

CURVES ON POWERS OF HYPERELLIPTIC JACOBIANS

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ABSTRACT. Let X be a curve of genus ≥ 4 that is either very general or very general hyperelliptic. We classify all ways in which a power $(JX)^k$ of the Jacobian of X can be isogenous to a product of Jacobians of curves. As an application, we show that if A is a very general principally polarized abelian variety of dimension ≥ 4 or the intermediate Jacobian of a very general cubic threefold, then no power A^k is isogenous to a product of Jacobians of curves. This confirms some cases of the Coleman–Oort conjecture. We further deduce from our results some progress on the question whether the integral Hodge conjecture fails for A as above.

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Date: January 17, 2024.

2010 Mathematics Subject Classification. primary 14K02, 14H40, 14H10; secondary 14C25, 14C30.

Key words and phrases. Jacobians, intermediate Jacobians, abelian varieties, Coleman–Oort conjecture, integral Hodge conjecture.

1. INTRODUCTION

In this paper we work over the field of complex numbers. For a positive integer g , let \mathcal{M}_g be the moduli space of smooth projective connected curves of genus g . If C is a smooth projective curve, let JC denote its Jacobian. Our main result is the following theorem.

Theorem 1.1. *Let $Z \subset \mathcal{M}_g$ be an irreducible subvariety containing the hyperelliptic locus and let X be a curve that defines a very general point in Z . Assume that there is an isogeny*

$$\varphi: (JX)^k \longrightarrow JC_1 \times \cdots \times JC_n$$

for some positive integers k and n and some smooth projective connected curves C_1, \dots, C_n of positive genus. If $g \geq 4$, then $k = n$ and there is an isomorphism $C_i \simeq X$ for each $i \in \{1, \dots, n\}$.

Theorem 1.1 generalizes in various directions the main result of Bardelli and Pirola in [BP89], which says that the Jacobian JX of a very general curve of genus $g \geq 4$ is not isogenous to the Jacobian of any other curve C with $C \not\simeq X$.

The case $n = 1$ of the above theorem implies for instance that for a smooth projective genus g curve X that is very general or very general hyperelliptic, the k -th power $(JX)^k$ of its Jacobian does for $k > 1$ not contain any irreducible curve of geometric genus kg that generates $(JX)^k$.

The case $k = 1$ of Theorem 1.1 is due to Naranjo and Pirola, see [NP18, Theorem 1.1]. By proving Theorem 1.1, we fix a gap in the proof of the Naranjo–Pirola theorem in *loc. cit.* (cf. Remark 4.5) and generalize their theorem to arbitrary powers $(JX)^k$ with $k \geq 1$. For $k \geq 2$, additional difficulties appear and the proof requires new ingredients, most notably work of Kneser [Kne57] on the classification of integral inner product spaces of dimension at most 16, work of Lu and Zuo [LZ19] on Shimura curves in the Torelli locus, and a recent result of Lazarsfeld and Martin [LM23] that grew out from their study of various measures of irrationality.

The assumptions that $Z \subset \mathcal{M}_g$ contains the hyperelliptic locus $\mathcal{H}_g \subset \mathcal{M}_g$ and that $[X] \in Z$ is very general quickly reduce the proof of Theorem 1.1 to the case where $n = 1$ and $Z = \mathcal{H}_g$. Thus, we may assume X is a very general hyperelliptic curve and $(JX)^k$ is isogenous to the Jacobian of a smooth projective connected curve C . We then have to show $k = 1$ and $C \simeq X$. At the heart of the proof of that statement lies a degeneration of X to four different one-nodal hyperelliptic curves X_{01}, X_{02}, X_{03} and X_{04} . A close analysis of the resulting isogenies $J\tilde{C}_{0i} \rightarrow (J\tilde{X}_{0i})^k$, where \tilde{C}_{0i} and \tilde{X}_{0i} are the normalizations of the limit curves C_{0i} and X_{0i} , reveals enough information to conclude that $JC \simeq (JX)^k$ as unpolarized abelian varieties. In other words, this reduces Theorem 1.1 to the case where φ is an isomorphism. To finish the proof, we then establish the following result of independent interest; for more general versions of the statement, see Theorem 6.1 and Corollary 6.2 below.

