

ZEROS OF ONE-FORMS AND HOMOLOGICALLY TRIVIAL DEFORMATIONS

STEFAN SCHREIEDER AND RUIJIE YANG

ABSTRACT. We show that a conjecture of Kotschick about one-forms without zeros on compact Kähler manifolds follows in the case of simple Albanese torus from a conjecture of Bobadilla and Kollár about homologically trivial deformations. As an application, we prove Kotschick's conjecture for compact Kähler manifolds X with $b_1(X) \geq 2 \dim X - 2$, whose Albanese torus is simple.

1. INTRODUCTION

In [7], Kotschick made the following

Conjecture 1.1. For a compact Kähler manifold X , the following conditions are equivalent:

- (A) X admits a holomorphic one-form without zeros;
- (B) X admits a real closed one-form without zeros; or by Tischler's theorem [13] equivalently, the underlying differential manifold of X is a \mathcal{C}^∞ -fiber bundle over the circle.

We propose the following stronger conjecture, which implies Conjecture 1.1 when we apply it to the Albanese morphism $X \rightarrow \text{Alb}(X)$.

Conjecture 1.2. Let X be a compact Kähler manifold and let $f : X \rightarrow A$ be a morphism to a complex torus A . Then the following conditions are equivalent:

- (A) X admits a holomorphic one-form w without zeros such that

$$[w] \in f^*H^0(A, \Omega_A^1).$$

- (B) X admits a real closed one-form α without zeros such that $[\alpha] \in f^*H^1(A, \mathbb{R})$; or, by Tischler's argument [13] equivalently, the underlying differential manifold of X is a \mathcal{C}^∞ -fiber bundle $g : X \rightarrow S^1$ over the circle with

$$g^*H^1(S^1, \mathbb{R}) \subseteq f^*H^1(A, \mathbb{R}).$$

In both conjectures, it is easy to see that Condition (A) implies Condition (B). The converse direction is the non-trivial part. These conjectures are known for surfaces [7, 12] and for projective threefolds [6]. (In *loc. cit.*, Conjecture 1.1 is considered, but the arguments prove in fact the slightly stronger assertion from Conjecture 1.2, cf. [6, Theorem 1.4].)

The observation of this paper is that we can relate Conjecture 1.2 to the following conjecture of Bobadilla and Kollár [3, Conjecture 3]. For a commutative ring R , a proper morphism $f : X \rightarrow Y$ between complex analytic spaces is called an R -homology fiber bundle if Y has an open cover $Y = \cup_i U_i$ such that for every i and every $y \in U_i$, the inclusion

$$H_*(f^{-1}(y), R) \hookrightarrow H_*(f^{-1}(U_i), R) \quad \text{is an isomorphism.}$$

Conjecture 1.3. Let $f : X \rightarrow Y$ be a proper morphism between complex analytic spaces, where X and Y are both smooth. If f is a \mathbb{Z} -homology fiber bundle, then f is smooth.

Our main result is

Theorem A. *Let X be a compact Kähler manifold and let $f : X \rightarrow A$ be a morphism to a simple complex torus A . Assume that Conjecture 1.3 holds for the morphism f , then Conjecture 1.2 holds for f .*

To prove the theorem above, we show that if $f : X \rightarrow A$ is a morphism to a simple complex torus A such that there is a closed real 1-form α on X without zeros and such that $[\alpha] \in f^*H^1(A, \mathbb{R})$, then f is a \mathbb{Z} -homology fiber bundle, see Proposition 3.1 and Corollary 3.4. This generalizes a recent result of Dutta–Hao–Liu [2, Corollary 1.6], who proved that f is a \mathbb{C} -homology fiber bundle (under the assumption that X is projective). In order to obtain the integral statement, one essentially has to prove that f is a \mathbb{K} -homology fiber bundle for any algebraically closed field \mathbb{K} . To this end we use a different method than in [2]: instead of the Kashiwara estimate for \mathbb{C} -perverse sheaves, we use the generic vanishing theorem for \mathbb{K} -perverse sheaves by Bhatt–Schnell–Scholze and a result of Krämer and Weissauer on classification of \mathbb{K} -perverse sheaves with vanishing Euler characteristics.

