# ZEROS OF HOLOMORPHIC ONE-FORMS AND TOPOLOGY OF KÄHLER MANIFOLDS

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APPENDIX WRITTEN JOINTLY WITH HSUEH-YUNG LIN

Abstract. A conjecture of Kotschick predicts that a compact Kähler manifold $X$ fibres smoothly over the circle if and only if it admits a holomorphic one-form without zeros. In this paper we develop an approach to this conjecture and verify it in dimension two. In a joint paper with Hao [HS19], we use our approach to prove Kotschick's conjecture for smooth projective threefolds.

## 1. Introduction

This paper is motivated by the following conjecture of Kotschick [Ko13].

**Conjecture 1.1.** *For a compact Kähler manifold $X$, the following are equivalent.*

*(A) $X$ admits a holomorphic one-form without zeros;*

*(B) $X$ admits a real closed 1-form without zeros; or, by Tischler's theorem [Ti70] equivalently, the underlying differentiable manifold is a $C^\infty$-fibre bundle over the circle.*

The implication (A) ⇒ (B) is clear; the possibility of the converse implication (B) ⇒ (A) is asked in [Ko13]. Condition (B) is equivalent to asking that the smooth manifold that underlies $X$ is a quotient $M \times [0,1]/\sim$, where $M$ is a closed real manifold of odd dimension and $M \times 0$ is identified with $M \times 1$ via some diffeomorphism of $M$. Kotschick's conjecture relates this purely topological condition with the complex geometric condition that $X$ has a holomorphic one-form without zeros.

The purpose of this paper is to related Kotschick's conjecture to the following condition

(C) there is a holomorphic one-form $\omega \in H^0(X, \Omega_X^1)$, such that for any finite étale cover $\tau : X' \to X$, the sequence

$$H^{i-1}(X', \mathbb{C}) \xrightarrow{\wedge\omega'} H^i(X', \mathbb{C}) \xrightarrow{\wedge\omega'} H^{i+1}(X', \mathbb{C}),$$

given by cup product with $\omega' := \tau^*\omega$, is exact for all $i$.

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This is motivated by a theorem of Green and Lazarsfeld [GL87, Proposition 3.4], who proved the implication (A) $\Rightarrow$ (C). Our first result is the following, which in view of Green and Lazarsfeld's theorem yields some positive evidence for Conjecture 1.1.

**Theorem 1.2.** *For any compact Kähler manifold $X$, we have (B) $\Rightarrow$ (C).*

By the above theorem, in order to prove Kotschick's conjecture, it would be enough to show that (C) implies (A). Compared to the original implication (B) $\Rightarrow$ (A), this has the major advantage that (C) and (A) are complex geometric conditions, while (B) is not. More precisely, it is natural to wonder whether a one-form $\omega \in H^0(X, \Omega^1_X)$ which satisfies condition (C) must be without zeros. This would have the remarkable implication that the question whether $\omega$ has zeros depends only on the de Rham class of $\omega$ and the homotopy type of $X$. We show that this is true for surfaces.

**Theorem 1.3.** *Let $X$ be a compact Kähler surface. If $\omega \in H^0(X, \Omega^1_X)$ satisfies condition (C), then it has no zeros. In particular, Conjecture 1.1 holds for compact Kähler surfaces.*

The proof of Theorem 1.3 uses classification of surfaces. In the Appendix to this paper, written jointly with Lin, we give however a more general and direct argument which does not rely on classification results, see Theorem A.1 below.

In joint work with Hao [HS19], we use the approach developed here to prove Conjecture 1.1 for smooth projective threefolds.

The following theorem proves some partial results in arbitrary dimension.

**Theorem 1.4.** *Let $X$ be a compact connected Kähler manifold with a holomorphic one-form $\omega$ such that the complex $(H^*(X, \mathbb{C}), \wedge\omega)$ given by cup product with $\omega$ is exact. Then the analytic space $Z(\omega)$ given by the zeros of $\omega \in H^0(X, \Omega^1_X)$ has the following properties.*

*(1) For any connected component $Z \subset Z(\omega)$ with $d = \dim Z$,*

$$H^d(Z, \omega_X|_Z) = 0.$$

*In particular, $\omega$ does not have any isolated zero.*

*(2) If $f : X \to A$ is a holomorphic map to a complex torus $A$ such that $\omega \in f^* H^0(A, \Omega^1_A)$, then $f(X) \subset A$ is fibred by tori.*

Ein and Lazarsfeld [EL97, Theorem 3] showed that the image of a morphism $f : X \to A$ to a complex torus $A$ is fibred by tori if $\chi(X, \omega_X) = 0$ and $\dim f(X) = \dim X$. In item (2) above we obtain the same conclusion without any assumption on $f$, but where we replace $\chi(X, \omega_X) = 0$ by the stronger condition on the exactness of $(H^*(X, \mathbb{C}), \wedge\omega)$.