Theorem 1.2. *Let $g \in \mathbb{Z}_{\geq 1}$. Let X be a curve of genus g such that X is either very general, or very general hyperelliptic. Assume that, for some positive integers k and n , there is an isomorphism of unpolarized abelian varieties $JC_1 \times \cdots \times JC_n \simeq (JX)^k$ for some smooth projective connected curves C_1, \dots, C_n of positive genus. Then $k = n$ and $C_i \simeq X$ for each i .*

We emphasize that Theorem 1.2 works in any genus $g \geq 1$, while the assumption $g \geq 4$ in Theorem 1.1 is necessary. Moreover, the assumption in Theorem 1.2 cannot be weakened to only ask that $(JX)^k$ is isogenous to a product of Jacobians of curves; for instance, the third power of a very general elliptic curve is isogenous to countably many Jacobians of smooth projective connected curves of genus three. The subtlety of the result lies in the fact that for $k \geq 2$, the abelian variety $(JX)^k$ carries various principal polarizations and so the Torelli theorem can a priori not be applied directly. Besides the aforementioned results from [Kne57; LZ19], our proof depends on various fortunate numerical coincidences, see Section 6.2.

1.1. Applications. Theorem 1.1 has several consequences. For instance, we obtain:

Corollary 1.3. *Let A be either the intermediate Jacobian of a very general cubic threefold, or a very general principally polarized abelian variety of dimension $g \geq 4$. Then there exists no integer $k \geq 1$ such that A^k is isogenous to a product of Jacobians of smooth projective curves.*

Lu and Zuo prove in [LZ19] that no power E^k with $k \geq 12$ of a very general elliptic curve E is isogenous to the Jacobian of a smooth projective connected curve. Corollary 1.3 implies in turn that no power of a very general principally polarized abelian variety of dimension $g \geq 4$ is isogenous to the Jacobian of a smooth projective curve (as we pointed out before, the two results are not independent, as we use [LZ19] in our proof of Theorem 1.1).

For $g \geq 8$, the locus of abelian varieties isogenous to a non-trivial power of an abelian variety of dimension ≥ 4 is, if non-empty, a dense countable union of special subvarieties in the moduli space \mathcal{A}_g of principally polarized abelian varieties of dimension g . Corollary 1.3 implies that the irreducible components \mathcal{Z} of this locus are not contained in the closure of the Torelli locus. In particular, these positive dimensional special subvarieties $\mathcal{Z} \subset \mathcal{A}_g$ have no dense open subset contained in the Torelli locus, as predicted by the Coleman–Oort conjecture [MO13, Expectation 4.2]. We thus get:

Corollary 1.4. *Let $g \geq 8$ be an integer which is divisible by some integer h with $4 \leq h < g$. Let $\mathcal{S} \subset \mathcal{A}_g$ be the locus of abelian varieties that are isogenous to a non-trivial power of an abelian variety of dimension at least four. Then \mathcal{S} is a dense countable union of special subvarieties of \mathcal{A}_g all of which satisfy the Coleman–Oort conjecture.*

The assertion of Corollary 1.3 that concerns intermediate Jacobians of cubics seems to go beyond what is predicted by the Coleman–Oort conjecture.

Another consequence of Theorem 1.1 is the following:

Corollary 1.5. *Let A be either the intermediate Jacobian of a very general cubic threefold, or a very general principally polarized abelian variety of dimension $g \geq 4$. Let A_1 be an abelian variety isogenous to a power of A , and let A_2 be an abelian variety with $\text{Hom}(A, A_2) = 0$. Then $A_1 \times A_2$ is not isomorphic (as unpolarized abelian varieties) to a product of Jacobians of curves.*

In the above corollary, we cannot exclude that $A_1 \times A_2$ is isogenous to a product of Jacobians. In fact, for any abelian variety A , a sufficiently general complete intersection curve $C \subset A$ has the property that JC is isogenous to $A \times A_2$ for some abelian variety A_2 with $\text{Hom}(A, A_2) = 0$.

