ARBEITSSEMINAR TROPICAL GEOMETRY

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The aim of the seminar is to give an overview of (some parts of) tropical geometry. We will assume no prior knowledge about the subject; since we will view the area from an algebro-geometric perspective, knowledge in algebraic geometry will be assumed in most talks (usually roughly equivalent to a two semester course, sometimes some more specialized knowledge).

The talks will be given by participants. To aid the choice, there is a rough estimate of the accessibility of the literature: [T] indicates that the material is written up in a textbook or detailed survey, [R] stands for research literature, and [T/R] either some parts of each, or it depends on the choice of the material covered by the speaker. In what detail proofs are discussed is left to the speaker; but given that we aim for an overview, it usually will not be possible to cover all of the details. Also, most talks could be cut or extended to two talks if people are more interested in one subject than others.

Program:

(1) **Introduction .** In the first meeting we will discuss the plan and distribute the talks.

(2) **Valuations (Balkan, 21.4.).** [T] Recall the definitions related to a (real) valuation and discuss some properties, including: a valuation ring is Noetherian iff it is discretely valued. An algebraically closed valued field is not discretely valued. Discuss how to extend valuations on algebraic extensions and the role completeness plays here. Discuss that the completion of the algebraic closure of a valued field is algebraically closed. Discuss some examples, in particular the different versions of power series and their properties (Laurent, Puisseux, etc.).

    This is standard material, most of it is covered on the stacks project (some of the mentioned properties are stacks 10.50.18 and stacks 15.108.6). See [Temb, §1] for lecture notes with a view towards Berkovich analytification; some material is also covered in [MS15, §2.1], which will be used in the next talk.

(3) **Tropical algebra and tropical hypersurfaces (Lange, 28.4.).** [T] Discuss some basic aspects of embedded tropicalization, including: Introduce the tropical semiring and tropical hypersurfaces/varieties. Define the tropicalization of a closed subset of the torus over a valued field. Discuss the Fundamental and Structure Theorems. Explain all these concepts explicitly for curves and give some examples.

    This is covered in [MS15, §3.1 – §3.3]. If you prefer, you can restrict the formulations of the two theorems to hypersurfaces [MS15, 3.1.3 and 3.1.6]. See also [MS15, §§1.1 and 1.3] for an informal introduction.

(4) **Severi varieties I (Paulsen, 5.5.).** [T/R]

Set up the enumerative problem of [Mik05]: counting the number of curves with a fixed number of nodes through points in general position (that is, calculating the degree of the Severi varieties). Describe the two parts of the tropical proof: a tropical count, which is essentially combinatorial; and a correspondence theorem, relating the tropical count to the algebro-geometric one.

The general approach and the combinatorial argument are described in a couple of surveys; see, e.g., [MS15, §1.7] and [Rau17] for informal introductions, and [BIMS15, §4] for some more details. You should explain the combinatorial argument using floor decomposed curves, as introduced in [BM09] and described in the above surveys. The proof of the correspondence theorem is more involved; if you decide to present the basic idea, besides [Mik05], see [Shu05] for an algebro-geometric approach.

(5) **Toric varieties and tropical degenerations (Pavic, 12.5.).** [T]

Recall the parts of the dictionary toric varieties $\leftrightarrow$ fans, and polarized toric varieties $\leftrightarrow$ polytopes that you need. Describe how the tropicalization of a hypersurface in a toric variety gives rise to a degeneration of the pair. Discuss this for curves in an example.

Toric geometry is well-covered, the standard reference is [Ful93]; alternatively, there is a description in [MS15, §6.1] and one option would be to also describe the connection to tropicalization explained in [MS15, §6.2]. For the second part, follow [NO20, §3.6] and the references there, since this talk sets up the next one. See [MS15, §6.6] and the references there for a slightly different perspective. Another option here would be to explain the relation to the correspondence theorem in the previous talk described in [Shu05] (see also [Tyo12]).