Theorem 1.2 and item (2) in the above theorem imply for instance that a Kähler manifold $X$ with simple Albanese torus $\mathrm{Alb}(X)$ and with $b_1(X) > 2\dim(X)$ does not admit a $C^\infty$-fibration over the circle. Similarly, we obtain the following corollary in the projective case.

**Corollary 1.5.** *Let $X$ be a smooth complex projective variety such the manifold which underlies $X$ fibres smoothly over the circle. Then there is a surjective holomorphic morphism $f : X \to A$ to a positive-dimensional abelian variety $A$.*

The following example of Debarre, Jiang and Lahoz shows that the étale covers in condition (C) are necessary to make Theorem 1.3 true.

**Example 1.6** ([DJL17, Example 1.11])**.** *Let $C_1, C_2$ be smooth projective curves with $g(C_1) > 1$ and $g(C_2) = 1$ and automorphisms $\varphi_i \in \mathrm{Aut}(C_i)$ of order two such that $C_i/\varphi_i$ has genus one for $i = 1, 2$. Then the quotient*

$$X := (C_1 \times C_2)/(\varphi_1 \times \varphi_2)$$

*has the same rational cohomology ring as an abelian surface, and so $\wedge\omega$ is exact on cohomology for any non-zero $\omega \in H^0(X, \Omega^1_X)$. However, if $\omega$ is obtained as pullback via the map $\pi : X \to C_1/\varphi_1$, then it vanishes along the multiple fibres of $\pi$, which lie above the branch points of $C_1 \to C_1/\varphi_1$.*

**Remark 1.7.** *This paper raises the question whether condition (C) implies (A). In view of* [GL87, Proposition 3.4] *it is natural to wonder whether more generally, a holomorphic one-form $\omega \in H^0(X, \Omega^1_X)$ such that for any finite étale cover $\tau : X' \to X$*

$$H^{i-1}(X', \mathbb{C}) \xrightarrow{\wedge\tau^*\omega} H^i(X', \mathbb{C}) \xrightarrow{\wedge\tau^*\omega} H^{i+1}(X', \mathbb{C})$$

*is exact for all $i < c$ implies that $\mathrm{codim}_X(Z(\omega)) \geq c$. This goes back to* [BWY16]*, where it is asked whether equality always holds in* [BWY16, Theorem 1.1]*. However, blowing-up a point in $Z(\omega)$ easily produces counterexamples to this conjecture.*

**Why the Kähler assumption?** The Kähler assumption in Conjecture 1.1 is essential. For instance, a Hopf surface $X$ is a compact complex surface with $H^0(X, \Omega^1_X) = 0$, whose underlying differentiable manifold is diffeomorphic to $S^1 \times S^3$, and so it satisfies (B) but not (A).

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**Notation.** For a holomorphic one-form $\omega$ on a Kähler manifold $X$, we denote by $Z(\omega)$ the (possibly non-reduced) analytic space given by the zeros of $\omega$, viewed as a section of the vector bundle $\Omega^1_X$.

## 2. Proof of Theorem 1.2

Let $X$ be a smooth connected manifold. We denote by $\mathrm{Loc}(X)$ the group of local systems on $X$ whose stalks are one-dimensional $\mathbb{C}$-vector spaces. Since local systems on the interval are trivial, the choice of a base point $s \in S^1$ induces a canonical isomorphism $\mathrm{Loc}(S^1) \cong \mathbb{C}^*$. Hence, if we fix a base point $x \in X$, then for any $L \in \mathrm{Loc}(X)$, any continuous map $\gamma : S^1 \to X$ with $\gamma(s) = x$ yields a canonical element $\gamma^* L \in \mathrm{Loc}(S^1) \cong \mathbb{C}^*$, which, as one checks, depends only on the homotopy class of $\gamma$. This construction gives rise to the so called monodromy representation, which (since $X$ is connected) induces an isomorphism between $\mathrm{Loc}(X)$ and the character variety

$$\mathrm{Char}(X) := \mathrm{Hom}(\pi_1(X, x), \mathbb{C}^*) \cong H^1(X, \mathbb{C}^*).$$

If $L \in \mathrm{Loc}(X)$, then the associated complex line bundle has locally constant transition functions, hence it admits a flat connection and so the first Chern class $c_1(L)$ must be torsion. The long exact sequence associated to the short exact sequence $0 \to \mathbb{Z} \to \mathbb{C} \to \mathbb{C}^* \to 0$ of locally constant sheaves on $X$ thus shows that $\mathrm{Loc}(X)$ is isomorphic to an extension of a finite group by the connected subgroup $\mathrm{Loc}^0(X) \subset \mathrm{Loc}(X)$ which contains the trivial local system. Moreover,

$$\mathrm{Loc}^0(X) \cong \frac{H^1(X, \mathbb{C})}{H^1(X, \mathbb{Z})} \cong (\mathbb{C}^*)^{b_1(X)}.$$

coincides with the subgroup $\{L \in \mathrm{Loc}(X) \mid c_1(L) = 0\}$.

