OPTIMALITY OF THE PRYM–TYURIN CONSTRUCTION FOR $\mathcal{A}_6$

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Abstract. We prove that on a very general principally polarized abelian 6-fold, the smallest multiple of the minimal curve class which can be represented by an algebraic cycle is 6.

1. Introduction

This paper is a sequel to [EGFS25b], where we gave counterexamples to the integral Hodge conjecture for abelian varieties, arising from certain combinatorial invariants of regular matroids.

It was classically known by Riemann that a general principally polarized abelian variety $(X, \Theta) \in \mathcal{A}_g$ of dimension $g = 1, 2, 3$ is the Jacobian of smooth genus g curve C. Thus, the integral Hodge class $\Theta^{g-1}/(g-1)!$ is represented by an algebraic cycle; the Abel–Jacobi image of C. For $g = 4, 5$, the general member $(X, \Theta) \in \mathcal{A}_g$ was shown by Wirtinger in 1895 to be the Prym variety of an étale double cover $\widetilde{C} \to C$ of a curve C of genus $g+1$. Thus $2 \cdot \Theta^{g-1}/(g-1)!$ is represented by an algebraic curve; the image of $\widetilde{C}$ under its natural map to the Prym variety.

In fact, the multiple 2 is optimal, for very general principally polarized abelian varieties of dimension $g = 4, 5$. By [EGFS25b, Thm. 1.1], the minimal algebraic multiple of the minimal curve class $\Theta^{g-1}/(g-1)!$ on a very general $(X, \Theta) \in \mathcal{A}_g$ is divisible by 2, for any $g \geq 4$.

Question 1.1. Let $(X, \Theta) \in \mathcal{A}_g$ be very general. What is the smallest positive integer $N_g \in \mathbb{N}$ for which $N_g \cdot \Theta^{g-1}/(g-1)!$ is algebraic?

A simple observation is that $N_g \mid (g-1)!$ since the algebraic multiples of $\Theta^{g-1}/(g-1)!$ form an ideal in $\mathbb{Z}$, and Θ^{g-1} is visibly algebraic. Furthermore $N_g \mid N_{g+1}$ follows by specializing to a product of an elliptic curve with a very general principally polarized abelian variety of lower dimension, see [EGFS25b, Sec. 8.3, Proof of Thm. 1.1]. The purpose of this paper is to add one entry to the list of known values of N_g. In tabular form:

g	1	2	3	4	5	6
N_g	1	1	1	2	2	6

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Theorem 1.2. *Let $(X, \Theta) \in \mathcal{A}_6$ be a very general principally polarized abelian 6-fold. The smallest positive multiple N_6 of the minimal curve class $\Theta^5/5!$ which can be represented by an algebraic cycle is exactly 6.*

In [EGFS25b, Corollary 1.5], we showed that not every principally polarized abelian variety admits a split embedding into a product of Jacobians (to admit such an embedding is equivalent to satisfying the integral Hodge conjecture for curve classes, see [BGF23; Voi24]). This leaves open the question whether every principally polarized abelian variety admits a split embedding into a product of Prym varieties. Via [BGF23], Theorem 1.2 implies that this is not the case:

Corollary 1.3. *A very general principally polarized abelian variety of dimension $g \geq 6$ does not admit a split embedding (of unpolarized abelian varieties) into a product of Prym varieties.*

By work of Alexeev–Donagi–Farkas–Izadi–Ortega [ADFIO20], it is known that $N_6 \,|\, 6$. To prove Theorem 1.2, it thus suffices to show that also $6 \,|\, N_6$. For this, we consider a certain positive integer-valued combinatorial invariant $d(G)$ of a graph G, introduced in [EGFS25b]. In the case $G = K_{g+1}$ of the complete graph on $g+1$ vertices, the results in *loc.cit.* imply that this invariant satisfies $d(K_{g+1}) \,|\, N_g$, see Theorem 2.13 below. Moreover, by [EGFS25b, Theorem 1.1], we know that $2 \,|\, N_6$. Hence to prove that $6 \,|\, N_6$ it suffices to show $3 \,|\, d(K_7)$. With the initial version of the program we wrote for computing such invariants, checking that $3 \,|\, d(K_7)$ was computationally infeasible. Here, we develop some theoretical tools, which allow us to "average" with respect to certain symmetries of the graph K_7, in turn reducing the computation to a manageable size. See Remark 3.10 for further details.

Prym–Tyurin constructions [Wel87; Kan87; BL92; Sal07] explicitly produce families of abelian varieties A on which some multiple of the minimal curve class, called the *exponent*, is the class of a curve $C \subset A$. Upper bounds for N_g arise from Prym–Tyurin constructions that dominate $\mathcal{A}_g$. It is unknown whether for large g, the minimal exponent of such a Prym–Tyurin construction is less than $(g-1)!$. Conversely, our invariants of regular matroids give lower bounds for N_g. For $g \leq 6$, we are lucky enough to achieve equality between the upper and lower bounds.

As discussed in [EGFS25b, Prop. 8.10], we can (independently of this paper) show that $N_7 \geq 6$ by computing $6 \,|\, d(K_{3,5})$. But arriving at an exact value of N_7 will likely require (1) techniques that give obstructions to the algebraicity of prime *power* multiples of the minimal class, and (2) a new Prym–Tyurin construction of small exponent, dominating $\mathcal{A}_7$.

Question 1.4. What is the value of N_7?

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2. Matroids and their solutions

2.1. **Solutions of matroids in graphs.** There are many equivalent formulations of the notion of a regular matroid, such as a matroid realizable over any field, or a matroid realizable by a totally unimodular matrix. For the purposes of this note, we declare:

Definition 2.1. A *regular matroid* R on a finite *ground set* S is a matroid realized by a map $S \to U^* = \mathrm{Hom}(U, \mathbb{Z})$, $s \mapsto x_s$, where U is a free abelian group of finite rank, for which:

(1) the x_s generate U^*, and
(2) any subset of the x_s generates a saturated sublattice of U^*.

We call $g = \mathrm{rank}(U)$ the *rank* of R. By condition (2), the base changed map $S \to U^* \otimes \mathbb{F}$ realizes the same matroid over any field $\mathbb{F}$. The map $S \to U^*$ defines canonically a map $\mathbb{Z}^S \to U^*$ sending $e_s \mapsto x_s$. Dually, we have an injective map $U \hookrightarrow (\mathbb{Z}^S)^* \simeq \mathbb{Z}^S$.

Let H denote an oriented graph and let $E(H)$ denote the set of oriented edges. An S-*coloring* of H is a partition $E(H) = \sqcup_{s\in S} E_s(H)$ of the edges, indexed by the set S. We say that an edge has color s if $e \in E_s(H)$. Following [EGFS25b, Def. 5.8]:

Definition 2.2. A $\mathbb{F}_p$-*solution of* R *in an* S-*colored graph* H is the data $b = (b_s)_{s\in S}$ of a collection of 1-chains $b_s \in C_1(H, \mathbb{F}_p)$, for $s \in S$, for which

(A) b_s is supported on edges $E_s(H) \subset E(H)$ of color $s \in S$, and
(B) whenever $\sum_{s\in S} a_s e_s \in U \subset \mathbb{Z}^S$, $a_s \in \mathbb{Z}$, we have that $\sum_{s\in S} a_s b_s \in H_1(H, \mathbb{F}_p)$ is a cycle.

