TORSION IN THE GRIFFITHS GROUP OF PRODUCTS WITH ENRIQUES SURFACES

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Abstract. We show that the torsion subgroup of the Griffiths group of a smooth complex projective variety is in general not finitely generated; in fact, there may be infinite 2-torsion. The main new ingredient is refined unramified cohomology, recently introduced in [Sch20a].

1. Introduction

The Griffiths group $\mathrm{Griff}^i(X)$ of a smooth complex projective variety X is the group of homologically trivial cycles of codimension i modulo algebraic equivalence. This is a countable abelian group which is a basic invariant of X that measures the failure of injectivity of the cycle class map. However, detecting whether a given homologically trivial cycle is nontrivial in the Griffiths group is a subtle problem.

Griffiths [Gri69] used his transcendental Abel–Jacobi maps to construct the first example of a smooth complex projective variety with nontrivial Griffiths group. Clemens [Cle83] combined Griffiths' approach with a degeneration argument to show that in fact $\mathrm{Griff}^i(X) \otimes \mathbb{Q}$ may be an infinite dimensional $\mathbb{Q}$-vector space for any $i \geq 2$. This showed that Griffiths groups are in general not finitely generated modulo torsion.

Improving earlier results of Schoen [Schoe02] and Rosenschon–Srinivas [RS10], Totaro [Tot16] showed that $\mathrm{Griff}^i(X)/\ell$ may for $i \geq 2$ be infinite for any prime ℓ (including in particular $\ell = 2$). These results rely on Griffiths' method, a theorem of Bloch–Esnault [BE96] and ideas from Nori's proof [Nor89] of Clemens' theorem.

Schoen [Schoe92] used Griffiths' method to show that Griffiths groups may contain nontrivial torsion. The first non-trivial torsion classes with trivial transcendental Abel–Jacobi invariants have been constructed by Totaro [Tot97] via a topological method; non-torsion classes with that property had earlier been constructed by Nori [Nor93] via his Hodge theoretic connectivity theorem.

This paper produces the first examples of infinitely many torsion classes in the Griffiths group. This shows that the torsion subgroup of Griffiths groups is not finitely generated

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in general, which, despite the above mentioned breakthroughs, was arguably one of the basic remaining open problems about Griffiths groups. All our classes have trivial Abel–Jacobi invariants and it seems they cannot be detected via previous methods.

Theorem 1.1. *Let JC be the Jacobian of a very general quartic curve $C \subset \mathbb{P}^2_{\mathbb{C}}$. Then for any very general Enriques surface X over $\mathbb{C}$, $\mathrm{Griff}^3(X \times JC)$ has infinite 2-torsion. In fact, infinitely many of those 2-torsion classes are linearly independent modulo 2.*

As mentioned above, Theorem 1.1 shows that the torsion subgroup of Griffiths groups of smooth complex projective varieties is in general not finitely generated. Note that the analogous problem is trivial for Chow groups, as any elliptic curve E has infinite torsion in $\mathrm{CH}_0(E)$. However, even for Chow groups, the problem becomes interesting if we restrict to ℓ-torsion for a given prime ℓ. That problem has been solved by Schoen [Schoe00, Schoe02] and Rosenschon–Srinivas [RS10] for infinitely many and almost all primes, respectively. While those results excluded in particular the prime $\ell = 2$, Totaro [Tot16] settled the problem completely by dealing with all primes ℓ, including 2. However, in all those examples all involved torsion cycles are algebraically equivalent to zero.

The infinitely many different 2-torsion classes in $\mathrm{Griff}^3(X \times JC)$ from Theorem 1.1 are given by products $D \times z$, where $D \in \mathrm{Pic}\, X$ is the unique 2-torsion class on the Enriques surface X and where z is the Ceresa cycle $C - C^-$ on JC (see [Cer83]), or a pullback of the Ceresa cycle by one of infinitely many isogenies. We will show that infinitely many of the cycles $D \times z$ are linearly independent modulo 2. To this end we will use Totaro's result [Tot16], who showed that infinitely many of the classes z are nonzero modulo 2 in $\mathrm{Griff}^2(JC)$. Totaro used among other ingredients that the Ceresa cycle has nontrivial Abel–Jacobi invariant, cf. [Hai93, Hai95]. In contrast, $D \times z$ has trivial Abel–Jacobi invariant.

Taking products with projective spaces, Theorem 1.1 implies the following.

Corollary 1.2. *For any $n \geq 5$ and any $3 \leq i \leq n-2$, there is a smooth complex projective n-fold X with infinite torsion (in fact 2-torsion) in $\mathrm{Griff}^i(X)$.*

It remains open whether $\mathrm{Griff}^i(X)$ may have infinite torsion for $i = 2$ or $n-1$. But note that Merkurjev–Suslin [MS83] proved that $\mathrm{Griff}^2(X)$ has finite ℓ-torsion for any prime ℓ.

The main new ingeredient used in the present paper is refined unramified cohomology, recently introduced in [Sch20a]. The idea is that the cohomological interpretations of several cycle groups given in [Sch20a] allow one to use cohomological tools (most notably Poincaré duality) that are not available within the framework of Chow groups.

1.1. **An injectivity theorem.** By [Sch20a, Corollary 8.3], there is for any smooth complex projective variety Y a canonical extension

$$0 \longrightarrow \mathrm{Griff}^i(Y)/\ell^r \longrightarrow E^i_{\ell^r}(Y) \longrightarrow \frac{Z^{2i}(Y)[\ell^r]}{H^{2i}(Y,\mathbb{Z}(i))[\ell^r]} \longrightarrow 0, \tag{1.1}$$

where $Z^{2i}(Y)[\ell^r]$ denotes the ℓ^r-torsion subgroup of $\operatorname{coker}(\mathrm{cl}^i : \mathrm{CH}^i(Y) \to H^{i,i}(Y,\mathbb{Z}(i)))$. Theorem 1.1 will be deduced from the following, which is the main result of this paper.

Theorem 1.3. *Let Y be any smooth complex projective variety and let X be an Enriques surface that is very general with respect to Y. Then there is a natural injective map*

$$E^i_2(Y) \hookrightarrow \mathrm{Griff}^{i+1}(X \times Y)[2],$$

whose image lies in the kernel of the transcendental Abel–Jacobi map. Moreover, no nontrivial class in the image is divisible by 2 in $\mathrm{CH}^{i+1}(X \times Y)/\sim_{\mathrm{alg}}$.

The inclusion in Theorem 1.3 is defined and proven on the level of refined unramified cohomology. Nonetheless, the map in question has a concrete geometric description as follows. If $z \in \mathrm{Griff}^i(Y)/2$, then z is mapped to $[D \times z]$, where $D \in \mathrm{Pic}(X)$ is the unique nontrivial 2-torsion class. If $\alpha \in H^{i,i}(Y,\mathbb{Z}(i))$ is a non-algebraic Hodge class, which is either non-torsion or torsion of order 4, and such that $2\alpha = \mathrm{cl}^i(z)$ for some $z \in \mathrm{CH}^i(Y)$, then z is unique up to an element of $\mathrm{Griff}^i(Y)/2$ and $[D \times z]$ is homologically trivial. Hence, $[D \times z] \in \mathrm{Griff}^{i+1}(X \times Y)[2]$ and Theorem 1.3 asserts that this class is nonzero modulo 2, cf. Theorem 10.1 below.

In [SV05], Soulé and Voisin constructed for any prime $\ell > 5$ an indivisible ℓ-torsion class in the Griffiths group of $X \times Y$ where $Y \subset \mathbb{P}^4$ is a carefully chosen hypersurface of degree ℓ^3 which carries a non-algebraic non-torsion Hodge class by Kollár's argument [BCC92], and where X is a surface with $H^2(X,\mathbb{Z})[\ell] \neq 0$. In [Voi12], Voisin constructed similar examples for $\ell = 2$, where Y is a $\mathbb{Z}/5$-quotient of a carefully chosen hypersurface of bidegree $(3,4)$ in $\mathbb{P}^1 \times \mathbb{P}^3$. Theorem 1.3 can be seen as a generalization of these results in the case where X is a very general Enriques surface: Y may be arbitrary and the contributions of $\mathrm{Griff}^i(Y)/2$ and $Z^{2i}(Y)[2]$ are taken into account in a very precise way.

1.2. **Griffiths groups of varieties with small CH_0.** In [Voi12, Ma17], Voisin and Ma showed that for any smooth complex projective variety X whose Chow group of zero-cycles is supported on a threefold, there is a short exact sequence

$$0 \longrightarrow \left(\frac{H^5(X,\mathbb{Z})}{N^2H^5(X,\mathbb{Z})}\right)_{\mathrm{tors}} \longrightarrow H^4_{nr}(X,\mathbb{Q}/\mathbb{Z}) \longrightarrow \mathcal{T}^3(X) \longrightarrow 0,$$

where $N^*H^5(X,\mathbb{Z})$ denotes the coniveau filtration and where $\mathcal{T}^3(X) \subset \mathrm{Griff}^3(X)$ denotes the subgroup of torsion classes with trivial transcendental Abel–Jacobi invariants. All three terms in the above sequence are birational invariants of X. In [Voi12, §4], Voisin considered the question whether it can happen that the first term vanishes, while the unramified cohomology group in the middle is nonzero. In *loc. cit.,* Voisin showed that the generalized Hodge conjecture implies the existence of such examples; by the generalized Bloch conjecture, her examples should actually satisfy $\mathrm{CH}_0(X) = \mathbb{Z}$. By a result of Ottem and Suzuki [OS20], Theorem 1.3 implies the existence of such examples unconditionally.

Corollary 1.4. *For any $n \geq 5$, there is a smooth complex projective n-fold X with $\mathrm{CH}_0(X) \simeq \mathbb{Z}$ such that*

$$N^2H^5(X,\mathbb{Z}) = H^5(X,\mathbb{Z}) \quad \text{and} \quad H^4_{nr}(X,\mathbb{Q}/\mathbb{Z}) \simeq \mathcal{T}^3(X) = \mathrm{Griff}^3(X)_{\mathrm{tors}} \neq 0.$$

The condition $\mathrm{CH}_0(X) \simeq \mathbb{Z}$ means that X admits a rational decomposition of the diagonal. It is known (see e.g. [BO74, 7.4] and [CTV12, Proposition 3.3]), that

$$\mathrm{CH}_0(X) \simeq \mathbb{Z} \quad \Longrightarrow \quad \mathrm{Griff}^2(X) = 0.$$

This implication fails for Griff^3, because we can always blow-up varieties with $\mathrm{CH}_0 = \mathbb{Z}$ along a smooth subvariety with nontrivial Griff^2. We are however not aware of any other construction that would yield varieties with small Chow groups of zero-cycles but nontrivial Griffiths groups. For instance, to the best of our knowledge, it was previously not known whether varieties with a rational decomposition of the diagonal always admit a birational model with trivial Griffiths groups. The following consequence of the above corollary solves that problem.

Corollary 1.5. *For any $n \geq 5$, there is a smooth complex projective n-fold X such that any smooth complex projective variety X' that is birational to X satisfies*

$$\mathrm{CH}_0(X') \simeq \mathbb{Z} \quad \text{and} \quad \mathrm{Griff}^3(X') \neq 0.$$

1.3. **Detecting nontrivial cycles without knowing the cycle.** Diaz [Dia20] showed that the Kummer variety $Y = \widetilde{JC}/\pm$ associated to the Jacobian of a smooth quartic $C \subset \mathbb{P}^2_{\mathbb{C}}$ with good reduction at 2 admits a non-torsion class $\alpha \in H^4(Y,\mathbb{Z})$ which is not algebraic but such that 2α is algebraic. Theorem 1.3 thus implies the following.

Corollary 1.6. *Let JC be the Jacobian of a smooth quartic $C \subset \mathbb{P}^2_{\mathbb{C}}$ defined over $\mathbb{Q}$, with good reduction at 2 and with associated Kummer variety $Y = \widetilde{JC}/\pm$. Then for any very general Enriques surface X over $\mathbb{C}$:*

$$\mathrm{Griff}^3(X \times Y)[2] \neq 0 \quad \text{and} \quad \mathrm{Griff}^3(X \times Y)/2 \neq 0.$$

Diaz [Dia20] uses [CTV12] to reduce the problem to a computation of a nontrivial class in $H^3_{nr}(Y,\mathbb{Z}/2)$. As a consequence, the non-algebraic Hodge class on $Y = \widetilde{JC}/\pm$ is not explicitly known. In particular, Corollary 1.6 shows the existence of a nontrivial element in the Griffiths group without knowing the involved cycle explicitly.

1.4. **Degenerations of Enriques surfaces.** Despite the theory of refined unramified cohomology, Theorem 1.3 relies on the following geometric input, which might be of independent interest.

Theorem 1.7. *There is a regular flat proper scheme $\mathcal{X} \to \operatorname{Spec} R$ over a discrete valuation ring R whose residue field is an algebraically closed field κ of characteristic zero, such that:*

- *the geometric generic fibre $X_{\overline{\eta}}$ is an Enriques surface;*
- *the special fibre $X_0 = \mathcal{X} \times \kappa$ is a union of ruled surfaces;*
- *the restriction map $\operatorname{Br}(\mathcal{X}) \longrightarrow \operatorname{Br}(X_{\overline{\eta}})$ is surjective.*

The geometric meaning of the theorem is as follows: the unique nonzero class in $\operatorname{Br}(X_{\overline{\eta}}) \simeq \mathbb{Z}/2$ corresponds to a smooth (i.e. unramified) conic bundle $P \to X_{\overline{\eta}}$ and the above theorem says that this conic bundle extends to a smooth (!) conic bundle $\mathcal{P} \to \mathcal{X}$. That is, while the Enriques surface breaks up into ruled components, the conics in the fibration $P \to X_{\overline{\eta}}$ remain smooth and do, possibly in contrast to our intuition, not break up into the union of two lines. This result is nontrivial, because the obstruction group for extending P without ramification across the central fibre of $\mathcal{X} \to \operatorname{Spec} R$ is nontrivial in type II as well as in type III degenerations of Enriques surfaces.

For us, the crucial consequence of the above theorem will be as follows: The unique nonzero Brauer class on the geometric generic fibre extends to Brauer class on the whole family, but since the components of the special fibre are ruled surfaces over an algebraically closed field, the restriction of that Brauer class to any component of the special fibre must be zero.

It may be surprising that the Brauer class of the Enriques surface plays a role for us, as $\operatorname{Br}(X)$ is transcendental in the sense that it is the quotient of $H^2(X, \mathbb{Q}/\mathbb{Z})$ by the subspace of algebraic classes. Here the main point is that the Brauer class on X is Poincaré dual to the class in $H^2(X, \mathbb{Z}/2)$, given as the reduction modulo 2 of the unique 2-torsion class in $H^2(X, \mathbb{Z})$. The latter is the first Chern class of the unique 2-torsion class in $\operatorname{Pic}(X)$, and so the relation to the Brauer class comes from Poincaré duality.

2. Notation

2.1. **Conventions.** For an abelian group G, we denote by $G[\ell^r]$ the subgroup of ℓ^r-torsion elements. Whenever G and H are abelian groups so that there is a canonical map $H \to G$ (and there is no reason to confuse this map with a different map), we write G/H as a short hand for $\operatorname{coker}(H \to G)$.

All schemes are separated. An algebraic scheme X is a scheme of finite type over a field. Its Chow group of codimension i cycles modulo rational equivalence is denoted by $\operatorname{CH}^i(X)$; the quotient of $\operatorname{CH}^i(X)$ modulo algebraic equivalence is denoted by $A^i(X)$.

A variety is an integral scheme of finite type over a field. For an algebraic scheme X, we denote by $X^{(i)}$ the set of all codimension i points of X. For a scheme X over a field k, we write for any field extension K of k the scheme given by extension of scalars by $X_K := X \times_k K$. A very general point of a scheme over $\mathbb{C}$ is a closed point outside a countable union of proper closed subsets.

If R is an integral local ring with residue field κ and fraction field K, with algebraic closure $\overline{K}$, then for any flat R-scheme $\mathcal{X} \to \operatorname{Spec} R$, we write $X_0 := \mathcal{X} \times_R \kappa$ for the special fibre and $X_\eta := \mathcal{X} \times_R K$ (resp. $X_{\overline{\eta}} := \mathcal{X} \times_R \overline{K}$) for the generic (resp. geometric generic) fibre.

2.2. **Strictly semi-stable schemes.** Let R be a discrete valuation ring. An irreducible flat R-scheme $\mathcal{X}$ is called strictly semi-stable, if $\mathcal{X}$ is regular, the generic fibre X_η is smooth and the special fibre X_0 is a geometrically reduced simple normal crossing divisor on $\mathcal{X}$, i.e. the components of X_0 are smooth and the scheme-theoretic intersection of r different components of X_0 is either empty or smooth and equi-dimensional of codimension r in $\mathcal{X}$.

2.3. **Cohomology.** Let X be a variety over a field k and let ℓ be a prime invertible in k. For any integer n, $A(n)$ will always denote one out of the following coefficients: $\mu_{\ell^r}^{\otimes n}$ for some $r \geq 1$, $\mathbb{Z}_\ell(n)$, $\mathbb{Q}_\ell(n)$ or $\mathbb{Q}_\ell/\mathbb{Z}_\ell(n)$. We further use the notation

$$H^i(X, A(n)) := H^i_{\text{ét}}(X, A(n)).$$

If $X = \operatorname{Spec} B$ for some ring B, we also write $H^i(B, A(n))$ in place of $H^i(X, A(n))$. If B is a field and $A(n) = \mu_{\ell^r}^{\otimes n}$, then the above étale cohomology group coincides with Galois cohomology. If $B = k(X)$ is the function field of a variety over k, then

$$H^i(k(X), A(n)) \simeq \varinjlim_{\emptyset \neq U \subset X} H^i(U, A(n)),$$

where U runs through all non-empty open subsets of X.

3. Preliminaries

3.1. **Classical unramified cohomology.** Let X be an integral regular scheme and let $x \in X^{(1)}$ be a codimension one point such that ℓ is invertible in the residue field $\kappa(x)$. Then there is a residue map in Galois cohomology

$$\partial_x : H^i(k(X), \mu_{\ell^r}^{\otimes n}) \longrightarrow H^{i-1}(\kappa(x), \mu_{\ell^r}^{\otimes n-1}), \tag{3.1}$$

where $k(X)$ denotes the residue field of the generic point of X. If ℓ is invertible in each residue field of X, then the unramified cohomology of X is defined as

$$H^i_{nr}(X, \mu_{\ell^r}^{\otimes n}) := \left\{ \alpha \in H^i(k(X), \mu_{\ell^r}^{\otimes n}) \mid \partial_x \alpha = 0 \ \ \forall x \in X^{(1)} \right\}.$$

There is always a natural map $H^i(X, \mu_{\ell^r}^{\otimes n}) \to H^i_{nr}(X, \mu_{\ell^r}^{\otimes n})$. If X is a smooth variety over an algebraically closed field k, then

$$H^2(X, \mu_{\ell^r}^{\otimes n}) \twoheadrightarrow H^2_{nr}(X, \mu_{\ell^r}^{\otimes n}) \tag{3.2}$$

is surjective, see e.g. [CT95] or [Sch20a, Lemma 4.11]. Passing to direct limits, we see that the above result still holds true if X is regular of finite type over the local ring of a smooth k-variety.

Let $f : X' \to X$ be a dominant morphism between regular integral schemes on which ℓ is invertible. Let $x \in X^{(1)}$ and $x' \in (X')^{(1)}$ be codimension one points with $f(x') = x$. Let π be a uniformizer of $\mathcal{O}_{X,x}$ and let $e = \nu_{x'}(f^*\pi)$ be the valuation of $f^*\pi$ in the valuation ring $\mathcal{O}_{X',x'}$. Then we have a commutative diagram

$$\begin{tikzcd} H^i(k(X'), \mu_{\ell^r}^{\otimes n}) \arrow[r, "\partial_{x'}"] & H^{i-1}(\kappa(x'), \mu_{\ell^r}^{\otimes n-1}) \ , \\ H^i(k(X), \mu_{\ell^r}^{\otimes n}) \arrow[u, "f^*"] \arrow[r, "\partial_x"] & H^{i-1}(\kappa(x), \mu_{\ell^r}^{\otimes n-1}) \arrow[u, "e \cdot f^*"'] \end{tikzcd} \tag{3.3}$$

see e.g. [CTO89, p. 143].