2.1. **Local systems associated to closed 1-forms and Novikov's inequality.** If $\alpha$ is a closed complex valued 1-form on $X$, then we can construct a local system $L(\alpha) \in \mathrm{Loc}^0(X)$ as follows. Consider the twisted de Rham complex $(\mathcal{A}^*_{X,\mathbb{C}}, d + \wedge\alpha)$, where $\mathcal{A}^k_{X,\mathbb{C}}$ denotes the sheaf of complex valued $C^\infty$-differential $k$-forms on $X$, and where $\wedge\alpha$ acts on a $k$-form $\beta$ via $\beta \mapsto \alpha \wedge \beta$. There is an open covering $\mathcal{U} = \{U_i\}_{i\in I}$ of $X$ such that $\alpha|_{U_i} = dg_i$ for some smooth function $g_i$ on $U_i$. For a $k$-form $\beta$ on $U_i$, we then have $(d + \wedge\alpha)(\beta) = 0$ if and only if $d(e^{g_i}\beta) = 0$. This shows that the twisted de Rham complex $(\mathcal{A}^*_{X,\mathbb{C}}, d + \wedge\alpha)$ is exact in positive degrees and it resolves a sheaf $L(\alpha)$ whose sections above $U_i$ are given by all smooth functions $f$ with $d(e^{g_i} f) = 0$, i.e. $f = e^{-g_i}c$ for some constant $c \in \mathbb{C}$. Hence, $L(\alpha) \in \mathrm{Loc}(X)$ is a local system with stalk $\mathbb{C}$ on $X$. Moreover, $c_1(L(\alpha)) = 0$ because the cocycle $(g_i - g_j) \in \check{C}^1(\mathcal{U}, (\mathcal{A}^0_X)^\times)$ maps to zero in $H^2(X, \mathbb{Z})$ and so

$$L(\alpha) \in \mathrm{Loc}^0(X), \tag{1}$$

as we want.

Since $L(\alpha)$ is resolved by the $\Gamma$-acyclic complex $(\mathcal{A}^*_{X,\mathbb{C}}, d + \wedge\alpha)$, we find that

$$H^k(X, L(\alpha)) = H^k((A^*(X,\mathbb{C}), d + \wedge\alpha)), \tag{2}$$

where $A^k(X,\mathbb{C}) = \Gamma(X, \mathcal{A}^k_{X,\mathbb{C}})$. In view of (2), we can define the Novikov Betti numbers $b_i(\alpha)$ of $\alpha$ as follows, cf. [Pa87] or [Fa04]:

$$b_k(\alpha) := \dim_{\mathbb{C}} H^k(X, L(\alpha)).$$

A closed 1-form $\alpha$ on $X$ is Morse if locally at each zero $x \in Z(\alpha)$ of $\alpha$, $\alpha = dh$ for some Morse function $h$. If $\alpha$ is Morse, its Morse index at a zero $x$ is defined as the Morse index of $h$ and we denote by $m_i(\alpha)$ the number of zeros of $\alpha$ of Morse index $i$. The Novikov inequalities then state the following, see [Pa87, Theorem 1]:

**Theorem 2.1** (Novikov's inequalities)**.** *Let $X$ be a closed manifold and let $\alpha$ be a closed 1-form on $X$. Suppose that $\alpha$ is Morse in the above sense. Then for sufficiently large $t \in \mathbb{R}$, $m_i(\alpha) \geq b_i(t\alpha)$. In particular, if $\alpha$ has no zeros, then for $t \gg 0$,*

$$H^i(X, L(t\alpha)) = 0 \quad \textit{for all } i.$$

2.2. **Local systems associated to holomorphic 1-forms.** Let now $X$ be a compact Kähler manifold. For any holomorphic 1-form $\omega$ on $X$, $\omega$ is closed and so we get a local system $L(\omega)$ as above. This induces a short exact sequence

$$0 \longrightarrow H^0(X, \Omega^1_X) \longrightarrow \mathrm{Loc}^0(X) \longrightarrow \mathrm{Pic}^0(X) \longrightarrow 0, \tag{3}$$

where $\mathrm{Loc}^0(X) \to \mathrm{Pic}^0(X)$ is given by $L \mapsto L \otimes_{\mathbb{C}} \mathcal{O}_X$.