In [EGFS25b], we considered the more general notion of Λ-*solution* of R, for a $\mathbb{Z}_{(p)}$-algebra Λ; in this paper, all solutions are $\mathbb{F}_p$-solutions. We use the term "solution" to mean "$\mathbb{F}_p$-solution" when no confusion can arise. Since H is oriented, we have a canonical identification $C_1(H, \mathbb{F}_p) = \mathbb{F}_p^{E(H)}$. Since $E(H) = \sqcup_{s\in S} E_s(H)$, the sum map $\mathbb{F}_p^{E_s(H)} \to \mathbb{F}_p$ which sends a vector to the sum of its coefficients thus induces a color profile map $\lambda\colon C_1(H, \mathbb{F}_p) \to \mathbb{F}_p^S$.

Definition 2.3. The *color profile* of a solution of R in H is the vector $\lambda(b) := \lambda(\sum_{s\in S} b_s) \in \mathbb{F}_p^S$. We say that the color profile is *constant* if $\lambda(b) = (c, \dots, c) \in \mathbb{F}_p^S$ for some $c \in \mathbb{F}_p$.

If R admits a $\mathbb{F}_p$-solution in some graph H, then it in fact admits a solution with the same color profile in a certain "universal graph" called the *mod* p *Albanese graph* $\mathrm{Alb}_p(R)$, see [EGFS25b, Sec. 5] (also notated $\mathrm{Alb}_{p,1}(R)$ in *loc.cit.*). We recall its construction:

Definition 2.4. The vertices of $\mathrm{Alb}_p(R)$ are the elements of the $\mathbb{F}_p$-vector space $\mathbb{F}_p^S/U_{\mathbb{F}_p}$. The edges of $\mathrm{Alb}_p(R)$ are given by connecting $v \to v + \overline{e}_s$ for any vertex $v \in \mathbb{F}_p^S/U_{\mathbb{F}_p}$ and $\overline{e}_s$ the image of any basis vector $e_s \in \mathbb{F}_p^S$. In this way, $\mathrm{Alb}_p(R)$ is a directed graph, which is naturally S-colored, by declaring the edge $v \to v + \overline{e}_s$ to have color $s \in S$.

The *Cayley graph* of a group with a distinguished set of generators is the graph whose vertices are the elements of the group and whose directed edges are given by (left) multiplication of the generators. The basis vectors of $\mathbb{Z}^S$ mapping to the quotient $\mathbb{Z}^S/U$ realize the *dual matroid* R^*, defined on the same ground set S. Thus, in concise terms: $\mathrm{Alb}_p(R)$ is the Cayley graph of the mod p realization $S \to \mathbb{F}_p^S/U_{\mathbb{F}_p}$ of the dual matroid: $\mathrm{Alb}_p(R) = \mathrm{Cay}_p(R^*)$ with respect to the defining vectors of the dual matroid.

Remark 2.5. All of the above definitions make sense over $\mathbb{Z}$, at the expense of the graph $\mathrm{Alb}_{\mathbb{Z}}(R) = \mathrm{Cay}_{\mathbb{Z}}(R^*)$ being infinite.

Notation 2.6. We use the term *solution of* R *mod* p, or simply, *solution of* R, without reference to a graph, to mean an $\mathbb{F}_p$-solution of R in $\mathrm{Alb}_p(R)$. The solutions of R mod p form an $\mathbb{F}_p$-vector space, which we denote by $\mathrm{Sol}_p(R)$. By Definition 2.2(A), the supports of the 1-chains $(b_s)_{s\in S}$ are disjoint. So we can view $\mathrm{Sol}_p(R) \subset C_1(\mathrm{Alb}_p(R), \mathbb{F}_p)$ as a subspace of the space of 1-chains of its mod p Albanese graph, by taking the sum $\sum_{s\in S} b_s$.

Remark 2.7. The graph $\mathrm{Alb}_p(R)$ satisfies a certain universal property [EGFS25b, Prop. 5.7]: Any connected, oriented S-colored graph H whose color profile map $\lambda\colon C_1(H, \mathbb{F}_p) \to \mathbb{F}_p^S$ satisfies $\lambda(H_1(H, \mathbb{F}_p)) \subset U_{\mathbb{F}_p} \subset \mathbb{F}_p^S$ admits, for any choice of vertex $v_0 \in V(H)$, a canonical map of oriented S-colored graphs $\mathrm{alb}\colon H \to \mathrm{Alb}_p(R)$ which does not contract any edge. In particular, the pushforward of a solution of R in H yields a solution in $\mathrm{Alb}_p(R)$ with the same color profile.

2.2. **Graphic matroids and the minimal class.** Let G be an oriented graph with edge set $E(G) = E$. We will specialize the discussion of the previous section as follows:

(1) $S = E(G) = E$ is the set of edges of G.
(2) $U \to \mathbb{Z}^S$ is the inclusion $B^1(G,\mathbb{Z}) \to C^1(G,\mathbb{Z}) = \mathbb{Z}^E$ of 1-coboundaries into 1-cochains.
(3) $\mathbb{Z}^S \to U^*$ is the quotient map $\mathbb{Z}^E = C_1(G,\mathbb{Z}) \to C_1(G,\mathbb{Z})/H_1(G,\mathbb{Z}) = B^1(G,\mathbb{Z})^*$.

Here, we observe that $H_1(G,\mathbb{Z})$ embeds naturally into $C_1(G,\mathbb{Z}) \simeq \mathbb{Z}^E$ by viewing a cycle as a $\mathbb{Z}$-linear combination of oriented edges $i \in E(G)$.

For an edge $i \in E(G)$, let $e_i \in \mathbb{Z}^{E(G)}$ denote the associated basis element.

Definition 2.8. The *graphic matroid* $M(G)$ is the matroid with ground set the edges $E = E(G)$, realized by the map $i \mapsto x_i := \overline{e}_i \in \mathbb{Z}^E/H_1(G,\mathbb{Z})$.

To define the dual matroid of $M(G)$, we first dualize the map $C_1(G,\mathbb{Z}) \to C_1(G,\mathbb{Z})/H_1(G,\mathbb{Z})$, giving the map $B^1(G,\mathbb{Z}) \to C^1(G,\mathbb{Z})$. Then the dual matroid of the graphic matroid, the *cographic matroid* $M^*(G)$, is realized by the map

$$E(G) \to H^1(G,\mathbb{Z}) = C^1(G,\mathbb{Z})/B^1(G,\mathbb{Z})$$

which assigns to the ith edge the linear functional $e_i^* \in H^1(G,\mathbb{Z}) = H_1(G,\mathbb{Z})^*$ whose value on any 1-cycle is the coefficient of the edge e_i in it.

Applying Definition 2.4 and the ensuing discussion, the Albanese graph $\mathrm{Alb}_p(M(G))$ of the graphic matroid $M(G)$ is the Cayley graph of $H^1(G,\mathbb{F}_p)$ under the generators $e_i^* \in H^1(G,\mathbb{F}_p)$ defining the cographic matroid $M^*(G)$.

Consider a graphic matroid $M(G)$. Its rank is $|E(G)|-h^1(G) = |V(G)|-h^0(G)$. The number of connected components of G can always be arranged to equal 1, by identifying a vertex on each connected component to a single vertex. This does not change the edge set or the first homology, and so leaves the graphic and cographic matroids the same. We always arrange for G to have one connected component, so that the rank of $M(G)$ is $|V(G)|-1$.