Assume in the above notation that $f : X' \to X$ is proper and generically finite. Let $x \in X^{(1)}$ and let $x'_1, \dots, x'_r \in (X')^{(1)}$ be the preimages of x. Then we have a commutative diagram

$$\begin{tikzcd} H^i(k(X'), \mu_{\ell^r}^{\otimes n}) \arrow[r, "\sum_{j=1}^r \partial_{x'_j}"] \arrow[d, "f_*"] & \bigoplus_{j=1}^r H^{i-1}(\kappa(x'_j), \mu_{\ell^r}^{\otimes n-1}) \ , \arrow[d, "\sum_{j=1}^r (f_{x'_j})_*"] \\ H^i(k(X), \mu_{\ell^r}^{\otimes n}) \arrow[r, "\partial_x"] & H^{i-1}(\kappa(x), \mu_{\ell^r}^{\otimes n-1}) \end{tikzcd} \tag{3.4}$$

where $f_{x'_j} : \operatorname{Spec} \kappa(x'_i) \to \kappa(x)$ is induced by f, see e.g. [Sch20b, §3.2].

3.2. **Gysin sequence.** Let k be an algebraically closed field and let ℓ be a prime invertible in k. For any smooth variety X over k and any smooth subvariety $Z \subset X$ of pure codimension c and with complement $U = X \setminus Z$, there is a Gysin sequence

$$H^i(X, \mu_{\ell^r}^{\otimes n}) \longrightarrow H^i(U, \mu_{\ell^r}^{\otimes n}) \xrightarrow{\partial} H^{i+1-c}(Z, \mu_{\ell^r}^{\otimes n-c}) \longrightarrow H^{i+1}(X, \mu_{\ell^r}^{\otimes n}), \tag{3.5}$$

see e.g. [Mil13, 16.2]. Passing to direct limits, the map ∂ above induces for any codimension one point $x \in X^{(1)}$ residue maps as in (3.1). Unfortunately, the residue map defined this way is exactly the negative of the residue maps from (3.1) that are defined in the framework of Galois cohomology, see e.g. [CT95, §3.3].

The Gysin sequence is functorial with respect to cup products in the following sense. If $\alpha \in H^j(X, \mu_{\ell^r}^{\otimes n})$, then cup product with α induces a commutative diagram

$$\begin{tikzcd} H^i(X,\mu_{\ell^r}^{\otimes n}) \arrow[r] \arrow[d, "\cup\alpha"] & H^i(U,\mu_{\ell^r}^{\otimes n}) \arrow[r, "\partial"] \arrow[d, "\cup\alpha|_U"] & H^{i+1-c}(Z,\mu_{\ell^r}^{\otimes n-c}) \arrow[d, "\cup\alpha|_Z"] \\ H^{i+j}(X,\mu_{\ell^r}^{\otimes n}) \arrow[r] & H^{i+j}(U,\mu_{\ell^r}^{\otimes n}) \arrow[r, "\partial"] & H^{i+j+1-c}(Z,\mu_{\ell^r}^{\otimes n-c}) \end{tikzcd} \tag{3.6}$$

Passing to direct limits, we see that (3.5) and (3.6) still exist if $X \to \operatorname{Spec} R$ is a regular scheme of finite type over a local ring R of a smooth k-variety. In particular, it holds if X is a smooth R-scheme, or a regular proper R-scheme, where R is the local ring of a smooth k-variety, with k algebraically closed as above.

3.3. **Refined unramified cohomology.** We fix an algebraically closed field k and a prime ℓ invertible in k, and recall the most important definitions from [Sch20a]. For a variety X over k, we denote by F_*X the increasing filtration on X given by

$$F_0X \subset F_1X \subset \cdots \subset F_{\dim X}X = X, \quad \text{where} \quad F_jX := \{x \in X \mid \operatorname{codim}_X(x) \leq j\}.$$

In particular, F_0X is the generic point of X, F_1X is the union of the generic point with all codimension one points, and so on.

Definition 3.1. *For any smooth variety X over k and for any j, we define*

$$H^i(F_jX, A(n)) := \varinjlim_{F_jX\subset U\subset X} H^i(U, A(n)),$$

where U runs through all open subsets of X that contain F_jX.

The following lemma is a direct consequence of the Gysin exact sequence (3.5), cf. [Sch20a, Lemma 4.2].

Lemma 3.2. *Let X be a smooth variety over k. Then for any $j, n \in \mathbb{Z}$, there is a long exact sequence*

$$\dots \longrightarrow \bigoplus_{x\in X^{(j)}} H^{i-2j}(\kappa(x), A(n-j)) \xrightarrow{\iota_*} H^i(F_jX, A(n)) \longrightarrow H^i(F_{j-1}X, A(n))$$
$$\xrightarrow{\partial} \bigoplus_{x\in X^{(j)}} H^{i+1-2j}(\kappa(x), A(n-j)) \xrightarrow{\iota_*} H^{i+1}(F_jX, A(n)) \longrightarrow \dots,$$

where ι_* (resp. ∂) is induced by the pushforward (resp. residue) map from (3.5).

As an immediate consequence of the above lemma, we see that

$$H^i(F_jX, A(n)) \simeq H^i(X, A(n)) \quad \text{for all } j \geq \lceil i/2 \rceil,$$

see [Sch20a, Lemma 4.3].

Definition 3.3. *For any smooth variety X over k, we define a decreasing filtration F^* on $H^i(F_jX, A(n))$ by*

$$F^m H^i(F_jX, A(n)) := \operatorname{im}\big(H^i(F_mX, A(n)) \longrightarrow H^i(F_jX, A(n))\big)$$

whenever $m \geq j$ and by $F^m H^i(F_jX, A(n)) := H^i(F_jX, A(n))$ for $m < j$.

The j-th refined unramified cohomology of X with values in $A(n)$ is then defined by

$$H^i_{j,nr}(X, A(n)) := F^{j+1} H^i(F_jX, A(n)).$$

For $j = 0$, this coincides with classical refined unramified cohomology

$$H^i_{0,nr}(X, A(n)) = H^i_{nr}(X, A(n)),$$

see [Sch20a, Corollary 12.1].

Definition 3.4. *For any smooth variety X over k, we define a decreasing filtration G^* on $H^i(F_jX, \mu_{\ell^r}^{\otimes n})$ by*

$$\alpha \in G^m H^i(F_jX, \mu_{\ell^r}^{\otimes n}) \quad \Longleftrightarrow \quad \delta(\alpha) \in F^m H^{i+1}(F_jX, \mathbb{Z}_\ell(n)),$$

where δ denotes the Bockstein map associated to $0 \to \mathbb{Z}_\ell(n) \to \mathbb{Z}_\ell(n) \to \mu_{\ell^r}^{\otimes n} \to 0$.

For $m \geq j$, G^* induces the following decreasing filtration on $F^m H^i(F_jX, \mu_{\ell^r}^{\otimes n})$:

$$G^h F^m H^i(F_jX, \mu_{\ell^r}^{\otimes n}) := \operatorname{im}(G^h H^i(F_mX, \mu_{\ell^r}^{\otimes n}) \to F^m H^i(F_jX, \mu_{\ell^r}^{\otimes n})).$$

In this paper we will also need the following variant of G^*.

Definition 3.5. *For any smooth variety X over k, we define a decreasing filtration $\widetilde{G}^*$ on $H^i(F_jX, \mu_{\ell^r}^{\otimes n})$ by*

$$\alpha \in \widetilde{G}^m H^i(F_jX, \mu_{\ell^r}^{\otimes n}) \quad \Longleftrightarrow \quad \tilde{\delta}(\alpha) \in F^m H^{i+1}(F_jX, \mu_{\ell^r}^{\otimes n}).$$

where $\tilde{\delta}$ denotes the Bockstein map associated to $0 \to \mu_{\ell^r}^{\otimes n} \to \mu_{\ell^{2r}}^{\otimes n} \to \mu_{\ell^r}^{\otimes n} \to 0$.

The compatibility of δ and $\tilde{\delta}$ implies directly:

Lemma 3.6. *We have: $G^m H^i(F_jX, \mu_{\ell^r}^{\otimes n}) \subset \widetilde{G}^m H^i(F_jX, \mu_{\ell^r}^{\otimes n})$.*

4. Cycle groups in terms of refined unramified cohomology

4.1. **Indivisible torsion classes with trivial transcendental Abel–Jacobi invariant.** Let X be a smooth variety over an algebraically closed field k and let ℓ be a prime invertible in k. Then there is a transcendental Abel–Jacobi map on torsion cycles

$$\lambda_{tr} : \operatorname{Griff}^i(X)[\ell^r] \longrightarrow H^{2i-1}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(i))/N^{i-1}H^{2i-1}(X, \mathbb{Q}_\ell(i)),$$

where N^* denotes the coniveau filtration, see [Sch20a, Section 9.1] where an elementary construction of this map, which in the proper case goes back to Griffiths [Gri69] and Bloch [Blo79], is given. Following [Voi12], we denote the kernel of λ_{tr} by

$$\mathcal{T}^i(X)[\ell^r] \subset \mathrm{Griff}^i(X)[\ell^r].$$

Theorem 4.1 ([Sch20a, Theorem 9.4, Proposition 9.8])**.** *Let X be a smooth variety over an algebraically closed field k and let ℓ be a prime invertible in k. Then there are canonical isomorphisms*

$$\mathcal{T}^i(X)[\ell^r] \simeq \frac{H^{2i-2}(F_{i-2}X, \mu_{\ell^r}^{\otimes i})}{G^i H^{2i-2}(F_{i-2}X, \mu_{\ell^r}^{\otimes i})} \simeq \frac{F^{i-2}H^{2i-2}(F_{i-3}X, \mu_{\ell^r}^{\otimes i})}{G^i F^{i-2} H^{2i-2}(F_{i-3}X, \mu_{\ell^r}^{\otimes i})}$$

Proposition 4.2. *Let X be a smooth variety over an algebraically closed field k and let ℓ be a prime invertible in k. Then the kernel of the canonical surjection*

$$\mathcal{T}^i(X)[\ell^r] \simeq \frac{H^{2i-2}(F_{i-2}X, \mu_{\ell^r}^{\otimes i})}{G^i H^{2i-2}(F_{i-2}X, \mu_{\ell^r}^{\otimes i})} \twoheadrightarrow \frac{H^{2i-2}(F_{i-2}X, \mu_{\ell^r}^{\otimes i})}{\widetilde{G}^i H^{2i-2}(F_{i-2}X, \mu_{\ell^r}^{\otimes i})}$$

is given by all classes in $\mathcal{T}^i(X)[\ell^r]$ that are ℓ^r-divisible in $A^i(X)$.

Proof. By [Sch20a, Theorem 8.1], there is a canonical isomorphism

$$\mathrm{Griff}^i(X) \simeq \frac{F^{i-1}H^{2i-1}(F_{i-2}X, \mathbb{Z}_\ell(i))}{H^{2i-1}(X, \mathbb{Z}_\ell(i))}.$$

The natural inclusion

$$\mathcal{T}^i(X)[\ell^r] \hookrightarrow \mathrm{Griff}^i(X)$$

corresponds via the isomorphism in Theorem 4.1 to the map

$$\frac{H^{2i-2}(F_{i-2}X, \mu_{\ell^r}^{\otimes i})}{G^i H^{2i-2}(F_{i-2}X, \mu_{\ell^r}^{\otimes i})} \longrightarrow \frac{F^{i-1}H^{2i-1}(F_{i-2}X, \mathbb{Z}_\ell(i))}{H^{2i-1}(X, \mathbb{Z}_\ell(i))}, \quad [\alpha] \longmapsto [\delta(\alpha)],$$

where δ denotes the Bockstein associated to $0 \to \mathbb{Z}_\ell(i) \to \mathbb{Z}_\ell(i) \to \mu_{\ell^r}^{\otimes i} \to 0$. Here the fact that $\delta(\alpha) \in H^{2i-1}(F_{i-2}X, \mathbb{Z}_\ell(i))$ lies in $F^{i-1}H^{2i-1}(F_{i-2}X, \mathbb{Z}_\ell(i))$ follows from Lemma 3.2, because $\delta(\alpha)$ is torsion while $\bigoplus_{x \in X^{(i)}} H^2(\kappa(x), \mathbb{Z}_\ell(i))$ is torsion-free, see e.g. [Sch20a, Lemma 4.12].

Let now $[\alpha] \in \mathcal{T}^i(X)[\ell^r]$ with $H^{2i-2}(F_{i-2}X, \mu_{\ell^r}^{\otimes i})$. As we have seen above, the class

$$\delta(\alpha) \in H^{2i-1}(F_{i-2}X, \mathbb{Z}_\ell(i))$$

admits a lift

$$\delta(\alpha)' \in H^{2i-1}(F_{i-1}X, \mathbb{Z}_\ell(i)). \tag{4.1}$$

By Lemma 3.2 the lift is unique up to classes in $\bigoplus_{x\in X^{(i-1)}} H^1(\kappa(x),\mathbb{Z}_\ell(1))$. By Lemma 3.2, there is an exact sequence

$$H^{2i-1}(X,\mathbb{Z}_\ell(i))\longrightarrow H^{2i-1}(F_{i-1}X,\mathbb{Z}_\ell(i)) \xrightarrow{\partial} \bigoplus_{x\in X^{(i)}}[x]\mathbb{Z}_\ell.$$

The class

$$\partial(\delta(\alpha)')\in \bigoplus_{x\in X^{(i)}}[x]\mathbb{Z}_\ell$$

is unique up to an element of the image of

$$\partial\circ\iota_* : \bigoplus_{x\in X^{(i-1)}} H^1(\kappa(x),\mathbb{Z}_\ell(1))\longrightarrow \bigoplus_{x\in X^{(i)}}[x]\mathbb{Z}_\ell.$$

The cokernel of the above map is isomorphic to $A^i(X)\otimes\mathbb{Z}_\ell$, see [Sch20a, Lemma 6.2]. Since ∂ is trivial on classes that lift to $H^{2i-1}(X,\mathbb{Z}_\ell(i))$, we get a well-defined map

$$\frac{H^{2i-2}(F_{i-2}X,\mu_{\ell^r}^{\otimes i})}{G^iH^{2i-2}(F_{i-2}X,\mu_{\ell^r}^{\otimes i})}\longrightarrow A^i(X)\otimes\mathbb{Z}_\ell,\quad [\alpha]\longmapsto[\partial(\delta(\alpha)')].$$

This map identifies via the isomorphism in Theorem 4.1 to the inclusion $\mathcal{T}^i(X)[\ell^r]\hookrightarrow A^i(X)\otimes\mathbb{Z}_\ell$.

Let us first assume that

$$[\partial(\delta(\alpha)')]\in A^i(X)\otimes\mathbb{Z}_\ell$$

is divisible by ℓ^r. Then up to a suitable choice of the lift $\delta(\alpha)'$, we may assume that $\partial(\delta(\alpha)')$ is zero modulo ℓ^r. By Lemma 3.2, there is an exact sequence

$$H^{2i-1}(X,\mu_{\ell^r}^{\otimes i})\longrightarrow H^{2i-1}(F_{i-1}X,\mu_{\ell^r}^{\otimes i}) \xrightarrow{\partial} \bigoplus_{x\in X^{(i)}}[x]\mathbb{Z}/\ell^r.$$

We thus conclude that the reduction $\tilde{\delta}(\alpha)'$ modulo ℓ^r of $\delta(\alpha)'$ lifts to a class in $H^{2i-1}(X,\mu_{\ell^r}^{\otimes i})$. Since $\tilde{\delta}(\alpha)'$ is a lift of $\tilde{\delta}(\alpha)$, this implies $\alpha\in\widetilde{G}^iH^{2i-1}(F_{i-2}X,\mu_{\ell^r}^{\otimes i})$.

Conversely, assume that $\alpha\in H^{2i-2}(F_{i-2}X,\mu_{\ell^r}^{\otimes i})$ lies in $\widetilde{G}^iH^{2i-2}(F_{i-2}X,\mu_{\ell^r}^{\otimes i})$. That is,

$$\tilde{\delta}(\alpha)\in H^{2i-1}(F_{i-2}X,\mu_{\ell^r}^{\otimes i})$$

lifts to a class in $H^{2i-1}(X,\mu_{\ell^r}^{\otimes i})$. Consider the lift $\delta(\alpha)'\in H^{2i-1}(F_{i-1}X,\mathbb{Z}_\ell(i))$ of $\delta(\alpha)$ from above. The reduction $\overline{\delta(\alpha)'}\in H^{2i-1}(F_{i-1}X,\mu_{\ell^r}^{\otimes i})$ modulo ℓ^r of the lift $\delta(\alpha)'$ is a lift of $\tilde{\delta}(\alpha)$. Since $\tilde{\delta}(\alpha)$ lifts to $H^{2i-1}(X,\mu_{\ell^r}^{\otimes i})$, we know that up to replacing $\overline{\delta(\alpha)'}$ by another lift of $\tilde{\delta}(\alpha)$, the class extends to all of X. By Lemma 3.2, this means that there is a class

$$\xi\in\bigoplus_{x\in X^{(i-1)}} H^1(\kappa(x),\mu_{\ell^r}),$$

such that

$$\partial\left(\overline{\delta(\alpha)'}-\iota_*\xi\right)=0\in\bigoplus_{x\in X^{(i)}}[x]\mathbb{Z}/\ell^r.$$

Since ξ lifts by Hilbert's Theorem 90 to a class in $\bigoplus_{x\in X^{(i-1)}} H^1(\kappa(x),\mathbb{Z}_\ell)$, and because $\partial\left(\overline{\delta(\alpha)'}\right)$ is the reduction modulo ℓ^r of $\partial(\delta(\alpha)')$, we conclude that

$$[\partial(\delta(\alpha)')]\in A^i(X)\otimes\mathbb{Z}_\ell$$

is zero modulo ℓ^r. Hence, the class $[\alpha]\in\mathcal{T}^i(X)[\ell^r]$ is divisible by ℓ^r in $A^i(X)$, as we want. This concludes the proposition. $\square$

4.2. **The extension.**

Theorem 4.3 ([Sch20a, Corollary 8.3])**.** *Let X be a smooth variety over an algebraically closed field k and let ℓ be a prime that is invertible in k. Let*

$$E^i_{\ell^r}(X):=\frac{F^{i-1}H^{2i-1}(F_{i-2}X,\mu_{\ell^r}^{\otimes i})}{H^{2i-1}(X,\mu_{\ell^r}^{\otimes i})}$$

Then there is a canonical short exact sequence

$$0\longrightarrow \mathrm{Griff}^i(X)_{\mathbb{Z}_\ell}\otimes\mathbb{Z}/\ell^r\longrightarrow E^i_{\ell^r}(X)\longrightarrow\frac{Z^{2i}(X)[\ell^r]}{H^{2i}(X,\mathbb{Z}_\ell(i))[\ell^r]}\longrightarrow 0.$$

where $Z^{2i}(X)[\ell^r]:=\operatorname{coker}\left(\mathrm{cl}^i:\mathrm{CH}^i(X)_{\mathbb{Z}_\ell}\to H^{2i}(X,\mathbb{Z}_\ell(i))\right)[\ell^r]$.

For us the following geometric interpretation of this extension will be useful.

Lemma 4.4. *Let X be a smooth variety over an algebraically closed field k in which ℓ is invertible. There is a canonical exact sequence*

$$0\longrightarrow E^i_{\ell^r}(X)\longrightarrow A^i(X)/\ell^r\longrightarrow H^{2i}(X,\mathbb{Z}_\ell(i))/\ell^r,$$

where the last arrow is given by reduction modulo ℓ^r of the cycle class map $\mathrm{cl}^i:A^i(X)\to H^{2i}(X,\mathbb{Z}_\ell(i))$.

