ZEROS OF ONE-FORMS AND HOMOLOGICALLY TRIVIAL FIBRATIONS

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Abstract. We show that a conjecture of Kotschick about one-forms without zeros on compact Kähler manifolds follows in the case of simple Albanese torus from a conjecture of Bobadilla and Kollár about homologically trivial fibrations. As an application, we prove Kotschick's conjecture for compact Kähler manifolds X with $b_1(X) \geq 2\dim X - 2$, whose Albanese torus is simple.

1. Introduction

In [7], Kotschick made the following

Conjecture 1.1. For a compact Kähler manifold X, the following conditions are equivalent:

(A) X admits a holomorphic one-form without zeros;
(B) X admits a real closed one-form without zeros; or by Tischler's theorem [13] equivalently, the underlying differential manifold of X is a $\mathcal{C}^\infty$-fiber bundle over the circle.

We propose the following stronger conjecture, which implies Conjecture 1.1 when we apply it to the Albanese morphism $X \to \mathrm{Alb}(X)$.

Conjecture 1.2. Let X be a compact Kähler manifold and let $f : X \to A$ be a morphism to a complex torus A. Then the following conditions are equivalent:

(A) X admits a holomorphic one-form w without zeros such that
$$[w] \in f^*H^0(A, \Omega^1_A).$$
(B) X admits a real closed one-form α without zeros such that $[\alpha] \in f^*H^1(A, \mathbb{R})$; or, by Tischler's argument [13] equivalently, the underlying differential manifold of X is a $\mathcal{C}^\infty$-fiber bundle $g : X \to S^1$ over the circle with
$$g^*H^1(S^1, \mathbb{R}) \subseteq f^*H^1(A, \mathbb{R}).$$

In both conjectures, it is easy to see that Condition (A) implies Condition (B). The converse direction is the non-trivial part. These conjectures are known for surfaces [7, 12] and for projective threefolds [6]. (In *loc. cit.*, Conjecture 1.1 is considered, but the arguments prove in fact the slightly stronger assertion from Conjecture 1.2, cf. [6, Theorem 1.4].)

The observation of this paper is that we can relate Conjecture 1.2 to the following conjecture of Bobadilla and Kollár [3, Conjecture 3]. For a commutative ring R, a proper morphism $f : X \to Y$ between complex analytic spaces is called an R-homology fiber bundle if Y has an open cover $Y = \cup_i U_i$ such that for every i and every $y \in U_i$, the map induced by inclusion
$$H_*(f^{-1}(y), R) \to H_*(f^{-1}(U_i), R) \quad \text{is an isomorphism.}$$

Conjecture 1.3. Let $f : X \to Y$ be a proper morphism between complex analytic spaces, where X and Y are both smooth. If f is a $\mathbb{Z}$-homology fiber bundle, then f is smooth.

Our main result is

Theorem A. *Let X be a compact Kähler manifold and let $f : X \to A$ be a morphism to a simple complex torus A. Assume that Conjecture 1.3 holds for the morphism f, then Conjecture 1.2 holds for f.*

To prove the theorem above, we show that if $f : X \to A$ is a morphism to a simple complex torus A such that there is a closed real 1-form α on X without zeros and such that $[\alpha] \in f^*H^1(A, \mathbb{R})$, then f is a $\mathbb{Z}$-homology fiber bundle, see Proposition 3.1 and Corollary 3.4. This generalizes a recent result of Dutta–Hao–Liu [2, Corollary 1.6], who proved that f is a $\mathbb{C}$-homology fiber bundle (under the assumption that X is projective). In order to obtain the integral statement, one essentially has to prove that f is a $\mathbb{K}$-homology fiber bundle for any infinite field $\mathbb{K}$. To this end we use a different method than in [2]: instead of the Kashiwara estimate for $\mathbb{C}$-perverse sheaves, we use the generic vanishing theorem for $\mathbb{K}$-perverse sheaves by Bhatt–Schnell–Scholze and a result of Krämer and Weissauer on classification of $\mathbb{K}$-perverse sheaves with vanishing Euler characteristics.

In the case of non-simple tori, Conjectures 1.2 does not directly follow from 1.3. Indeed, there are smooth complex projective threefolds X such that for any morphism $f : X \to A$ to a positive-dimensional complex torus A, f is not even a $\mathbb{Q}$-homology fiber bundle (e.g. this happens for the blow-up of $E_1 \times E_2 \times \mathbb{P}^1$ along the union of $E_1 \times 0 \times 0$ and $0 \times E_2 \times \infty$, where E_1, E_2 denote non-isogeneous elliptic curves).