Definition 2.9. A regular matroid R with ground set S is *simple* if no $x_s = 0 \in U^*$ and no $x_s = \pm x_t \in U^*$ for $s \neq t$.

In terms of graphs, $M(G)$ is simple if G has no loops or parallel edges. Thus, when $M(G)$ is simple, G is a minor of the complete graph on $|V(G)|$ vertices $K_{|V(G)|}$ because G arises by deleting some subset of the edges of $K_{|V(G)|}$. We deduce the following (very simple) fact:

Fact 2.10. *The complete graph K_{g+1} defines the unique rank g graphic matroid $M(K_{g+1})$ that is maximal, under the "minors" partial ordering of simple graphic matroids of rank g.*

Definition 2.11. Let $d(G)$ be the product of all primes $p \in \mathbb{N}$ for which $M(G)$ admits **no** mod p solution whose color profile $(c,\dots,c)$ is nonzero and equal to a fixed constant $c \in \mathbb{F}_p^*$.

It follows [EGFS25b, Rem. 8.5] that $d(G) \leq \prod_{p<g \text{ prime}} p$. If a regular matroid R has a mod p solution with constant, nonzero color profile, then all minors of R have a mod p solution with the same property, see [EGFS25b, Prop. 7.2]. Combined with Fact 2.10, we deduce

Proposition 2.12. *For all $g \in \mathbb{N}$, the following holds:*

(1) for any connected, simple graph G on $g+1$ vertices, $d(G) \mid d(K_{g+1})$;
(2) $d(K_g) \mid d(K_{g+1})$.

Theorem 2.13 ([EGFS25b])**.** *The integer $d(K_{g+1})$ divides the minimal multiple N_g of the minimal class $\Theta^{g-1}/(g-1)!$ which is algebraic on a very general $(X,\Theta) \in \mathcal{A}_g$.*

Proof. This follows from [EGFS25b, Thm. 8.9] and [EGFS25b, Lem. 2.8]. □

We can view $d(K_{g+1})$ as the "graphic contribution" to the matroidal obstruction of [EGFS25b]. In terms of matroidal monodromy cones of degenerations [EGFS25a, Def. 4.6], the relevant cone is "Voronoi's principal domain of the first type" [EGFS25a, Ex. 6.7], and the base of the corresponding degeneration of abelian varieties dominates $\mathcal{A}_g$.

2.3. **Solutions of graphic matroids.** We now analyze in more detail the linear algebraic set-up of $\mathbb{F}_p$-solutions for the graphic matroid $M(G)$, associated to an oriented graph G. As noted,

$\mathrm{Alb}_p(M(G)) = \mathrm{Cay}_p(M^*(G))$ and thus, the vertex set of $\mathrm{Alb}_p(M(G))$ is $H^1(G, \mathbb{F}_p)$ while the $E(G)$-colored edges are of the form $v \to v + e_i^*$. So we have an isomorphism

$$C_1(\mathrm{Alb}_p(M(G)), \mathbb{F}_p) \simeq \mathbb{F}_p[H^1(G, \mathbb{F}_p)] \otimes C^1(G, \mathbb{F}_p) \tag{1}$$

where $\mathbb{F}_p[\cdot]$ denotes the group algebra. This isomorphism sends the edge $v \to v + e_i^*$ to $[v] \otimes e_i^*$. Then the boundary map $\partial \colon C_1(\mathrm{Alb}_p(M(G)), \mathbb{F}_p) \to C_0(\mathrm{Alb}_p(M(G)), \mathbb{F}_p)$ is identified with

$$\begin{aligned} \partial \colon \mathbb{F}_p[H^1(G, \mathbb{F}_p)] \otimes C^1(G, \mathbb{F}_p) &\to \mathbb{F}_p[H^1(G, \mathbb{F}_p)], \\ [v] \otimes e_i^* &\mapsto [v + e_i^*] - [v]. \end{aligned} \tag{2}$$

Here, we view $C^1(G, \mathbb{F}_p) = (\mathbb{F}_p^E)^*$, via the orientation of G.

Proposition 2.14. *Let G be an oriented graph with edges $E = E(G)$. The space of $\mathbb{F}_p$-solutions $b = (b_i)_{i \in E} \in \mathrm{Sol}_p(M(G))$ of the graphic matroid $M(G)$ is identified with space of the maps*

$$C^1(G, \mathbb{F}_p) \xrightarrow{\beta} \mathbb{F}_p[H^1(G, \mathbb{F}_p)] \otimes C^1(G, \mathbb{F}_p),$$

satisfying the following two conditions:

(A) *$\beta(e_i^*) \in - \otimes e_i^*$ for all $i \in E$, and*
(B) *$\beta(B^1(G, \mathbb{F}_p)) \subset \ker(\partial)$,*

where ∂ is defined in (2). Furthermore, define a map

$$\begin{aligned} \lambda \colon \mathbb{F}_p[H^1(G, \mathbb{F}_p)] \otimes C^1(G, \mathbb{F}_p) &\to C^1(G, \mathbb{F}_p), \\ [v] \otimes e_i^* &\mapsto e_i^*. \end{aligned}$$

Then $\lambda \circ \beta$ is a diagonal matrix $\oplus_{i \in E} \mathbb{F}_p e_i^* \to \oplus_{i \in E} \mathbb{F}_p e_i^*$ and the color profile of a solution $b \leftrightarrow \beta$ is $\lambda(\beta(e_i^*))_{i \in E} \in C^1(G, \mathbb{F}_p)$, via the isomorphism $C^1(G, \mathbb{F}_p) \simeq \mathbb{F}_p^E$. In particular, the condition that the color profile be constant is the same as saying $\lambda \circ \beta = c \cdot \mathrm{Id}_{C^1(G, \mathbb{F}_p)}$ for some $c \in \mathbb{F}_p$.

Proof. Definition 2.2(A) is easily seen to translate into condition (A) in the statement of the proposition, by taking $\beta(e_i^*) := b_i$ under the isomorphism (1). Then, Definition 2.2(B) exactly says that, in case of a graphic matroid, β sends $U = B^1(G, \mathbb{F}_p) \subset C^1(G, \mathbb{F}_p)$ into $H_1(\mathrm{Alb}_p(M(G)), \mathbb{F}_p) = \ker(\partial)$.

Furthermore, $(\lambda \circ \beta)(e_i^*)$ computes the i-th entry of the color profile because it is the sum of the coefficients of $b_i = \beta(e_i^*)$ in $\mathbb{F}_p$. □

2.4. **Symmetrizing solutions.** We now consider "symmetrization" of solutions. Let G be an oriented graph with edges $E = E(G)$ and let $\mathrm{Aut}(G)$ be its automorphism group, not necessarily preserving the chosen orientation of the edges of G. Then $\mathrm{Aut}(G)$ acts canonically on the spaces $C^1(G, \mathbb{F}_p)$ and $H^1(G, \mathbb{F}_p)$ via pullback of simplicial cochains and cohomology. It also acts on the E-colored graph $\mathrm{Alb}_p(M(G))$, i.e. the Cayley graph of $H^1(G, \mathbb{F}_p)$ with respect to the generators $e_i^* \in H^1(G, \mathbb{F}_p)$, because this set of generators is preserved, up to sign, by the action of $\mathrm{Aut}(G)$. In turn, it acts canonically on the spaces of 1-chains and 0-chains of $\mathrm{Alb}_p(M(G))$.