Proof. By Lemma 3.2, we have exact sequences

$$H^{2i-1}(X,\mu_{\ell^r}^{\otimes i})\longrightarrow H^{2i-1}(F_{i-1}X,\mu_{\ell^r}^{\otimes i})\xrightarrow{\partial}\bigoplus_{x\in X^{(i)}}[x]\mathbb{Z}/\ell^r\xrightarrow{\iota_*}H^{2i}(X,\mu_{\ell^r}^{\otimes i})$$

and

$$\bigoplus_{x\in X^{(i-1)}}H^1(\kappa(x),\mu_{\ell^r})\xrightarrow{\iota_*}H^{2i-1}(F_{i-1}X,\mu_{\ell^r}^{\otimes i})\longrightarrow H^{2i-1}(F_{i-2}X,\mu_{\ell^r}^{\otimes i}).$$

Combining these two sequences, we find that $E^i_{\ell^r}(X)$ is isomorphic to

$$\operatorname{coker}\left(\partial\circ\iota_*:\bigoplus_{x\in X^{(i-1)}}H^1(\kappa(x),\mu_{\ell^r})\longrightarrow\ker\left(\iota_*:\bigoplus_{x\in X^{(i)}}[x]\mathbb{Z}/\ell^r\to H^{2i}(X,\mu_{\ell^r}^{\otimes i})\right)\right).$$

By Hilbert's theorem 90,

$$H^1(\kappa(x),\mu_{\ell^r})\simeq\kappa(x)^*/(\kappa(x)^*)^{\ell^r}.$$

Moreover, the composition $\partial \circ \iota$ maps a class that is represented by a rational function $\varphi \in \kappa(x)^*$ to the pushforward of the divisor of zeros and poles of φ (on some normal projective model of $\overline{\{x\}} \subset X$). Thus the above description shows by [Sch20a, Lemma 4.13] that

$$\mathrm{CH}^i(X)/\ell^r \simeq \operatorname{coker}\left(\partial \circ \iota_* : \bigoplus_{x \in X^{(i-1)}} H^1(\kappa(x), \mu_{\ell^r}) \longrightarrow \bigoplus_{x \in X^{(i)}} [x]\mathbb{Z}/\ell^r\right).$$

Since the group of algebrically trivial cycles is ℓ^r-divisible, we find that

$$\mathrm{CH}^i(X)/\ell^r \simeq A^i(X)/\ell^r.$$

We thus get an exact sequence

$$0 \longrightarrow E^i_{\ell^r}(X) \longrightarrow A^i(X)/\ell^r \longrightarrow H^{2i}(X, \mu_{\ell^r}^{\otimes i}),$$

where the last arrow is given by reduction modulo ℓ^r of the cycle class map $\mathrm{cl}^i : A^i(X) \to H^{2i}(X, \mathbb{Z}_\ell(i))$. This concludes the lemma. □

4.3. Product map on refined unramified cohomology.

Lemma 4.5. *Let k be an algebraically closed field and let ℓ be a prime invertible in k. Let X and Y be smooth varieties over k. Then there is a well-defined linear map*

$$\Lambda : \tilde{\delta}\left(H^1(X, \mu_{\ell^r})\right) \otimes \frac{F^{i-1}H^{2i-1}(F_{i-2}Y, \mu_{\ell^r}^{\otimes i})}{H^{2i-1}(Y, \mu_{\ell^r}^{\otimes i})} \longrightarrow \frac{H^{2i}(F_{i-1}(X \times Y), \mu_{\ell^r}^{\otimes i+1})}{\widetilde{G}^{i+1}H^{2i}(F_{i-1}(X \times Y), \mu_{\ell^r}^{\otimes i+1})}$$

which on elementary tensors is given by

$$\Lambda([\tilde{\delta}(\alpha)] \otimes [\beta]) := [p^*\alpha \cup q^*\beta]$$

where $p : X \times Y \to X$ and $q : X \times Y \to Y$ denote the natural projections and where $\alpha \in H^1(X, \mu_{\ell^r})$ and $\beta \in H^{2i-1}(F_{i-1}Y, \mu_{\ell^r}^{\otimes i})$.

Proof. Cup product yields a natural map

$$H^1(X, \mu_{\ell^r}) \otimes \frac{F^{i-1}H^{2i-1}(F_{i-2}Y, \mu_{\ell^r}^{\otimes i})}{H^{2i-1}(Y, \mu_{\ell^r}^{\otimes i})} \longrightarrow \frac{F^{i-1}H^{2i}(F_{i-2}(X \times Y), \mu_{\ell^r}^{\otimes i+1})}{H^{2i}(X \times Y, \mu_{\ell^r}^{\otimes i+1})}.$$

By definition, $G^{i+1}F^{i-1}H^{2i}(F_{i-2}(X \times Y), \mu_{\ell^r}^{\otimes i+1})$ denotes the image of

$$G^{i+1}H^{2i}(F_{i-1}(X \times Y), \mu_{\ell^r}^{\otimes i+1}) \longrightarrow H^{2i}(F_{i-2}(X \times Y), \mu_{\ell^r}^{\otimes i+1}).$$

By [MS83], [Sch20a, Proposition 9.8] asserts that the natural map

$$\frac{H^{2i}(F_{i-1}(X \times Y), \mu_{\ell^r}^{\otimes i+1})}{G^{i+1}H^{2i}(F_{i-1}(X \times Y), \mu_{\ell^r}^{\otimes i+1})} \xrightarrow{\simeq} \frac{F^{i-1}H^{2i}(F_{i-2}(X \times Y), \mu_{\ell^r}^{\otimes i+1})}{G^{i+1}F^{i-1}H^{2i}(F_{i-2}(X \times Y), \mu_{\ell^r}^{\otimes i+1})}$$

is an isomorphism. Composing the above cup product map with the quotient map

$$\frac{F^{i-1}H^{2i}(F_{i-2}(X\times Y),\mu_{\ell^r}^{\otimes i+1})}{H^{2i}(X\times Y,\mu_{\ell^r}^{\otimes i+1})}\twoheadrightarrow \frac{F^{i-1}H^{2i}(F_{i-2}(X\times Y),\mu_{\ell^r}^{\otimes i+1})}{G^{i+1}F^{i-1}H^{2i}(F_{i-2}(X\times Y),\mu_{\ell^r}^{\otimes i+1})}$$

followed by the inverse of the above isomorphism, we thus obtain a map

$$H^1(X,\mu_{\ell^r})\otimes \frac{F^{i-1}H^{2i-1}(F_{i-2}Y,\mu_{\ell^r}^{\otimes i})}{H^{2i-1}(Y,\mu_{\ell^r}^{\otimes i})}\longrightarrow \frac{H^{2i}(F_{i-1}(X\times Y),\mu_{\ell^r}^{\otimes i+1})}{G^{i+1}H^{2i}(F_{i-1}(X\times Y),\mu_{\ell^r}^{\otimes i+1})}.$$

By Lemma 3.6, this induces a map

$$H^1(X,\mu_{\ell^r})\otimes \frac{F^{i-1}H^{2i-1}(F_{i-2}Y,\mu_{\ell^r}^{\otimes i})}{H^{2i-1}(Y,\mu_{\ell^r}^{\otimes i})}\longrightarrow \frac{H^{2i}(F_{i-1}(X\times Y),\mu_{\ell^r}^{\otimes i+1})}{\widetilde{G}^{i+1}H^{2i}(F_{i-1}(X\times Y),\mu_{\ell^r}^{\otimes i+1})}.$$

The map is explicitly given by

$$\alpha\otimes[\beta]\longmapsto [p^*\alpha\cup q^*\beta],$$

where $\alpha\in H^1(X,\mu_{\ell^r})$ and $\beta\in H^{2i-1}(F_{i-1}Y,\mu_{\ell^r}^{\otimes i})$. If $\tilde{\delta}(\alpha)=0$, then by the derivation property of the Bockstein $\tilde{\delta}$ (see [Hat02, p. 304]) together with its functoriality, we find

$$\tilde{\delta}(p^*\alpha\cup q^*\beta)=-p^*\alpha\cup q^*\tilde{\delta}(\beta).$$

By [Sch20a, Corollary 7.4], $\tilde{\delta}(\beta)$ extends to a class on Y and so

$$p^*\alpha\cup q^*\beta\in \widetilde{G}^{i+1}H^{2i}(F_{i-1}(X\times Y),\mu_{\ell^r}^{\otimes i+1}).$$

Altogether, we find that the above cup product map induces a well-defined map

$$\tilde{\delta}(H^1(X,\mu_{\ell^r}))\otimes \frac{F^{i-1}H^{2i-1}(F_{i-2}Y,\mu_{\ell^r}^{\otimes i})}{H^{2i-1}(Y,\mu_{\ell^r}^{\otimes i})}\longrightarrow \frac{H^{2i}(F_{i-1}(X\times Y),\mu_{\ell^r}^{\otimes i+1})}{\widetilde{G}^{i+1}H^{2i}(F_{i-1}(X\times Y),\mu_{\ell^r}^{\otimes i+1})}$$

as we want. This proves the lemma. □

4.4. **The exterior product map on cycles.** Let X and Y be smooth varieties over an algebraically closed field k and let ℓ be a prime invertible in k. There is a natural exterior product map

$$A^j(X)[\ell^r]\otimes A^i(Y)/\ell^r\longrightarrow A^{i+j}(X\times Y)[\ell^r],\quad [z_1]\otimes[z_2]\longmapsto[z_1\times z_2]. \tag{4.2}$$

If $\mathrm{cl}^i(z_2)$ is trivial modulo ℓ^r, then $[z_1\times z_2]$ is homologically trivial. The restriction of the above map thus yields by Lemma 4.4 a map

$$A^j(X)[\ell^r]\otimes E^i_{\ell^r}(Y)\longrightarrow \mathrm{Griff}^{i+j}(X\times Y)[\ell^r].$$

Using the fact that the cycle class of elements in $E^i_{\ell^r}(Y)$ is zero modulo ℓ^r, while classes in $A^j(X)[\ell^r]$ are ℓ^r-torsion, a straightforward calculation shows that the image of the above map is contained in the kernel of the transcendental Abel–Jacobi mapping and so we have a map

$$A^j(X)[\ell^r]\otimes E^i_{\ell^r}(Y)\longrightarrow \mathcal{T}^{i+j}(X\times Y)[\ell^r]. \tag{4.3}$$

(If $k=\mathbb{C}$, this follows directly from the fact that the transcendental Deligne cycle class of elements in $A^j(X)[\ell^r]$ is ℓ^r-torsion, while it is divisible by ℓ^r for elements in $E^i_{\ell^r}(Y)$.)

From now on, we concentrate on the case $j=1$. By Theorem 4.1,

$$\mathcal{T}^{i+1}(X\times Y)[\ell^r]\simeq \frac{H^{2i}(F_{i-1}(X\times Y),\mu_{\ell^r}^{\otimes i+1})}{G^{i+1}H^{2i}(F_{i-1}(X\times Y),\mu_{\ell^r}^{\otimes i+1})}.$$

We may thus consider the quotient map

$$\mathcal{T}^{i+1}(X\times Y)[\ell^r]\longrightarrow \frac{H^{2i}(F_{i-1}(X\times Y),\mu_{\ell^r}^{\otimes i+1})}{\widetilde{G}^{i+1}H^{2i}(F_{i-1}(X\times Y),\mu_{\ell^r}^{\otimes i+1})}.$$

By Proposition 4.2, any class that is divisible by ℓ^r in $A^{i+1}(X\times Y)$ lies in the kernel of that map. We thus conclude from (4.3) that the exterior product map (4.2) induces a map

$$\frac{A^1(X)[\ell^r]}{\ell^r A^1(X)[\ell^{2r}]}\otimes E^i_{\ell^r}(Y)\longrightarrow \frac{H^{2i}(F_{i-1}(X\times Y),\mu_{\ell^r}^{\otimes i+1})}{\widetilde{G}^{i+1}H^{2i}(F_{i-1}(X\times Y),\mu_{\ell^r}^{\otimes i+1})}. \tag{4.4}$$

Since algebraic and homological equivalence coincides for divisors on smooth varieties,

$$A^1(X)[\ell^r]\simeq H^2(X,\mathbb{Z}_\ell(1))[\ell^r]\simeq \delta(H^1(X,\mu_{\ell^r})),$$

where we use that any class in $H^2(X,\mathbb{Z}_\ell(1))[\ell^r]\simeq\delta(H^1(X,\mu_{\ell^r}))$ is algebraic as it comes from a μ_{ℓ^r}-torsor. By the compatibility of δ and $\tilde{\delta}$, we find that

$$\frac{A^1(X)[\ell^r]}{\ell^r A^1(X)[\ell^{2r}]}\simeq \tilde{\delta}(H^1(X,\mu_{\ell^r})).$$

Hence, (4.4) identifies to a map

$$\tilde{\delta}(H^1(X,\mu_{\ell^r}))\otimes E^i_{\ell^r}(Y)\longrightarrow \frac{H^{2i}(F_{i-1}(X\times Y),\mu_{\ell^r}^{\otimes i+1})}{\widetilde{G}^{i+1}H^{2i}(F_{i-1}(X\times Y),\mu_{\ell^r}^{\otimes i+1})}. \tag{4.5}$$

Recall from Theorem 4.3 that

$$E^i_{\ell^r}(X)=\frac{F^{i-1}H^{2i-1}(F_{i-2}X,\mu_{\ell^r}^{\otimes i})}{H^{2i-1}(X,\mu_{\ell^r}^{\otimes i})}$$

and so the exterior product map in (4.5) has a description purely in terms of refined unramified cohomology.

Proposition 4.6. *Let k be an algebraically closed field and let ℓ be a prime invertible in k. Let X and Y be smooth varieties over k. Then the map in (4.5) that is induced by the exterior product map in (4.2) coincides with the map*

$$\Lambda:\tilde{\delta}\left(H^1(X,\mu_{\ell^r})\right)\otimes\frac{F^{i-1}H^{2i-1}(F_{i-2}Y,\mu_{\ell^r}^{\otimes i})}{H^{2i-1}(Y,\mu_{\ell^r}^{\otimes i})}\longrightarrow\frac{H^{2i}(F_{i-1}(X\times Y),\mu_{\ell^r}^{\otimes i+1}))}{\widetilde{G}^{i+1}H^{2i}(F_{i-1}(X\times Y),\mu_{\ell^r}^{\otimes i+1}))}$$

from Lemma 4.5 which on elementary tensors is given by

$$\Lambda([\tilde{\delta}(\alpha)]\otimes[\beta]):=[p^*\alpha\cup q^*\beta]$$

where $\alpha \in H^1(X,\mu_{\ell^r})$ *and* $\beta \in H^{2i-1}(F_{i-1}Y,\mu_{\ell^r}^{\otimes i})$.

Proof. Let $\alpha \in H^1(X,\mu_{\ell^r})$ and $\beta \in H^{2i-1}(F_{i-1}Y,\mu_{\ell^r}^{\otimes i})$. By the derivation property of $\tilde{\delta}$ (see [Hat02, p. 304]) and functoriality of the Bockstein $\tilde{\delta}$, we find

$$\tilde{\delta}(p^*\alpha \cup q^*\beta) = p^*(\tilde{\delta}(\alpha)) \cup q^*\beta - p^*\alpha \cup q^*(\tilde{\delta}(\beta)) \in H^{2i+1}(F_{i-1}(X\times Y),\mu_{\ell^r}^{\otimes i+1}))$$

By [Sch20a, Corollary 7.4], $\tilde{\delta}(\beta)$ extends to a class on Y. Moreover, $\tilde{\delta}(\alpha)$ is an algebraic class (hence vanishes away from a codimension one subset of X) and β is a class which is defined away from some codimension i subset of Y. Altogether, we find that $p^*(\tilde{\delta}(\alpha))\cup q^*\beta$ and $p^*\alpha \cup q^*(\tilde{\delta}(\beta))$ both admit a lift to

$$H^{2i+1}(F_i(X\times Y),\mu_{\ell^r}^{\otimes i+1})).$$

The difference of these lifts yields a lift of $\tilde{\delta}(p^*\alpha \cup q^*\beta)$. The image of that lift via the residue map

$$\partial : H^{2i+1}(F_i(X\times Y),\mu_{\ell^r}^{\otimes i+1}))\longrightarrow \bigoplus_{x\in (X\times Y)^{(i+1)}} [x]\mathbb{Z}/\ell^r$$

is given by the class $D\times \partial\beta$, where $D \in A^1(X)[\ell^r]$ is the class with $\tilde{\delta}(\alpha) = \mathrm{cl}^1(D)$. By the proof of Lemma 4.4, we see that $\partial\beta$ is the cycle that corresponds to β via the inclusion

$$\frac{F^{i-1}H^{2i-1}(F_{i-2}Y,\mu_{\ell^r}^{\otimes i})}{H^{2i-1}(Y,\mu_{\ell^r}^{\otimes i})} \simeq E^i_{\ell^r}(Y) \hookrightarrow A^i(Y)/\ell^r.$$

This shows that Λ coincides with the map induced by the exterior product map in (4.2). This concludes the proposition. ☐

5. Smooth specialization of refined unramified cohomology

Let κ be an algebraically closed field and let $R = \mathcal{O}_{B,0}$ be the local ring of a smooth pointed curve $(B,0)$ over κ. Let $K = \operatorname{Frac} R$ and let ℓ be a prime invertible in κ.

5.1. **Specialization of ordinary cohomology in smooth families.** Let $\mathcal{U} \to \operatorname{Spec} R$ be a smooth (not necessarily proper) morphism with special fibre $U_0 = \mathcal{U}\times_R \kappa$ and generic fibre $U_\eta = \mathcal{U}\times_R K$. Then $\mathcal{U}$ extends to a smooth κ-variety over some open neighbourhood of $0\in B$ and so the Gysin sequence (3.5) induces a sequence

$$H^i(\mathcal{U},\mu_{\ell^r}^{\otimes n})\longrightarrow H^i(U_\eta,\mu_{\ell^r}^{\otimes n}) \xrightarrow{\partial} H^{i-1}(U_0,\mu_{\ell^r}^{\otimes n-1})\longrightarrow H^{i+1}(\mathcal{U},\mu_{\ell^r}^{\otimes n}). \tag{5.1}$$

Let $\pi \in R$ be a uniformizer. Then π gives rise to a class in $H^1(K,\mu_{\ell^r}) \simeq K^*/(K^*)^{\ell^r}$ and so we get a class $(\pi) \in H^1(U_\eta,\mu_{\ell^r}^{\otimes n})$ via pullback. We define a specialization map

$$sp : H^i(U_\eta,\mu_{\ell^r}^{\otimes n})\longrightarrow H^i(U_0,\mu_{\ell^r}^{\otimes n}),\quad \alpha \longmapsto -\partial((\pi)\cup\alpha),$$

where ∂ is the residue map in the Gysin sequence (5.1). The minus sign is due to the aforementioned fact that the residue map in Galois cohomology (3.1) is the negative of the map induced via the Gysin sequence.

Lemma 5.1. *In the above notation, if $\alpha \in H^i(U_\eta, \mu_{\ell^r}^{\otimes n})$ extends to a class $\tilde{\alpha} \in H^i(\mathcal{U}, \mu_{\ell^r}^{\otimes n})$, then*

$$sp(\alpha) = \tilde{\alpha}|_{U_0} \in H^i(U_0, \mu_{\ell^r}^{\otimes n}).$$

Proof. By (3.6), we have

$$-\partial((\pi) \cup \alpha) = -(\partial(\pi)) \cup \widetilde{\alpha}|_Z$$

and so the result follows from the fact that $\partial(\pi) = -1 \in H^0(Z, \mu_{\ell^r}^{\otimes 0}) = \mathbb{Z}/\ell^r$, because π is a uniformizer of the local ring $\mathcal{O}_{\mathcal{X},X_0}$ of $\mathcal{X}$ at the generic point of X_0 and so the residue in Galois cohomology, which agrees with $-\partial(\pi)$, is equal to 1. $\square$

5.2. **Specialization of refined unramified cohomology.** Let $\mathcal{X} \to \operatorname{Spec} R$ be a smooth (not necessarily proper) morphism and let $X_0 = \mathcal{X} \times_R \kappa$ and $X_\eta = \mathcal{X} \times_R K$. For any $j \geq i$, we define a specialization map

$$sp : H^i(F_j X_\eta, \mu_{\ell^r}^{\otimes n}) \longrightarrow H^i(F_j X_0, \mu_{\ell^r}^{\otimes n})$$

as follows.