The assumption that A is simple is automatic in the case where A is an elliptic curve and so the above theorem gives good evidence that Conjecture 1.1 may hold in the case where $b_1(X) = 2$; an interesting wide open special case.

It is straightforward to see that Conjecture 1.3 holds for any proper morphism of relative dimension ≤ 0. By [3, Proposition 10], Conjecture 1.3 also holds for proper morphisms of relative dimension 1. Therefore we have

Corollary B. *Let X be a compact Kähler manifold and let $f : X \to A$ be a morphism to a simple complex torus A. Assume that* $\dim A \geq \dim X - 1$. *Then Conjecture 1.2 holds for f.*

Corollary C. *Let X be a compact Kähler manifold such that* $\mathrm{Alb}(X)$ *is simple and* $b_1(X) \geq 2\dim X - 2$. *Then Conjecture 1.1 holds for X.*

By a result of Popa and Schnell [11], smooth projective varieties of general type do not admit nowhere vanishing holomorphic one-forms. If the Albanese variety is simple, this had earlier been proven in [5, Theorem 1.4]. Therefore we have

Corollary D. *Let X be a smooth projective variety of general type such that* $\mathrm{Alb}(X)$ *is simple and* $b_1(X) \geq 2\dim X - 2$. *Then the underlying differential manifold of X cannot be a $\mathcal{C}^\infty$-fiber bundle over the circle.*

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2. Perverse sheaves on abelian varieties

In this section, let us review some results of Bhatt-Schnell-Scholze [1] and Krämer-Weissauer [8] about perverse sheaves on abelian varieties.

Let A be a compact complex torus and let $\mathbb{K}$ be any field.

Theorem 2.1. [1, Theorem 1.1] *Let $\mathcal{P}$ be a $\mathbb{K}$-perverse sheaf on A. Then for a generic rank one $\mathbb{K}$-local system L on A, we have*

$$H^i(A, \mathcal{P} \otimes_{\mathbb{K}} L) = 0, \quad \text{for all } i \neq 0.$$

Definition 2.2. For any $\mathbb{K}$-perverse sheaf $\mathcal{P}$ on A, the *Euler characteristic* of $\mathcal{P}$ is defined by

$$\chi(\mathcal{P}) := \sum_i (-1)^i \dim_{\mathbb{K}} H^i(A, \mathcal{P}).$$

Under the additional assumption that A is simple, we have the following important result.

Theorem 2.3 (Krämer-Weissauer)**.** *Let $\mathcal{P}$ be a simple $\mathbb{K}$-perverse sheaf on A. Suppose that $\chi(\mathcal{P}) = 0$ and A is simple, then $\mathcal{P}$ is a shift of a local system.*

If A is a simple abelian variety, the above theorem follows from [8, Proposition 10.1], cf. [9, Proposition 5.11] (for more details see Corollary 5.15 in the first arXiv version of [9]). The argument extends to the case of complex tori and we include some details in the appendix of this paper, see Proposition A.4 below.

Corollary 2.4. *Let $\mathcal{P}$ be a $\mathbb{K}$-perverse sheaf on A. Suppose that $\chi(\mathcal{P}) = 0$ and A is simple, then $\mathcal{P}$ is a shift of a local system.*

Proof. First, Theorem 2.1 implies that the Euler characteristic of any perverse sheaf $\mathcal{P}$ is non-negative: let L be a generic rank one local system, then

$$\chi(\mathcal{P}) = \chi(\mathcal{P} \otimes L) = \dim H^0(A, \mathcal{P} \otimes L) \geq 0.$$

The first equality comes from the invariance of Euler characteristic under twisting by a rank one local system and the second equality uses Theorem 2.1. Now suppose $\chi(\mathcal{P}) = 0$. We can write $\mathcal{P}$ as a successive extension of simple perverse sheaves $\mathcal{P}_\alpha$. Since the Euler characteristic is additive in short exact sequences and Euler characteristic for any perverse sheaf $\mathcal{P}_\alpha$ is always non-negative, we see that $\chi(\mathcal{P}_\alpha) = 0$. By Theorem 2.3, we know that $\mathcal{P}_\alpha$ is a shift of a local system by the same constant $\dim X$. Therefore we conclude that $\mathcal{P}$ is a shift of a local system. □

3. The proof

In this section, we prove Theorem A and deduce its corollaries.