Remark 2.15. The isomorphism (1) is not $\mathrm{Aut}(G)$-equivariant in general. The action of $a \in \mathrm{Aut}(G)$ on edges of $\mathrm{Alb}_p(M(G))$ sends $v \to v + e_i^*$ to the directed edge $a^*v \to a^*(v+e_i^*)$ (whose direction may not agree with the one given by the orientation of $\mathrm{Alb}_p(M(G))$), whereas the action on $\mathbb{F}_p[H^1(G,\mathbb{F}_p)] \otimes C^1(G,\mathbb{F}_p)$ sends $[v] \otimes e_i^*$ to $[a^*v] \otimes (a^*(e_i^*))$. Note that $a^*(e_i^*) = \pm e^*_{a^{-1}(i)}$ with the sign depending on whether $a\colon a^{-1}(i) \mapsto i$ preserves or reverses orientation of the edge $i \in E(G)$. Hence the actions of $a \in \mathrm{Aut}(G)$ on the two sides of the isomorphism (1) agree only when a preserves the orientation of G.

By Proposition 2.14, solutions $b = (b_i)_{i\in E}$ of G mod p can be interpreted as elements of the morphism space $\beta \in \mathrm{Hom}(C^1(G,\mathbb{F}_p),\, C_1(\mathrm{Alb}_p(M(G)),\mathbb{F}_p))$. Then $\mathrm{Aut}(G)$ acts canonically on this morphism space via conjugation. That is, $a \cdot \beta = a \circ \beta \circ a^{-1}$, for $a \in \mathrm{Aut}(G)$.

Proposition 2.16. *Solutions of $M(G)$ mod p form an $\mathrm{Aut}(G)$-invariant subspace*

$$\mathrm{Sol}_p(M(G)) \subset \mathrm{Hom}(C^1(G,\mathbb{F}_p),\, C_1(\mathrm{Alb}_p(M(G)),\mathbb{F}_p)).$$

Given $\beta \in \mathrm{Sol}_p(M(G))$, the color profile satisfies $\lambda \circ (a\cdot\beta) = a \cdot (\lambda \circ \beta)$ for any $a \in \mathrm{Aut}(G)$. In particular, if β has a constant color profile $\lambda \circ \beta = c \cdot \mathrm{Id}_{C^1(G,\mathbb{F}_p)}$, then $a \cdot \beta$ has a constant color profile, for the same constant $c \in \mathbb{F}_p$.

Proof. Condition (A) in Proposition 2.14 is preserved, because the action of $a \in \mathrm{Aut}(G)$ permutes the E-colorings of G and $\mathrm{Alb}_p(M(G))$ by the same permutation. Thus, conjugation by a respects colors, i.e. $(a\cdot\beta)(e_i^*) = (a\beta a^{-1})(e_i^*)$ is a 1-chain supported on the edges of $\mathrm{Alb}_p(M(G))$ of color i. Condition (B) in Proposition 2.14 is preserved, because the action of $\mathrm{Aut}(G)$ on the source and target of β stabilizes the subspaces $B^1(G,\mathbb{F}_p)$ and $H_1(\mathrm{Alb}_p(M(G)),\mathbb{F}_p)$, respectively. The first statement follows.

We have $a \cdot (\lambda \circ \beta) = (a\cdot\lambda) \circ (a \cdot \beta)$. Note that $a \cdot \lambda = a\lambda a^{-1} = \lambda$ because λ is seen to be $\mathrm{Aut}(G)$-equivariant. Thus, $\lambda \circ (a \cdot \beta) = a \cdot (\lambda \circ \beta)$. Therefore, if $\lambda \circ \beta = c \cdot \mathrm{Id}_{C^1(G,\mathbb{F}_p)}$,

$$a \cdot (\lambda \circ \beta) = a(c \cdot \mathrm{Id}_{C^1(G,\mathbb{F}_p)})a^{-1} = c \cdot \mathrm{Id}_{C^1(G,\mathbb{F}_p)}.$$

So $\lambda \circ (a \cdot \beta) = \lambda \circ \beta$, i.e. β and $a \cdot \beta = a\beta a^{-1}$ have the same constant color profile. □

More generally, the conjugation action permutes the color profile of a solution β, according to the underlying permutation of a. We may unwind the above proposition in terms of collections of 1-cycles $b = (b_i)_{i\in E}$ supported on the edges of color $i \in E$, as follows:

Corollary 2.17. *Let $b = (b_i)_{i\in E}$ be a solution of $M(G)$ mod p. Then so is $b' := (b_i')_{i\in E}$, where $b_i' := \pm a \cdot b_{a(i)}$ with sign determined by whether $a \in \mathrm{Aut}(G)$ respects or reverses the orientation of the edge $i \mapsto a(i)$. If b has constant color profile, so does b', for the same constant.*

Proof. Suppose $b \leftrightarrow \beta$ under the identification of Proposition 2.14. Let $\beta' := a \cdot \beta = a\beta a^{-1}$. On $C^1(G,\mathbb{F}_p)$, the canonical action is $a^{-1} \cdot e_i^* = \pm e^*_{a(i)}$ with the sign determined as in the statement of the corollary. By Proposition 2.16, $b_i' := \beta'(e_i^*) = \pm a\beta(e^*_{a(i)}) = \pm a \cdot b_{a(i)}$ defines a solution $b' \leftrightarrow \beta'$. The last statement concerning color profiles also follows from Proposition 2.16. □

The graph $\mathrm{Alb}_p(M(G))$ has an involution ι, which acts on vertices by the map

$$\iota \colon H^1(G, \mathbb{F}_p) \to H^1(G, \mathbb{F}_p), \qquad v \mapsto -v,$$

and sends the edge $v \to v + e_i^*$ to the edge $-v \to -v - e_i^*$ (which is the edge $-v - e_i^* \to -v$, but with the wrong orientation). So ι respects the colors of all edges, but also reverses the orientations of all edges of $\mathrm{Alb}_p(M(G))$. Define an action of ι on $\mathrm{Sol}_p(M(G))$ by

$$\iota \cdot \beta := \iota_* \circ \beta \circ (\text{negation}) = -\iota_* \circ \beta, \tag{3}$$

where negation acts on the source $C^1(G, \mathbb{F}_p)$ of β. It preserves solutions because the action of ι respects colors and because negation and the pushforward map ι_* on 1-chains both preserve the homology $H_1(\mathrm{Alb}_p(M(G)), \mathbb{F}_p)$. From the action of ι on edges, we see that $\lambda \circ (\iota \cdot \beta) = \lambda \circ \beta$ and so the action (3) preserves the color profile. Furthermore, ι commutes with the action of $\mathrm{Aut}(G)$ on the space of solutions. To prove this, note that ι_* and negation commute with the actions of $a \in \mathrm{Aut}(G)$ on the source and target of β, respectively.

Definition 2.18. Let $A \subset \mathrm{Aut}(G) \times \mathbb{Z}/2$ be any subgroup. The *symmetrization map* is

$$\begin{aligned} \sigma_A \colon \mathrm{Sol}_p(M(G)) &\to \mathrm{Sol}_p(M(G)) \\ \beta &\mapsto \textstyle\sum_{(a,\,r)\,\in\,A} \iota^r \cdot a \cdot \beta. \end{aligned} \tag{4}$$

Corollary 2.19. *If β has a constant color profile, then $\lambda \circ \sigma_A(\beta) = |A| \cdot (\lambda \circ \beta)$. Moreover, for any $(a, r) \in A$, we have $\iota^r \cdot a \cdot \sigma_A(\beta) = \sigma_A(\beta)$.*

In other words, the symmetrization map sends a solution β to one which is A-invariant. The corollary alleviates some of the computational difficulties involved in determining the existence of a solution with constant nonzero color profile, by reducing the solutions under consideration to a smaller space of invariant solutions.

