Prof. Dr. Stefan Schreieder Wintersemester 2020/21

Unramified cohomology

Wednesday 12-2 pm, F442

The program and dates are provisional and open for discussion. Please contact schreieder@... if you are willing to give a talk.

1. **Introduction (Schreieder, Oct 21)**

2. **Review of étale cohomology (Overkamp, Nov 4)**

   Recall briefly the definition of étale cohomology and discuss briefly the following topics: Kummer sequence and Hilbert theorem 90 for function fields; Cohomology with support and Gysin sequence, Comparison theorem to singular cohomology (for complex varieties). A good source is [13]; more details can be found in [12].

3. **Brauer groups (Mezzedimi + Torelli, Nov 11 + 18)**

   Discuss some basic facts about Brauer groups, including the Brauer group of a field and the (cohomological) Brauer group of a scheme, or possibly of a complex analytic space. A reference for Brauer groups of varieties from a complex geometric point of view is [1]. The algebraic story of Brauer groups of a field is for instance treated in the book [10], or in Grothendiecks original paper(s) [11].

   You should also introduce residue maps for Brauer groups and give a geometric interpretation for them. Use this to define unramified Brauer groups and explain in particular the idea that the Brauer group of a smooth projective variety over an algebraically closed field can be computed by looking at unramified elements of the Brauer group of its function field.

4. **Definition and basic properties of unramified cohomolgy (Christ, Nov 25)**

   You may use the surveys [14] or [5] to introduce unramified cohomology of a finitely generated field extension K/k. Prove some basic properties: it is a stable invariant (does not change if we replace K by $K(t)$), it has some functoriality properties, etc. Prove also that $H^1(k(X)/k, \mu_m^{\otimes n}) \cong H^1(X, \mu_m^{\otimes n})$ whenever X is smooth projective over k (an algebraically closed field in which m is invertible).

5. **Merkurjev's pairing and the degeneration method (Valloni, Dec 2)**

   Recall briefly the definition of Chow groups of a scheme of finite type over a field k (see e.g. Fulton's book). Introduce Merkurjev's pairing [14, §5] (but skip its generalizaton to snc schemes in [14, §6]).

   Introduce the notion of decompositions of the diagonal and prove some basic facts about it (e.g. stably rational varieties admit a decomposition of the diagonal), see e.g. [4], [14, §7] and the references therein. Explain the specialization method, see e.g. [14, §8] and the references given there.

6. **Computation of unramified cohomology and applications (Pavic, Dec 16)**

   Discuss some of the examples mentioned in [14, §9].

7. **Connection to the IHC (Paulsen, Jan 6)**

   Follow the approach in [15, §6] to prove the following theorem of Colliot-Thélène and Voisin [8], which says that for a smooth complex projective variety X, the failure of the integral Hodge conjecture for codimension two cycles is detected by

   $$\operatorname{coker}(H^3_{nr}(X,\mathbb{Q}) \longrightarrow H^3_{nr}(X,\mathbb{Q}/\mathbb{Z})).$$

   Discuss also the generalization given in [15] and in particular the application [15, Theorem 1.2].

8. **Applications of the CTV-theorem (Schreieder, Jan 20)**

   Discuss some applications of [8], including the main result of [5], where Colliot-Thélène proves a generalization of a beauiful result of Benoist Ottem [2], asserting that the integral Hodge conjecture fails for the product of an Enriques surface with a very general elliptic curve.

9. **Theorem of Bloch-Esnault (Valloni, Jan 27)**

   Discuss the main result of [3] and possibly the application in [9].

Literatur

[1] A. Beauville, *The Lüroth problem*, `http://math1.unice.fr/~beauvill/conf/Cime.pdf`.

[2] O. Benoist and J.C. Ottem, *Failure of the integral Hodge conjecture for threefolds of Kodaira dimension zero*, Commentarii Mathematici Helvetici **95** (2020), 27-35.

[3] S. Bloch and H. Esnault, *The coniveau filtration and non-divisibility for algebraic cycles*, Math. Ann. **304** (1996), 303–314.

[4] S. Bloch and V. Srinivas, *Remarks on Correspondences and Algebraic Cycles*, American J. Math. **105** (1983), 1235–1253.

[5] J.-L. Colliot-Thélène, *Birational invariants, purity and the Gersten conjecture*, K-theory and algebraic geometry: connections with quadratic forms and divison algebras (Santa Barbara, CA, 1992), 1–64, Proc. Sympos. Pure Math. **58**, AMA, Providence,RI, 1995.

[6] J.-L. Colliot-Thélène, *Cohomologie non ramifiée dans le produit avec une courbe elliptique*, Manuscripta Math. **160** (2019), 561–565.

[7] J.-L. Colliot-Thélène and M. Ojanguren, *Variétés unirationnelles non rationnelles: au-delà de l'exemple d'Artin et Mumford*, Invent. Math. **97** (1989), 141–158.

[8] J.-L. Colliot-Thélène and C. Voisin, *Cohomologie non ramifiée et conjecture de Hodge entière*, Duke Math. J. **161** (2012), 735–801.

[9] H.A. Diaz, *On the unramified cohomology of certain quotient varieties*, Preprint 2019, arXiv:1906.06598v1.

[10] P. Gille and T. Szamuely, *Central simple algebras and Galois cohomology*, Cambridge Studies in Advanced Mathematics **101**, 2006.

[11] A. Grothendieck, *Le groupe de Brauer I, II, III*, in: Dix exposés sur la cohomologie des schémas, Advanced Studies in Pure Mathematics vol. 3, Masson et North-Holland, 1968.

[12] J.S. Milne, *Étale cohomology*, Princeton University Press, Princeton, NJ, 1980.

[13] J.S. Milne, *Lectures on étale cohomology*, 2013, https://www.jmilne.org/math/CourseNotes/LEC.pdf

[14] S. Schreieder, *Unramified cohomology, algebraic cycles and rationality*, Preprint 2020, https://www.iag.uni-hannover.de/fileadmin/iag/homepages/schreieder/publications/survey.pdf

[15] S. Schreieder, *Algebraic cycles and refined unramified cohomology*, Preprint 2020, https://www.iag.uni-hannover.de/fileadmin/iag/homepages/schreieder/publications/refined.pdf