(6) **Stable rationality (Schreieder, 19.5.).** [R]

Discuss the approach of [NO20] to deduce new cases of stable irrationality of a hypersurface in the toric variety $\mathbb{P}^n$ from combinatorics of its Newton polytope plus known rationality results. Discuss at least one of the hypersurfaces treated in [NO20] in some detail.

Apart from [NO20], ask Stefan about the first part.

(7) **Abstract and parametrized tropical curves (Torelli, 2.6.).** [T] Recall some properties of nodal models of curves, and the different reduction theorems (stable/semistable/nodal). Define the tropicalization of an abstract marked curve and some of its properties. Discuss how the tropical curve depends on the model. Explain how abstract and embedded tropical curves can be related via parametrized curves: Define a parametrized curve in a toric variety. Define its tropicalization and discuss some of its properties, including: show that the balancing condition corresponds to the fact that zeroes and poles of a rational function sum to zero, and that the degree of the tropicalization is dual to the polygon of the toric surface. Describe the tropical moduli space corresponding to parametrized tropical curves.

The different reduction theorems for curves are well covered, e.g. in [HM98, §3.C]; for all the details see the stacks project (§§53.19 + 20, 50, and 107.24). For the remaining part see [Tyo12, §2], [CHT20, §§3.1 and 4.2] and the references there.

(8) **Severi varieties II (Christ, 9.6.)** [R]

Discuss the general approach to proving that Severi varieties are irreducible. Describe the tropicalization of a one-parameter family of parametrized tropical curves. Discuss the induced map to the tropical moduli space and its properties. Describe the combinatorial argument that allows one to prove the degeneration part in the proof of the irreducibility of Severi varieties.

This is covered in [CHT20, §§3.1, 4.3, and 5], but ask Karl if there are any issues.

(9) **Tropicalization of line bundles (Sertöz, 16.6.)** [T]

Define divisors and linear equivalence on finite graphs/abstract tropical curves, via chip-firing/rational equivalence. Define the rank of a divisor and give some examples. State the tropical Riemann Roch Theorem. Describe the tropicalization of a divisor and of a line bundle on curves. Discuss Baker's specialization lemma. Explain that the dimension of the tropical linear system is not the same as the rank and discuss a bit the issue of liftability of tropical sections.

Most of the topics are covered in [BJ16]. A good description of liftability of sections is pretty much open; in addition to [BJ16, §10] and the references there, you can have a look at [ABBR15] or [MUW21] for additional discussions on liftability.

(10) **Applications of the specialization to tropical linear series (Mezzedimi, 30.6.)** [T/R]

Choose some applications of the circle of ideas discussed in the previous talk. A couple of them are described in [BJ16], and for example [JR21] or [FJP20] give more recent applications.

(11) **Berkovich analytification (Valloni, 7.7.)**[T/R] Introduce the Berkovich analytification of a variety (it is probably enough to do so as a topological space). Describe the points of the analytification in terms of valued field extensions. Describe some properties (for example: topological properties; local charts; some GAGA principles). Describe the Berkovich projective line in terms of its points and its topology.

There are a couple of surveys, e.g. the lecture notes [Tema].

(12) **Tropicalizations as Skeletons (Floccari, 14.7.).** [T/R]

Describe how any model of a curve gives rise to a skeleton of its Berkovich analytification. Explain that this can be identified with the tropicalization of the curve, and describe the reduction and tropicalization maps. Generalize this construction to higher dimensional degenerations.

For the first part you can find a detailed account in [BPR13]. For the second part, [Nic16] gives a survey.

(13) **Moduli of tropical curves (Ma, 21.7.).** [R] Recall $\overline{M}_{g,n}$ and its stratification by dual graphs; explain that $M_{g,n} \to \overline{M}_{g,n}$ is a toroidal embedding and how we can use this to define a tropicalization. Describe the moduli space of abstract tropical curves. Explain that these constructions give isomorphic spaces – the tropicalization of the moduli space is the tropical moduli space.

See [ACP15] and the references there.

References

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