It is a famous open problem to decide whether the integral Hodge conjecture for curve classes holds on any principally polarized abelian variety. This is partly motivated by [Voi17], where it is shown that a smooth cubic threefold is not stably rational, if the minimal class of its intermediate Jacobian is not algebraic. The question on the stable rationality of cubic threefolds is in turn open since the work of Clemens–Griffiths in the nineteen-seventies [CG72].

It is shown in [BGF23; Voi22] that a principally polarized abelian variety A satisfies the integral Hodge conjecture for curve classes if and only if there is an abelian variety B such that $A \times B$ is isomorphic to a product of Jacobians of curves. If $\dim(A) \geq 4$ and A is very general or the intermediate Jacobian of a very general cubic threefold, then Corollary 1.5 excludes several possibilities for the abelian variety B , as follows. Since A is simple, we know that B is an extension $0 \rightarrow B_1 \rightarrow B \rightarrow B_2 \rightarrow 0$ of an abelian variety B_2 with $\text{Hom}(A, B_2) = 0$ (i.e. B_2 does not have A as isogeny factor) by an abelian variety B_1 that is isogenous to a power of A . Corollary 1.5 shows that this extension cannot be split:

Corollary 1.6. *Let A be either the intermediate Jacobian of a very general cubic threefold, or a very general principally polarized abelian variety of dimension $g \geq 4$. Let B be an abelian variety. Assume that $B = B_1 \times B_2$, where B_1 is isogenous to a power of A and B_2 does not have A as an isogeny factor. Then $A \times B$ is not isomorphic (as unpolarized abelian varieties) to a product of Jacobians of curves.*

The above corollary implies for instance that $A \times B$ is not isomorphic to a product of Jacobians whenever A is as in the corollary and B is a product of simple abelian varieties.

A natural strategy to prove the integral Hodge conjecture for curve classes on an abelian variety A is to construct isogenies of coprime degrees from Jacobians of curves to k -th powers of A . This approach for $k = 1$ allowed Voisin to prove the property for special families of intermediate Jacobians of cubic threefolds in [Voi17]. Similarly, Beckmann and the first named author used this approach in [BGF23] to prove that the locus of principally polarized abelian varieties that satisfy the integral Hodge conjecture for curve classes is dense in moduli. Corollary 1.3 above shows that such an approach does not work for the very general abelian variety of dimension at least four, nor for the intermediate Jacobian of the very general cubic threefold.

Remark 1.7. A simple dimension count shows that there are complex abelian varieties of dimension ≥ 4 that are not isogenous to the Jacobian of a curve. While such an argument cannot work over countable fields, the statement remains true over $\overline{\mathbb{Q}}$ by work of Chai–Oort [CO12] and Tsimerman [Tsi12], and, via a different method, by Masser–Zannier [MZ20]. Our paper shows that there are abelian varieties A over \mathbb{C} such that no power of A is isogenous to the Jacobian of a curve, see Corollary 1.3. For this reason it is now reasonable to ask whether such examples exist over $\overline{\mathbb{Q}}$ as well. Our arguments apply to the algebraic closure of finitely generated field extensions of \mathbb{Q} of sufficiently large transcendence degree, but not to $\overline{\mathbb{Q}}$.