**Lemma 2.2.** *Let $X$ be a compact Kähler manifold and let $\omega \in H^0(X, \Omega^1_X)$ be a holomorphic 1-form. Let $c \in \mathbb{Z} \cup \{\infty\}$ be maximal such that*

$$H^{i-1}(X,\mathbb{C}) \xrightarrow{\wedge\omega} H^i(X,\mathbb{C}) \xrightarrow{\wedge\omega} H^{i+1}(X,\mathbb{C})$$

*is exact for all $i < c$. Then the local system $L(\omega)$ associated to $\omega$ satisfies $H^i(X, L(\omega)) = 0$ for all $i < c$. Moreover, if $c \neq \infty$, then $H^c(X, L(\omega)) \neq 0$.*

*Proof.* The local system $L(\omega)$ is resolved by the following complex

$$(\Omega^*_X, \partial + \wedge\omega) := 0 \longrightarrow \Omega^0_X \xrightarrow{\partial+\wedge\omega} \Omega^1_X \xrightarrow{\partial+\wedge\omega} \dots \Omega^n_X{-1} \xrightarrow{\partial+\wedge\omega} \Omega^n_X \to 0.$$

To see that this complex is exact in positive degrees, one uses that locally $\omega = dh$ and so for any local holomorphic form $\beta$, we have $de^h\beta = e^h(d\beta + dh \wedge \beta)$ and so $\partial\beta + \omega \wedge \beta = 0$ if and only if $de^h\beta = 0$ and we can use the holomorphic Poincaré lemma to prove the claim. Hence,

$$H^i(X, L(\omega)) = \mathbb{H}^i(X, (\Omega^*_X, \partial + \wedge\omega)).$$

There is a spectral sequence

$$'E_1^{p,q} := H^p(X,\Omega_X^q) \Rightarrow \mathbb{H}^{p+q}(X,(\Omega_X^*,\partial+\wedge\omega)).$$

The differential $d_1 :' E_1^{p,q} \to' E_1^{p,q+1}$ is induced by $\partial+\wedge\omega$. Since $\partial$ acts trivially on $'E_1^{p,q} := H^p(X,\Omega_X^q)$, we find that $d_1=\wedge\omega$. It thus follows from [GL87, Proposition 3.7] that the above spectral sequence degenerates at the second page, i.e. $'E_2={}'E_\infty$.

Our assumption implies $'E_2^{p,q}=0$ for $p+q<c$ and so $H^i(X,L(\omega))=0$ for $i<c$. Let us now assume $c\neq\infty$. By the definition of $c$,

$$H^{c-1}(X,\mathbb{C}) \xrightarrow{\wedge\omega} H^c(X,\mathbb{C}) \xrightarrow{\wedge\omega} H^{c+1}(X,\mathbb{C})$$

is not exact. Since $\omega\in H^{1,0}(X)$ is of type $(1,0)$, the above complex respects the Hodge decomposition and so we find that there must be some $j$ such that

$$H^{j-1,c-j}(X) \xrightarrow{\wedge\omega} H^{j,c-j}(X) \xrightarrow{\wedge\omega} H^{j+1,c-j}(X)$$

is not exact. Hence $'E_2^{j,c-j}\neq 0$. Since $'E_2^{j,c-j}={}'E_\infty^{j,c-j}$, we get $H^c(X,L(\omega))\neq 0$, as we want. This concludes the lemma. □

2.3. **Proof of Theorem 1.2.** Let $X$ be a compact Kähler manifold which admits a real closed one-form $\alpha$ without zeros, i.e. condition (B) in Conjecture 1.1 holds. Since the pullback of $\alpha$ via a finite étale cover is again a real closed one-form without zeros, in order to prove (C), it suffices to show that $X$ carries a holomoprhic one-form $\omega$ such that $\wedge\omega$ is exact on cohomology. For this, we may without loss of generality assume that $X$ is connected.

Since $\alpha$ has no zero on $X$, Theorem 2.1 implies that there is a local system $L\in \mathrm{Loc}^0(X)$ that has no cohomology. By the generic vanishing theorems [GL87, GL91, Ar92, Si93], the locus of those local systems that have some cohomology are subtori, translated by torsion points, see [Wa16, Theorem 1.3]. It follows that for general $\omega\in H^0(X,\Omega_X^1)$, the local system $L(\omega)$ has no cohomology. It thus follows from Lemma 2.2 that

$$H^{i-1}(X,\mathbb{C}) \xrightarrow{\wedge\omega} H^i(X,\mathbb{C}) \xrightarrow{\wedge\omega} H^{i+1}(X,\mathbb{C})$$

is exact for all $i$, as we want. This finishes the proof of Theorem 1.2.