Let $\alpha \in H^i(F_j X_\eta, \mu_{\ell^r}^{\otimes n})$. Then there is a closed subset $Z_\eta \subset X_\eta$ of codimension $> j$ such that $\alpha = [\alpha_{U_\eta}]$ is represented by a class in

$$\alpha_{U_\eta} \in H^i(U_\eta, \mu_{\ell^r}^{\otimes n}),$$

where $U_\eta = X_\eta \setminus Z_\eta$. The closure $\mathcal{Z} \subset \mathcal{X}$ of Z_η is automatically flat over R and so the special fibre Z_0 has codimension $> j$ in X_0. Let $\mathcal{U} := \mathcal{X} \setminus \mathcal{Z}$ with special fibre $U_0 = X_0 \setminus Z_0$. By Section 5.1, we get a class

$$\alpha_{U_0} := sp(\alpha_{U_\eta}) \in H^i(U_0, \mu_{\ell^r}^{\otimes n}).$$

We then define

$$sp(\alpha) = [\alpha_{U_0}] \in H^i(F_j X_0, \mu_{\ell^r}^{\otimes n}).$$

Functoriality of the Gysin sequence with respect to open immersions immediately shows that this definition is well-defined.

Lemma 5.2. *The specialization map*

$$sp : H^i(F_j X_\eta, \mu_{\ell^r}^{\otimes n}) \longrightarrow H^i(F_j X_0, \mu_{\ell^r}^{\otimes n}), \quad \alpha \longmapsto [\alpha_{U_0}]$$

defined above is compatible with the filtrations F^* and $\widetilde{G}^*$ on both sides.

Proof. Let $m \geq j$ and assume in the above notation that $\alpha \in F^m H^i(F_j X_\eta, \mu_{\ell^r}^{\otimes n})$. This means that we may choose $\alpha_{U_\eta} \in H^i(U_\eta, \mu_{\ell^r}^{\otimes n})$ in such a way that Z_η has actually codimension $> m$ in X_η. But then the above construction immediately shows that $sp(\alpha)$ lifts to a class in $H^i(F_m X_0, \mu_{\ell^r}^{\otimes n})$ and hence lies in $F^m H^i(F_j X_0, \mu_{\ell^r}^{\otimes n})$.

Similarly, if $\alpha \in \widetilde{G}^m H^i(F_j X_\eta, \mu_{\ell^r}^{\otimes n})$, then $\tilde{\delta}(\alpha) \in H^{i+1}(F_j X_\eta, \mu_{\ell^r}^{\otimes n})$ extends to $F_m X_\eta$ and the compatibility of the Gysin sequence with the Bockstein map shows that $\tilde{\delta}(sp(\alpha)) \in H^{i+1}(F_j X_0, \mu_{\ell^r}^{\otimes n})$ extends to $F_m X_0$. Hence, $sp(\alpha) \in \widetilde{G}^m H^i(F_j X_0, \mu_{\ell^r}^{\otimes n})$, as claimed. □

Lemma 5.3. *Let $\alpha \in H^i(F_j X_\eta, \mu_{\ell^r}^{\otimes n})$. Assume in the above notation that $\alpha = [\alpha_{U_\eta}]$ where α_{U_η} extends to a class $\alpha_{\mathcal{U}} \in H^i(\mathcal{U}, \mu_{\ell^r}^{\otimes n})$. Then*

$$sp(\alpha) = [\alpha_{U_0}]$$

where α_{U_0} denotes the restriction of $\alpha_{\mathcal{U}}$ to U_0.

Proof. This follows directly from the definition and Lemma 5.1. □

Remark 5.4. *The above specialization maps yield as a special case (where $j = 0$) specialization maps*

$$sp : H^i(K(X_\eta), \mu_{\ell^r}^{\otimes n}) \longrightarrow H^i(\kappa(X_0), \mu_{\ell^r}^{\otimes n}) \tag{5.2}$$

that are well-known. Lemma 5.2 contains as a special case the assertion that for any $j \geq 0$ there are specialization maps

$$sp : H^i_{j,nr}(X_\eta, \mu_{\ell^r}^{\otimes n}) \longrightarrow H^i_{j,nr}(X_0, \mu_{\ell^r}^{\otimes n}) \tag{5.3}$$

between the j-th refined unramified cohomology groups. This seems new even in the case $j = 0$, where the groups in question coincide with traditional unramified cohomology. In fact, the situation is subtle: Unramified classes may on proper flat families specialize via (5.2) to ramified classes and this was the main technique to prove nontriviality of certain unramified classes in [Sch19, Section 6]. *While (5.2) and (5.3) are compatible for $j = 0$, the point is that the families $\mathcal{X} \to \operatorname{Spec} R$ considered in* loc. cit. *are flat but not smooth and the ramification of the class in question will lie on the singular locus of $\mathcal{X}$ over R.*

6. A vanishing result

Let κ be an algebraically closed field and let ℓ be a prime that is invertible in κ. Let R be the local ring of a smooth pointed curve $(B, 0)$ over κ with fraction field $K = \operatorname{Frac} R$. Let $\mathcal{X}$ and $\mathcal{Y}$ be integral regular flat proper R-schemes and assume that $\mathcal{Y} \to \operatorname{Spec} R$ is smooth. Let $\mathcal{W} \to \operatorname{Spec} R$ be an integral flat proper R-scheme with proper R-morphisms

$$p : \mathcal{W} \longrightarrow \mathcal{X} \quad \text{and} \quad q : \mathcal{W} \longrightarrow \mathcal{Y},$$

where q is generically finite onto its image $q(\mathcal{W}) \subset \mathcal{Y}$. We denote the codimension of the image by

$$c = \operatorname{codim}_{\mathcal{Y}}(q(\mathcal{W})).$$

Proposition 6.1. *In the above notation, let $\alpha \in H^i_{nr}(\mathcal{X}, \mu_{\ell^r}^{\otimes n})$ be an unramified class on $\mathcal{X}$ whose restriction to the generic point of any component of the special fibre X_0 vanishes. Then for any $\xi \in H^j(K(W_\eta), \mu_{\ell^r}^{\otimes m})$, the class*

$$q_*(p^*\alpha \cup \xi) \in H^{i+j}(F_cY_\eta, \mu_{\ell^r}^{\otimes n+m})$$

lies in the kernel of the specialization map

$$sp : H^{i+j}(F_cY_\eta, \mu_{\ell^r}^{\otimes n+m}) \longrightarrow H^{i+j}(F_cY_0, \mu_{\ell^r}^{\otimes n+m})$$

from Section 5.2.

We will deduce the above proposition from the following lemma.

Lemma 6.2. *In the above notation, let $\tau : \mathcal{Z} \to q(\mathcal{W})$ be the normalization and let $\tilde{q} : \mathcal{W} \to \mathcal{Z}$ be the factorization of q into $q = \tau \circ \tilde{q}$. Let $Z_{0r} \subset (Z_0)^{\mathrm{red}}$ be a component of the reduced special fibre of $\mathcal{Z} \to \operatorname{Spec} R$ and let $\pi \in K(Z_\eta)$ be a uniformizer of the local ring $\mathcal{O}_{\mathcal{Z},Z_{0r}}$ of $\mathcal{Z}$ at the generic point of Z_{0r}. Let $s \in R$ be a uniformizer of R. Then the residue map ∂_π associated to the local ring of $\mathcal{Z}$ at the generic point of Z_{0r} from (3.1) satisfies*

$$\partial_\pi((s) \cup \tilde{q}_*(p^*\alpha \cup \xi)) = 0 \in H^{i+j}(\kappa(Z_{0r}), \mu_{\ell^r}^{\otimes m+n}).$$

Proof. By the projection formula, we have

$$(s) \cup \tilde{q}_*(p^*\alpha \cup \xi) = \tilde{q}_*(\tilde{q}^*(s) \cup p^*\alpha \cup \xi).$$

Let $W_{01}, \dots, W_{0m}$ be the components of W_0^{red} that dominate Z_{0r}. Let further $\varphi_h \in \mathcal{O}_{\mathcal{W},w_h}$ be a uniformizer of the local ring of $\mathcal{W}$ at the generic point w_h of W_{0h}. By the compatibility of the residue maps and pushforwards (see (3.4)), we then get

$$\begin{aligned} \partial_\pi((s) \cup \tilde{q}_*(p^*\alpha \cup \xi)) &= \partial_\pi(\tilde{q}_*(\tilde{q}^*(s) \cup p^*\alpha \cup \xi)) \\ &= \sum_{h=1}^{m} \tilde{q}_*\left(\partial_{\varphi_h}\left(\tilde{q}^*(s) \cup p^*\alpha \cup \xi\right)\right) \in H^{i+j}(\kappa(Z_{0r}), \mu_{\ell^r}^{\otimes n}). \end{aligned}$$

It thus suffices to show that

$$\partial_{\varphi_h}\left(\tilde{q}^*(s) \cup p^*\alpha \cup \xi\right) = 0 \in H^{i+j}(\kappa(W_{0h}), \mu_{\ell^r}^{\otimes n}) \tag{6.1}$$

for all $h = 1, \dots, m$. The residue map ∂_{φ_h} factors through the completion $\widehat{\mathcal{O}_{\mathcal{W},w_h}}$ of the local ring $\mathcal{O}_{\mathcal{W},w_h}$. It thus suffices to show that the image of $p^*\alpha$ in

$$H^i(\widehat{\mathcal{O}_{\mathcal{W},w_h}}, \mu_{\ell^r}^{\otimes n})$$

vanishes. Since R is the local ring of a smooth curve B over κ, $\mathcal{X}$ extends to variety $\mathcal{X}_B$ over κ with a flat proper map $\mathcal{X}_B \to B$. Since $\mathcal{X}$ is regular and κ is algebraically closed, we may up to shrinking B assume that $\mathcal{X}_B$ is smooth over κ. Since $\alpha \in H^i(K(X_\eta), \mu_{\ell^r}^{\otimes n})$ is unramified on $\mathcal{X}$, we may up to shrinking B further assume that

$$\alpha \in H^i_{nr}(\mathcal{X}_B, \mu_{\ell^r}^{\otimes n}).$$

Let $x_h := p(w_h)$ be the image in $\mathcal{X}$ of the generic point w_h of $W_{0h} \subset W_0$. Since α is unramified on the smooth κ-variety $\mathcal{X}_B$, the codimension one purity theorem in étale cohomology (see e.g. [CT95, 3.8.1 and 3.8.2]) implies that

$$\alpha \in H^i(\mathcal{O}_{X,x_h}, \mu_{\ell^r}^{\otimes n}) \subset H^i(K(X), \mu_{\ell^r}^{\otimes n}).$$

The image of $p^*\alpha$ in

$$H^i(\operatorname{Frac} \widehat{\mathcal{O}_{W,w_h}}, \mu_{\ell^r}^{\otimes n})$$

coincides with the image of α via the natural composition

$$H^i(K(X), \mu_{\ell^r}^{\otimes n}) \longrightarrow H^i(\operatorname{Frac} \widehat{\mathcal{O}_{X,x_h}}, \mu_{\ell^r}^{\otimes n}) \longrightarrow H^i(\operatorname{Frac} \widehat{\mathcal{O}_{W,w_h}}, \mu_{\ell^r}^{\otimes n}),$$

where $\widehat{\mathcal{O}_{X,x_h}}$ denotes the completion of $\mathcal{O}_{X,x_h}$. By [Mil80, Corollary VI.2.7], the restriction map

$$H^i(\widehat{\mathcal{O}_{X,x_h}}, \mu_{\ell^r}^{\otimes n}) \xrightarrow{\simeq} H^i(\kappa(x_h), \mu_{\ell^r}^{\otimes n})$$

is an isomorphism. Our assumptions imply that α restricts to zero on $\kappa(x_h)$ and so the image of α in $H^i(\widehat{\mathcal{O}_{X,x_h}}, \mu_{\ell^r}^{\otimes n}) \subset H^i(\operatorname{Frac} \widehat{\mathcal{O}_{X,x_h}}, \mu_{\ell^r}^{\otimes n})$ vanishes. This implies that the image of $p^*\alpha$ in $H^i(\operatorname{Frac} \widehat{\mathcal{O}_{W,w_h}}, \mu_{\ell^r}^{\otimes n})$ vanishes and so (6.1) holds, because ∂_φ factors through the cohomology of $\operatorname{Frac} \widehat{\mathcal{O}_{W,w_h}}$. This concludes the lemma. ☐

6.1. **Proof of Proposition 6.1.**

Proof of Proposition 6.1. Let Z_{0r} denote the components of the reduced special fibre of $\mathcal{Z} \to \operatorname{Spec} R$. Let π_r be a uniformizer of the local ring $\mathcal{O}_{\mathcal{Z},Z_{0r}}$ of $\mathcal{Z}$ at the generic point of Z_{0r}. Let $s \in R$ be a uniformizer of R. The definition of the specialization map from Section 5.2 yields

$$\begin{aligned} sp(q_*(p^*\alpha \cup \xi)) &= -\partial((s) \cup q_*(p^*\alpha \cup \xi)) \\ &= -\partial((s) \cup \tau_*\tilde{q}_*(p^*\alpha \cup \xi)) \\ &= -\partial(\tau_*\tilde{q}_*((s) \cup p^*\alpha \cup \xi)) \in H^{i+j}(F_cY_0, \mu_{\ell^r}^{\otimes n+m}), \end{aligned}$$

where we used in the last line the projection formula and the fact that $s \in R$, (i.e. s is implicitly a pullback). By the compatibility of residues and pushforward maps (see (3.4)), we find

$$sp(q_*(p^*\alpha \cup \xi)) = \tau_* \sum_r \partial_{\pi_r}(\tilde{q}_*((s) \cup p^*\alpha \cup \xi)) \in H^{i+j}(F_cY_0, \mu_{\ell^r}^{\otimes n+m}),$$

where the minus sign disappears because we pass from the residue map induced by the Gysin sequence (see Section 3.2) to the residue map (3.1) from Galois cohomology. Using the projection formula once again, we get

$$sp(q_*(p^*\alpha \cup \xi)) = \tau_* \sum_r \partial_{\pi_r}((s) \cup \tilde{q}_*(p^*\alpha \cup \xi)) \in H^{i+j}(F_c Y_0, \mu_{\ell^r}^{\otimes n+m}).$$

This class vanishes by Lemma 6.2, as we want. This concludes the proof. $\square$

7. An injectivity theorem

Theorem 7.1. *Let κ be an algebraically closed field and let R be the local ring of a smooth curve over κ. Let k be an algebraic closure of $\operatorname{Frac} R$ and let ℓ be a prime that is invertible in κ. Assume that there is a proper strictly semi-stable R-scheme $\mathcal{X} \to \operatorname{Spec} R$ with connected fibres of relative dimension two, such that the following holds, where $X := \mathcal{X} \times_R k$ denotes the geometric generic fibre:*

- ***C1*** *the restriction map $H^2_{nr}(\mathcal{X}, \mu_{\ell^r}) \to H^2_{nr}(X, \mu_{\ell^r})$ is surjective;*
- ***C2*** *for each component X_{0i} of X_0, the restriction map $H^2_{nr}(\mathcal{X}, \mu_{\ell^r}) \to H^2_{nr}(X_{0i}, \mu_{\ell^r})$ is trivial;*
- ***C3*** *$r = 1$ or the cup product pairing on $H^2(X, \mu_{\ell^r})$ is perfect.*

Then for any smooth projective variety Y_κ over κ with base change $Y = Y_\kappa \times k$, the map Λ from Lemma 4.5 induces an injection

$$\tilde{\delta}\left(H^1(X, \mu_{\ell^r})\right) \otimes \frac{F^{i-1}H^{2i-1}(F_{i-2}Y_\kappa, \mu_{\ell^r}^{\otimes i})}{H^{2i-1}(Y_\kappa, \mu_{\ell^r}^{\otimes i})} \hookrightarrow \frac{H^{2i}(F_{i-1}(X \times Y), \mu_{\ell^r}^{\otimes i+1}))}{\widetilde{G}^{i+1}H^{2i}(F_{i-1}(X \times Y), \mu_{\ell^r}^{\otimes i+1}))}.$$

Proof. Note first that $\kappa \subset k$ is an extension of algebraically closed fields and so the natural map

$$\frac{F^{i-1}H^{2i-1}(F_{i-2}Y_\kappa, \mu_{\ell^r}^{\otimes i})}{H^{2i-1}(Y_\kappa, \mu_{\ell^r}^{\otimes i})} \longrightarrow \frac{F^{i-1}H^{2i-1}(F_{i-2}Y, \mu_{\ell^r}^{\otimes i})}{H^{2i-1}(Y, \mu_{\ell^r}^{\otimes i})}$$

is injective (this follows by a standard specialization argument, or by Lemma 4.4 and the fact that $A^i(Y_\kappa)/\ell^r \hookrightarrow A^i(Y)/\ell^r$, since κ and k are both algebraically closed). By definition in Lemma 4.5, Λ is on elementary tensors given by

$$\Lambda([\tilde{\delta}(\alpha)] \otimes [\beta]) = [p^*\alpha \cup q^*\beta],$$

where $p : X \times Y \to X$ and $q : X \times Y \to Y$ denote the natural projections and where $\alpha \in H^1(X, \mu_{\ell^r})$ and $\beta \in H^{2i-1}(F_{i-1}Y, \mu_{\ell^r}^{\otimes i})$.

We need to show that Λ is injective when restricted to the subspace

$$\tilde{\delta}\left(H^1(X, \mu_{\ell^r}^{\otimes 1})\right) \otimes \frac{F^{i-1}H^{2i-1}(F_{i-2}Y_\kappa, \mu_{\ell^r}^{\otimes i})}{H^{2i-1}(Y_\kappa, \mu_{\ell^r}^{\otimes i})}.$$

For this let $\alpha_1,\dots,\alpha_n \in H^1(X,\mu_{\ell^r})$ and $\beta_1,\dots,\beta_n \in F^{i-1}H^{2i-1}(F_{i-2}Y,\mu_{\ell^r}^{\otimes i})$ such that each β_i is defined over κ, i.e.

$$\beta_i \in \operatorname{im}(F^{i-1}H^{2i-1}(F_{i-2}Y_\kappa,\mu_{\ell^r}^{\otimes i}) \to F^{i-1}H^{2i-1}(F_{i-2}Y,\mu_{\ell^r}^{\otimes i})) \tag{7.1}$$

and such that

$$\sum_{j=1}^n \tilde{\delta}(\alpha_j)\otimes[\beta_j] \neq 0 \in \tilde{\delta}\left(H^1(X,\mu_{\ell^r})\right)\otimes \frac{F^{i-1}H^{2i-1}(F_{i-2}Y,\mu_{\ell^r}^{\otimes i})}{H^{2i-1}(Y,\mu_{\ell^r}^{\otimes i})}. \tag{7.2}$$

We assume for a contradiction that this class lies in the kernel Λ, which by the derivation property for $\tilde{\delta}$ (see [Hat02, p. 304]) is equivalent to saying that

$$\sum_{j=1}^n p^*(\tilde{\delta}\alpha_j)\cup q^*\beta_j - \sum_{j=1}^n p^*(\alpha_j)\cup q^*(\tilde{\delta}\beta_j) \in F^{i+1}H^{2i+1}(F_{i-1}(X\times Y),\mu_{\ell^r}^{\otimes i+1}), \tag{7.3}$$

which means that the above class in $H^{2i+1}(F_{i-1}(X\times Y),\mu_{\ell^r}^{\otimes i+1})$ extends to a class on $X\times Y$. The theorem will be proven if we derive a contradiction from (7.2) and (7.3).

