}_{\mathbb{P}^N}(1)|_{\mathcal{X}}$. Assume that for all closed points $t \in T$, the fibre $X_t := \mathcal{X} \times_R \kappa(t)$ contains a surface $S_t \subset X_t$ whose normalization is $\mathbb{P}^2$ and such that

$$H_t^2 \cdot S_t \leq d$$

for some integer d that does not depend on t.

Show that X contains a rational surface.

Exercise 4. *Let X be a projective variety over a field k. Let $R \subset k$ be a ring and let*

$$\pi : \mathcal{X} \longrightarrow T := \operatorname{Spec} R$$

be a projective R-scheme with $X \cong \mathcal{X} \times_R k$. Assume that X contains a rational curve. Show that there is a non-empty open subset $U \subset T$ such that for any $t \in U$, the geometric fibre

$$X_{\bar{t}} := X_t \times_{\kappa(t)} \overline{\kappa(t)}$$

contains a rational curve.

You can hand in your solutions via email to schreieder@math.uni-hannover.de before **Monday, June 29th, 10:00**. It is preferable if you submit solutions as a single (pdf) file, e.g. by using Latex or by converting pictures of your handwritten solutions into a single (pdf) file.