First, let us study the topological implication of the existence of a nowhere vanishing real one-form. It exploits the interplay of two structures: a $\mathcal{C}^\infty$-fiber bundle structure over S^1 and a morphism to a compact torus.

Proposition 3.1. *Let $f : X \to A$ be a morphism from a compact Kähler manifold X to a simple complex torus A. Suppose that the condition (B) of Conjecture 1.2 holds, i.e. there is a closed real 1-form α with $[\alpha] \in f^*H^1(A, \mathbb{R})$ on X without zeros. Then we have*

$$\chi({}^pR^j f_* \mathbb{K}_X) = 0,$$

for any $j \in \mathbb{Z}$ and any infinite field $\mathbb{K}$.

Proof. Up to perturbing α slightly, we may assume that $[\alpha] \in f^*H^1(A, \mathbb{Q})$. Multiplying by a suitbale integer, we thus reduce to the case where $[\alpha] \in f^*H^1(A, \mathbb{Z})$. Integration over α then yields as in [13] a submersive $\mathcal{C}^\infty$-map $g : X \to S^1$ to the circle with $g^*d\theta = \alpha$, where θ denotes the angular coordinate on S^1. In particular, g is a $\mathcal{C}^\infty$-fiber bundle with $g^*H^1(S^1, \mathbb{R}) \subseteq f^*H^1(A, \mathbb{R})$.

Let $\mathbb{K}$ be any infinite field. We first mimic the proof of [6, Theorem A.1] to produce a $\mathbb{K}$-local system on X with no cohomology. Let L_λ be a generic $\mathbb{K}$-local system on S^1 with monodromy given by $\lambda \in \mathbb{K}$. Set

$$L = g^*L_\lambda.$$

Consider the Leray spectral sequence with E_2-term

$$E_2^{p,q} = H^p(S^1, R^q g_* g^* L_\lambda) = H^p(S^1, L_\lambda \otimes_{\mathbb{K}} R^q g_* \mathbb{K}_X) \Longrightarrow H^{p+q}(X, L).$$

Since g is a $\mathcal{C}^\infty$-fiber bundle, the sheaf $R^q g_* \mathbb{K}_X$ is a local system on S^1 with the stalk V^q being a finite dimensional $\mathbb{K}$-vector space. Since $\mathbb{K}$ is infinite and so we can choose an element $\lambda \in \mathbb{K}$ such that λ^{-1} is different from any of the eigenvalues of the natural monodromy operator on V^q for all q. Therefore we conclude that

$$H^0(S^1, L_\lambda \otimes_{\mathbb{K}} R^q g_* \mathbb{K}_X) = 0, \quad \text{for all } q.$$

Since the Euler characteristic of any local system of finite rank on S^1 is zero, we also know that

$$H^1(S^1, L_\lambda \otimes_{\mathbb{K}} R^q g_* \mathbb{K}_X) = 0, \quad \text{for all } q.$$

Therefore, $E_2^{p,q} = 0$ for all p, q and we obtain $H^k(X, L) = 0$ for all k.

Since $g^*H^1(S^1, \mathbb{R}) \subseteq f^*H^1(A, \mathbb{R})$, the local system L above is isomorphic to the pull-back of some $\mathbb{K}$-local system on A. By the semicontinuity of cohomology in families, we deduce that a generic $\mathbb{K}$-local system L_A of rank 1 on A satisfies:

$$H^i(X, f^*L_A) = 0 \quad \text{for all } i. \tag{1}$$

We now set $L_X := f^*L_A$ and consider the perverse Leray spectral sequence with E_2-term

$$\begin{aligned} E_2^{j,\ell} &= H^j(\mathrm{Alb}(X), {}^pR^\ell f_* L_X) = H^j(\mathrm{Alb}(X), {}^pR^\ell f_*(f^*L_A)) \\ &= H^j(\mathrm{Alb}(X), {}^pR^\ell f_* \mathbb{K}_X \otimes_{\mathbb{K}} L_A) \Longrightarrow H^{j+\ell}(X, L_X). \end{aligned}$$