**Remark 2.3.** *Botong Wang points out that one can bypass the use of Theorem 2.1 in the above argument by showing directly that if $X$ is a $C^\infty$-fibre bundle over the circle, then the pullback of a general local system on the circle has no cohomology on $X$.*

**Remark 2.4.** *Let $X$ be a compact connected Kähler manifold. As we have used above, the results in* [GL87] *imply that $(H^*(X,\mathbb{C}),\wedge\omega)$ is exact if and only if $L(\omega)$ has no cohomology. The locus of such local systems is well understood by generic vanishing theory. In particular,* [Wa16, Theorem 1.3] *implies that the locus of those holomorphic* *one-forms $\omega \in H^0(X, \Omega^1_X)$ for which $(H^*(X,\mathbb{C}), \wedge\omega)$ is not exact is a finite union of linear subspaces of the form $f_i^* H^0(T_i, \Omega^1_{T_i})$, where $f_i : X \to T_i$ is a finite collection of holomorphic maps to complex tori $T_i$. As a special case we see that if there is one holomorphic one-form $\omega$ on $X$ which makes $(H^*(X,\mathbb{C}), \wedge\omega)$ exact, then this holds for all forms in a non-empty Zariski open subset of $H^0(X, \Omega^1_X)$.*

## 3. The case of surfaces

*Proof of Theorem 1.3.* Let $X$ be a compact Kähler surface with a one-form $\omega \in H^0(X, \Omega^1_X)$ such that for any finite étale cover $\tau : X' \to X$,

$$H^{i-1}(X',\mathbb{C}) \xrightarrow{\wedge\omega'} H^i(X',\mathbb{C}) \xrightarrow{\wedge\omega'} H^{i+1}(X',\mathbb{C}) \tag{4}$$

is exact for all $i$, where $\omega' := \tau^*\omega$. This implies $\chi(X, \Omega^p_X) = 0$ for all $p$ and so $c_2(X) = 0$.

Replacing $X$ by its connected components, we may without loss of generality assume that $X$ is connected. The classification of surfaces (see [BHPV04, Chapter VI.1]) thus shows that only the following cases occur.

**Case 1.** $X$ is birational to a ruled surface over a curve $C$ of positive genus.

**Case 2.** $X$ is a minimal bi-elliptic surface or a complex 2-torus.

**Case 3.** $X$ is a minimal properly elliptic surface.

In Case 1, exactness of (4) implies that $X$ is birational to a ruled surface over an elliptic curve $C$. This implies $b_1(X) = 2$. Since $e(X) = 0$, we conclude $b_2(X) = 2$ and so $X$ is a minimal ruled surface over an elliptic curve. In particular, since $\omega$ is nonzero, it must be a holomorphic one-form without zeros.

In Case 2, any nontrivial holomorphic one-form on $X$ has no zeros and so we are done because exactness of (4) implies $\omega \neq 0$, as before.

In Case 3, the condition $c_2(X) = 0$ implies by [BHPV04, Proposition III.11.4] that $X$ admits a fibration $\pi : X \to C$ to a curve $C$ such that the reduction of any fibre of $\pi$ is isomorphic to a smooth elliptic curve, but where multiple fibres are allowed. Let $F$ be a general fibre of $\pi : X \to C$. Suppose for the moment that the one-form $\omega$ restricts to a nonzero form on $F$. In particular, the Albanese map $a : X \to \mathrm{Alb}(X)$ does not contract $F$ and the reduction of any fibre of $a$ is isomorphic to $F$. Moreover, the restriction of $\omega$ to $F$ does not depend on the fibre and so it is nonzero everywhere. That is, $\omega$ has no zeros.

It remains to deal with the case where $\omega$ restricts to zero on the fibres of $\pi : X \to C$. In this case, $\omega = \pi^*\alpha$ for a one-form $\alpha$ on $C$. Since cup product with $\omega$ is exact, $C$ must be an elliptic curve. If $\pi$ is smooth, then $\omega$ has no zeros. Otherwise, $\omega$ vanishes along the multiple fibres of $\pi$. We may thus assume that $\pi$ has at least one multiple fibre.

The multiple fibres of $\pi$ give rise to a orbifold structure on $C$. Since $C$ is an elliptic curve, this orbifold is good and so there is a finite orbifold covering $C' \to C$ such that the orbifold structure on $C'$ is trivial, see e.g. [CHK00, Corollary 2.29]. Let $X'$ be the normalization of the base change $X \times_C C'$. Then, $X'$ is a smooth surface, $X' \to X$ is étale and $X' \to C'$ is an elliptic surface without singular fibres, see e.g. [BHPV04, Proposition III.9.1]. Since $\tau : X' \to X$ is finite étale, $(H^*(X', \mathbb{C}), \wedge\omega')$ is exact for $\omega' := \tau^*\omega$ by assumptions. On the other hand, since $\pi$ has singular fibres, $C' \to C$ is a branched covering with nontrivial branch locus and so $C'$ is a curve of genus $\geq 2$. This is a contradiction, because $\omega'$ is a pullback of a one-form from $C'$. This finishes the proof of Theorem 1.3. □