The case of non-simple tori is more complicated, and we hope to treat it in a later paper.

It is straightforward to see that Conjecture 1.3 holds for any proper morphism of relative dimension ≤ 0 . By [3, Proposition 10], Conjecture 1.3 also holds for proper morphisms of relative dimension 1. Therefore we have

Corollary B. *Let X be a compact Kähler manifold and let $f : X \rightarrow A$ be a morphism to a simple complex torus A . Assume that $\dim A \geq \dim X - 1$. Then Conjecture 1.2 holds for f .*

Corollary C. *Let X be a compact Kähler manifold such that $\text{Alb}(X)$ is simple and $b_1(X) \geq 2 \dim X - 2$. Then Conjecture 1.1 holds for X .*

By a result of Popa and Schnell [11], smooth projective varieties of general type do not admit nowhere vanishing holomorphic one-forms. If the Albanese variety is simple, this had earlier been proven in [5, Theorem 1.4]. Therefore we have

Corollary D. *Let X be a smooth projective variety of general type such that $\text{Alb}(X)$ is simple and $b_1(X) \geq 2 \dim X - 2$. Then the underlying differential manifold of X cannot be a C^∞ -fiber bundle over the circle.*

Acknowledgement. We would like to thank Thomas Krämer and Botong Wang for helpful discussion, a special thank to Botong Wang for pointing out the reference [10] and the first arxiv version of [9]. SS is supported by the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation programme under grant agreement No 948066. RY is supported by a Riemann Fellowship of Riemann Center for Geometry and Physics at Leibniz Universität Hannover during the preparation of this paper.

2. PERVERSE SHEAVES ON ABELIAN VARIETIES

In this section, let us review some results of Bhatt–Schnell–Scholze [1] and Krämer–Weissauer [8] about perverse sheaves on abelian varieties.

Let A be a compact complex torus and let \mathbb{K} be any field.

Theorem 2.1. [1, Theorem 1.1] *Let \mathcal{P} be a \mathbb{K} -perverse sheaf on A . Then for a generic rank one \mathbb{K} -local system L on A , we have*

$$H^i(A, \mathcal{P} \otimes_{\mathbb{K}} L) = 0, \quad \text{for all } i \neq 0.$$

Definition 2.2. For any \mathbb{K} -perverse sheaf \mathcal{P} on A , the *Euler characteristic* of \mathcal{P} is defined by

$$\chi(\mathcal{P}) := \sum_i (-1)^i \dim_{\mathbb{K}} H^i(A, \mathcal{P}).$$

Under the additional assumption that A is simple, we have the following important result.

Theorem 2.3 (Krämer-Weissauer). *Let \mathcal{P} be a simple \mathbb{K} -perverse sheaf on A . Suppose that $\chi(\mathcal{P}) = 0$ and A is simple, then \mathcal{P} is a shift of a local system.*

If A is a simple abelian variety, the above theorem follows from [8, Proposition 10.1], cf. [9, Proposition 5.11] (for more details see Corollary 5.15 in the first arXiv version of [9]). The argument extends to the case of complex tori and we include some details in the appendix of this paper, see Proposition A.4 below.

3. THE PROOF

In this section, we prove Theorem A and deduce its corollaries.

First, let us study the topological implication of the existence of a nowhere vanishing real one-form. It exploits the interplay of two structures: a C^∞ -fiber bundle structure over S^1 and a morphism to a compact torus.

