Birational Geometry

Sheet 09

Exercise 1. *Let X be a smooth projective variety and let $D \subset X$ be a smooth hypersurface. Assume that there is an integral curve $C \subset X$ with*

$$K_X \cdot C = 0 \quad \text{and} \quad D \cdot C < 0.$$

Show that X contains a rational curve.

Exercise 2. *Let X be a smooth projective variety over an algebraically closed field. Let $R \subset \overline{\mathrm{NE}}(X)$ be an extremal K_X-negative ray. Assume that the contraction $cont_R : X \to Y$ of R exists and that it is of fibre type, i.e. $\dim X > \dim Y$. Show that X is uniruled.*

Exercise 3. *Let X be a smooth projective surface over an algebraically closed field and let $C \subset X$ be a closed scheme of pure dimension one. Define $p_a(C) := 1 - \chi(C, \mathcal{O}_C)$. Use the Riemann–Roch formula on X to deduce that*

$$2p_a(C) - 2 = (K_X + C) \cdot C.$$

Recall: *The RR formula on X says that for a line bundle L on X,*

$$\chi(X, L) = \frac{1}{2} L \cdot (L - K_X) - \chi(X, \mathcal{O}_X)$$

Exercise 4. *Let C be an integral projective curve over an algebraically closed field k. Show that $p_a(C) = 0$ if and only if $C \cong \mathbb{P}^1$.*

Hint: *Let $\tau : C' \to C$ be the normalization and consider the short exact sequence*

$$0 \to \mathcal{O}_C \to \tau_* \mathcal{O}_{C'} \to \delta \to 0$$

where δ is a skyscraper sheaf whose support coincides with the singular locus of C.

You can hand in your solutions via email to schreieder@math.uni-hannover.de before **Monday, July 6th, 10:00**. It is preferable if you submit solutions as a single (pdf) file, e.g. by using Latex or by converting pictures of your handwritten solutions into a single (pdf) file.