1.2. Outline. In Section 2 we gather various preliminary results. The goal of Section 3 is to study the degeneration of an isogeny $JC \rightarrow (JX)^k$ between the Jacobian JC of a connected smooth projective curve C and the k -th power of the Jacobian JX of a very general hyperelliptic

curve X , when X degenerates to a very general one-nodal hyperelliptic curve X_0 . We proceed in Section 4 by stating Theorem 4.1, due to Naranjo and Pirola, which is the analogue of Theorem 1.1 in the case $k = n = 1$, $C = C_1$ is hyperelliptic, and $g \geq 3$. Theorem 4.1 allows us to prove the more general Theorem 4.2, whose statement is the adaption of Theorem 1.1 obtained by weakening the assumption $g \geq 4$ to $g \geq 3$ but adding the assumption that C_i is hyperelliptic for all i . Apart from the $k = n = 1$ case, the proof relies on a recent result of Lazarsfeld and Martin [LM23]. In Section 5, we study polarizations on abelian varieties which are isogenous to a power of a principally polarized abelian variety. The results of this section will be used in the proof of our main theorem, but also in Section 6 where we prove Theorem 1.2. In Section 7, we collect several lattice theoretic results. In Section 8 we combine the results from Sections 2–7 to prove our main theorem, Theorem 1.1. In Section 9, we prove Corollaries 1.3, 1.4, 1.5 and 1.6. Finally, in Appendix A, we give a counterexample to the statement in [Mar13, Proposition 3.6] (versions of which seem to be used without proof in [BP89; NP18]). We further prove a weaker version of that statement that suffices for the application in [BP89] and for the proof of Theorem 4.1 given in [NP18], see Propositions A.1 and A.7 below.

1.3. Conventions. We work over the field of complex numbers. Varieties are integral separated schemes of finite type over \mathbb{C} . A very general point of a complex variety (or complex analytic space) is a point outside a countable union of proper Zariski closed subsets. A curve is a reduced projective scheme of pure dimension one over \mathbb{C} , or the analytification of such a scheme. A curve X of arithmetic genus $g \geq 2$ is semi-stable (resp. stable) if its singularities are at most nodal and each rational connected component of its normalization contains at least two (resp. three) points lying over nodes of X . A curve is of compact type if its dual graph is a tree. A family of curves is a proper flat morphism $p: \mathcal{X} \rightarrow B$ of finite type schemes, or complex analytic spaces, such that for each $b \in B$, the fibre $X_b = p^{-1}(b)$ is a curve which is (unless mentioned otherwise) connected. We say that a family of curves $p: \mathcal{X} \rightarrow B$ is a family of semi-stable (resp. stable, smooth, compact type, nodal) curves if the curve X_b is semi-stable (resp. stable, smooth, of compact type, has at most nodal singularities) for each $b \in B$.

1.4. Acknowledgements. Starting with the influential work of Bardelli–Pirola [BP89], many authors have studied irreducible curves of low genus on (sufficiently general) g -dimensional abelian varieties, see e.g. [NP94; LS02; Mar13; MNP16; NP18]. We were greatly inspired by these works.

We would like to thank Frans Oort for his valuable comments on this paper.

This project has received funding from the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation programme under grant agreement N° 948066 (ERC-StG RationAlgie).

2. PRELIMINARIES

In this section, we gather various preliminary results.

2.1. Isogenies between powers of abelian schemes and cohomology. We start with:

Lemma 2.1. *Let $k \in \mathbb{Z}_{\geq 1}$. Let S be a connected complex manifold, $f: \mathcal{A} \rightarrow S$ and $g: \mathcal{B} \rightarrow S$ families of compact complex tori, and $\varphi: \mathcal{B} \rightarrow \mathcal{A}^k$ a family of isogenies over S , where \mathcal{A}^k denotes the k -fold fibre product of \mathcal{A} over S . Suppose that, for some $t \in S$, the induced map $(\varphi_t)_*: H^1(B_t, \mathbb{Z}) \rightarrow H^1(A_t, \mathbb{Z})^{\oplus k}$ satisfies $(\varphi_t)_*(H^1(B_t, \mathbb{Z})) = \alpha \cdot H^1(A_t, \mathbb{Z})^{\oplus k} \subset H^1(A_t, \mathbb{Z})^{\oplus k}$ for some $\alpha \in M_k(\mathbb{Z})$. Then there is an isomorphism $\mathcal{B} \simeq \mathcal{A}^k$ over S .*

Proof. The case $k = 1$ goes back to [BP89]; the more general version stated above is similar.