Theorem 2.1 implies that $E_2^{j,\ell} = 0$ for $j > 0$, and so the spectral sequence degenerates at E_2-page. On the other hand, $H^{j+\ell}(X, L_X) = 0$ for all j, ℓ by (1) and so

$$E_2^{j,\ell} = H^j(\mathrm{Alb}(X), {}^pR^\ell f_* \mathbb{K}_X \otimes_{\mathbb{K}} L_A) = 0$$

for all j, ℓ. Hence,

$$\chi(\mathrm{Alb}(X), {}^pR^\ell f_* \mathbb{K}_X \otimes L_A) = 0, \quad \text{for all } \ell.$$

Using the invariance of Euler characteristic under twisting by a rank one local system, we conclude that

$$\chi(\mathrm{Alb}(X), {}^pR^j f_* \mathbb{K}_X) = \chi(\mathrm{Alb}(X), {}^pR^j f_* \mathbb{K}_X \otimes_{\mathbb{K}} L_A) = 0,$$

for all j and any infinite field $\mathbb{K}$. □

Before deriving a consequence of Proposition 3.1, we recall

Definition 3.2. [9, Definition 4.1] Let R be any commutative ring. An R-constructible complex $\mathcal{F}$ is *locally constant* if the cohomology sheaves $\mathcal{H}^j(\mathcal{F})$ are local systems for all j.

The following lemma is a version of [9, Lemma 5.9].

Lemma 3.3. *Let $\mathcal{F}$ be a $\mathbb{Z}$-constructible complex on a complex manifold. If $\mathcal{F} \otimes^L_{\mathbb{Z}} \mathbb{K}$ is locally constant for any infinite field $\mathbb{K}$, then $\mathcal{F}$ is locally constant.*

Proof. The proof of [9, Lemma 5.9] is reduced to [9, Lemma 5.8]. But to show a morphism of bounded complexes of free $\mathbb{Z}$-modules being quasi-isomorphic, we just need to check it still holds after tensoring with any infinite field (in fact one field per characteristic is enough). □

Corollary 3.4. *With the same assumption as in Proposition 3.1, assume in addition that A is simple. Then $f : X \to A$ is a $\mathbb{Z}$-homology fiber bundle.*

Proof. Note first that f is a $\mathbb{Z}$-homology fiber bundle if and only if

$$\mathcal{H}^j(Rf_*\mathbb{Z}_X) = R^j f_*\mathbb{Z}_X$$

are local systems for all j, i.e. the $\mathbb{Z}$-constructible complex $Rf_*\mathbb{Z}_X$ is locally constant. By Lemma 3.3, it suffices to show that for any infinite field $\mathbb{K}$, the $\mathbb{K}$-constructible sheaf

$$Rf_*\mathbb{Z}_X \otimes^L_{\mathbb{Z}} \mathbb{K} = Rf_*\mathbb{K}_X$$

is locally constant. Then [9, Proposition 4.3] says that we only need to check that the perverse cohomology sheaf

$$^pR^j f_*\mathbb{K}_X \text{ is a shift of a local system for each } j \text{ and any infinite } \mathbb{K}.$$

Since A is simple, by Corollary 2.4, it suffices to show that

$$\chi(^pR^j f_*\mathbb{K}_X) = 0, \text{ for each } j \text{ and any infinite } \mathbb{K},$$

which follows from Proposition 3.1. Therefore we conclude that $f : X \to A$ is a $\mathbb{Z}$-homology fiber bundle. □

Proof of Theroem A. It suffices to prove (B) $\Longrightarrow$ (A). Starting from (B), using Corollary 3.4 and the Bobadilla-Kollár conjecture 1.3 for the morphism $f : X \to A$, we deduce that $f : X \to A$ is smooth. Therefore any pullback of holomorphic 1-form has no zeros on X and thus (B) $\Longrightarrow$ (A). □

Proof of Corollary B. Let $f : X \to A$ be a proper map, where A is a simple complex torus with $\dim A \geq \dim X - 1$. Assume that condition (B) in Conjecture 1.2 holds. By Corollary 3.4, f is a $\mathbb{Z}$-homology fibration and in particular surjective. The Bobadilla–Kollár conjecture is clearly true for proper maps of relative dimension zero and it holds for proper maps of relative dimension one by [3, Proposition 10]. This concludes the argument because $\dim A \geq \dim X - 1$. □

Proof of Corollary C. This is a direct consequence of Corollary B. □

Appendix A. Degenerate perverse sheaves on complex tori

In this appendix, we provide a proof of Theorem 2.3.