Since X is a smooth proper connected surface, we have a canonical isomorphism

$$H^4(X,\mu_{\ell^r}^{\otimes 2}) = \mathbb{Z}/\ell^r\cdot \mathrm{cl}^2(pt.)$$

where $\mathrm{cl}^2(pt.)$ denotes the cycle class of a point on X.

Step 1. Up to a change of the presentation of the class in (7.2), we may assume that for any $j_0\in\{1,\dots,n\}$, there is a class $\overline{\tilde{\delta}(\alpha_{j_0})}\in H^2(X,\mu_{\ell^r})$ with

$$\overline{\tilde{\delta}(\alpha_{j_0})}\cup\tilde{\delta}(\alpha_j) = \begin{cases} \mathrm{cl}^2(pt.) & \text{if } j=j_0;\\ 0 & \text{otherwise.}\end{cases}$$

Moreover, the above property does not change if we add to $\overline{\tilde{\delta}(\alpha_{j_0})}$ a class that lifts to $H^2(X,\mathbb{Z}_\ell(1))$.

Proof. If $r>1$, then our assumption in (**C3**) says that the cup product pairing on $H^2(X,\mu_{\ell^r})$ is perfect. For $r=1$, the pairing in question is always perfect by Poincaré duality. This yields the existence of the class $\overline{\tilde{\delta}(\alpha_{j_0})}$.

By the compatibility of the integral Bockstein and $\tilde{\delta}$, the class $\tilde{\delta}(\alpha_j)$ is the reduction modulo ℓ^r of a torsion class in $H^2(X,\mathbb{Z}_\ell(1))$. Since any class in $H^2(X,\mathbb{Z}_\ell(1))$ has trivial cup product pairing with a torsion class in $H^2(X,\mathbb{Z}_\ell(1))$, we find that the properties in question do not change if we add to $\overline{\tilde{\delta}(\alpha_{j_0})}$ a class that lifts to $H^2(X,\mathbb{Z}_\ell(1))$. This concludes step 1. $\square$

Since $\beta_j\in F^{i-1}H^{2i-1}(F_{i-2}Y,\mu_{\ell^r}^{\otimes i})$ for all $j=1,\dots,n$, there is an open subset $U\subset Y$ such that $\operatorname{codim}_Y(Y\setminus U)>i-1$ and such that β_j is represented by a class in $H^{2i-1}(U,\mu_{\ell^r}^{\otimes i})$

for all $i = 1, \dots, n$. By slight abuse of notation, we will from now on denote the lift of β_j to $H^{2i-1}(U, \mu_{\ell^r}^{\otimes i})$ by the same letter.

Assumption (7.3) together with Lemma 3.2 then implies that there is a class

$$\xi \in \bigoplus_{x \in (X \times Y)^{(i)}} H^1(\kappa(x), \mu_{\ell^r})$$

such that

$$\gamma := \sum_{j=1}^{n} p^*(\tilde{\delta}\alpha_j) \cup q^*\beta_j - \sum_{j=1}^{n} p^*(\alpha_j) \cup q^*(\tilde{\delta}\beta_j) + \iota_*\xi \in H^{2i+1}(F_i(X \times U), \mu_{\ell^r}^{\otimes i+1}) \tag{7.4}$$

extends to a class on $X \times Y$.

Step 2. There is a well-defined pushforward map

$$q_* : H^{2i+3}(F_i(X \times U), \mu_{\ell^r}^{\otimes i+2}) \longrightarrow H^{2i-1}(F_{i-2}U, \mu_{\ell^r}^{\otimes i})$$

and for any $j_0 \in \{1, \dots, n\}$, we have

$$q_*\left(p^*\left(\overline{\tilde{\delta}(\alpha_{j_0})}\right) \cup \iota_*\xi\right) = q_*\left(p^*\left(\overline{\tilde{\delta}(\alpha_{j_0})}\right) \cup \gamma\right) - \beta_{j_0} \in H^{2i-1}(F_{i-2}U, \mu_{\ell^r}^{\otimes i}). \tag{7.5}$$

Proof. The existence of the pushforward map is a direct consequence of the fact that the projection $X \times U \to U$ is proper of relative dimension two, because X is a smooth proper surface.

The claimed equality is then a consequence of (7.4), the computation in Step 1 and the fact that

$$q_*\left(p^*\left(\overline{\tilde{\delta}(\alpha_{j_0})} \cup \alpha_j\right) \cup q^*(\tilde{\delta}\beta_j)\right) = 0,$$

because $\overline{\tilde{\delta}(\alpha_{j_0})} \cup \alpha_j$ is of degree three. This concludes step 2. $\square$

Recall that R is the local ring of a smooth pointed curve $(B, 0)$ over κ. From now one we will repeatedly need to make base changes. By this we will mean a base change corresponding to a ring map $R \to R'$, where R' is the local ring of a smooth pointed curve $(B', 0')$ over κ, where $B' \to B$ is a finite morphism that maps $0'$ to 0. When we perform such a base change, the model $\mathcal{X}$ becomes singular, but it follows from [Har01, Proposition 2.2] that $\mathcal{X} \times_R R'$ can be made into a strictly semi-stable R'-scheme $\mathcal{X}' \to \operatorname{Spec} R'$ by repeatedly blowing up all non-Cartier components of the special fibre. The exceptional divisors introduced in these blow-ups are ruled surfaces over the algebraically closed field κ and so they have trivial second unramified cohomology. For this reason, assumptions (**C1**), (**C2**) and (**C3**) remain true after such a base change.

In what follows, by the phrase ‘ performing a suitable base change’ will mean a base change via some ring map $R \to R'$ as above, where we replace $\mathcal{X} \times_R R'$ by the strictly semi-stable model $\mathcal{X}' \to \operatorname{Spec} R'$ constructed above.

Step 3. Up to a suitable base change, we may assume that $H^1(X_\eta, \mu_{\ell^r}) \to H^1(X, \mu_{\ell^r})$ and $H^2(X_\eta, \mu_{\ell^r}) \to H^2(X, \mu_{\ell^r})$ are surjective, where X_η denotes the generic fibre of $\mathcal{X} \to \operatorname{Spec} R$. In particular, α_j and $\overline{\tilde{\delta}(\alpha_j)}$ can be defined on X_η.

Proof. Since $H^1(X, \mu_{\ell^r})$ is a finite group, there is a normal finite index subgroup $H \subset \mathrm{Gal}(k/K)$ that acts trivially on $H^1(X, \mu_{\ell^r})$. Since R is the local ring of a smooth pointed curve $(B, 0)$ over κ, this means that up to replacing B by a finite cover and 0 by a point in the preimage, we may assume that $\mathrm{Gal}(k/K)$ acts trivially on $H^1(X, \mu_{\ell^r})$.

By the Hochschild–Serre spectral sequence, we then get an exact sequence

$$0 \longrightarrow H^1(K, \mu_{\ell^r}) \longrightarrow H^1(X_\eta, \mu_{\ell^r}) \longrightarrow H^1(X, \mu_{\ell^r}) \xrightarrow{d_2} H^2(K, H^1(X, \mu_{\ell^r})) = 0$$

where the last vanishing follows from the fact that K has cohomological dimension one. This shows that $H^1(X_\eta, \mu_{\ell^r}) \to H^1(X, \mu_{\ell^r})$ may be assumed to be surjective and the assertion concerning the surjectivity of $H^2(X_\eta, \mu_{\ell^r}) \to H^2(X, \mu_{\ell^r})$ is proven exactly the same way. This concludes step 3. $\square$

Up to a further base change, we may assume that $U = U_K \times k$ for some K-variety U_K.

Step 4. Up to a suitable base change, we may assume that for $j = 1, \ldots, m$ there are flat proper normal R-schemes $\mathcal{W}_j \to \operatorname{Spec} R$ with proper morphisms

$$p : \mathcal{W}_j \longrightarrow \mathcal{X} \quad \text{and} \quad q : \mathcal{W}_j \longrightarrow \mathcal{Y},$$

such that the following holds. There are classes $\xi_j \in H^1(K(W_{j,\eta}), \mu_{\ell^r})$, such that

$$\sum_j (p \times q)_*(\xi_j) \in H^{2i+1}(F_i(X_\eta \times U_K), \mu_{\ell^r}^{\otimes i+1})$$

pulls back to $\iota_* \xi \in H^{2i+1}(F_i(X \times U), \mu_{\ell^r}^{\otimes i+1})$ over the algebraic closure k of K.

Proof. We may write $\xi = \sum \xi_w$, where $\xi_w \in H^1(\kappa(w), \mu_{\ell^r}) \simeq \kappa(w)^*/(\kappa(w)^*)^{\ell^r}$ for some codimension i-points $w \in (X \times Y)^{(i)}$. Up to a suitable base change, we may assume that each w and each class ξ_w is already defined over K. The result in question is then clear. $\square$

By (7.1), β_j is defined over κ and hence over K. By steps 3 and 4, the class γ in (7.4) can thus be defined over K. That is, there is a class

$$\gamma_K \in H^{2i+1}(F_i(X_\eta \times_K U_K), \mu_{\ell^r}^{\otimes i+1})$$

that restricts to γ over $k = \overline{K}$.

Step 5. Up to a suitable base change, we may assume that γ_K extends to a class on $H^{2i+1}(X_\eta \times_K Y_K, \mu_{\ell^r}^{\otimes i+1})$.

Proof. We know that there is a class $\gamma' \in H^{2i+1}(X\times Y, \mu_{\ell^r}^{\otimes i+1})$ that extends γ. The result in question thus follows if we can show that up to a suitable base change, γ' is Galois invariant and

$$H^{2i+1}(X_\eta \times_K Y_K, \mu_{\ell^r}^{\otimes i+1}) \longrightarrow H^{2i+1}(X \times Y, \mu_{\ell^r}^{\otimes i+1})^{\mathrm{Gal}(k/K)}$$

is surjective. The former follows from the fact that $H^{2i+1}(X\times Y, \mu_{\ell^r}^{\otimes i+1})$ is a finite group and so there is a normal finite index subgroup of $\mathrm{Gal}(k/K)$ that acts trivially. The latter follows similarly as in Step 3 from the Hochschild–Serre spectral sequence and the fact that K has cohomological dimension one. This concludes step 5. □

Since $U \subset Y$ is defined over K, it extends to a smooth R-scheme

$$\mathcal{U} \longrightarrow \operatorname{Spec} R$$

such that the special fibre $U_0 = \mathcal{U} \times \kappa$ is an open subset of Y_0 whose complement has dimension at least i. By steps 3, 4, and 5, we may assume that α_j, $\overline{\tilde{\delta}(\alpha_{j_0})}$, ξ and γ, as well as the lift of γ to $H^{2i+1}(X\times Y, \mu_{\ell^r}^{\otimes i+1})$ are all defined over K. By (7.1), β_j is defined over κ and hence in particular over K. By slight abuse of notation, we will denote the respective classes over k and over K with the same symbol.

Step 6. We have

$$q_*\left(p^*\left(\overline{\tilde{\delta}(\alpha_{j_0})}\right) \cup \iota_*\xi\right) = q_*\left(p^*\left(\overline{\tilde{\delta}(\alpha_{j_0})}\right) \cup \gamma\right) - \beta_{j_0} \in H^{2i-1}(F_{i-2}U_K, \mu_{\ell^r}^{\otimes i}).$$

Proof. The equality in question is an equality in the cohomology of some open subset of U_K whose complement has codimension at least $i-1$. By (7.5) in step 2, there is an open subset $V \subset U$ over k whose complement has codimension at least $i-1$ such that the equality in question holds on V. Up to a suitable base change, we may assume that V is defined over K. By the Hochschild–Serre spectral sequence, there is an exact sequence

$$0 \longrightarrow H^{2i-1}(K, \mu_{\ell^r}^{\otimes i}) \longrightarrow H^{2i-1}(V_K, \mu_{\ell^r}^{\otimes i}) \longrightarrow H^{2i-1}(V, \mu_{\ell^r}^{\otimes i}).$$

Any class in $H^{2i-1}(K, \mu_{\ell^r}^{\otimes i})$ vanishes after a suitable base change (in fact it vanishes on the nose for $i \geq 2$, because K has cohomological dimension one) and so the claimed equality in step 6 follows. □

Since $\mathcal{U}$ is smooth over R, we may apply the specialization map from Section 5.2 to the identity in step 6. We then find

(7.6)
$$sp\left(q_*\left(p^*\left(\overline{\tilde{\delta}(\alpha_{j_0})}\right) \cup \iota_*\xi\right)\right) = sp\left(q_*\left(p^*\left(\overline{\tilde{\delta}(\alpha_{j_0})}\right) \cup \gamma\right) - \beta_{j_0}\right) \in H^{2i-1}(F_{i-2}U_0, \mu_{\ell^r}^{\otimes i}).$$

Step 7. We have

$$sp\left(q_*\left(p^*\left(\overline{\tilde{\delta}(\alpha_{j_0})}\right)\cup \iota_*\xi\right)\right)=0\in H^{2i-1}(F_{i-2}U_0,\mu_{\ell^r}^{\otimes i}).$$

Proof. Let $w\in (X\times Y)^{(i)}$ and let $\xi_w\in H^1(\kappa(w),\mu_{\ell^r})$. Consider the two projections $p:X\times Y\to X$ and $q:X\times Y\to Y$. If $q(x)\in Y$ is not of codimension $i-2$, then

$$q_*\left(p^*\left(\overline{\tilde{\delta}(\alpha_{j_0})}\right)\cup \iota_*\xi_w\right)=0\in H^{2i-1}(F_{i-2}U,\mu_{\ell^r}^{\otimes i}).$$

In particular, we may without loss of generality assume that in the notation of step 4, $\operatorname{codim}_{\mathcal{Y}}(q(\mathcal{W}_j))=i-2$ for each j. We then deduce the vanishing in question from step 4 and Proposition 6.1. ☐

By (7.6) and step 7,

$$sp\left(q_*\left(p^*\left(\overline{\tilde{\delta}(\alpha_{j_0})}\right)\cup \gamma\right)\right)=sp(\beta_{j_0})\in H^{2i-1}(F_{i-2}U_0,\mu_{\ell^r}^{\otimes i}).$$

By steps 3 and 5, $q_*\left(p^*\left(\overline{\tilde{\delta}(\alpha_{j_0})}\right)\cup \gamma\right)$ extends to a class on Y_K and so

$$sp(\beta_{j_0})\in F^iH^{2i-1}(F_{i-2}U_0,\mu_{\ell^r}^{\otimes i})$$

by Lemma 5.2. On the other hand, β_{j_0} is already defined over κ by (7.1) and so

$$sp(\beta_{j_0})=\beta_{j_0}\in H^{2i-1}(F_{i-2}U_0,\mu_{\ell^r}^{\otimes i})$$

by Lemma 5.3 (where we again abuse notation slightly as our notation does not distinguish between β_{j_0} and its pullback via some some field extension of κ). In particular, we deduce that β_{j_0} extends to a class on Y_κ. Hence, β_{j_0} is zero in the quotient

$$H^{2i-1}(F_{i-2}Y_\kappa,\mu_{\ell^r}^{\otimes i})/H^{2i-1}(Y_\kappa,\mu_{\ell^r}^{\otimes i}).$$

This holds for any index j_0, which contradicts the assumption in (7.2). This concludes the proof of the theorem. ☐

Remark 7.2. *Looking back at the proof of Theorem 7.1, we see that conditions (**C1**) and (**C2**) can be slightly weakened as follows. The surjectivity in (**C1**) is only needed to lift the classes $\overline{\tilde{\delta}(\alpha_{j_0})}$ from Step 2 to $\mathcal{X}$. These classes may by Step 2 be modified by the image of integral classes and so (**C1**) may be weakened to only ask that the composition*

$$H^2_{nr}(\mathcal{X},\mu_{\ell^r}^{\otimes 2})\longrightarrow H^2_{nr}(X,\mu_{\ell^r}^{\otimes 2})\longrightarrow H^2_{nr}(X,\mu_{\ell^r}^{\otimes 2})/H^2_{nr}(X,\mathbb{Z}_\ell(2))$$

is surjective.

*Item (**C2**) is only used to ensure that the lifts of the classes $\overline{\tilde{\delta}(\alpha_{j_0})}$ to $\mathcal{X}$ restrict trivially to the components of the special fibre of $\mathcal{X}$ and it would be enough to replace (**C2**) by this more precise condition.*

In section 4.4 (see (4.3)), we have constructed an exterior product map

$$\times : A^1(X)[\ell^r] \otimes E^i_{\ell^r}(Y) \longrightarrow \mathcal{T}^{i+1}(X \times Y)[\ell^r], \quad [z_1] \otimes [z_2] \longmapsto [z_1 \times z_2].$$

In the notation of Theorem 7.1, we may restrict that map to get a map

$$\times : A^1(X)[\ell^r] \otimes E^i_{\ell^r}(Y_\kappa) \longrightarrow \mathcal{T}^{i+1}(X \times Y)[\ell^r],$$

where we used that $\kappa \subset k$ is an extension of algebraically closed fields and so $E^i_{\ell^r}(Y_\kappa) \subset E^i_{\ell^r}(Y)$.

Corollary 7.3. *In the notation of Theorem 7.1, the natural composition*

$$A^1(X)[\ell^r] \otimes E^i_{\ell^r}(Y_\kappa) \xrightarrow{\times} \mathcal{T}^{i+1}(X \times Y)[\ell^r] \longrightarrow \mathrm{Griff}^{i+1}(X \times Y)/\ell^r \longrightarrow A^{i+1}(X \times Y)/\ell^r$$

has as kernel the subspace

$$\big(\ell^r \cdot A^1(X)[\ell^{2r}]\big) \otimes E^i_{\ell^r}(Y_\kappa).$$

Proof. It is clear that any class in $(\ell^r \cdot A^1(X)[\ell^{2r}]) \otimes E^i_{\ell^r}(Y_\kappa)$ maps to zero in $A^{i+1}(X \times Y)/\ell^r$. The converse implication follows from Proposition 4.6 and Theorem 7.1. $\square$

8. Degenerations of surfaces with $p_g = 0$ and $H_1 = \mathbb{Z}/2$

The purpose of this section is to give an explicit criterion that allows to construct degenerations $\mathcal{X} \to \operatorname{Spec} R$ of (Enriques) surfaces with the properties needed in Theorem 7.1.

Let k be an algebraically closed field (for simplicity) of characteristic zero and let $B = \operatorname{Spec} k[s,t]_{(s,t)}$ be the spectrum of the local ring of $\mathbb{A}^2_k$ at the origin. Let $B_0 := \{t = 0\} \subset B$ be the fibre above 0 of the natural projection $p : B \to \operatorname{Spec} k[t]_{(t)}$. Assume that there is a regular flat proper B-scheme $\mathcal{Y} \to B$ with the following properties:

A1 the geometric generic fibre $Y_{\overline{\eta}}$ is a smooth projective surface with $p_g(Y_{\overline{\eta}}) = 0$, $H^1(Y_{\overline{\eta}}, \mu_2) \simeq \mathbb{Z}/2$ and such that the pullback map

$$H^1(\mathcal{Y}, \mu_2) \longrightarrow H^1(Y_{\overline{\eta}}, \mu_2)$$

is surjective;

A2 the composition $\mathcal{Y} \to B \to \operatorname{Spec} k[t]_{(t)}$ is smooth away from 0 and the central fibre $\mathcal{Y}_{B_0}$ is a union

$$\mathcal{Y}_{B_0} = \mathcal{Y}_{0,1} \cup \mathcal{Y}_{0,2} \cup \cdots \cup \mathcal{Y}_{0,r},$$

where each $\mathcal{Y}_{0,i}$ is a geometrically irreducible regular flat proper surface over B_0 whose geometric generic fibre is a ruled surface over $\overline{k(s)}$ and such that any three different components of $\mathcal{Y}_{B_0}$ have trivial intersection;

A3 if $\mathcal{C}_{ij} := \mathcal{Y}_{0,i} \cap \mathcal{Y}_{0,j}$ for $i \neq j$ is non-empty, then it is a geometrically irreducible regular flat proper curve over B_0 with the property that the pullback map

$$H^1(\mathcal{C}_{ij}, \mu_2) \longrightarrow H^1(\overline{C}_{ij}, \mu_2)$$

is zero, where $\overline{C}_{ij} := \mathcal{C}_{ij} \times \overline{k(s)}$ denotes the geometric generic fibre of $\mathcal{C}_{ij} \to B_0$.