First of all, in order to prove $\mathcal{B} \simeq \mathcal{A}^k$ over S , it suffices to show that the variation of integral Hodge structure $R^1 g_* \mathbb{Z}$ on S is isomorphic to $(R^1 f_* \mathbb{Z})^{\oplus k}$. The matrix α has full rank, because φ_t is an isogeny and $(\varphi_t)_*(H^1(B_t, \mathbb{Z})) = \alpha \cdot H^1(A_t, \mathbb{Z})^{\oplus k}$ by assumption. Hence, fibrewise multiplication by α yields an embedding of integral Hodge structures $\alpha: (R^1 f_* \mathbb{Z})^{\oplus k} \rightarrow (R^1 g_* \mathbb{Z})^{\oplus k}$. Similarly, fibrewise pushforward along the family of isogenies φ yields an embedding $\varphi_*: R^1 g_* \mathbb{Z} \rightarrow (R^1 f_* \mathbb{Z})^{\oplus k}$. To prove that $R^1 g_* \mathbb{Z} \simeq (R^1 f_* \mathbb{Z})^{\oplus k}$, it thus suffices to prove that the images of the above embeddings coincide, i.e. that the following equality holds:

$$\mathrm{im}(\varphi_*: R^1 g_* \mathbb{Z} \rightarrow (R^1 f_* \mathbb{Z})^{\oplus k}) = \mathrm{im}(\alpha: (R^1 f_* \mathbb{Z})^{\oplus k} \rightarrow (R^1 g_* \mathbb{Z})^{\oplus k}).$$

Since S is connected, the above identity can be checked at the single point $t \in S$, where it holds by assumption. This concludes the proof of the lemma. \square

As a corollary, we obtain the following useful criterion.

Lemma 2.2. *Let S be a smooth connected complex manifold and let $f: \mathcal{A} \rightarrow S$ and $g: \mathcal{B} \rightarrow S$ be families of compact complex tori. Let $\varphi: \mathcal{B} \rightarrow \mathcal{A}^k$ an isogeny of complex tori over S . Suppose that, for some $t \in S$, we have $\mathrm{End}(A_t) = \mathbb{Z}$ and $B_t \simeq A_t^k$ for some $k \in \mathbb{Z}_{\geq 1}$. Then there is an isomorphism $\mathcal{B} \simeq \mathcal{A}^k$ over S .*

Proof. By Lemma 2.1, it suffices to show that the induced map $(\varphi_t)_*: H^1(B_t, \mathbb{Z}) \rightarrow H^1(A_t, \mathbb{Z})^{\oplus k}$ has image $\alpha \cdot H^1(A_t, \mathbb{Z})^{\oplus k} \subset H^1(A_t, \mathbb{Z})^{\oplus k}$ for some $\alpha \in M_k(\mathbb{Z})$. This is clear: as $\mathrm{End}(A_t) = \mathbb{Z}$, the composition

$$A_t^k \xrightarrow{\varphi_t} B_t \simeq A_t^k$$

is given by a matrix $\alpha \in M_k(\mathbb{Z})$, hence the same is true on cohomology. \square

2.2. Gauß maps. The goal of this section is to prove Proposition 2.6 below. This lemma says that if C is a smooth connected non-hyperelliptic curve and $m \in \mathbb{Z}_{\geq 1}$, then the surface

$$m(C - C) \subset JC$$

contains no hyperelliptic curves.

Let A be an abelian variety of dimension $g \geq 2$ and let $V \subset A$ be a closed subvariety of dimension k , with $1 \leq k \leq g - 1$. Let $T_0 A$ be the tangent space of A at the origin, and consider the canonical trivialization $TA \simeq T_0 A \times A$ of the tangent bundle TA of A . Define $\mathrm{Gr}(k, T_0 A)$ as the Grassmannian of k -planes in $T_0 A$. Recall that, in this setting, the *Gauß map*

$$\mathcal{G}_{V,A}: V \dashrightarrow \mathrm{Gr}(k, T_0 A) = \mathrm{Gr}(k - 1, \mathbb{P}(T_0 A))$$

is the rational map defined as follows. For a point x in the smooth locus of V , the induced map on tangent spaces $T_x V \rightarrow T_x A$ is an embedding, whose image can be identified with a k -plane in $T_0 A$ via t ; we let $\mathcal{G}_{V,A}(x) \in \text{Gr}(k, T_0 A)$ be the induced point of the Grassmannian.