We now consider a special case, relevant to the computations in the following section. Let G be a graph, and choose an orientation. Let $A \subset \mathrm{Aut}(G) \times \mathbb{Z}/2$ be a subgroup such that, for any element $(a, r) \in A$, $a \in \mathrm{Aut}(G)$ preserves or reverses the orientation of every edge of G. Let

$$\chi \colon A \to \{\pm 1\} \tag{5}$$

be the character satisfying $\chi(e, 1) = -1$ and $\chi(a, 0) = 1$ if and only if $a \in \mathrm{Aut}(G)$ preserves the orientation of every edge of G.

Definition 2.20. With the above notation and assumptions, we say that a 1-chain $b \in C_1(\mathrm{Alb}_p(M(G)), \mathbb{F}_p)$ is *(A, χ)-equivariant* if $\iota^r \cdot a \cdot b = \chi(a, r) b$ for every $(a, r) \in A$.

Proposition 2.21. *Let G be an oriented graph and $A \subset \mathrm{Aut}(G) \times \mathbb{Z}/2$ be a subgroup such that for any $(a, r) \in A$, a either preserves or reverses the orientation of every edge of G. Then, on* *the level of* 1*-chains* $b = \sum_{e \in E(\mathrm{Alb}_p(M(G)))} b_e e$ *of* $\mathrm{Alb}_p(M(G))$*, the symmetrization map is*

$$\begin{aligned}\sigma_A \colon C_1(\mathrm{Alb}_p(M(G)), \mathbb{F}_p) &\to C_1(\mathrm{Alb}_p(M(G)), \mathbb{F}_p), \\ b &\mapsto \textstyle\sum_{(a,r) \in A} \chi(a,r)(\iota^r \cdot a \cdot b).\end{aligned} \tag{6}$$

The element $\sigma_A(b)$ *is a solution whenever* $b \in C_1(\mathrm{Alb}_p(M(G)), \mathbb{F}_p)$ *is a solution. Its color profile is* $\lambda(\sigma_A(b)) = |A| \cdot \lambda(b)$ *if the color profile of* b *is constant.*

Finally, when $|A|$ *and* p *are coprime, the image of the map* σ_A *in* (6) *is the space of* (A,χ)*-equivariant* 1*-chains* $b \in C_1(\mathrm{Alb}_p(M(G)), \mathbb{F}_p)$*.*

Proof. The fact that σ_A preserves solutions follows from Corollary 2.17 and the ensuing discussion of ι. The discrepancy in signs between (4) and (6) is due to the additional global negation arising from Corollary 2.17, when an automorphism reverses all edge orientations, and the sign appearing in the definition (3) of the action of ι. Corollary 2.19 implies the color profile multiplies by $|A|$ when the color profile of b is constant. For the last statement, note that the averaging σ_A produces a 1-chain which is A-equivariant with respect to the character χ and that conversely, $\sigma_A(|A|^{-1}b) = b$ for any (A,χ)-equivariant 1-chain $b \in C_1(\mathrm{Alb}_p(M(G)), \mathbb{F}_p)$. $\square$

Remark 2.22. The symmetrization map can alternatively be described as follows. For a positively oriented edge $e \in E(G)$, put

$$|\pm e| \coloneqq e \in C_1(G, \mathbb{F}_p) = \mathbb{F}_p^{E(G)}. \tag{7}$$

That is, $|e| = e$ and $|-e| = e \in C_1(G, \mathbb{F}_p)$. This "absolute value" is only defined for 1-chains of the form $\pm e \in C_1(G, \mathbb{F}_p)$. Let $A \subset \mathrm{Aut}(G) \times \mathbb{Z}/2$ be a subgroup such that, for $(a,r) \in A$, a preserves or reverses the orientation of every edge of G. Then the symmetrization map sends an edge $e \in C_1(\mathrm{Alb}_p(M(G)), \mathbb{F}_p)$ to the element $\sum_{(a,r) \in A} |\iota^r \cdot a \cdot e|$.

3. Proof of the main result

Here we prove the following theorem.

Theorem 3.1. *Let* K_7 *be the complete graph on* 7 *vertices. Let* $A = D_7 \times \mathbb{Z}/2$ *where* $D_7 \subset \mathrm{Aut}(K_7)$ *is the standard dihedral action. Then every* A*-invariant element* $\beta \in \mathrm{Sol}_3(M(K_7))$ *with constant color profile satisfies* $\lambda \circ \beta = 0$*.*

The proof will be reduced to a computer calculation, detailed in Section 3.4 below. The code is available in the repository [EGFS25c]. We now give the theoretical framework underpinning the computer computation.

3.1. **Lexicographic vertex fundamental domain.** We set up the computation in the generality of a complete graph K_{2n+1} and an arbitrary prime $p \nmid 2n+1$. It is convenient to label the vertices by $\{0, 1, \ldots, 2n-1, 2n\} = \mathbb{Z}/(2n+1)$ and the oriented edges as

$$E(K_{2n+1}) = \{e_{i,i+1},\, e_{i,i+2},\, e_{i,i+3},\, \ldots,\, e_{i,i+n}\}_{i \bmod 2n+1}.$$

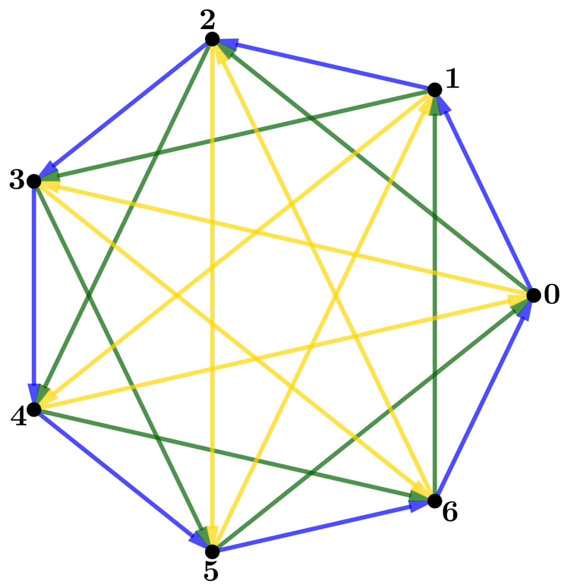

FIGURE 1. Orientation of K_7 as a regular tournament with step-1 edges (blue), step-2 edges (green), and step-3 edges (yellow).

That is, we orient K_{2n+1} as a regular tournament, see Figure 1 for the example $2n+1 = 7$. We call $e_{i,\,i+k}$ the *step-k edges*. We require:

Lemma 3.2. *For any prime $p \nmid 2n+1$, the linear functionals $\{e^*_{i,\,i+k}\}_{i=0,\,\ldots,\,2n,\,k=1,\,\ldots\,,\,n-1} \cup \{s^*\}$ form a basis of $H^1(K_{2n+1}, \mathbb{F}_p)$, where $s^* \coloneqq \sum_{i \bmod 2n+1} e^*_{i,\,i+n}$ is the* star dual.