Proposition 3.1. *Let $f : X \rightarrow A$ be a morphism from a compact Kähler manifold X to a simple complex torus A . Suppose that the condition (B) of Conjecture 1.2 holds, i.e. there is a closed real 1-form α with $[\alpha] \in f^*H^1(A, \mathbb{R})$ on X without zeros. Then we have*

$$\chi(R^j f_* \mathbb{K}_X) = 0,$$

for any $j \in \mathbb{Z}$ and any algebraically closed field \mathbb{K} .

Proof. Up to perturbing α slightly, we may assume that $[\alpha] \in f^*H^1(A, \mathbb{Q})$. Multiplying by a suitable integer, we thus reduce to the case where $[\alpha] \in f^*H^1(A, \mathbb{Z})$. Integration over α then yields as in [13] a submersive C^∞ -map $g : X \rightarrow S^1$ to the circle with $g^*d\theta = \alpha$, where θ denotes the angular coordinate on S^1 . In particular, g is a C^∞ -fiber bundle with $g^*H^1(S^1, \mathbb{R}) \subseteq f^*H^1(A, \mathbb{R})$.

Let \mathbb{K} be an algebraically closed field of arbitrary characteristic. We first mimic the proof of [6, Theorem A.1] to produce a \mathbb{K} -local system on X with no cohomology. Let L_λ be a generic \mathbb{K} -local system on S^1 with monodromy given by $\lambda \in \mathbb{K}$. Set

$$L = g^* L_\lambda.$$

Consider the Leray spectral sequence with E_2 -term

$$E_2^{p,q} = H^p(S^1, R^q g_* g^* L_\lambda) = H^p(S^1, L_\lambda \otimes_{\mathbb{K}} R^q g_* \mathbb{K}_X) \implies H^{p+q}(X, L).$$

Since g is a C^∞ -fiber bundle, the sheaf $R^q g_* \mathbb{K}_X$ is a local system on S^1 with the stalk V^q being a finite dimensional \mathbb{K} -vector space. Since \mathbb{K} is algebraically closed, it is infinite and so we can choose an element $\lambda \in \mathbb{K}$ such that λ^{-1} is different from any of the eigenvalues of the natural monodromy operator on V^q for all q . Therefore we conclude that

$$H^0(S^1, L_\lambda \otimes_{\mathbb{K}} R^q g_* \mathbb{K}_X) = 0, \quad \text{for all } q.$$

Since the Euler characteristic of any local system of finite rank on S^1 is zero, we also know that

$$H^1(S^1, L_\lambda \otimes_{\mathbb{K}} R^q g_* \mathbb{K}_X) = 0, \quad \text{for all } q.$$

Therefore, $E_2^{p,q} = 0$ for all p, q and we obtain $H^k(X, L) = 0$ for all k .

Since $g^*H^1(S^1, \mathbb{R}) \subseteq f^*H^1(A, \mathbb{R})$, the local system L above is isomorphic to the pull-back of some \mathbb{K} -local system on A . By the semicontinuity of cohomology in families, we deduce that a generic \mathbb{K} -local system L_A of rank 1 on A satisfies:

$$(1) \quad H^i(X, f^*L_A) = 0 \quad \text{for all } i.$$

We now set $L_X := f^*L_A$ and consider the perverse Leray spectral sequence with E_2 -term

$$\begin{aligned} E_2^{j,\ell} &= H^j(\text{Alb}(X), {}^pR^\ell f_* L_X) = H^j(\text{Alb}(X), {}^pR^\ell f_*(f^*L_A)) \\ &= H^j(\text{Alb}(X), {}^pR^\ell f_* \mathbb{K}_X \otimes_{\mathbb{K}} L_A) \implies H^{j+\ell}(X, L_X). \end{aligned}$$