For a dominant morphism of varieties $f: X \rightarrow Y$ with $\dim(X) = \dim(Y)$, let the branch locus $B(f) \subset Y$ be the reduced closed subscheme which is the complement of the largest open subset of Y over which f is étale. Similarly, define the ramification locus $R(f) \subset X$ of f as the complement of the largest open subset of X over which f is étale. These definitions readily extend to the case of a dominant rational map $f: X \dashrightarrow Y$ of varieties of the same dimension. Namely, if $U \subset X$ is a non-empty open subset on which f restricts to a morphism $f|_U: U \rightarrow Y$, then we define $R(f)$ as the closure of $R(f|_U)$ in X , and $B(f)$ as the closure of $B(f|_U)$ in Y .

Let C be a smooth connected non-hyperelliptic curve (hence of genus $g \geq 3$), and let

$$\phi_{K_C}: C \longrightarrow \mathbb{P}(H^0(C, K_C)) = \mathbb{P}(T_0 J C)$$

be the canonical embedding. This gives a morphism

$$s: C \times C \longrightarrow \text{Gr}(1, \mathbb{P}(T_0 J C)), \quad (p, q) \longmapsto (\text{line spanned by } \phi_{K_C}(p) \text{ and } \phi_{K_C}(q)).$$

Lemma 2.3. *In the above notation, we have*

$$\deg(s) = \begin{cases} 2 & \text{if } g \geq 4, \\ 12 & \text{if } g = 3. \end{cases}$$

The ramification locus $R(s)$ of s is the diagonal $\Delta_C \subset C \times C$, and the branch locus $B(s)$ of s is the curve D in $\text{Gr}(1, \mathbb{P}(T_0 J C))$ parametrizing lines which are tangent to the canonical curve $\phi_{K_C}(C) \subset \mathbb{P}(T_0 J C)$. Moreover, the induced map $C = \Delta_C = R(s) \rightarrow B(s) = D$ is birational.

Proof. This well-known (see e.g. [BP89, Remark 3.1.2]). □

Define a morphism

$$a: C \times C \longrightarrow J C, \quad a(x, y) = x - y. \quad (2.1)$$

For $m \in \mathbb{Z}_{\geq 1}$, let $m(C - C) \subset J C$ be the image of the morphism $m \cdot a: C \times C \rightarrow J C$, and put $C - C = 1(C - C) \subset J C$. Consider the Gauß map $G_m = \mathcal{G}_{m(C-C), J C}$, which is a rational map

$$G_m: m(C - C) \dashrightarrow \text{Gr}(1, \mathbb{P}(T_0 J C)). \quad (2.2)$$

By [BP89, Lemma 3.1.1 & Remark 3.1.2], the following diagram commutes:

$$\begin{array}{ccccc} C \times C & \xrightarrow{a} & C - C & \xrightarrow{m} & m(C - C) \\ \downarrow s & & \downarrow G_1 & & \downarrow G_m \\ \text{Gr}(1, \mathbb{P}(T_0 J C)) & \xlongequal{\quad} & \text{Gr}(2, T_0 J C) & \xlongequal{\quad} & \text{Gr}(2, T_0 J C). \end{array} \quad (2.3)$$

In particular, we obtain the following rational maps, the first of which is a morphism:

$$s: C \times C \longrightarrow s(C \times C), \quad G_1: C - C \dashrightarrow s(C \times C), \quad G_m: m(C - C) \dashrightarrow s(C \times C). \quad (2.4)$$

Lemma 2.4. *Let C be a smooth connected non-hyperelliptic curve. Consider the commutative diagram (2.3) above, and the resulting morphisms (2.4). The following assertions are true.*