First, we adapt [4, §2] to the analytic setting, where one studys the generic degree of a meromorphic map to a Grassmannian (e.g. the Gauss map). Let M be a connected k-dimensional complex manifold. Let V be an n-dimensional complex vector space, and let

$$f : M \dashrightarrow G(k, V)$$

be a meromorphic map to the Grassmannian of k-dimensional linear subspaces of V. Up to replacing M by a dense Zariski open subset, we can assume that f is regular. Consider the flag variety $F(k, n-1, V)$ and its projections

$$\begin{array}{ccc} & F(k,n-1,V) & \\ \swarrow^{p} & & \searrow^{q} \\ G(k,V) & & G(n-1,V). \end{array}$$

Consider the fiber product diagram

$$\begin{array}{ccc} \tilde{M} := M \times_{G(k,V)} F(k,n-1,V) & \longrightarrow & F(k,n-1,V) \\ \downarrow & & \downarrow p \\ M & \xrightarrow{\quad f \quad} & G(k,V) \end{array}$$

Since p is a smooth map with $(n-k-1)$-dimensional fibers, $\tilde{M}$ is a connected $(n-1)$-dimensional complex manifold. Set-theoretically, we have

$$\tilde{M} = \{(x, V_0, W) \in M \times F(k, n-1, V) \mid f(x) = V_0, V_0 \subseteq W\}.$$

The map q induces a holomorphic map between complex manifolds of the same dimension

$$\begin{aligned} q' : & \tilde{M} \to F(k, n-1, V) \xrightarrow{q} G(n-1, V), \\ & (x, V_0, W) \mapsto (V_0, W) \mapsto W. \end{aligned}$$

Note that we have $f(x) \in G(k, W)$.

Definition A.1. We define $\deg f$ to be the degree of the map q'.

With the set-up above, it is direct to deduce the following

Proposition A.2. *If $W \subseteq V$ is a generic hyperplane, then $\deg f$ is equal to the number of $x \in M$ such that $f(x) \in G(k, W) \subseteq G(k, V)$. If $\deg f > 0$, then for any such x, the map f is locally an embedding near x and the intersection $f(M) \cap G(k, W)$ is transversal at x.*

Now we apply the discussion above to the Gauss map associated to a complex torus. We find it is easier to work in a more general setting. Let G be a complex Lie group, let $\mathfrak{g}$ be its Lie algebra. If $Z \subseteq G$ is an irreducible k-dimensional closed analytic subset, we have the meromorphic map (called Gauss map)

$$\Gamma_Z : Z \dashrightarrow G(k, \mathfrak{g})$$

defined as follows. For $x \in G$, let

$$\ell_x : G \to G, \quad \ell_x(y) = xy$$

be the left multiplication by x. Then, for a smooth point $z \in Z$, we have $\Gamma_Z(z) = z^{-1}(T_z Z)$, which is the image of the differential map $\ell_{z^{-1}}$

$$d_z \ell_{z^{-1}} : T_z Z \to T_e G = \mathfrak{g}.$$

Let $\Lambda_Z \subseteq T^*G$ denote the conic Lagrangian variety associated to Z, which is the closure in T^*G of the conormal bundle in G to the smooth locus of Z. For $\gamma \in \mathfrak{g}^*$, let $\Omega_\gamma \subseteq T^*G$ be the graph of the left invariant 1-form ω_γ on G associated to γ.

Proposition A.3. *Let $\gamma \in \mathfrak{g}^*$ be a generic linear functional. Then $\Lambda_Z \cap \Omega_\gamma$ consists of finitely many points that are smooth on Λ_Z and in which the intersection is transverse. Moreover,*

$$\deg \Gamma_Z = \#|\Lambda_Z \cap \Omega_\gamma|.$$

Proof. Apply Proposition A.2 where $M = Z, V = \mathfrak{g}, f = \Gamma_Z$. We can view $\gamma \in \mathfrak{g}^*$ as a generic hyperplane in $\mathfrak{g}$ and $\Gamma_Z(z) \subseteq \gamma \subseteq \mathfrak{g}$ if and only if $(z, \omega_\gamma(z)) \in \Lambda_Z \cap \Omega_\gamma$. □

Finally, we can prove Theorem 2.3.