Theorem 2.1 implies that $E_2^{j,\ell} = 0$ for $j > 0$, and so the spectral sequence degenerates at E_2 -page. On the other hand, $H^{j+\ell}(X, L_X) = 0$ for all j, ℓ by (1) and so

$$E_2^{j,\ell} = H^j(\text{Alb}(X), {}^pR^\ell f_* \mathbb{K}_X \otimes_{\mathbb{K}} L_A) = 0$$

for all j, ℓ . Hence,

$$\chi(\text{Alb}(X), {}^pR^\ell f_* \mathbb{K}_X \otimes L_A) = 0, \quad \text{for all } \ell.$$

Using the invariance of Euler characteristic under twisting by a rank one local system, we conclude that

$$\chi(\text{Alb}(X), {}^pR^j f_* \mathbb{K}_X) = \chi(\text{Alb}(X), {}^pR^j f_* \mathbb{K}_X \otimes_{\mathbb{K}} L_A) = 0,$$

for all j and any algebraically closed field \mathbb{K} . \square

Before deriving a consequence of Proposition 3.1, we recall

Definition 3.2. [9, Definition 4.1] Let R be any commutative ring. An R -constructible complex \mathcal{F} is *locally constant* if the cohomology sheaves $\mathcal{H}^j(\mathcal{F})$ are local systems for all j .

The following lemma is a version of [9, Lemma 5.9].

Lemma 3.3. *Let \mathcal{F} be a \mathbb{Z} -constructible complex on a complex manifold. If $\mathcal{F} \otimes_{\mathbb{Z}}^L \mathbb{K}$ is locally constant for any algebraically closed field \mathbb{K} , then \mathcal{F} is locally constant.*

Proof. The proof of [9, Lemma 5.9] is reduced to [9, Lemma 5.8]. But to show a morphism of bounded complexes of free \mathbb{Z} -modules being quasi-isomorphic, we just need to check it still holds after tensoring with any algebraically closed field (in fact one field per characteristic is enough). \square

Corollary 3.4. *With the same assumption as in Proposition 3.1. In addition, assume that A is simple. Then $f : X \rightarrow A$ is a \mathbb{Z} -homology fiber bundle.*

Proof. Note first that f is a \mathbb{Z} -homology fiber bundle if and only if

$$\mathcal{H}^j(Rf_* \mathbb{Z}_X) = R^j f_* \mathbb{Z}_X$$

are local systems for all j , i.e. the \mathbb{Z} -constructible complex $Rf_* \mathbb{Z}_X$ is locally constant. By Lemma 3.3, it suffices to show that for any algebraically closed field \mathbb{K} , the \mathbb{K} -constructible sheaf

$$Rf_* \mathbb{Z}_X \otimes_{\mathbb{Z}}^L \mathbb{K} = Rf_* \mathbb{K}_X$$

is locally constant. Then [9, Proposition 4.3] says that we only need to check that the perverse cohomology sheaf

$${}^pR^j f_* \mathbb{K}_X \text{ is a shift of a local system for each } j \text{ and any } \mathbb{K} = \overline{\mathbb{K}}.$$

Since A is simple, by Theorem 2.3, it suffices to show that

$$\chi({}^pR^j f_* \mathbb{K}_X) = 0, \text{ for each } j \text{ and any } \mathbb{K} = \overline{\mathbb{K}},$$

which follows from Proposition 3.1. Therefore we conclude that $f : X \rightarrow A$ is a \mathbb{Z} -homology fiber bundle. \square

Proof of Theorem A. It suffices to prove (B) \implies (A). Starting from (B), using Corollary 3.4 and the Bobadilla-Kollár conjecture 1.3 for the morphism $f : X \rightarrow A$, we deduce that $f : X \rightarrow A$ is smooth. Therefore any pullback of holomorphic 1-form has no zeros on X and thus (B) \implies (A). \square

Proof of Corollary B. Let $f : X \rightarrow A$ be a proper map, where A is a simple complex torus with $\dim A \geq \dim X - 1$. Assume that condition (B) in Conjecture 1.2 holds. By Corollary 3.4, f is a \mathbb{Z} -homology fibration and in particular surjective. The Bobadilla-Kollár conjecture is clearly true for proper maps of relative dimension zero and it holds for proper maps of relative dimension one by [3, Proposition 10]. This concludes the argument because $\dim A \geq \dim X - 1$. \square

Proof of Corollary C. This is a direct consequence of Corollary B. \square

APPENDIX A. DEGENERATE PERVERSE SHEAVES ON COMPLEX TORI

In this appendix, we provide a proof of Theorem 2.3.