Proof. Observe that $h^1(K_{2n+1}, \mathbb{F}_p) = |E(K_{2n+1})| - |V(K_{2n+1})| + 1 = \binom{2n+1}{2} - (2n+1) + 1$ which agrees with the number of linear functionals provided. Thus, it suffices to check that any cycle $\gamma \in H_1(K_{2n+1}, \mathbb{F}_p)$ on which all the linear functional vanish, is necessarily $\gamma = 0$.

The vanishing of the step-k edge duals, $e^*_{i,\,i+k}(\gamma) = 0$, $k \in \{1, \ldots, n-1\}$, implies that γ is a cycle supported on the step-n edges. Up to taking a multiple, there is a unique such cycle, which is the sharpest $(2n+1)$-pointed star $s = \sum_{i \bmod 2n+1} e_{i,\,i+n} \in H_1(K_{2n+1}, \mathbb{F}_p)$, depicted in Figure 1 as the sum of the yellow edges. Hence $\gamma = m \cdot s$ for some $m \in \mathbb{F}_p$. But then we have $s^*(s) = 2n+1 \not\equiv 0 \bmod p$. Thus the additional condition $s^*(\gamma) = 0$ ensures $\gamma = 0$. □

We henceforth let $n \in \mathbb{N}$, and let p be a prime number that does not divide $2n+1$. We call the linear functionals of Lemma 3.2 the *step-star basis* of $H^1(K_{2n+1}, \mathbb{F}_p)$. The major advantage of the step-star basis is that it is naturally D_{2n+1}-equivariant. More precisely, given an element $a \in C_{2n+1} \subset D_{2n+1}$ of the cyclic subgroup, we have $a \cdot e^*_{i,\,i+k} = e^*_{i-a,\,i+k-a}$, and given an element $a \in D_{2n+1} \setminus C_{2n+1}$, the action of a also preserves the set of step-k edge duals and the star dual, after a global negation (such an a visibly reverses the orientation of every edge of the chosen orientation of K_{2n+1}, see Figure 1).

To set up the computation explicitly, we must find a fundamental domain for the action of $D_{2n+1} \times \mathbb{Z}/2$ (or more generally, a subgroup $A \subset D_{2n+1} \times \mathbb{Z}/2$), on $V(\mathrm{Alb}_p(M(K_{2n+1}))) = H^1(K_{2n+1}, \mathbb{F}_p)$ and elements of $E(\mathrm{Alb}_p(M(K_{2n+1})))$ representing every A-orbit.

The *lexicographic fundamental domain* $V_{\text{fund}} \subset V(\mathrm{Alb}_p(M(K_{2n+1}))) = H^1(K_{2n+1}, \mathbb{F}_p)$ is defined as follows. For $v \in H^1(K_{2n+1}, \mathbb{F}_p)$, take the unique expression of v in the step-star basis

$$v = cs^* + \sum_{k=1}^{n-1} \sum_{i=0}^{2n} c_{i,\, i+k} e^*_{i,\, i+k} \qquad \text{with} \qquad c, c_{i,\, i+k} \in \{0, 1, \ldots, p-1\}.$$

Consider now the action of $(a, r) \in D_{2n+1} \times \mathbb{Z}/2$ on v. We begin by taking the lexicographically minimal $D_{2n+1} \times \mathbb{Z}/2$ orbit-representative of the step-1 coefficients $(c_{01}, c_{12}, c_{23}, \ldots, c_{2n,0})$ with respect to the ordering $0 < 1 < \cdots < p-1$. This amounts to choosing the minimal representative of such a $\{0, \ldots, p-1\}$-string under the operations of cyclic rotation, reflections, and negations. Having found this minimal representative $c_{\text{step-1}}$, we set $A' := \mathrm{Stab}_A(c_{\text{step-1}})$.

Then we consider the A'-action on the step-2 coefficients $(c_{02}, c_{13}, c_{24}, \ldots, c_{2n-1,0}, c_{2n,1})$. We again find a lexicographically minimal representative under the A'-action $c_{\text{step-2}}$, and define $A'' := \mathrm{Stab}_{A'}(c_{\text{step-2}})$. Continuing inductively, we arrive at a subgroup $A^{(n)} \subset A$. It acts on the coefficient c of the star dual element s^* by the character $(a, r) \cdot c = (-1)^{\mathrm{sgn}(a)+r} c$. Here, sgn is the orientation character of the dihedral group, which preserves or reverses the orientation of every edge of K_{2n+1}. Again taking the lexicographically minimal representative for c, we reduce any vertex $v \in V(\mathrm{Alb}_p(M(K_{2n+1})))$ into a fundamental domain, expressible in the form

$$v_{\text{fund}} := (c_{\text{step-1}}, c_{\text{step-2}}, \ldots, c_{\text{step-}(n-1)}, c_{\text{star}}).$$

We additionally extract the stabilizer group $\mathrm{Stab}_A(v_{\text{fund}}) = \mathrm{Stab}_{A^{(n)}}(c_{\text{star}})$.

Definition 3.3. Define $V_{\text{fund}} \subset H^1(K_{2n+1}, \mathbb{F}_p)$, the *lexicographic fundamental domain* of the vertices under the action of A, as the set of all v_{fund} for $v \in H^1(K_{2n+1}, \mathbb{F}_p)$.

Example 3.4. For $\mathrm{Alb}_3(M(K_7))$, consider the vertex represented by $v = (1200220, 2200111, 1)$, with ternary strings representing the step-1, step-2, and star coefficients, respectively. Then we have $v_{\text{fund}} = (0011021, 0022211, 2)$ with a group element satisfying $a \cdot v = v_{\text{fund}}$ given by $a = (\mathrm{rot}^2, 1) \in D_7 \times \mathbb{Z}/2$ and $\mathrm{rot} \in C_7$ the cyclic rotation increasing the indices of vertices by 1. The stabilizer $\mathrm{Stab}_{D_7 \times \mathbb{Z}/2}(v_{\text{fund}})$ is trivial.

3.2. **Equivariant 1-chains of the Albanese graph.** Let $v = v_{\text{fund}} \in V_{\text{fund}}$ be a fundamental domain vertex. We consider, for each i mod $2n+1$, the incoming and outgoing edges of v of color i, given respectively by

$$e_{\text{in}}(v, i) := (v - e^*_i \to v) \quad \text{and} \quad e_{\text{out}}(v, i) := (v \to v + e^*_i) \in E(\mathrm{Alb}_p(M(K_{2n+1}))).$$

For each ingoing or outgoing edge e from $v \in V_{\text{fund}}$, we take the (A, χ)-average

$$\sigma_A(e) := \sum_{(a,r) \in A} |\iota^r \cdot a \cdot e| \in C_1(\mathrm{Alb}_p(M(K_{2n+1})), \mathbb{F}_p), \tag{8}$$

where the "absolute value" $|\cdot|$ is defined by (7). We define the *color j boundary* of $\sigma_A(e)$ as follows. Break $\sigma_A(e)$ into its color components

$$\sigma_A(e) = \sum_{j \in E(K_{2n+1})} \sigma_A(e)_j$$

with $\sigma_A(e)_j \in C_1(\mathrm{Alb}_p(M(K_{2n+1})), \mathbb{F}_p)$ supported on edges of color j, and take the boundary

$$\partial \sigma_A(e)_j \in C_0(\mathrm{Alb}_p(M(K_{2n+1}))) = \mathbb{F}_p^{V(\mathrm{Alb}_p(M(K_{2n+1})))}.$$

Define a *color weighting* as an element $w \in \mathbb{F}_p^{E(K_{2n+1})} = C^1(K_{2n+1}, \mathbb{F}_p)$. For a given color weighting w, define the *w-weighted boundary* by the following formula:

$$\partial_{A,w}(e) \coloneqq \sum_{j \in E(K_{2n+1})} w_j \partial_A(e)_j|_{V_{\mathrm{fund}}} \in \mathbb{F}_p^{V_{\mathrm{fund}}}.$$

That is, $\partial_{A,w}(e)$ represents the boundary of an averaged 1-chain (with respect to the action (8)), which has been further weighted by w_i on the i-th color, and then restricted to V_{fund}.