**Corollary 3.1.** *Let $X$ be a compact connected Kähler surface with a holomorphic one-form $\omega$ such that $(H^*(X, \mathbb{C}), \wedge\omega)$ is exact. Then $\omega$ has no zeros and $(X, \omega)$ is given by one of the following:*

*(a) $X$ is a minimal ruled surface over an elliptic curve;*
*(b) $X$ is a complex 2-torus;*
*(c) $X$ is a minimal elliptic surface $f : X \to C$ such that one of the following holds:*
  *(i) $f$ is smooth, $C$ is an elliptic curve and $\omega \in f^*H^0(C, \Omega^1_C)$;*
  *(ii) $f$ is quasi-smooth, i.e. all singular fibres are multiple fibres, and the restriction of $\omega$ to the reduction of any fibre of $f$ is nonzero.*

*Proof.* The classification into types (a), (b) and (c) follows directly from the proof of Theorem 1.3, where we note that bi-elliptic surfaces fall in the class (ci). The fact that $\omega$ has no zeros follows from this classification. □

**Corollary 3.2.** *In the notation of Corollary 3.1, assume that $X$ is projective. Then,*

*(d) $X$ admits a smooth morphism to a positive-dimensional abelian variety;*
*(e) if $\kappa(X) \geq 0$, then there is a finite étale cover $\tau : X' \to X$ which splits into a product $X' = A' \times S'$, where $A'$ is a positive-dimensional abelian variety and $S'$ is smooth projective.*

*Proof.* Note that item (d) is clear in cases (a), (b) and (ci) of Corollary 3.1. It remains to deal with case (cii). In this case, since $X$ and hence $\mathrm{Alb}(X)$ are projective, $\mathrm{Alb}(X)$ is isogeneous to $E \times \mathrm{Jac}(C)$, where $E$ is an elliptic curve which is isogeneous to the reduction of any fibre of $f$. It follows that there is a morphism $g : X \to E$ which restricts to an isogeny on the reduction of each fibre of $f : X \to C$. Since $\omega$ restricts non-trivially to the reduction of any fibre of $f$, the morphism $g : X \to E$ must be smooth, as we want.

It clearly suffices to prove item (e) in the case (c) of Corollary 3.1. In this case, there is a finite étale cover $X' \to X$, such that $\mathrm{Alb}(X') \cong E \times \mathrm{Jac}(C')$ for a smooth projective curve $C'$ which maps finitely to $C$. Moreover, the Albanese map identifies $X'$ to the product $E \times C'$, as we want. This concludes the corollary. □

## 4. Proof of Theorem 1.4

4.1. **Preliminaries.** We will use the following lemma.

**Lemma 4.1.** *Let $K^*$ be a bounded complex of sheaves on a manifold $X$. Let $Z, Z' \subset X$ be closed subsets with $Z \cap Z' = \emptyset$, such that*

$$\operatorname{supp} \mathcal{H}^i(K^*) \subset Z \cup Z'$$

*for all $i$. Then the differentials $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$ in the spectral sequence*

$$E_2^{p,q} = H^p(X, \mathcal{H}^q(K^*)) \Rightarrow \mathbb{H}^{p+q}(X, K^*)$$

*respect the natural decompositions*

$$E_2^{p,q} = H^p(Z, \mathcal{H}^q(K^*)|_Z) \oplus H^p(Z', \mathcal{H}^q(K^*)|_{Z'}).$$

*Proof.* Let $i : Z \to X$ and $j : Z' \to X$ be the inclusions. Then the natural map of complexes

$$K^* \twoheadrightarrow i_* i^{-1} K^* \oplus j_* j^{-1} K^*$$

is a quasi-isomorphism. This proves the lemma, because the spectral sequence depends only on the class of $K^*$ in the derived category of sheaves on $X$. □

4.2. **Item (1) of Theorem 1.4.** Let $X$ be a compact connected Kähler manifold and let $\omega$ be a holomorphic one-form on $X$ with associated local system $L(\omega)$. Recall the isomorphism

$$H^k(X, L(\omega)) \cong \mathbb{H}^k(X, (\Omega_X^*, \omega \wedge -)).$$

The above hypercohomology is computed by a spectral sequence with $E_2$-page

$$E_2^{p,q} := H^p(X, \mathcal{H}^q(K^*)) \Rightarrow H^{p+q}(X, L(\omega)), \tag{5}$$

where $K^* := (\Omega_X^*, \omega \wedge -)$ and $\mathcal{H}^q(K^*)$ denotes the $q$-th cohomology sheaf of that complex. In particular, $\mathcal{H}^q(K^*) = 0$ if $\omega \wedge -$ is exact on holomorphic $q$-forms and the latter holds if $\omega$ has no zeros. More precisely, this shows that $\mathcal{H}^q(K^*)$ are sheaves that are supported on the zero locus $Z(\omega)$ of $\omega$.