First, we adapt [4, §2] to the analytic setting, where one studies the generic degree of a meromorphic map to a Grassmannian (e.g. the Gauss map). Let M be a connected k -dimensional complex manifold. Let V be an n -dimensional complex vector space, and let

$$f : M \dashrightarrow G(k, V)$$

be a meromorphic map to the Grassmannian of k -dimensional linear subspaces of V . Up to replacing M by a dense Zariski open subset, we can assume that f is regular. Consider the flag variety $F(k, n-1, V)$ and its projections

$$\begin{array}{ccc} & F(k, n-1, V) & \\ & \swarrow p \quad \searrow q & \\ G(k, V) & & G(n-1, V). \end{array}$$

Consider the fiber product diagram

$$\begin{array}{ccc} \tilde{M} := M \times_{G(k, V)} F(k, n-1, V) & \longrightarrow & F(k, n-1, V) \\ \downarrow & & \downarrow p \\ M & \xrightarrow{f} & G(k, V) \end{array}$$

Since p is a smooth map with $(n-k-1)$ -dimensional fibers, \tilde{M} is a connected $(n-1)$ -dimensional complex manifold. Set-theoretically, we have

$$\tilde{M} = \{(x, V_0, W) \in M \times F(k, n-1, V) \mid f(x) = V_0, V_0 \subseteq W\}.$$

The map q induces a holomorphic map between complex manifolds of the same dimension

$$\begin{aligned} q' : \tilde{M} &\rightarrow F(k, n-1, V) \xrightarrow{q} G(n-1, V), \\ (x, V_0, W) &\mapsto (V_0, W) \mapsto W. \end{aligned}$$

Note that we have $f(x) \in G(k, W)$.

Definition A.1. We define $\deg f$ to be the degree of the map q' .

With the set-up above, it is direct to deduce the following

Proposition A.2. *If $W \subseteq V$ is a generic hyperplane, then $\deg f$ is equal to the number of $x \in M$ such that $f(x) \in G(k, W) \subseteq G(k, V)$. If $\deg f > 0$, then for any such x , the map f is locally an embedding near x and the intersection $f(M) \cap G(k, W)$ is transversal at x .*

Now we apply the discussion above to the Gauss map associated to a complex torus. We find it is easier to work in a more general setting. Let G be a complex Lie group, let \mathfrak{g} be its Lie algebra. If $Z \subseteq G$ is an irreducible k -dimensional closed analytic subset, we have the meromorphic map (called Gauss map)

$$\Gamma_Z : Z \dashrightarrow G(k, \mathfrak{g})$$

defined as follows. For $x \in G$, let

$$\ell_x : G \rightarrow G, \quad \ell_x(y) = xy$$

be the left multiplication by x . Then, for a smooth point $z \in Z$, we have $\Gamma_Z(z) = z^{-1}(T_z Z)$, which is the image of the differential map $\ell_{z^{-1}}$

$$d_z \ell_{z^{-1}} : T_z Z \rightarrow T_e G = \mathfrak{g}.$$

Let $\Lambda_Z \subseteq T^*G$ denote the conic Lagrangian variety associated to Z , which is the closure in T^*G of the conormal bundle in G to the smooth locus of Z . For $\gamma \in \mathfrak{g}^*$, let $\Omega_\gamma \subseteq T^*G$ be the graph of the left invariant 1-form ω_γ on G associated to γ .