Consider the character $\chi\colon A \to \{\pm 1\}$, see equation (5).

Lemma 3.5. *Let $w^{(1)}, \ldots, w^{(2n)}$ be a basis of $B^1(K_{2n+1}, \mathbb{F}_p)$. A 1-chain*

$$b = \sum_{v \in V_{\mathrm{fund}}} \sum_{i \in E(K_{2n+1})} \sum_{\substack{e = e_{\mathrm{in}}(v,i) \\ \text{or } e_{\mathrm{out}}(v,i)}} c_e \sigma_A(e) \in C_1(\mathrm{Alb}_p(M(K_{2n+1})), \mathbb{F}_p) \tag{9}$$

with $c_e \in \mathbb{F}_p$ is a solution if and only if $\partial_{A,w^{(k)}}(b) = 0$ for all $w^{(k)}$. When $|A|$ and p are coprime, every (A,χ)-equivariant 1-chain on $\mathrm{Alb}_p(M(G))$ is of the form (9).

Proof. The condition that $\partial_{A,w^{(k)}}(b) = 0$ is equivalent to the condition that $\sum_{i \in E(K_{2n+1})} w_i^{(k)} b_i \in H_1(\mathrm{Alb}_p(M(G)), \mathbb{F}_p)$ is a 1-cycle, because for an equivariant 1-chain, it suffices to check that the boundary vanishes on an A-fundamental domain of vertices. Thus, if this condition holds for $w^{(k)}$ ranging over a basis of $B^1(K_{2n+1}, \mathbb{F}_p)$, then b defines a solution.

If $|A|$ and p are coprime, then any (A,χ)-equivariant 1-chain b is expressible as a linear combination of $\sigma_A(e)$, for e ranging over $e_{\mathrm{in}}(v,i)$, $e_{\mathrm{out}}(v,i)$ for all $v \in V_{\mathrm{fund}}$ and $i \in E$. This holds by Proposition 2.21, as every edge of $\mathrm{Alb}_p(M(G))$ can be reduced (possibly non-uniquely) via A, to an edge incident to a fundamental domain vertex. Hence, b is of the form (9). □

3.3. Building the equivariant solution matrix.

We now define a matrix whose kernel admits a natural surjection onto the space of (A,χ)-equivariant solutions $b \in C_1(\mathrm{Alb}_p(M(G)), \mathbb{F}_p)$.

Definition 3.6. Define a matrix $S_A(p, 2n+1)$ over $\mathbb{F}_p$, as follows. It is a vertical stacking of $\operatorname{rank} B^1(K_{2n+1}, \mathbb{F}_p) = 2n$ matrices, each one having proportions $|V_{\mathrm{fund}}| \times (2n+1)(2n) \cdot |V_{\mathrm{fund}}|$. The column of the k-th matrix in the vertical stack corresponding to an ingoing or outgoing edge e to $v \in V_{\mathrm{fund}}$ is given by the entries of $\partial_{A,w^{(k)}}(e) \in \mathbb{F}_p^{V_{\mathrm{fund}}}$. The result is a matrix with proportions $(2n) \cdot |V_{\mathrm{fund}}| \times (4n^2 + 2n) \cdot |V_{\mathrm{fund}}|$, that we call $S_A(p, 2n+1)$.

Remark 3.7. It follows from Lemma 3.5 that the right kernel of $S_A(p, 2n+1)$ is the space of expressions of the form (9) which represent an A-invariant (in β form) element of $\mathrm{Sol}_p(M(K_{2n+1}))$. In particular, $\ker(S_A(p, 2n+1))$ surjects onto $\mathrm{Sol}_p(M(K_{2n+1}))^A$. But, in general, there may be a kernel, because the columns of $S_A(p, 2n+1)$ do not represent a basis of the space of (A, χ)-equivariant 1-chains (in b form) on $\mathrm{Alb}_p(M(K_{2n+1}))$. The issue is that $e_{\mathrm{in}}(v, i)$, $e_{\mathrm{out}}(v, i)$ for $v \in V_{\mathrm{fund}}$ and $i \in E(K_{2n+1})$ do not form a fundamental domain for the A-action on $E(\mathrm{Alb}_p(M(K_{2n+1})))$, though they do contain all A-orbit representatives. The averaging $\sigma_A(e)$ will identify averaged 1-chains associated to e, e' if and only if they lie in the same A-orbit. As such, we only know $\dim \ker(S_A(p, 2n+1)) \geq \dim \mathrm{Sol}_p(M(K_{2n+1}))$.

The condition that an (A, χ)-invariant solution b has color profile zero can be detected from an expression of the form (9). To detect such solutions, define a matrix $S_A^{\text{step-1}}(p, 2n+1)$ by appending a single row to $S_A(p, 2n+1)$ which has an entry 1 on all columns indexed by $e_{\mathrm{in}}(v, i)$, $e_{\mathrm{out}}(v, i)$ for $i \in E(K_{2n+1})$ a step-1 edge, and a 0 entry for all other columns. Similarly, we define $S_A^{\text{step-1,2}}(p, 2n+1)$ by appending to $S_A^{\text{step-1}}(p, 2n+1)$ a single row which is an indicator on all step-2 edges, etc.

Lemma 3.8. *If all (A, χ)-equivariant solutions are colorless, we have*

$$\operatorname{rank} S_A(p, 2n+1) = \operatorname{rank} S_A^{\text{step-1}}(p, 2n+1) = S_A^{\text{step-1,2}}(p, 2n+1) = \cdots$$

and conversely, when $|A|$ and p are coprime, and there is an (A, χ)-equivariant solution with constant nonzero color profile, then $\operatorname{rank} S_A^{\text{step-1}}(p, 2n+1) = \operatorname{rank} S_A(p, 2n+1) + 1$.

Proof. Indeed, $\ker(S_A^{\text{step-1}}(p, 2n+1))$ surjects onto the space of (A, χ)-equivariant solutions which have color profile zero on all step-1 edges, $\ker(S_A^{\text{step-1,2}}(p, 2n+1))$ surjects onto the space of (A, χ)-equivariant solutions which have color profile 0 on step-1 and 2 edges, etc. (Here we used that the number of step-k edges is $2n+1$, which is coprime to p.)

The converse follows from Proposition 2.21. □

This concludes the theoretical setup for the computation proving Theorem 3.1.

3.4. **Computational aspects of the proof of Theorem 3.1.** Following the logic of the computation laid out in Sections 3.1–3.3, the code `SymmetrizedSolutionsKn` [EGFS25c] executes the following steps:

(V_{fund}) Build the vertices $V(\mathrm{Alb}_p(K_{2n+1}))$ using the step-star basis, and extract the lexicographic fundamental domain $V_{\mathrm{fund}} \subset V(\mathrm{Alb}_p(K_{2n+1}))$ with respect to the action of $A \subset D_{2n+1} \times \mathbb{Z}/2$ labeled as $A =$ `DnZ2`, `Dn`, `CnZ2`, `Cn`, `Z2`, `Triv`. Nota bene: The program uses n for $2n+1$.