- (1) *The morphisms $a: C \times C \rightarrow C - C$ and $m: C - C \rightarrow m(C - C)$ are birational.*
- (2) *The morphism $s: C \times C \rightarrow s(C \times C)$, and the rational maps $G_1: C - C \dashrightarrow s(C \times C)$ and $G_m: m(C - C) \dashrightarrow s(C \times C)$, are generically finite.*

Proof. For item (1), see [Mar13, Proposition 2.8]. By item (1) and the commutativity of (2.3), item (2) follows from the fact that $s: C \times C \rightarrow s(C \times C)$ is generically finite, see Lemma 2.3. \square

Remark 2.5. Let C be a smooth non-hyperelliptic curve of genus $g \geq 3$. In [NP18], one can find an alternative proof of item (1) of Lemma 2.4. However, this proof only works for $g \geq 4$. Namely, the idea is to exploit the equality $G_m \circ m \circ a = s$ of maps $C \times C \rightarrow C - C$, which implies $\deg(G_m) \cdot \deg(m) \cdot \deg(a) = \deg(s)$, so that it would suffice to prove $\deg(G_m) = 2$ and $\deg(s) = 2$. The authors claim that $\deg(s) = 2$ for $g \geq 3$, which is false, see Lemma 2.3.

We are now in position to prove the following proposition.

Proposition 2.6. *Let C be a smooth connected curve of genus $g \geq 3$. Suppose that, for some $m \in \mathbb{Z}_{\geq 1}$, there is a non-constant morphism*

$$f: X \longrightarrow m(C - C) \subset JC,$$

where X is a smooth connected hyperelliptic curve. Then C is hyperelliptic.

Proof. Assume that C is non-hyperelliptic; our goal is to arrive at a contradiction. Recall that $m: C - C \rightarrow m(C - C)$ is birational, see item (1) of Lemma 2.4. Assume first that $f(X)$ is contained in the branch locus $B(m)$ of m . The commutativity of diagram (2.3) yields a rational map $X \dashrightarrow B(s)$ defined as the composition

$$X \xrightarrow{f} B(m) \xrightarrow{G_m} B(G_m \circ m) \longlongequal{\quad} B(G_1) \subset B(G_1 \circ a) \longlongequal{\quad} B(s).$$

This rational map is non-constant as $B(m)$ must be a curve (as f is non-constant) and G_m is generically finite, see item (2) in Lemma 2.4. As $B(s)$ is birational to C by Lemma 2.3, one obtains a non-constant morphism $X \rightarrow C$, proving that C is hyperelliptic (see e.g. [Sch90, Lemma 1.1]), which yields the desired contradiction.

Therefore, the curve $f(X) \subset m(C - C)$ is not contained in $B(m)$, and we obtain a non-constant rational map $X \dashrightarrow C - C$ defined as the composition

$$X \xrightarrow{f} m(C - C) \xrightarrow{m^{-1}} C - C.$$

Note that $R(a) = \Delta_C \subset C \times C$ and $B(a) = \{0\} \subset C - C$. Consequently, composing the non-constant rational map $m^{-1} \circ f$ with the rational map $a^{-1}: C - C \dashrightarrow C \times C$, one obtains a non-constant rational map $X \dashrightarrow C$ defined as the composition

$$a^{-1} \circ m^{-1} \circ f: X \dashrightarrow m(C - C) \dashrightarrow C - C \dashrightarrow C \times C.$$

Thus, X admits a non-constant morphism $X \rightarrow C$. This is a contradiction, and we are done. \square

Remarks 2.7. Let C be a non-hyperelliptic curve of genus $g \geq 3$. Essential in the proof of Proposition 2.6 above is to exploit the birational map $(m \circ a)^{-1}: m(C - C) \dashrightarrow C \times C$. This idea was inspired by [NP18, page 902]. We provided some additional details of the argument for convenience of the reader.

