Birational Geometry

Sheet 10

Exercise 1. *Let k be a field and let $X \subset \mathbb{A}_k^{n+1}$ be the affine cone over a smooth projective hypersurface $Z \subset \mathbb{P}^n$ of degree d. Show that K_X is $\mathbb{Q}$-Cartier and determine for which values of d and n X has terminal singularities.*

Exercise 2. *Let X be a smooth projective surface over an algebraically closed field. Let c be a positive constant. Show that there is a divisor E over X with $a(E, X) > c$.*

Exercise 3. *Let k be an algebraically closed field of characteristic zero. Let X be a quasi-projective variety over k. Assume that K_X is $\mathbb{Q}$-factorial and let $\tau : Y \to X$ be a resolution such that the discrepancy $a(E, X)$ of any τ-exceptional divisor $E \subset Y$ is positive. Show that then for any other resolution of singularities $\tau' : Y' \to X$ the discrepancy $a(E', X)$ of any τ'-exceptional divisor $E' \subset Y'$ is positive as well.*

Remark: *Note that this implies in particular that smooth varieties are terminal.*

Exercise 4. *Let X be a $\mathbb{Q}$-factorial normal projective variety over an algebraically closed field k. Let R be an extremal K_X-negative ray of $\overline{\mathrm{NE}}(X)$. Assume that the contraction $f := cont_R : X \to Z$ exists and assume that it is divisorial.*

(a) Assume that X is terminal and K_Z is $\mathbb{Q}$-Cartier (e.g. assume that Z is $\mathbb{Q}$-factorial). Show that under these assumptions Z is terminal.

(b) Show that $f^* : \operatorname{Pic} Z \to \operatorname{Pic} X$ is injective.

(c) Let $R = [C] \cdot \mathbb{R}_{\geq 0}$. Assume that there is an exact sequence

$$0 \longrightarrow \operatorname{Pic} Z \xrightarrow{f^*} \operatorname{Pic} X \xrightarrow{L \mapsto L \cdot C} \mathbb{Z}.$$

Conclude under this assumption that Z is $\mathbb{Q}$-factorial.

(d) Show that the assumption in (c) is satisfied if the following holds: Let $L \in \operatorname{Pic} X$ be a line bundle with $L \cdot C = 0$ and let $M = f^*\mathcal{O}_Z(1)$, where $\mathcal{O}_Z(1)$ is ample on Z. Then for $m \gg 0$:

- *the line bundle $mM + L$ is base point free;*
- *any curve on X that intersect $mM + L$ trivially lies in R.*

  Remark: *Note that this last condition is essentially a condition on the shape of $\overline{\mathrm{NE}}(X)$ locally around R. In particular, it is not hard to see that it holds if Mori's cone theorem holds for $\overline{\mathrm{NE}}(X)$, which we already know if X is smooth.*

You can hand in your solutions via email to schreieder@math.uni-hannover.de before **Monday, July 13th, 10:00**. It is preferable if you submit solutions as a single (pdf) file, e.g. by using Latex or by converting pictures of your handwritten solutions into a single (pdf) file.