Lemma 8.1. *Assume that $\mathcal{Y} \to B$ is a regular flat proper B-scheme that satisfies (**A1**), (**A2**) and (**A3**). Let $m \geq 1$ be an integer and consider the base change $\mathcal{Y} \times_B B$, where $B \to B$ is given by $s \longmapsto s$ and $t \longmapsto t^m$. Then there is a resolution $\mathcal{Y}' \to \mathcal{Y} \times_B B$, given by repeatedly blowing up the singular locus of $\mathcal{Y} \times_B B$, which is again a regular flat proper B-scheme that satisfies (**A1**), (**A2**) and (**A3**).*

Proof. Condition (**A2**) and (**A3**) imply that étale locally at a point of $\mathcal{Y}$ where $\mathcal{Y} \to B$ is not smooth, $\mathcal{Y}$ looks like $xy = t$. This shows that a resolution of $\mathcal{Y}' \to \mathcal{Y} \times_B B$ can be constructed by repeatedly blowing up the singular locus of $\mathcal{Y} \times_B B$. The exceptional divisors E will be $\mathbb{P}^1$-bundles over non-empty intersections $\mathcal{C}_{ij}$ (with $i \neq j$) which admit two disjoint sections such that E meets exactly two of the other components – one along one sections and another one along the other section of $E \to \mathcal{C}_{ij}$. This description shows that (**A2**) and (**A3**) are satisfied for $\mathcal{Y}'$, which proves the lemma, as (**A1**) is clearly true for $\mathcal{Y}'$ as it holds for $\mathcal{Y}$. □

Theorem 8.2. *Let $\mathcal{Y} \to B$ be a regular flat proper B-scheme which satisfies (**A1**), (**A2**) and (**A3**). Then up to a base change $t \longmapsto t^m$ and replacing $\mathcal{Y}$ by the model $\mathcal{Y}'$ from Lemma 8.1, the following holds.*

*Let $\overline{k(s)}$ be an algebraic closure of $k(s)$ and consider $\overline{R} = \overline{k(s)}[[t]]$. Let $\overline{\mathcal{X}} \longrightarrow \operatorname{Spec} \overline{R}$ be the regular flat proper $\overline{R}$-scheme, given as base change of $\mathcal{Y}$. Then the assumptions (**C1**) and (**C2**) from Theorem 7.1 are satisfied for $\overline{\mathcal{X}} \longrightarrow \operatorname{Spec} \overline{R}$. More precisely, for any $n \in \mathbb{Z}$, the following hold:*

1. *the restriction map $H^2_{nr}(\overline{\mathcal{X}}, \mu_2^{\otimes n}) \longrightarrow H^2_{nr}(\overline{X}_{\overline{\eta}}, \mu_2^{\otimes n})$ is surjective;*
2. *any component $\overline{X}_{0i}$ of the special fibre of $\overline{\mathcal{X}} \to \operatorname{Spec} \overline{R}$ is a ruled surface over an algebraically closed field; in particular, the restriction map*

$$H^2_{nr}(\overline{\mathcal{X}}, \mu_2^{\otimes n}) \longrightarrow H^2_{nr}(\overline{X}_{0i}, \mu_2^{\otimes n})$$

is zero.

Proof. The second item follows from the fact that $\overline{X}_{0i}$ is by item (**A2**) a ruled surface over an algebraically closed field and so $H^2_{nr}(\overline{X}_{0i}, \mu_2^{\otimes 2}) = 0$. Moreover, $\mu_2^{\otimes n} \simeq \mathbb{Z}/2$ for all n and so it suffices to prove the conclusion in the theorem in the case where $n = 2$.

To prove the first item, let $R := k[s]_{(s)}[[t]]$ with natural map $\operatorname{Spec} R \to B$. The base change

$$\mathcal{X} := \mathcal{Y} \times_B \operatorname{Spec} R \longrightarrow \operatorname{Spec} R$$

is a regular flat proper R-scheme with

$$\overline{\mathcal{X}} = \mathcal{X} \times_R \overline{R}.$$

We split the proof into several steps.

Step 1. We have

$$H^2_{nr}(Y_{\overline{\eta}}, \mu_2^{\otimes 2}) \simeq \mathrm{Br}(Y_{\overline{\eta}}) \simeq \mathbb{Z}/2 \quad \text{and} \quad H^2_{nr}(X_{\overline{\eta}}, \mu_2^{\otimes 2}) \simeq \mathrm{Br}(X_{\overline{\eta}}) \simeq \mathbb{Z}/2.$$

Proof. Since $X_{\overline{\eta}}$ is given by base changing $Y_{\overline{\eta}}$ to a larger algebraically closed field, it clearly suffices to prove the claimed identity for $Y_{\overline{\eta}}$. By (**A1**), $p_g(Y_{\overline{\eta}}) = 0$ and $H^1(Y_{\overline{\eta}}, \mu_2) \simeq \mathbb{Z}/2$. This implies

$$H^2_{nr}(Y_{\overline{\eta}}, \mu_2^{\otimes 2}) \simeq \mathrm{Br}(Y_{\overline{\eta}}) \simeq H^2(Y_{\overline{\eta}}, \mathbb{G}_m) \simeq \mathbb{Z}/2,$$

where we use the assumption that k has characteristic zero, which in particular implies by the Lefschetz $(1,1)$-theorem that

$$H^2_{nr}(Y_{\overline{\eta}}, \mu_2^{\otimes 2}) \simeq H^2(Y_{\overline{\eta}}, \mu_2^{\otimes 2})/H^2(Y_{\overline{\eta}}, \mathbb{Z}_2(2)).$$

This concludes the proof of Step 1. □

Step 2. Up to a base change $t \longmapsto t^m$ and up to replacing $\mathcal{Y}$ by the model $\mathcal{Y}'$ from Lemma 8.1, we may assume that the pullback map

$$H^2_{nr}(X_\eta, \mu_2^{\otimes 2}) \longrightarrow H^2_{nr}(X_{\overline{\eta}}, \mu_2^{\otimes 2})$$

is surjective.

Proof. Let $G = \mathrm{Gal}(\overline{k(s,t)}/k(s,t))$, then there is a Hochschild–Serre spectral sequence

$$E_2^{p,q} = H^p(G, H^q(Y_{\overline{\eta}}, \mathbb{G}_m)) \Longrightarrow H^{p+q}(Y_\eta, \mathbb{G}_m),$$

see [Mil80, III.2.20]. Since $k(s,t)$ has cohomological dimension two, $E_2^{p,q} = 0$ for all $p > 2$ and so $E_3 = E_\infty$. The Hochschild–Serre spectral sequence thus yields an exact sequence

$$H^2(k(s,t), \mathbb{G}_m) \longrightarrow H^2(Y_\eta, \mathbb{G}_m) \longrightarrow H^2(Y_{\overline{\eta}}, \mathbb{G}_m)^G \xrightarrow{d_2} H^2(G, H^2(Y_{\overline{\eta}}, \mathbb{G}_m)).$$

Since $H^2(Y_{\overline{\eta}}, \mathbb{G}_m) \simeq \mathbb{Z}/2$ and $\overline{k(t)}(s)$ has cohomological dimension one, we see that there is a finite cover $C \to \operatorname{Spec} k[t]$ such that the differential d_2 in the above sequence vanishes if we pass from $k(s,t)$ to $k(C)(s)$. Let A be the local ring of $C \times \mathbb{A}^1$ at a point that maps to the origin in $\mathbb{A}^2_k$ and let $\mathcal{Y}'' := \mathcal{Y} \times A$. Then the above sequence shows that

$$H^2(Y''_\eta, \mathbb{G}_m) \longrightarrow H^2(Y''_{\overline{\eta}}, \mathbb{G}_m)^G$$

is surjective. By step 1,

$$H^2(Y''_{\overline{\eta}}, \mathbb{G}_m) \simeq \mathbb{Z}/2.$$

The Galois group G acts linearly on the above group and so it must act trivially. Hence, the restriction map

$$H^2(Y''_\eta, \mathbb{G}_m) \longrightarrow H^2(Y''_{\overline{\eta}}, \mathbb{G}_m)^G = H^2(Y''_{\overline{\eta}}, \mathbb{G}_m)$$

is surjective.

The finite cover $C \to \operatorname{Spec} k[t]$ corresponds, after base change to $k[[t]]$, to a disjoint union of covers of the form $t \longmapsto t^m$. We conclude from this that up to a base change $t \longmapsto t^m$ and up to replacing $\mathcal{Y}$ by the model $\mathcal{Y}'$ from Lemma 8.1, we may assume that the natural map

$$H^2(X_\eta, \mathbb{G}_m) \longrightarrow H^2(X_{\overline{\eta}}, \mathbb{G}_m)$$

is surjective. This concludes the proof of step 2. □

Let $B_0 := \{t = 0\} \subset \operatorname{Spec} R$ and consider the base change $\mathcal{X}_{B_0} := \mathcal{X} \times_R B_0$.

Step 3. Up to a base change $t \longmapsto t^2$ and up to replacing $\mathcal{Y}$ by $\mathcal{Y}'$ from Lemma 8.1, the natural pullback map

$$H^2_{nr}(\mathcal{X} \setminus \mathcal{X}_{B_0}, \mu_2^{\otimes 2}) \longrightarrow H^2_{nr}(X_{\overline{\eta}}, \mu_2^{\otimes 2})$$

is surjective.

Proof. By steps 1 and 2, there is a class $\alpha \in H^2_{nr}(X_\eta, \mu_2^{\otimes 2})$ that restricts to a generator of $H^2_{nr}(X_{\overline{\eta}}, \mu_2^{\otimes 2}) \simeq \mathbb{Z}/2$. Consider $\{s = 0\} \subset \operatorname{Spec} R$ with base change $\mathcal{X}_{\{s=0\}}$, which is a flat proper generically smooth scheme over $\{s = 0\} \simeq \operatorname{Spec} k[[t]]$. The claim in step 3 will follow if we can show that we may arrange that α has trivial residue at the generic point of $\mathcal{X}_{\{s=0\}}$. Since $\mathcal{Y}$ extends to a smooth k-variety over $\mathbb{A}^2$, the Gysin sequence (3.5) shows that the residue in question lifts to a class

$$\partial_{\mathcal{X}_{\{s=0\},\eta}}(\alpha) \in H^1(\mathcal{X}_{\{s=0\},\eta}, \mu_2),$$

where $\mathcal{X}_{\{s=0\},\eta}$ denotes the generic fibre of $\mathcal{X}_{\{s=0\}} \to \{s = 0\}$. The Hochschild–Serre spectral sequence yields an exact sequence

$$0 \longrightarrow H^1(k((t))\mu_2) \longrightarrow H^1(\mathcal{X}_{\{s=0\},\eta}, \mu_2) \longrightarrow H^1(\mathcal{X}_{\{s=0\},\overline{\eta}}, \mu_2) \simeq \mathbb{Z}/2$$

where the latter isomorphism follows from item (**A1**), which by the smooth and proper base change theorem in étale cohomology implies that $H^1(\mathcal{X}_{\{s=0\},\overline{\eta}}, \mu_2) \simeq \mathbb{Z}/2$ is Galois invariant. By (**A1**), the image of $\partial_{\mathcal{X}_{\{s=0\},\eta}}(\alpha)$ in $H^1(\mathcal{X}_{\{s=0\},\overline{\eta}}, \mu_2)$ lifts to a class $\gamma \in H^1(\mathcal{X}, \mu_2)$. Hence, up to replacing α by $\alpha - (s, \gamma)$ (which does not change the restriction to the geometric generic fibre and does not change any other residue on $\mathcal{X}$, because γ is unramified on $\mathcal{X}$), we may assume that

$$\partial_{\mathcal{X}_{\{s=0\},\eta}}(\alpha) \in H^1(k[[t]], \mu_2) \subset H^1(\mathcal{X}_{\{s=0\},\eta}, \mu_2).$$

Since $H^1(k((t)),\mu_2)\simeq k((t))^*/(k((t))^*)^2\simeq \mathbb{Z}/2$, the residue in question vanishes after a base change $t\longmapsto t^2$. This concludes step 3. $\square$

Note that $\mathcal{X}_{B_0}=\mathcal{Y}_{B_0}$ and so (**A2**) implies that

$$\mathcal{X}_{B_0}=\mathcal{X}_{0,1}\cup\mathcal{X}_{0,2}\cup\cdots\cup\mathcal{X}_{0,r},$$

where each $\mathcal{X}_{0,i}$ is a geometrically irreducible regular flat proper surface over B_0 whose geometric generic fibre is a ruled surface over $\overline{k(s)}$ and such that any three different components of $\mathcal{X}_{B_0}$ have trivial intersection.

By Step 3, we may and will from now on fix a class

$$\alpha\in H^2_{nr}(\mathcal{X}\setminus\mathcal{X}_{B_0},\mu_2^{\otimes 2})$$

that restricts to a generator of $H^2_{nr}(X_{\overline{\eta}},\mu_2^{\otimes 2})\simeq\mathbb{Z}/2$.

Step 4. Up to a base change $t\longmapsto t^2$ and up to replacing $\mathcal{Y}$ by $\mathcal{Y}'$ from Lemma 8.1, we may assume that the residues

$$\gamma_i:=\partial_{\mathcal{X}_{0,i}}(\alpha)\in H^1(k(\mathcal{X}_{0i}),\mu_2)$$

have the following properties:

- for any indices $i\neq j$ such that $\mathcal{C}_{ij}=\mathcal{X}_{0,i}\cap\mathcal{X}_{0,j}$ is non-empty, γ_i or γ_j vanishes;
- if $\gamma_i\neq 0$, then $\mathcal{X}_{0i}$ is a $\mathbb{P}^1$-bundle over one of the curves $\mathcal{C}_{ij}=\mathcal{X}_{0,i}\cap\mathcal{X}_{0,j}$ from (**A2**).

Proof. Let us perform a base change $t\longmapsto t^2$ and let $\mathcal{Y}'$ be the model from Lemma 8.1. By the commutative diagram (3.3), this base change has the effect that the pullback of α to $\mathcal{Y}'\setminus\mathcal{Y}'_{B_0}$ must have trivial residues at all components of $\mathcal{Y}'_{B_0}$ that are not introduced in the blow-up $\mathcal{Y}'\to\mathcal{Y}\times_R R$, where R maps to R via $t\longmapsto t^2$. By assumptions, three different components of $\mathcal{X}_{B_0}$ do not meet. The exceptional divisors of $\mathcal{Y}'\to\mathcal{Y}\times_R R$ are thus disjoint, because the singularities of $\mathcal{Y}\times_R R$ look étale locally like $xy=t^2$ and so they are resolved by a single blow-up. This implies the claim in step 4. $\square$

Let $\mathcal{U}\subset\mathcal{X}$ be the maximal open subset such that α is unramified on $\mathcal{U}$. By step 4, we may assume that the complement $\mathcal{X}\setminus\mathcal{U}$ is a disjoint union of $\mathbb{P}^1$-bundles $\mathcal{T}_i\to\mathcal{C}_i$, where $\mathcal{C}_i\to B_0$ is a regular proper flat curve which by item (**A3**) has the property that

$$H^1(\mathcal{C}_i,\mu_2)\longrightarrow H^1(\overline{C}_i,\mu_2)$$

is zero, where $\overline{C}_i=\mathcal{C}_i\times\overline{k(s)}$. Since α is unramified on $\mathcal{U}$, it actually lifts by (3.2) to a honest class $\alpha_{\mathcal{U}}\in H^2(\mathcal{U},\mu_2^{\otimes 2})$. Since $\mathcal{Y}$ extends to a smooth k-variety, the Gysin sequence (see (3.5)) then shows that the residue

$$\partial_{\mathcal{T}_i}(\alpha)\in H^1(k(\mathcal{T}_i),\mu_2)$$

lifts to a class in $H^1(\mathcal{T}_i,\mu_2)\simeq H^1(\mathcal{C}_i,\mu_2)$. On the other hand, the restriction map

$$H^1(\mathcal{T}_i,\mu_2)\simeq H^1(\mathcal{C}_i,\mu_2)\longrightarrow H^1(\mathcal{T}_{i,\overline{\eta}},\mu_2)\simeq H^1(\overline{C}_i,\mu_2)$$

is zero by assumption (**A3**). This implies that the pullback of α to

$$\overline{\mathcal{X}}=\mathcal{X}\times \operatorname{Spec}\overline{k(s)}[[t]]$$

is unramified on $\overline{\mathcal{X}}$ and it restricts to the unique generator of $H^2_{nr}(\overline{X}_{\overline{\eta}},\mu_2^{\otimes 2})\simeq \mathbb{Z}/2$. This concludes the proof of the theorem. □

9. A two-dimensional degeneration of Enriques surfaces

The purpose of this section is to prove the following theorem, which implies Theorem 1.7 from the introduction.

Theorem 9.1. *There is a discrete valuation ring R with algebraically closed residue field κ of characteristic zero and fraction field K, together with a flat proper strictly semi-stable R-scheme $\mathcal{X}\to \operatorname{Spec} R$ whose geometric generic fibre $X_{\overline{\eta}}=\mathcal{X}\times\overline{K}$ is an Enriques surface with the following properties:*

- *the restriction map $H^2_{nr}(\mathcal{X},\mu_2)\longrightarrow H^2_{nr}(X_{\overline{\eta}},\mu_2)$ is surjective;*
- *any component X_{0i} of the central fibre $X_0=\mathcal{X}\times_R\kappa$ is ruled; in particular, the restriction map $H^2_{nr}(\mathcal{X},\mu_2)\longrightarrow H^2_{nr}(X_{0i},\mu_2)$ is zero.*

By Theorem 8.2, the proof of the above theorem is reduced to the following: For some algebraically closed field k of characteristic zero, we need to construct a regular flat proper scheme $\mathcal{Y}\to B=\operatorname{Spec} k[s,t]_{(s,t)}$ whose geometric generic fibre is an Enriques surface and such that items (**A1**), (**A2**) and (**A3**) from Section 8 hold.

9.1. An elliptic curve over the function field of $\mathbb{P}^1$.

Lemma 9.2 ([MP86])**.** *Let k be an algebraically closed field of characteristic zero. There is a minimal rational elliptic surface $Y\to\mathbb{P}^1$ over k with the following properties:*

- *there is a singular fibre Y_0 of type I_2;*
- *the elliptic fibration $Y\to\mathbb{P}^1$ admits exactly two sections and for any of the two components of Y_0 there is exactly one section that meets that component.*

Proof. By [MP86, Table 5.2], the rational elliptic surface $Y:=X_{321}$ in *loc. cit.* has exactly two sections and a singular fibre of type I_2. A Weierstrass equation for Y is given by

$$y^2=x^3-(s+3)x+s+2,$$

where s denotes an affine coordinate on $\mathbb{P}^1$. The singular fibre of type I_2 lies above $s=0$, where we get the Weierstrass equation

$$y^2=x^3-3x+2=(x-1)^2(x+2) \tag{9.1}$$

of a nodal rational curve. The two sections are given by the point at ∞ and by the point given by $y = 0$ and $x = 1$. The former lies in the smooth locus of (9.1), while the latter passes through the singular point of that nodal curve. This shows that the two sections of $Y \to \mathbb{P}^1$ pass through different components of the I_2-fibre, which concludes the proof. □

Let $0, \tau \subset Y$ denote the two sections of $Y \to \mathbb{P}^1$. Since Y is relatively minimal over $\mathbb{P}^1$, translation by τ induces a regular involution $\iota : Y \to Y$. Since 0 and τ pass through different components of the I_2-type fibre Y_0, the involution ι is fixed point free locally around Y_0 (see e.g. [MP89, 5.1.2]) and so the quotient $Y' := Y/\iota \to \mathbb{P}^1$ is regular locally around the fibre Y'_0, which is a singular fibre of type I_1.

