Raymond Cheng and Stefan Schreieder Winter term 2024/25

Working Seminar: Rational curves

Thursday 14:15-15:45 in F 107

This seminar discusses the theory of rational curves on algebraic varieties. The program is provisional and open to discussions. The talks will be distributed during the first meeting on Oct 17. Please contact schreieder@math.uni-hannover.de if you cannot attend the first meeting but would like to give a talk.

1. **Introduction and distribution of talks** (Schreieder)

   Date: 17.10.2024

   Reading: [AK03, §1],

2. **Deforming parameterized curves and bend-and-break** (Friesen)

   Date: 24.10.2024

   Reading: [AK03, §2], [Deb01, §§2.1–3.4]

   Contents: Define the morphism scheme $\mathrm{Mor}(X, Y)$, and discuss its local structure via deformation theory of maps: [Deb10, §1.3–1.6]. Apply this to prove Mori's bend-and-break, as stated in [Deb10, Proposition 1.14]. Use this to produce rational curves in, for example, Fano varieties: see [Deb10, Theorems 1.16 and 1.18].

3. **Free and very free curves** (Pietig)

   Date: 7.11.2024

   Reading: [AK03, §§3–4 and peek at §7], [Deb01, §§4.1–4.3]

   Content: Define the notions of r-free curves as in [Deb10, Definition 2.25, 2.10] and give examples. State [Deb10, Proposition 2.27], and explain the proof for $s = 1$ as in [Deb10, Proposition 2.12]. Define the notions of separable uniruledness and separable rational connectednes, and explain how they are related to existence of free and very free curves, respectively: see [Deb10, 2.13–2.17, 2.28–2.31, and 2.34].

4. **Smoothing trees, combs, and chains** (Lahni)

   Date: 14.11.2024

   Reading: [AK03, §§5 and 7], [Deb01, §§4.5–4.7]

   Content: Explain how to smooth trees and combs of rational curves: [Deb10, §§2.7.1–2.7.3]. Apply this to show that, over a field of characteristic 0, rational chain connectedness is equivalent to (separable) rational connectedness: [Deb10, Theorem 2.49].

5. **Smoothing combs as subschemes and constructing sections** (Fong)

   Date: 28.11.2024

   Reading: [AK03, §6], [GHS03, dJS03], [Deb03]

   Content: Comment on the local structure of Hilbert schemes: [Deb10, 1.7]. Explain the embedded smoothing technique, as in [Deb10, Theorem 2.43], and apply this to prove that separably rationally connected fibrations over curves have sections: see [Deb10, §3.2].

6. **Lifting algebraic to deformation equivalence, I** (Mattei)

   Date: 5.12.2024

   Reading: [KT23, §§0–2]

   Content: State and explain the statement of [KT23, Theorem 1]. Construct examples to show that this is nontrivial: for example, when components are contracted; when a component of $\pi\colon C \to X$ has degree > 1 onto its image; preserving nodal families and the issue with simply running semistable reduction. Sketch the strategy by looking at the proof of [KT23, Proposition 34], which is the essential case. Explain the glueing step, the content of [KT23, Theorem 29], and defer modification of the auxiliary curve.

7. **Lifting algebraic to deformation equivalence, II** (Suzuki)

   Date: 12.12.2024

   Reading: [KT23]

   Content: Explain the modification the auxiliary curve step: essentially [KT23, §2 and Theorem 31]. Try to relate the main results of §2 with the $\dim X = 1$ case of the main Theorem. If possible, try to summarize the results of §2 in terms of the global structure of the (generalized) Hurwitz spaces. Explain [KT23, Theorem 31].

8. **Geometric Manin conjecture** (Cheng)

   Date: 19.12.2024

   Reading:

   Content:

9. **Stable curves and stable maps** (Gräfnitz)

   Date: 16.1.2025

   Reading: [Cav16] Lecture 2; [Gro11] §2.1.1, §4.1; [BCM20] §3, §4; [EH16] §8.5

   Content: Introduce the moduli spaces of stable curves and stable maps. Sketch the intersection theory on these moduli spaces and the definition of Gromov-Witten invariants. Prove the Kontsevich-Manin recursive formula for plane rational curves through general points. If time permits, talk about degeneration formulas for stable maps and tropical correspondence.

10. **Counting (real) rational curves** (Viergever)

    Date: 23.1.2025

    Reading:

    Content:

References

[AK03] C. Araujo and J. Kollár, *Rational Curves on Varieties*, Higher dimensional varieties and rational points (Budapest, 2001), Bolyai Soc. Math. Stud., vol. 12, Springer, Berlin, 2003, pp. 13–68.

[BCM20] L. Batistella, F. Carocci, C. Manolache, *Virtual Classes for the Working Mathematician*, SIGMA 16 (2020), 026, 38 pages, 2020.

[Cav16] R. Cavalieri, *Moduli Spaces of Pointed Rational Curves*, Lecture Notes, Graduate Student School Combinatorial Algebraic Geometry program at the Fields Institute, July 18-22, 2016.

[Deb01] O. Debarre, *Higher-dimensional algebraic geometry*, Universitext, Springer, New York, 2001.

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[Deb10] O. Debarre, *Rational curves on algebraic varieties*, lecture notes available online.

[dJS03] A. J. de Jong and J. M. Starr, Every rationally connected variety over the function field of a curve has a rational point, Amer. J. Math. **125** (2003), no. 3, 567–580.

[EH16] D. Eisenbud, J. Harris, *3264 & All That - Intersection Theory in Algebraic Geometry*, Cambridge University Press, 2016.

[GHS03] T. Graber, J. D. Harris and J. M. Starr, Families of rationally connected varieties, J. Amer. Math. Soc. **16** (2003), no. 1, 57–67.

[Gro11] M. Gross, *Mirror Symmetry and Tropical Geometry*, Springer, Regional Conference Series in Mathematics 144, ISBN: 978-0-8218-5232-3, 2011.

[Kol96] J. Kollár, *Rational curves on algebraic varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, 32, Springer, Berlin, 1996.

[KT23] J. Kollár and Z. Tian, *Stable maps of curves and algebraic equivalence of 1-cycles*, arXiv:2302.07069, to appear in Duke Math. J.