Proposition A.3. *Let $\gamma \in \mathfrak{g}^*$ be a generic linear functional. Then $\Lambda_Z \cap \Omega_\gamma$ consists of finitely many points that are smooth on Λ_Z and in which the intersection is transverse. Moreover,*

$$\deg \Gamma_Z = \#|\Lambda_Z \cap \Omega_\gamma|.$$

Proof. Apply Proposition A.2 where $M = Z, V = \mathfrak{g}, f = \Gamma_Z$. We can view $\gamma \in \mathfrak{g}^*$ as a generic hyperplane in \mathfrak{g} and $\Gamma_Z(z) \subseteq \gamma \subseteq \mathfrak{g}$ if and only if $(z, \omega_\gamma(z)) \in \Lambda_Z \cap \Omega_\gamma$. \square

Finally, we can prove Theorem 2.3.

Proposition A.4. *Let A be a complex torus, which is simple as a torus. Let \mathbb{K} be any field and let \mathcal{P} be a simple \mathbb{K} -perverse sheaf on A . If the Euler characteristic of \mathcal{P} vanishes, i.e.*

$$\chi(\mathcal{P}) = \sum_i (-1)^i \dim_{\mathbb{K}} H^i(A, \mathcal{P}) = 0,$$

then \mathcal{P} is a shift of a local system.

Proof. We adapt the proof of [8, Proposition 10.1]. For $Z \subseteq A$ closed and irreducible, let $\Lambda_Z \subseteq T^*A$ denote the closure in T^*A of the conormal bundle in A to the smooth locus of Z . By [10, Definition 3.34], the characteristic cycle associated to the \mathbb{K} -perverse sheaf \mathcal{P} on the complex manifold A is a finite formal sum

$$CC(\mathcal{P}) = \sum_{Z \subseteq A} n_Z \cdot \Lambda_Z, \quad \text{with } n_Z \in \mathbb{Z},$$

where Z runs through all closed irreducible subsets of A ,

$$n_Z := (-1)^{\dim Z} \cdot \chi(\text{NMD}(\mathcal{P}, Z)),$$

and $\text{NMD}(\mathcal{P}, Z)$ is the sheaf-theoretic counterpart of the normal Morse data defined in [10, §3.1, (26)]. By [10, Example 3.26], we have $n_Z \geq 0$ for \mathbb{K} -perverse sheaves. The Dubson-Kashiwara's microlocal index formula still holds in this setting: apply [10, Theorem 3.38] when f is a constant function. Therefore we have

$$\chi(\mathcal{P}) = \sum_{Z \subseteq A} n_Z \cdot d_Z,$$

where

$$d_Z = \langle [\Lambda_A] \cdot [\Lambda_Z] \rangle_{T^*A} \in \mathbb{N}.$$

Moreover, d_Z can be computed as the intersection number of Λ_Z with Ω_γ inside T^*A , where Ω_γ is the graph of a generic differential one-form γ on A . Since A is a torus, the cotangent bundle $T^*A = A \times \mathbb{C}^g$ is trivial of rank $g = \dim A$, with two projection maps

$$\begin{array}{ccc} & T^*A & \\ \swarrow \text{pr}_1 & & \searrow \text{pr}_2 \\ A & & \mathbb{C}^g \end{array}$$

Projecting from $\Lambda_Z \subseteq T^*A$ onto the second factor \mathbb{C}^g induces a map

$$p : \Lambda_Z \rightarrow \mathbb{C}^g.$$

It is easy to see that degree of p is equal to the degree of the Gauss map associated to Z and A as discussed above. By Proposition A.3, the intersection number $d_Z = \#|\Lambda_Z \cap \Omega_\gamma|$ is the generic degree of the Gauss map, which is equal to the generic degree of p .