Let now $A := \mathcal{O}_{\mathbb{P}^1,0}$ and consider the base changes

$$\mathcal{C} := Y \times \operatorname{Spec} A \longrightarrow \operatorname{Spec} A \quad \text{and} \quad \mathcal{C}' := Y' \times \operatorname{Spec} A \longrightarrow \operatorname{Spec} A. \tag{9.2}$$

Let $K = k(\mathbb{P}^1)$ and define

$$C := \mathcal{C} \times_A K \quad \text{and} \quad C' := \mathcal{C}' \times_A K. \tag{9.3}$$

Then $C(K) = \{0, \tau\}$ for the 2-torsion point τ above.

The key property for us will be as follows.

Lemma 9.3. *(1) The monodromy action of* $\operatorname{Gal}(\overline{K}/K)$ *on* $H^1(C_{\overline{K}}, \mu_2)$ *fixes exactly one nontrivial element* $\gamma \in H^1(C_{\overline{K}}, \mu_2)$*.*

(2) The class γ *does not lie in the image of the restriction map*

$$H^1(\mathcal{C}, \mu_2) \longrightarrow H^1(C_{\overline{K}}, \mu_2)$$

Proof. The Galois action on $H^1(C_{\overline{K}}, \mu_2)$ is isomorphic to the Galois action on the 2-torsion points of $C_{\overline{K}}$. Since $C(K) = \{0, \tau\}$, this action fixes exactly one nontrivial element $\gamma \in H^1(C_{\overline{K}}, \mu_2)$. Moreover, γ is given by the étale cover of C that corresponds to the 2-torsion line bundle $\tau - 0$ on C.

The K-points 0 and τ of C come from sections of $\mathcal{C} \to \operatorname{Spec} A$ that pass through different components of the special fibre $C_0 = C_{01} \cup C_{02}$. This description readily implies that $\tau - 0$ does not extend to a 2-torsion line bundle on $\mathcal{C}$. Hence, γ does not lie in the image of

$$H^1(\mathcal{C}, \mu_2) \longrightarrow H^1(C_{\overline{K}}, \mu_2),$$

as we want. This concludes the lemma. □

Corollary 9.4. *Let* $\overline{C} := C \times_K \overline{K}$*. Then the natural pullback map*

$$H^1(\mathcal{C}, \mu_2) \longrightarrow H^1(\overline{C}, \mu_2)$$

is zero.

Proof. A class in $H^1(\overline{C},\mu_2)$ that lies in the image of the map in question is invariant under the Galois group $\mathrm{Gal}(\overline{K}/K)$ and it lifts to $\mathcal{C}$. Any such class is zero by Lemma 9.3, which proves the corollary. $\square$

9.2. **Elliptic curves with involution.**

Lemma 9.5. *Let C be a smooth projective curve of genus one over a field K of characteristic zero. Assume that there is a fixed point free involution $\iota : C \to C$. Then there are two smooth quadrics $Q_1, Q_2 \subset \mathbb{P}^3$ with the following property:*

- *$Q_1, Q_2 \subset \mathbb{P}^3$ are both invariant under the involution*
$$\varphi : \mathbb{P}^3 \longrightarrow \mathbb{P}^3, \quad [x_0 : x_1 : y_0 : y_1] \longmapsto [y_0 : y_1 : x_0 : x_1];$$
- *$C \simeq Q_1 \cap Q_2$ and the restriction of φ coincides with ι.*

Proof. After the choice of a base point $0 \in C(\overline{K})$, the involution $\iota : C \to C$ corresponds over $\overline{K}$ to translation by a 2-torsion point τ. Since ι is defined over K, it follows that the set $\{0, \tau\} \subset C(\overline{K})$ is Galois invariant. This implies (by faithful flat descent) that the line bundle $\mathcal{O}_C(2\cdot[0]+2\cdot[\tau])$ is defined over K. By Riemann–Roch, this bundle has four linearly independent sections (over K). It is well-known that these sections yield an embedding
$$C \hookrightarrow \mathbb{P}^3 = \mathbb{P}(H^0(C, \mathcal{O}_C(2\cdot[0]+2\cdot[\tau]))^\vee).$$
Since $H^0(\mathbb{P}^3, \mathcal{O}(2))$ has dimension ten, while $H^0(C, \mathcal{O}_C(4\cdot[0]+4\cdot[\tau]))$ is of dimension eight, $H^0(\mathbb{P}^3, \mathcal{I}_C(2))$ is at least two-dimensional. Hence, C is contained on the intersection of two quadrics. These quadrics cannot have a plane in common (because C does not lie on a plane) and so C must be the complete intersection of two quadrics by degree reasons.

Note that $\iota^*\mathcal{O}_C(2\cdot[0]+2\cdot[\tau]) = \mathcal{O}_C(2\cdot[0]+2\cdot[\tau])$ and so the involution ι on C yields an action
$$\iota^* : H^0(C, \mathcal{O}_C(2\cdot[0]+2\cdot[\tau])) \longrightarrow H^0(C, \mathcal{O}_C(2\cdot[0]+2\cdot[\tau])), \quad s \longmapsto \iota^* s$$
This action induces an involution
$$\varphi : \mathbb{P}(H^0(C, \mathcal{O}_C(2\cdot[0]+2\cdot[\tau]))^\vee) \longrightarrow \mathbb{P}(H^0(C, \mathcal{O}_C(2\cdot[0]+2\cdot[\tau]))^\vee).$$
If we choose a basis of $H^0(C, \mathcal{O}_C(2\cdot[0]+2\cdot[\tau]))$ of the form $s_1, \iota^*s_1, s_2, \iota^*s_2$ for general $s_1, s_2 \in H^0(C, \mathcal{O}_C(2\cdot[0]+2\cdot[\tau]))$, then we see that the above action is in suitable coordinates given by
$$[x_0 : x_1 : y_0 : y_1] \longmapsto [y_0 : y_1 : x_0 : x_1],$$
as we want.

By construction, the action φ restricts to the involution ι on $C \subset \mathbb{P}^3$. Moreover, φ induces an involution on the space of quadrics $\mathbb{P}^1 \simeq \mathbb{P}H^0(\mathbb{P}^3, \mathcal{I}_C(2))$ that contain C and so there are two φ-invariant quadrics $Q_1, Q_2 \subset \mathbb{P}^3$ with $C = Q_1 \cap Q_2$. Since C is smooth, Q_i has rank at least 3 for $i = 1, 2$. Since φ acts on Q_i and since this action is free on C, Q_i cannot be a cone over a smooth conic (as φ would need to fix the tip of the cone and hence has at least two lines as fixed loci, contradicting the fact that it is free on C). Hence, Q_1 and Q_2 are smooth, as we want. ☐

Lemma 9.6. *If the curve C in Lemma 9.5 is the elliptic curve over $K = k(\mathbb{P}^1)$ from (9.3), then the quadrics $Q_1, Q_2 \subset \mathbb{P}^3_K$ from Lemma 9.5 may be chosen in such a way that they extend to proper flat A-schemes*

$$\mathcal{Q}_1, \mathcal{Q}_2 \subset \mathbb{P}^3_A$$

such that $\mathcal{C} \simeq \mathcal{Q}_1 \cap \mathcal{Q}_2$ is the model from (9.2).

Proof. Consider the divisor $[0] + [\tau]$ on $\mathcal{C}$ (which is defined over K). The involution ι of $\mathcal{C}$ acts on the linear series $H^0(\mathcal{C}, \mathcal{O}_{\mathcal{C}}(2[0] + 2[\tau]))$.

By (9.2), $\mathcal{C} = Y \times_k \operatorname{Spec} A$. Since Y is a rational surface, numerical and linear equivalence for divisors coincide on Y and hence also on $\mathcal{C}$. It follows that $2[0] + 2[\tau]$ is linearly equivalent to $4[0] + C_{01}$ and also to $4[\tau] + C_{02}$, where $C_0 = C_{01} \cup C_{02}$ and C_{01} is the component that meets $[0]$, while C_{02} meets $[\tau]$. In particular, the linear series $H^0(\mathcal{C}, \mathcal{O}_{\mathcal{C}}(2[0] + 2[\tau]))$ is base-point free and we obtain a morphism

$$\psi : \mathcal{C} \longrightarrow \mathbb{P}^3_A.$$

The restriction of ψ to the generic fibre coincides with the inclusion from Lemma 9.5. We claim that ψ is a closed embedding. To prove this, we need to study the restriction of ψ to the special fibre C_0.

To this end we consider the exact sequence

$$H^0(\mathcal{C}, \mathcal{O}_{\mathcal{C}}(2[0] + 2[\tau]) \longrightarrow H^0(C_0, \mathcal{O}_{\mathcal{C}}(2[0] + 2[\tau])|_{C_0}) \longrightarrow H^1(\mathcal{C}, \mathcal{O}_{\mathcal{C}}(2[0] + 2[\tau] - C_0)).$$

The line bundle $\mathcal{O}_{\mathcal{C}}(2[0] + 2[\tau] - C_0)$ is linearly equivalent to $\mathcal{O}_{\mathcal{C}}(2[0] + 2[\tau])$, which is ample over $\operatorname{Spec} A$. Since the relative canonical bundle of $\mathcal{C}$ is trivial over A, the relative Kodaira (resp. Kawamata–Vieweg) vanishing theorem implies that the above H^1-term vanishes. (More precisely, if $f : \mathcal{C} \to \operatorname{Spec} A$ denotes the structure morphism, then $R^1 f_* \mathcal{O}_{\mathcal{C}}(2[0] + 2[\tau]) = 0$, which implies $H^1(\mathcal{C}, \mathcal{O}_{\mathcal{C}}(2[0] + 2[\tau] - C_0)) = 0$.) Hence,

$$H^0(\mathcal{C}, \mathcal{O}_{\mathcal{C}}(2[0] + 2[\tau]) \longrightarrow H^0(C_0, \mathcal{O}_{\mathcal{C}}(2[0] + 2[\tau])|_{C_0})$$

is surjective. This shows that ψ embeds C_0 into $\mathbb{P}^3_k$. (More precisely, ψ embeds each of the components C_{0i} as a smooth plane conic contained, each contained in a different plane of $\mathbb{P}^3_k$.)

There is an exact sequence

$$0\longrightarrow H^0(C_0,\mathcal{O}_{C_0}(4[0]+4[\tau]))\longrightarrow \bigoplus_{i=1}^{2} H^0(C_{0i},\mathcal{O}(4))\longrightarrow k^{\oplus 2}\longrightarrow 0,$$

where $C_{0i}\simeq \mathbb{P}^1$ denote the components of C_0 and

$$\bigoplus_{i=1}^{2} H^0(C_{0i},\mathcal{O}(4))\longrightarrow k^{\oplus 2},\quad (s_1,s_2)\longmapsto (s_1(0)-s_2(0),s_1(\infty)-s_2(\infty)).$$

This implies that $H^0(C_0,\mathcal{O}_{C_0}(4[0]+4[\tau]))$ has dimension 8, while $H^0(\mathbb{P}^3_k,\mathcal{O}(2))$ has dimension 10. Hence, $\psi(C_0)\subset \mathbb{P}^3_k$ lies on two quadrics. These quadrics cannot have a plane in common, because $\psi(C_0)$ is not contained in the union of a plane and a line. Hence, $\psi(C_0)\subset \mathbb{P}^3_k$ and so it must be the complete intersection of two quadrics.

Let $Q_1,Q_2\subset \mathbb{P}^3_K$ be the quadrics from Lemma 9.5 with $C=Q_1\cap Q_2$. Let $\mathcal{Q}_i\subset \mathbb{P}^3_A$ be the closure of Q_i. Then $\mathcal{Q}_i$ is invariant under the action of φ, because Q_i is. Moreover, $\psi(\mathcal{C})\subset \mathcal{Q}_1\cap\mathcal{Q}_2$ and we claim that we may assume that equality holds scheme-theoretically:

$$\psi(\mathcal{C})=\mathcal{Q}_1\cap\mathcal{Q}_2.$$

This is true on the generic fibre by construction and so it suffices to show that the above equality holds on the special fibre. As we already know that $\psi(C_0)$ is the complete intersection of two quadrics, it suffices to show that the special fibres of $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are different φ-invariant quadrics. We have $\mathcal{Q}_i=\{q_i=0\}$ for quadratic forms $q_i\in A[x_0,x_1,y_0,y_1]$ that are invariant under the action of φ, given by $x_i\longmapsto y_i$ and $y_i\longmapsto x_i$. If $\mathcal{Q}_1$ and $\mathcal{Q}_2$ have the same special fibre, then there is a nonzero constant $\lambda_0\in k$ so that $q_1-\lambda_0q_2$ vanishes identically on the special fibre of $\mathbb{P}^3_A$. Hence, $q_1-\lambda_0q_2$ is divisible by the parameter π of A and so

$$q_{11}:=\frac{1}{\pi}(q_1-\lambda_0q_2)\in A[x_0,x_1,y_0,y_1].$$

If q_2 and q_{11} cut out different quadrics in the special fibre, then we replace q_1 by q_{11} and we are done. Otherwise there is some $\lambda_1\in k$ such that

$$q_{12}:=\frac{1}{\pi}(q_{11}-\lambda_1q_2)\in A[x_0,x_1,y_0,y_1].$$

Repeating this inductively, we are done if the process stops after finitely many steps. Otherwise, we get an element $\hat{\lambda}=\lambda_0+\pi\lambda_1+\pi^2\lambda_2+\dots$ in the completion $\widehat{A}$ of A with

$$q_1=\hat{\lambda}q_2\in \widehat{A}[x_0,x_1,y_0,y_1].$$

But then the images in $K[x_0,x_1,y_0,y_1]$ of the quadratic forms q_1 and q_2 from above must be multiples of each other (as this is true after passing from $K=\operatorname{Frac}A$ to the field extension $\operatorname{Frac}\widehat{A}$). This is a contradiction, which proves our claim, and hence the lemma. $\square$

9.3. **Construction of the model.** Let k be an algebraically closed field of characteristic zero and let $K = k(\mathbb{P}^1)$. Let C and C' be the elliptic curves over K from (9.3). Consider the involution

$$\varphi : \mathbb{P}^5_A \longrightarrow \mathbb{P}^5_A, \quad [x_0 : x_1 : x_2 : y_0 : y_1 : y_2] \longmapsto [y_0 : y_1 : y_2 : x_0 : x_1 : x_2].$$

Let $q_0, q_1, q_2 \in A[x_0, x_1, x_2, y_0, y_1, y_2]$ be homogeneous polynomials of degree two which are general subject to the following two conditions:

(1) We have

$$\mathcal{Q}_1 = \{q_1 = x_2 = y_2 = 0\} \subset \mathbb{P}^3_A \quad \text{and} \quad \mathcal{Q}_2 = \{q_2 = x_2 = y_2 = 0\} \subset \mathbb{P}^3_A,$$

where $\mathcal{Q}_1, \mathcal{Q}_2 \subset \mathbb{P}^3_A$ are the quadrics from Lemma 9.6.

(2) We have

$$q_i(y_0, y_1, y_2, x_0, x_1, x_2) = q_i(x_0, x_1, x_2, y_0, y_1, y_2)$$

for each $i = 0, 1, 2$.

Consider the 2-dimensional base $B := \operatorname{Spec} A[t]_{(t)}$. Since A is a discrete valuation ring, B has a unique closed point $0 \in B$. We further consider the curve $B_0 := \{t = 0\} \subset B$ and the projective B-scheme

$$\mathcal{Z} := \{x_2 y_2 + t q_0 = q_1 = q_2 = 0\} \subset \mathbb{P}^5_{A[t]_{(t)}}.$$

Lemma 9.7. *The morphism $\mathcal{Z} \to B$ is flat and smooth away from B_0. The fibre product $\mathcal{Z}_{B_0} := \mathcal{Z} \times_B B_0$ has two irreducible components*

$$\mathcal{Z}_{B_0,1} = \{x_2 = q_1 = q_2\} \subset \mathbb{P}^5_A \quad \textit{and} \quad \mathcal{Z}_{B_0,2} = \{y_2 = q_1 = q_2\} \subset \mathbb{P}^5_A,$$

each of which is regular and in fact smooth over $\operatorname{Spec} A$. *The intersection $\mathcal{Z}_{B_0,1} \cap \mathcal{Z}_{B_0,2}$ is isomorphic to $\mathcal{C}$ from (9.2). The involution*

$$\varphi : \mathbb{P}^5_B \longrightarrow \mathbb{P}^5_B, \quad [x_0 : x_1 : x_2 : y_0 : y_1 : y_2] \longmapsto [y_0 : y_1 : y_2 : x_0 : x_1 : x_2]$$

induces a fixed point free involution on $\mathcal{Z}$, whose restriction to $\mathcal{C}$ coincides with the involution ι.

Proof. Recall that $q_0, q_1, q_2 \in A[x_0, x_1, x_2, y_0, y_1, y_2]$ are general subject to the following two conditions:

- q_0, q_1 and q_2 are invariant under $x_i \longmapsto y_i$ for all $i = 0, 1, 2$;
- $\mathcal{Z}_{B_0,1} \cap \mathcal{Z}_{B_0,2} = \{x_2 = y_2 = q_1 = q_2\} \simeq \mathcal{C} \subset \mathbb{P}^3_A$.

We first show that $\mathcal{Z}_{B_0,i}$ is a smooth irreducible surface over $B_0 = \operatorname{Spec} A$ for each $i = 1, 2$ and it suffices to deal with $i = 1$ by symmetry. Since smoothness is an open condition, it suffices to show that the special fibre of $\mathcal{Z}_{B_0,1}$ above $0 \in B_0$ is an irreducible smooth projective surface. By construction, this is an intersection of two general quadrics in $\mathbb{P}^4_k$ subject to the condition that the quadrics are invariant under

$$\varphi'([x_0 : x_1 : y_0 : y_1 : y_2]) = [y_0 : y_1 : x_0 : x_1 : y_2]$$

and that they contain the curve $C_0 \subset \{y_2 = 0\} \subset \mathbb{P}^4_k$, which is a degree four curve of type I_2. More precisely, C_0 is the union $C_{01} \cup C_{02}$ of two smooth plane conics C_{01} and C_{02} contained on two different planes of $\{y_2 = 0\} \simeq \mathbb{P}^3_k$. Since φ' acts freely on C_0 and interchanges the two components of C_0, the condition that the φ'-invariant quadrics in question contain C_0 is equivalent to asking that they contain one of its components, say C_{01}.

Let $V \subset H^0(\mathbb{P}^4_k, \mathcal{O}(2))$ be the linear series of all quadrics that are φ'-invariant and contain C_{01}. Up to a change of coordinates, we may assume that $C_{01} \subset \{x_1 = y_2 = 0\}$ and $C_{02} \subset \{y_1 = y_2 = 0\}$. Let

$$q_{C_{01}} \in \left\langle x_0y_0,\, x_0^2 + y_0^2,\, x_1^2 + y_1^2,\, x_0y_1 + y_0x_1,\, y_0y_1 + x_0x_1 \right\rangle$$

be a φ'-invariant quadratic form that cuts out C_{01} on $\{x_1 = y_2 = 0\}$. Each element in V is then given by the sum of a multiple of $q_{C_{01}}$ with an element of

$$\left\langle x_1y_1,\, y_2^2,\, y_2(x_0 + y_0),\, y_2(x_1 + y_1) \right\rangle .$$

This description shows that the base locus of V is contained in the subscheme of $\mathbb{P}^4_k$ cut out by the ideal

$$I := (x_1y_1,\, y_2^2,\, y_2(x_0 + y_0),\, y_2(x_1 + y_1)).$$

We have $I = I_1 \cap I_2$ where

$$I_1 = (x_1y_1,\, y_2) \quad \text{and} \quad I_2 = (x_1y_1,\, y_2^2,\, x_0 + y_0,\, x_1 + y_1).$$

The ideal I_2 cuts out the subscheme $V(I_2) \subset \mathbb{P}^4_k$ which on the plane $\{x_0 + y_0 = x_1 + y_1 = 0\} \simeq \mathbb{P}^2_k$ is given by $\{x_1^2 = y_2^2 = 0\}$. Consider the reduced scheme

$$V(I_2)^{\mathrm{red}} = \{x_1 = y_1 = y_2 = x_0 + y_0 = 0\} \subset \mathbb{P}^4_k = \{x_2 = 0\} \subset \mathbb{P}^5_k.$$

Viewed as a point on $\mathbb{P}^5_k$, the reduced scheme $V(I_2)^{\mathrm{red}}$ is a point in the fixed locus of φ. Moreover, $V(I_2)^{\mathrm{red}}$ lies on the plane $\{x_1 = y_2 = 0\} \subset \mathbb{P}^4_k$. Since the linear series of quadrics V intersects $\{x_1 = y_2 = 0\}$ in the smooth conic C_{01} and φ does not fix any point of C_{01} (it acts freely on $C_0 = C_{01} \cup C_{02}$), we see that the base locus of V is disjoint from $V(I_2)$. Since $V(I) = V(I_1) \cup V(I_2)$, the base locus of V must thus be contained in

$$V(I_1) = \{x_1 = y_2 = 0\} \cup \{y_1 = y_2 = 0\},$$

which is the union of the two planes on which C_0 lies. This implies that the base locus of V is scheme-theoretically given by C_0.