**Lemma 4.2.** *We have $\mathcal{H}^n(K^*) \cong \Omega_X^n|_Z$.*

*Proof.* Locally $\omega = \sum_{i=1}^n f_i dx_i$. We are interested in the cokernel of

$$\Omega_X^{n-1} \to \Omega_X^n, \quad \alpha \mapsto \sum_{i=1}^n f_i dx_i \wedge \alpha.$$

The image of the above map is clearly spanned by $f_i dx_1 \wedge \cdots \wedge dx_n$ with $i = 1, \ldots, n$. Hence, $\mathcal{H}^n(K^*)$ is the quotient of $\Omega_X^n$ by the subsheaf $I_Z \otimes_{\mathcal{O}_X} \Omega_X^n$, where $I_Z$ denotes the ideal sheaf of $Z$. Hence,

$$\mathcal{H}^n(K^*) \cong \Omega_X^n \otimes_{\mathcal{O}_X} \mathcal{O}_Z = \Omega_X^n|_Z.$$

This proves the lemma. □

*Proof of item (1) in Theorem 1.4.* Let $Z \subset Z(\omega)$ be a connected component of the zero locus of $\omega$. Then we can write $Z(\omega) = Z \cup Z'$, where $Z$ and $Z'$ are disjoint closed subsets of $X$.

Consider the spectral sequence (5). By Lemma 4.2, we have

$$H^d(X, \Omega_X^n|_Z) \hookrightarrow E_2^{d,n}.$$

Using Lemma 4.1, one easily checks that this term survives on the infinity page and we get

$$H^d(X, \Omega_X^n|_Z) \hookrightarrow E_\infty^{d,n}.$$

By Lemma 2.2, exactness of $(H^*(X, \mathbb{C}), \wedge\omega)$ implies $H^i(X, L(\omega)) = 0$ for all $i$. Hence, $E_\infty^{d,n} = 0$, and so $H^d(Z, \Omega_X^n|_Z) = 0$, as we want. □

**Corollary 4.3.** *Let $X$ be a compact Kähler manifold and let $\omega \in H^0(X, \Omega_X^1)$ such that the complex $(H^*(X, \mathbb{C}), \wedge\omega)$ given by cup product with $\omega$ is exact. Let $Z \subset Z(\omega)$ be a connected component of the zero locus of $\omega$, and let $d = \dim Z$. Then*

$$H^d(Z', \omega_X|_{Z'}) = 0,$$

*for any irreducible component $Z'$ of the reduced scheme $Z^{\mathrm{red}}$.*

*Proof.* Consider the long exact sequence, associated to the short exact sequence

$$0 \longrightarrow \omega_X|_Z \otimes \mathcal{I}_{Z'} \longrightarrow \omega_X|_Z \longrightarrow \omega_X|_{Z'} \longrightarrow 0.$$

By item (1), $H^d(Z, \omega_X|_Z) = 0$. Moreover, $H^{d+1}(Z, \omega_X|_Z \otimes \mathcal{I}_{Z'}) = 0$ because of dimension reasons. This implies $H^d(Z', \omega_X|_{Z'}) = 0$, as we want. □

**Corollary 4.4.** *Let $X$ be a compact Kähler manifold with a holomorphic map $f : X \to A$ to a complex torus $A$. Let $\omega \in H^0(A, \Omega_A^1)$ such that the complex $(H^*(X, \mathbb{C}), \wedge f^*\omega)$ given by cup product with $f^*\omega$ is exact.*

*Then the restriction of $\omega$ to $f(X) \subset \mathrm{Alb}(X)$ does not vanish at a point $y \in f(X)$ such that the fibre $F := f^{-1}(y)$ is smooth with trivial normal bundle (the locus of such points $y \in f(X)$ is Zariski dense in $f(X)$).*

*Proof.* Assume that $\omega$ vanishes at a point $y \in f(X)$ such that the fibre $F := f^{-1}(y)$ is smooth with trivial normal bundle. Then $F \subset Z(f^*\omega)^{\mathrm{red}}$ is a connected component. This contradicts Corollary 4.3, because

$$H^{\dim F}(F, \omega_X|_F) = H^{\dim F}(F, \omega_F) \neq 0,$$

by Serre duality, where we used that $F$ has trivial normal bundle. □

4.3. **Item (2) of Theorem 1.4.** Let $f : X \to A$ be a holomorphic map to a complex torus $A$ and assume that there is a one-form $\omega \in f^*H^0(A, \Omega^1_A)$ such that $(H^*(A, \mathbb{C}), \wedge\omega)$ is exact. Since exactness is an open property, $(H^*(A, \mathbb{C}), \wedge\omega')$ is exact for any general $\omega' \in f^*H^0(A, \Omega^1_A)$.