Since n_Z, d_Z are both nonnegative, the assumption $\chi(P) = 0$ implies that $d_Z = 0$. Therefore p is not surjective and $\dim p(\Lambda_Z) < g$. Then for some cotangential vector $\omega \in p(\Lambda_Z)$, the fiber $p^{-1}(\omega)$ is positive-dimensional. If $Z \neq A$, we can assume $\omega \neq 0$. Let $Y \subseteq A$ be the image of $p^{-1}(\omega) \subseteq T^*A$ under the map $T^*A \rightarrow A$. Then $\dim Y > 0$, and up to a translation we can assume $0 \in Y$. By construction, ω is normal to Y in every smooth point of Y , so the preimage of Y under the universal covering $\mathbb{C}^g \rightarrow A = \mathbb{C}^g/\Lambda$ lies in the hyperplane of \mathbb{C}^g orthogonal to ω . Thus the subtorus of A generated by Y is strictly contained in A but non-zero, contradicting the assumption that A is simple. Therefore the characteristic cycle $\text{CC}(\mathcal{P})$ only contains the zero section of T^*A and hence \mathcal{P} is a shift of a local system, see e.g. Lemma 5.14 in the first arXiv version of [9]. \square

REFERENCES

- [1] B. Bhatt, C. Schnell, and P. Scholze. Vanishing theorems for perverse sheaves on abelian varieties, revisited. *Selecta Math. (N.S.)*, 24(1):63–84, 2018.
- [2] Y. Dutta, F. Hao, and Y. Liu. Generic vanishing, 1-forms, and topology of albanese maps. *arXiv preprint arXiv:2104.07074*, 2021.
- [3] J. Fernández de Bobadilla and J. Kollár. Homotopically trivial deformations. *J. Singul.*, 5:85–93, 2012.
- [4] J. Franecki and M. Kapranov. The Gauss map and a noncompact Riemann-Roch formula for constructible sheaves on semiabelian varieties. *Duke Math. J.*, 104(1):171–180, 2000.
- [5] C. D. Hacon and S. J. Kovács. Holomorphic one-forms on varieties of general type. *Ann. Sci. École Norm. Sup. (4)*, 38(4):599–607, 2005.
- [6] F. Hao and S. Schreieder. Holomorphic one-forms without zeros on threefolds. *Geom. Topol.*, 25(1):409–444, 2021.
- [7] D. Kotschick. Holomorphic one-forms, fibrations over the circle, and characteristic numbers of Kähler manifolds. *Math. Proc. Cambridge Philos. Soc.*, 172(1):95–103, 2022.
- [8] T. Krämer and R. Weissauer. Vanishing theorems for constructible sheaves on abelian varieties. *J. Algebraic Geom.*, 24(3):531–568, 2015.
- [9] Y. Liu, L. Maxim, and B. Wang. Aspherical manifolds, Mellin transformation and a question of Bobadilla-Kollár. *J. Reine Angew. Math.*, 781:1–18, 2021.
- [10] L. G. Maxim and J. Schürmann. Constructible sheaf complexes in complex geometry and applications. In *Handbook of Geometry and Topology of Singularities III*, pages 679–791. Springer, 2022.
- [11] M. Popa and C. Schnell. Kodaira dimension and zeros of holomorphic one-forms. *Ann. of Math. (2)*, 179(3):1109–1120, 2014.
- [12] S. Schreieder. Zeros of holomorphic one-forms and topology of Kähler manifolds. *IMRN*, 8:6169–6183, 2021.
- [13] D. Tischler. On fibering certain foliated manifolds over S^1 . *Topology*, 9:153–154, 1970.

INSTITUTE OF ALGEBRAIC GEOMETRY, LEIBNIZ UNIVERSITY HANNOVER, WELFENGARTEN 1, 30167 HANNOVER, GERMANY.

E-mail address: schreieder@math.uni-hannover.de

MAX-PLANCK-INSTITUT FÜR MATHEMATIK, VIVATSGASSE 7, 53111 BONN, GERMANY

E-mail address: ruijie.yang@hu-berlin.de