Since C_0 has two-dimensional tangent spaces at its singular points, and because we computed the base locus of V scheme-theoretically, we deduce from Bertini's theorem that the intersection of two general elements of V is an irreducible smooth projective surface in $\mathbb{P}^4_k$. This surface is the special fibre of $\mathcal{Z}_{B_0,1} \to B_0$. It follows that $\mathcal{Z}_{B_0,1}$ is smooth over $B_0 = \operatorname{Spec} A$, hence in particular regular because B_0 is smooth over k. By symmetry, the same holds for $\mathcal{Z}_{B_0,2}$. The intersection $\mathcal{Z}_{B_0,1} \cap \mathcal{Z}_{B_0,2}$ is isomorphic to $\mathcal{C}$ by construction.

Our description of Z_{B_0} shows that it is a complete intersection of three quadrics. This is an open condition and so $\mathcal{Z}$ must be flat over B.

Consider now the subspace

$$V_A \subset A[x_0, x_1, x_2, y_0, y_1, y_2]$$

of all quadratic forms q that are invariant under $x_i \longmapsto y_i$ for all $i = 0, 1, 2$ and such that

$$\mathcal{C} \subset \{x_2 = y_2 = q = 0\}.$$

We have seen above that scheme-theoretically, the base locus of this subspace of quadrics is over $t = 0$ given by C_0. Since this holds scheme-theoretically, we conclude that the base locus of V_A must be equal to $\mathcal{C}$ (it cannot be larger than it would than give rise to a larger base locus at $t = 0$ by specialization). Knowing this, we can apply Bertini's theorem to deduce that $\mathcal{Z}$ is smooth over $B \setminus B_0$.

Note that φ acts on $\mathcal{Z}$ and the restriction of this action to $\mathcal{C} = \mathcal{Z}_{B_0,1} \cap \mathcal{Z}_{B_0,2}$ coincides with the involution ι on $\mathcal{C}$ by Lemmas 9.5 and 9.6. Since φ has no fixed points on $\mathcal{C}$, Bertini's theorem implies that φ is in fact fixed point free on $\mathcal{Z}$. This concludes the lemma. $\square$

Corollary 9.8. *The non-regular locus of $\mathcal{Z}$ is given by*

$$\{x_2 = y_2 = q_0 = q_1 = q_2 = t = 0\} \subset \mathcal{Z}.$$

Proof. One inclusion is clear. For the converse, note that $\mathcal{Z} \to B$ is smooth away from B_0 and $\mathcal{Z}_{B_0} = \mathcal{Z}_{B_0,1} \cup \mathcal{Z}_{B_0,2}$ is the union of two smooth components whose intersection is regular and flat over B_0. It follows that a point $z \in \mathcal{Z}$ is regular if and only if $\mathcal{Z}_{B_0,1}$ and $\mathcal{Z}_{B_0,2}$ are both Cartier at z. Since $\mathcal{Z}_{B_0}$ is Cartier, this is equivalent to asking that either $\mathcal{Z}_{B_0,1}$ or $\mathcal{Z}_{B_0,2}$ is Cartier at z. This implies that the non-regular locus is contained in $\{x_2 = y_2 = q_0 = q_1 = q_2 = t = 0\}$, which concludes the proof. $\square$

The non-regular locus of $\mathcal{Z}$ is by the above corollary given by the intersection of $\mathcal{C} \subset \mathbb{P}^3_A$ with the quadric $\{q_0 = 0\}$. Since q_0 is general, the above locus meets C_0 transversely in 8 distinct smooth points. The non-regular locus of $\mathcal{Z}$ is therefore étale of degree 8 over $B_0 = \operatorname{Spec} A$.

We aim to construct a small φ-equivariant resolution of these singularities. To this end we need to pass to the base change

$$\{x_2y_2 + t^2q_0 = q_1 = q_2 = 0\} \subset \mathbb{P}^5_B.$$

Let $\mathcal{Z}'$ be the blow-up of the above B-scheme in

$$\{x_2 = y_2 = q_1 = q_2 = t = 0\} \subset \mathcal{Z}.$$

The fibre product $\mathcal{Z}'_{B_0} = \mathcal{Z}'' \times_B B_0$ is reduced with three irreducible components:

$$\mathcal{Z}'_{B_0} = \mathcal{Z}'_{B_0,1} \cup \mathcal{Q} \cup \mathcal{Z}'_{B_0,2},$$

where $\mathcal{Q}$ is a $\mathbb{P}^1$-bundle over $\mathcal{C}$ that has two distinct sections, and where $\mathcal{Z}'_{B_0,i} \simeq \mathcal{Z}_{B_0,i}$ for $i = 1, 2$. Each of the components $\mathcal{Z}'_{B_0,1}$ and $\mathcal{Z}'_{B_0,2}$ is glued to $\mathcal{Q}$ along one of these sections. In particular, the two end components $\mathcal{Z}'_{B_0,1}$ and $\mathcal{Z}'_{B_0,2}$ of $\mathcal{Z}'_{B_0}$ are disjoint and we may consider the blow-up $\mathcal{Z}'' \to \mathcal{Z}'$ of $\mathcal{Z}'$ along $\mathcal{Z}'_{B_0,1} \cup \mathcal{Z}'_{B_0,2}$. A local computation shows that $\mathcal{Z}''$ is a small resolution of $\mathcal{Z}'$ and the restriction to B_0 is given by

$$\mathcal{Z}''_{B_0} = \mathcal{Z}''_{B_0,1} \cup_{\mathcal{C}} \mathcal{Q} \cup_{\mathcal{C}} \mathcal{Z}''_{B_0,2}, \tag{9.4}$$

where $\mathcal{Q}$ is the $\mathbb{P}^1$-bundle over $\mathcal{C}$ from above which is glued to the two other components by disjoint sections of $\mathcal{Q} \to \mathcal{C}$. Moreover, $\mathcal{Z}''_{B_0,i}$ is isomorphic to the blow-up of $\mathcal{Z}_{B_0,i}$ along

$$\{x_2 = y_2 = q_0 = q_1 = q_2 = 0\} \subset \mathbb{P}^5_A,$$

which is given by the intersection of $\mathcal{C}$ with the general quadric $\{q_0 = 0\}$.

The action of φ on $\mathcal{Z}$ induces an action on $\mathcal{Z}''$. This action is fixed point free, because φ acts without fixed points on $\mathcal{Z}$. We consider the quotient

$$\mathcal{Y} := \mathcal{Z}''/\varphi \longrightarrow B. \tag{9.5}$$

Since $\mathcal{Z}''$ is regular and φ acts without fixed points, the quotient $\mathcal{Y}$ is regular as well.

9.4. **Proof of Theorem 9.1.**

Proof of Theorem 9.1. Consider the regular flat proper B-scheme $\mathcal{Y} \to B$ from (9.5). Its geometric generic fibre is by Lemma 9.7 the quotient of a smooth intersection of three quadrics in $\mathbb{P}^5$ (which is a K3 surface) by a fixed point free involution, hence it is an Enriques surface. By Theorem 8.2, it suffices to show that $\mathcal{Y} \to B$ satisfies (**A1**), (**A2**) and (**A3**) from Section 8.

By construction, the K3-cover of the geometric generic fibre extends to the finite étale cover $\mathcal{Z}'' \to \mathcal{Y}$ and so

$$H^1(\mathcal{Y}, \mu_2) \longrightarrow H^1(Y_{\overline{\eta}}, \mu_2)$$

is surjective. This proves (**A1**).

By (9.4), we have

$$\mathcal{Y}_{B_0} = (\mathcal{Z}''_{B_0,1} \cup_{\mathcal{C}} \mathcal{Q} \cup_{\mathcal{C}} \mathcal{Z}''_{B_0,2})/\varphi \simeq \mathcal{Z}''_{B_0,1} \cup_{\mathcal{C}} (\mathcal{Q}/\varphi)$$

where $\mathcal{Q}/\varphi$ is a $\mathbb{P}^1$-bundle over $\mathcal{C}' = \mathcal{C}/\iota$. In particular,

$$\mathcal{Y}_{B_0} = \mathcal{Y}_{B_0,1} \cup \mathcal{Y}_{B_0,2}$$

is a union of two geometrically irreducible regular flat proper surfaces over B_0. The geometric generic fibre of $\mathcal{Y}_{B_0,i} \to B_0$ is ruled, as it is either birational to a $(2,2)$-complete intersection in $\mathbb{P}^4$, hence it is rational, or it is a $\mathbb{P}^1$-bundle over the elliptic curve $\overline{C}'$. This proves (**A2**).

Finally, $\mathcal{C} = \mathcal{Y}_{B_0,1} \cap \mathcal{Y}_{B_0,2}$ is a regular flat proper curve over B_0 which by Corollary 9.4 has the property that

$$H^1(\mathcal{C}, \mu_2) \longrightarrow H^1(\overline{C}, \mu_2)$$

is zero. This proves (**A3**), which concludes the proof of Theorem 9.1. □

10. Proof of main results

Theorem 1.3 stated in the introduction follows from the following slightly more precise result. To state the result, recall from Lemma 4.4 that

$$E_2^i(Y) \simeq \ker\left(\mathrm{cl}^i : A^i(Y)/2 \longrightarrow H^{2i}(Y, \mathbb{Z}_2(i))/2\right).$$

Theorem 10.1. *Let Y be a any smooth complex projective variety and let X be an Enriques surface over $\mathbb{C}$ that is very general with respect to Y. Let $D \in \mathrm{Pic}(X)[2]$ be the unique nontrivial 2-torsion class. Then for any $i \geq 0$, the natural composition*

$$E_2^i(Y) \longrightarrow \mathcal{T}^{i+1}(X \times Y)[2] \longrightarrow \mathrm{Griff}^{i+1}(X \times Y)/2 \longrightarrow A^{i+1}(X \times Y)/2,$$

given by $[z] \longmapsto [D \times z]$ is injective.

Proof. By a straightforward specialization argument, it suffices to prove the result for some smooth projective Enriques surfaces X over $\mathbb{C}$.

Since Chow groups modulo algebraic equivalence are countable, there is a countable algebraically closed field $\kappa \subset \mathbb{C}$ such that $Y = Y_\kappa \times \mathbb{C}$ for some smooth projective variety Y_κ over κ and such that the natural map

$$E_2^i(Y_\kappa) \longrightarrow E_2^i(Y)$$

is an isomorphism.

Let $R := \kappa[[t]]$. By Theorem 9.1, we may up to enlarging κ assume that there is regular flat proper scheme $\mathcal{X} \to \operatorname{Spec} R$ whose generic fibre is a smooth Enriques surface and such that:

- there is a class $\alpha \in H^2_{nr}(\mathcal{X}, \mu_2^{\otimes 2})$ whose pullback to the geometric generic fibre is the unique nonzero Brauer class of the Enriques surface $X_{\overline{\eta}}$;
- the restriction of α to each component of the special fibre is trivial.

Let k be an algebraic closure of the fraction field of R. Since κ is countable, we may assume that $k \subset \mathbb{C}$.

It follows from Corollary 7.3 that the natural composition

$$E_2^i(Y_\kappa) \longrightarrow \mathcal{T}^{i+1}(X_{\overline{\eta}} \times Y_k)[2] \longrightarrow \operatorname{Griff}^{i+1}(X_{\overline{\eta}} \times Y_k)/2, \longrightarrow A^{i+1}(X \times Y)/2$$

given by $[z] \longmapsto [D \times z]$ is injective. Since $k \subset \mathbb{C}$, we may consider the base change $X := X_{\overline{\eta}} \times_k \mathbb{C}$. Since $k \subset \mathbb{C}$ is an extension of algebraically closed fields, a well-known and straightforward specialization argument shows that

$$A^{i+1}(X_{\overline{\eta}} \times Y_k)/2 \longrightarrow A^{i+1}(X \times Y)/2$$

is injective. Altogether, we thus see that the natural composition

$$E_2^i(Y) \simeq E_2^i(Y_\kappa) \longrightarrow \mathcal{T}^{i+1}(X \times Y)[2] \longrightarrow \operatorname{Griff}^{i+1}(X \times Y)/2 \longrightarrow A^{i+1}(X \times Y)/2,$$

given by $[z] \longmapsto [D \times z]$ is injective. Here the Enriques surface X is somewhat special, but as noted above, this also shows that the map in question is injective for a very general Enriques surface in place of X. This concludes the proof. $\square$

Proof of Theorem 1.1. Let $C \subset \mathbb{P}^2_{\mathbb{C}}$ be a very general quartic curve. By [Tot16], $\operatorname{Griff}^2(JC)/2$ is infinite; the nontrivial classes are explicitly given by pullbacks of the Ceresa cycle $C - C^-$ via suitable isogenies. Theorem 1.3 implies that for a very general Enriques surface X with nontrivial 2-torsion class $D \in \operatorname{Pic}(X)[2]$, the map

$$\operatorname{Griff}^2(JC)/2 \longrightarrow \operatorname{Griff}^3(X \times JC)[2], \quad [z] \longmapsto [D \times z]$$

is injective. Hence, $\operatorname{Griff}^3(X \times JC)$ has infinite 2-torsion, as we want. $\square$

Proof of Corollary 1.2. For $n \geq 5$, the projective bundle formula shows that the smooth complex projective variety $X \times JC \times \mathbb{P}^{n-5}$, where C and X are as in the proof of Theorem 1.1, has infinite 2-torsion in Griff^i for all $3 \leq i \leq n-2$. This proves the corollary. $\square$

Proof of Corollary 1.4. By [OS20], there is a smooth complex projective threefold Y, given as a pencil of Enriques surfaces, such that $\mathrm{CH}_0(Y) \simeq \mathbb{Z}$, $H^*(Y, \mathbb{Z})$ is torsion-free and Y admits a non-algebraic Hodge class $\alpha \in H^{2,2}(Y, \mathbb{Z})$ such that 2α is algebraic. Then Y has torsion–free cohomology, $\mathrm{CH}_0(Y) \simeq \mathbb{Z}$ and $H^4(Y, \mathbb{Z})$ contains a Hodge class α that is non-algebraic, but such that 2α is algebraic. It thus follows from Theorem 1.3 that for any very general Enriques surface X, the product

$$Z := X \times Y \times \mathbb{P}^{n-5}$$

contains a nonzero 2-torsion class in $\mathcal{T}^3(Z)$. Moreover, $\mathrm{CH}_0(Z) \simeq \mathbb{Z}$, because X has trivial Chow group of zero-cycles. Since Y has torsion-free cohomology, the Künneth formula applies and shows that

$$H^5(Z,\mathbb{Z}) \simeq H^5(X \times \mathbb{P}^{n-5},\mathbb{Z}) \oplus (H^2(X \times \mathbb{P}^{n-5},\mathbb{Z}) \otimes H^3(Y,\mathbb{Z})) \oplus H^5(Y,\mathbb{Z}).$$

Since $\mathrm{CH}_0(Y) \simeq \mathbb{Z}$, $b_1(Y) = b_5(Y) = 0$ and so $H^5(Y,\mathbb{Z}) = 0$, as the cohomology of Y is torsion-free. The cohomology group $H^2(X \times \mathbb{P}^{n-5},\mathbb{Z})$ consists of algebraic classes, so that each class in that group vanishes away from some divisor. Since $\mathrm{CH}_0(Y) = 0$, Y admits a rational decomposition of the diagonal and so $H^3_{nr}(Y,\mathbb{Z})$ is torsion, hence vanishes because $H^3(\mathbb{C}(Y),\mathbb{Z})$ is torsion–free by [MS83]. Hence, classes in $H^3(Y,\mathbb{Z})$ vanish away from a divisor on Y. This shows that

$$H^2(X \times \mathbb{P}^{n-5},\mathbb{Z}) \otimes H^3(Y,\mathbb{Z}) \subset N^2H^5(X \times Y \times \mathbb{P}^{n-5},\mathbb{Z}).$$

Finally,

$$H^5(X \times \mathbb{P}^{n-5},\mathbb{Z}) \simeq H^3(X,\mathbb{Z}) \otimes H^2(\mathbb{P}^{n-5},\mathbb{Z}).$$

Here, $H^3(X,\mathbb{Z}) \simeq \mathbb{Z}/2$ vanishes away from a divisor (in fact $H^3(\mathbb{C}(X),\mathbb{Z}) = 0$ by dimension reasons). Since $H^2(\mathbb{P}^{n-5},\mathbb{Z})$ vanishes away from a divisor on $\mathbb{P}^{n-5}$, we find that

$$H^5(X \times \mathbb{P}^{n-5},\mathbb{Z}) \subset N^2H^5(X \times Y \times \mathbb{P}^{n-5},\mathbb{Z}).$$

Altogether, we have thus seen that

$$N^2H^5(Z,\mathbb{Z}) = H^5(Z,\mathbb{Z}).$$

Since $\mathrm{CH}_0(Z) \simeq \mathbb{Z}$, this implies by [Voi12, Corollary 0.3] that

$$H^4_{nr}(Z,\mathbb{Q}/\mathbb{Z}) \simeq \mathcal{T}^3(Z).$$

Moreover, $N^2H^5(Z,\mathbb{Z}) = H^5(Z,\mathbb{Z})$ implies that the intermediate Jacobian $J^5(Z)$ of Z is generated by the images of $J^1(\widetilde{W})$, where $W \subset Z$ runs through all subvarieties of codimension 2 and $\widetilde{W}$ denotes a resolution of W. In particular, the transcendental intermediate Jacobian $J^5_{tr}(Z)$ vanishes, and so

$$\mathcal{T}^3(Z) = \mathrm{Griff}^3(Z)_{\mathrm{tors}}.$$

This concludes the proof of the corollary. □

Proof of Corollary 1.5. The result follows directly from Corollary 1.4 and the fact that $\mathcal{T}^3(Z)$ is a birational invariant of smooth complex projective varieties, see [Voi12, Lemma 2.2]. This concludes the proof. □

Proof of Corollary 1.6. Let C be a smooth plane curve of degree 4 defined over $\mathbb{Q}$ and with good reduction at 2. Then JC has good reduction at 2 and [Dia20, Corollary 2.13] implies that the Kummer variety $Y = \widetilde{JC/\pm}$ associated to JC is a smooth complex projective variety with $H^3_{nr}(Y, \mu_2) \neq 0$. As noted in *loc. cit.* the Chow group of Y is supported on a surface (see [BS83, §4, Example (1)]) and the integral cohomology of Y is torsion-free, so that [CTV12] implies that there is a non-algebraic non-torsion Hodge class $\alpha \in H^{2,2}(Y, \mathbb{Z})$ such that 2α is algebraic, cf. [Dia20, Corollary 3.3]. For any very general complex projective Enriques surface X, Theorem 1.3 thus implies that $\mathrm{Griff}^3(X \times Y)[2] \neq 0$ and $\mathrm{Griff}^3(X \times Y)/2 \neq 0$. This concludes the proof. □

Proof of Theorem 1.7. This follows directly from Theorem 9.1. □

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