(E_{orb}) Build the ingoing and outgoing directed edges $v - e_i^* \to v$ and $v \to v + e_i^*$ by ranging over all $v \in V_{\mathrm{fund}}$ in a fundamental domain and all colors i.

($w^{(k)}$) Choose a basis $(w^{(k)})_{k=1,\dots,2n}$ of the space $B^1(K_{2n+1}, \mathbb{F}_p)$.

($S_A(w^{(k)})$) For each color weighting $w^{(k)}$, build the $w^{(k)}$-weighted coboundary matrix, which is the map $S_A(w^{(k)})\colon \mathbb{F}_p^{E_{\text{orb}}} \to \mathbb{F}_p^{V_{\text{fund}}}$ whose kernel surjects onto the (A,χ)-equivariant 1-chains $b \in C_1(\mathrm{Alb}_p(M(K_{2n+1})), \mathbb{F}_p)$ for which $\sum_{i \in E(K_{2n+1})} w_i^{(k)} b_i$ are 1-cycles.

(S_A) Stack the matrices $S_A(w^{(k)})$, ranging over the basis $w^{(k)}$, $k = 1, \ldots, 2n$.

($S_A^{\text{step-1},\ldots,s}$) For $s = 1, \ldots, n$ add, for each step size $i \in \{1, \ldots, s\}$, a row to S_A, which is an indicator function on the edges of step size i.

(r, r^s) Compute the ranks $r := \mathrm{rank}(S_A)$, $r^s := \mathrm{rank}(S_A^{\text{step-1},\ldots,s})$, using sparse rank functions; run by `TestColorlessAveragedSolutions(p, 2n+1 : subgroup:="A")`.

(answer) If $r = r^1$, all (A,χ)-equivariant solutions with constant color profile have zero color profile, see Lemma 3.8.

Strictly speaking, we only need the computation up to the step $s = 1$; the higher values are built-in as a check. The computation is done in MAGMA [Bos+20] which is well-suited to rank computations of very sparse, large matrices over finite fields.

Proof of Theorem 3.1. Performing the above computation for $p = 3$ and $2n+1 = 7$, and $A = D_7 \times \mathbb{Z}/2$ (which indeed satisfies that $|A| = 28$ is coprime to 3 and that p and $2n+1$ are coprime), the outcomes are as follows: The size of the fundamental domain is $|V_{\text{fund}}| = 517{,}570$. There are $\binom{7}{2} = 21$ colors, and the number of color weighting vectors $w^{(k)}$ is rank $B^1(K_7, \mathbb{F}_3) = |V(K_7)| - 1 = 6$. We get that $S_A(3,7)$ is a $3{,}105{,}420 \times 21{,}737{,}940$ matrix over $\mathbb{F}_3$. As expected, this is $(2n)\cdot|V_{\text{fund}}| \times (2n+1)(2n)\cdot|V_{\text{fund}}|$. The matrices, $S_A^{\text{step-1}}(3,7)$, $S_A^{\text{step-1,2}}(3,7)$, $S_A^{\text{step-1,2,3}}(3,7)$ respectively have 1, 2, 3 more rows. We find that the ranks $r = r^1 = r^2 = r^3 = 3{,}060{,}117$ are all equal, though we only need the first equality for a proof. The total runtime of the program was 132,396 seconds (≈ 37 hours), executed on a Gemini HPC node of Utrecht University with 32 CPU cores and 8 GB of memory per core. The peak memory usage was 114 GB. □

Corollary 3.9. *Any element of* $\mathrm{Sol}_3(M(K_7))$ *with constant color profile has zero color profile.*

Proof. By Corollary 2.19, any solution with constant nonzero color profile can be $A = D_7 \times \mathbb{Z}/2$-averaged to a solution with constant nonzero color profile, since $|A|$ and 3 are coprime. We deduce the corollary from Theorem 3.1. □

Remark 3.10. One may wonder; what is the point in the A-averaging and reduction into a fundamental domain? The reason is that the rank computations are the rate-limiting step of the computation. Without symmetrizing at all, the relevant matrix has proportions

$$(2n) \cdot p^{2n^2-n} \times (2n+1)(n) \cdot p^{2n^2-n},$$

where $p^{2n^2-n} = |V(\mathrm{Alb}_p(M(K_{2n+1})))|$. (Whenever A preserves the orientations of all edges of $\mathrm{Alb}_p(M(K_{2n+1}))$, the outgoing edges alone from each fundamental domain vertex represent every A-orbit of edges. So the number of columns can be halved.) It has a similar sparsity to $S_A(p, 2n+1)$. For $p = 3$ and $2n+1 = 7$, we get a matrix over $\mathbb{F}_3$ with proportions

$$86{,}093{,}442 \times 301{,}327{,}047.$$

Finding the rank of such a matrix was beyond our computing capacity.

Proof of Theorem 1.2. It follows from [ADFIO20, Cor. 2] that, for any $(X, \Theta) \in \mathcal{A}_6$, the class $6 \cdot \Theta^5/5!$ is represented by an effective curve, arising from a Prym–Tyurin construction of exponent 6. The original construction is due to Kanev [Kan87] but dominance onto $\mathcal{A}_6$ was proven in [ADFIO20]. Thus, the minimal multiple of the minimal class which is represented by an algebraic cycle always divides 6. Let K_7 be the complete graph on 7 vertices. By the results of [EGFS25b], see Theorem 2.13, the combinatorial invariant $d(K_7) \in \mathbb{N}$ divides N_6. It follows from Corollary 3.9 that $3 \mid d(K_7)$. By Proposition 2.12, we have $d(K_5) \mid d(K_7)$. Finally, $2 \mid d(K_5)$ by [EGFS25b, Prop. 7.5]. So $6 \mid d(K_7) \mid N_6 \mid 6$ and hence $N_6 = 6$. □

Proof of Corollary 1.3. For each positive integer g, we have $N_g \mid N_{g+1}$ by specializing to a product of an elliptic curve with a very general principally polarized abelian variety of lower dimension, see [EGFS25b, Sec. 8.3, Proof of Thm. 1.1]. Thus $6 \mid N_g$ for $g \geq 6$ in view of Theorem 1.2. Let (X, Θ) be a very general principally polarized abelian variety of dimension $g \geq 6$. Assume for a contradiction that there exists an abelian variety Y and an isomorphism $X \times Y \simeq P$ where P is a product of Prym varieties. Since 2 times the minimal class on P is given by a reduced effective curve on P, we can apply [EGFS25b, Lemma 2.9] to see that the multiplication by 2 map $[2]\colon P \to P$ factors as $P \to JC \to P$ where $JC = \prod_i JC_i$ is the Jacobian of a stable curve $C = \cup_i C_i$ of compact type. Hence $[2]\colon X \to X$ admits a factorization $X \to JC \to X$. Since JC satisfies the integral Hodge conjecture for curve classes by [BGF23, Theorem 1.2], it follows that $2 \cdot \Theta^{g-1}/(g-1)! \in H^{2g-2}(X, \mathbb{Z})$ is algebraic, which contradicts the fact that $6 \mid N_g$. □

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