Let $Y := f(X)$ and fix a general point $y \in Y$. There are countably many non-trivial linear subspaces

$$\{0\} \neq W_i \subset T_{A,y}$$

such that there is a morphism of complex tori $\pi_i : A \to B_i$ with $\ker((d\pi_i)_y) = W_i$.

For a contradiction, we assume that $Y$ is not fibred by tori. This implies that the tangent space $T_{Y,y}$ does not contain any of the $W_i$. We may thus choose a one-form $\omega' \in H^0(A, \Omega^1_A)$, such that $\omega'$ vanishes on $T_{Y,y} \subset T_{A,y}$, but which is non-trivial on each $W_i$. Let $Z \subset Z(\omega')$ be an irreducible component which contains $y$. Then $\omega'$ vanishes on $Z$ and hence on the subtorus $\langle Z \rangle \subset A$, generated by $Z$. If $Z$ was positive-dimensional, then $T_{\langle Z\rangle,y} = W_i$ for some $i$, which contradicts the fact that $\omega'$ does not vanish on $W_i$. Hence, $Z$ is zero-dimensional and so $y$ is an isolated zero of $\omega'|_Y$. But this implies that a small perturbation of $\omega'|_Y$ has an isolated zero in some neighbourhood of $y$. Hence, a general one-form $\omega \in H^0(A, \Omega^1_A)$ has the property that $Z(\omega|_Y)$ contains a general point of $Y$ as a connected component. This contradicts Corollary 4.4, which finishes the proof.

## Appendix, written jointly with Hsueh-Yung Lin

In this appendix we prove the following.

**Theorem A.1.** *Let $X$ be a compact connected Kähler manifold. Assume that $\omega \in H^0(X, \Omega^1_X)$ satisfies condition (C). Then $\dim Z(\omega) \leq \dim X - 2$.*

By Theorem 1.4, we also have $1 \leq \dim Z(\omega)$. If $\dim X = 2$, the above theorem thus implies $Z(\omega) = \emptyset$, which yields a new proof of Theorem 1.3, without using the Enriques-Kodaira classification.

We start with the following auxiliary result; the same argument appeared in the last two paragraphs in the proof of Theorem 1.3, as well as in [HS19, Proposition 6.4].

**Lemma A.2.** *Let $X$ be a compact connected Kähler manifold with a morphism $f : X \to E$ to an elliptic curve $E$ with irreducible fibres. Assume that there is a one-form $\alpha \in H^0(E, \Omega^1_E)$ such that $\omega := f^*\alpha$ satisfies condition (C). Then $f$ has reduced fibres.*

*Proof.* Let $\Delta$ be the set of points $t \in E$ such that $f^{-1}(t)$ is a multiple fibre and let $m_t$ be its multiplicity. This gives rise to an orbifold structure on $E$. Since $E$ is an elliptic curve, this orbifold sutrcture is good (see e.g. [CHK00, Corollary 2.29]) and so there is a finite cover $C \to E$ which locally above each point of $t \in \Delta$ is ramified of order $m_t$. A local computation shows that the normalization $\tilde{X}$ of $X \times_E C$ is étale over $X$, cf. [BHPV04, Proposition III.9.1]. There is a natural map $\tilde{f}: \tilde{X} \to C$ and our assumptions imply that there is a one-form $\omega \in H^0(C, \Omega^1_C)$ such that $(H^*(\tilde{X}, \mathbb{C}), \wedge \tilde{f}^*\omega)$ is exact. This implies $g(C) = 1$ and so $\Delta = \emptyset$, as we want. □

*Proof of Theorem A.1.* Assume for the contrary that there is a prime divisor $D \subset Z(\omega)$. Let $f : X \to A$ be a morphism to a complex torus such that $\omega = f^*\alpha$ for some $\alpha \in H^0(A, \Omega^1_A)$, and assume that $\dim A$ is minimal with that property.

Since $\omega|_D = 0$, we have $\alpha|_{\langle f(D) \rangle} = 0$, where $\langle f(D) \rangle \subset A$ denotes the subtorus generated by $f(D)$. Hence, $\omega$ is the pullback of a one-form from $A/\langle f(D) \rangle$. Minimality of $\dim A$ thus shows that $f(D)$ is a point. It then follows from [HS19, Lemma 2.4] that $A$ is an elliptic curve. Moreover, up to replacing $f$ by its Stein factorization, we may by [HS19, Corollary 2.5] assume that all fibres of $f$ are irreducible. Hence, $f$ has reduced fibres by Lemma A.2. Since $A$ is an elliptic curve, $Z(\omega)$ is contained in the singular locus of $f$, which has codimension at least two, because the fibres of $f$ are reduced. This is a contradiction, which concludes the theorem. □

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