Proposition A.4. *Let A be a complex torus, which is simple as a torus. Let $\mathbb{K}$ be any field and let $\mathcal{P}$ be a simple $\mathbb{K}$-perverse sheaf on A. If the Euler characteristic of $\mathcal{P}$ vanishes, i.e.*

$$\chi(\mathcal{P}) = \sum_i (-1)^i \dim_{\mathbb{K}} H^i(A, \mathcal{P}) = 0,$$

then $\mathcal{P}$ is a shift of a local system.

Proof. We adapt the proof of [8, Proposition 10.1]. For $Z \subseteq A$ closed and irreducible, let $\Lambda_Z \subseteq T^*A$ denote the closure in T^*A of the conormal bundle in A to the smooth locus of Z. By [10, Definition 3.34], the characteristic cycle associated to the $\mathbb{K}$-perverse sheaf $\mathcal{P}$ on the complex manifold A is a finite formal sum

$$CC(\mathcal{P}) = \sum_{Z \subseteq A} n_Z \cdot \Lambda_Z, \qquad \text{with } n_Z \in \mathbb{Z},$$

where Z runs through all closed irreducible subsets of A,

$$n_Z := (-1)^{\dim Z} \cdot \chi(\mathrm{NMD}(\mathcal{P}, Z)),$$

and $\mathrm{NMD}(\mathcal{P}, Z)$ is the sheaf-theoretic counterpart of the normal Morse data defined in [10, §3.1, (26)]. By [10, Example 3.26], we have $n_Z \geq 0$ for $\mathbb{K}$-perverse sheaves. The Dubson-Kashiwara's microlocal index formula still holds in this setting: apply [10, Theorem 3.38] when f is a constant function. Therefore we have

$$\chi(\mathcal{P}) = \sum_{Z \subseteq A} n_Z \cdot d_Z,$$

where

$$d_Z = \langle [\Lambda_A] \cdot [\Lambda_Z] \rangle_{T^*A} \in \mathbb{N}.$$

Therefore, d_Z can be computed as the intersection number of Λ_Z with Ω_γ inside T^*A, where Ω_γ is the graph of a generic differential one-form γ on A. Since A is a torus, the cotangent bundle $T^*A = A \times \mathbb{C}^g$ is trivial of rank $g = \dim A$, with two projection maps

$$\begin{array}{ccc} & T^*A & \\ {}^{\mathrm{pr}_1}\swarrow & & \searrow^{\mathrm{pr}_2} \\ A & & \mathbb{C}^g \end{array}$$

Projecting from $\Lambda_Z \subseteq T^*A$ onto the second factor $\mathbb{C}^g$ induces a map

$$p : \Lambda_Z \to \mathbb{C}^g.$$

It is easy to see that degree of p is equal to the degree of the Gauss map associated to Z and A as discussed above. By Proposition A.3, the intersection number $d_Z = \#|\Lambda_Z \cap \Omega_\gamma|$ is the generic degree of the Gauss map, which is equal to the generic degree of p.

Since n_Z, d_Z are both nonnegative, the assumption $\chi(P) = 0$ implies that $d_Z = 0$. Therefore p is not surjective and $\dim p(\Lambda_Z) < g$. Then for some cotangential vector $\omega \in p(\Lambda_Z)$, the fiber $p^{-1}(\omega)$ is positive-dimensional. If $Z \neq A$, we can assume $\omega \neq 0$. Let $Y \subseteq A$ be the image of $p^{-1}(\omega) \subseteq T^*A$ under the map $T^*A \to A$. Then $\dim Y > 0$, and up to a translation we can assume $0 \in Y$. By construction, ω is normal to Y in every smooth point of Y, so the preimage of Y under the universal covering $\mathbb{C}^g \to A = \mathbb{C}^g/\Lambda$ lies in the hyperplane of $\mathbb{C}^g$ orthogonal to ω. Thus the subtorus of A generated by Y is strictly contained in A but non-zero, contradicting the assumption that A is simple.

Therefore the characteristic cycle $\mathrm{CC}(\mathcal{P})$ only contains the zero section of T^*A and hence $\mathcal{P}$ is a shift of a local system, see e.g. Lemma 5.14 in the first arXiv version of [9]. $\square$

Remark A.5. The proof of Proposition A.4 works verbatim for arbitrary possibly non-simple perverse sheaves. We presented an alternative argument to reduce to the non-simple case in Corollary 2.4 above for the convenience of the reader.

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