2.3. Extensions of abelian varieties. Let A be an abelian variety. Recall that, by the Barsotti–Weil formula, there is a canonical isomorphism

$$\mathrm{Ext}(A, \mathbb{G}_m) = \mathrm{Pic}^0(A) = A^\vee.$$

More generally, to give an extension of an abelian variety A by a torus T (i.e. an algebraic group isomorphic to $(\mathbb{C}^*)^r$ for some r) is equivalent to give a homomorphism $c^t: \mathrm{Hom}(T, \mathbb{G}_m) \rightarrow A^\vee = \mathrm{Pic}^0(A)$ from the character group of T to the dual abelian variety A^\vee of A . To see this, evaluating the functor $\mathrm{Hom}(-, \mathbb{G}_m)$ on such an extension

$$0 \longrightarrow T \longrightarrow G \longrightarrow A \longrightarrow 0, \quad (2.5)$$

one obtains a homomorphism

$$c^t: \mathrm{Hom}(T, \mathbb{G}_m) \longrightarrow \mathrm{Ext}(A, \mathbb{G}_m) = A^\vee.$$

Conversely, each homomorphism $c^t: \mathrm{Hom}(T, \mathbb{G}_m) \rightarrow A^\vee$ determines an exact sequence of the form (2.5), see [Cha85, Chapter II, Section 2]. We conclude (compare [Car80, Proposition 2]):

$$\mathrm{Ext}(A, T) = \mathrm{Hom}(\mathrm{Hom}(T, \mathbb{G}_m), A^\vee). \quad (2.6)$$

Lemma 2.8. *For $i \in \{1, 2\}$, let $0 \rightarrow T_i \rightarrow G_i \rightarrow A_i \rightarrow 0$ be an exact sequence of algebraic groups, where A_i is an abelian variety and $T_i \simeq (\mathbb{C}^*)^{r_i}$ a torus. Let $f: G_1 \rightarrow G_2$ be a morphism of algebraic groups. Then f restricts to a homomorphism $f|_{T_1}: T_1 \rightarrow T_2$ and hence induces a homomorphism $\bar{f}: A_1 \rightarrow A_2$. Moreover, if $c_i^t \in \mathrm{Hom}(\mathrm{Hom}(T_i, \mathbb{G}_m), A_i^\vee)$ is the homomorphism that corresponds to the class of G_i in $\mathrm{Ext}(A_i, T_i)$ via (2.6), then the following diagram commutes:*

$$\begin{array}{ccc} \mathrm{Hom}(T_2, \mathbb{G}_m) & \xrightarrow{(f|_{T_1})^*} & \mathrm{Hom}(T_1, \mathbb{G}_m) \\ \downarrow c_2^t & & \downarrow c_1^t \\ A_2^\vee & \xrightarrow{(\bar{f})^\vee} & A_1^\vee. \end{array} \quad (2.7)$$

Proof. The first statement follows from the fact that $\mathrm{Hom}(T_1, A_2) = 0$. For the second statement, consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & T_1 & \longrightarrow & G_1 & \longrightarrow & A_1 \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & T_2 & \longrightarrow & G_2 & \longrightarrow & A_2 \longrightarrow 0. \end{array}$$

Applying $\mathrm{Hom}(-, \mathbb{G}_m)$ to this diagram, and using the fact that for an abelian variety A , the isomorphism $\mathrm{Ext}(A, \mathbb{G}_m) = A^\vee$ is functorial in A , the commutativity of (2.7) follows. \square

2.4. Extension classes of nodal curves. Next, we recall some known results on extensions of Jacobians of smooth projective connected curves, following [Ale04, Sections 2.2–2.4]. We will make use of the following definition.

Definition 2.9. *Let X be a connected nodal curve, and let $\Gamma(X)$ be its dual graph. An orientation of $\Gamma(X)$ is the choice of an ordering (P^+, P^-) on every pair of points on the normalization \tilde{X} of X lying above the same node.*

Let X be a connected nodal curve. Let X_1, \dots, X_n be the irreducible components of X , and $\tilde{X}_i \rightarrow X_i$ their normalizations. We denote the dual graph of X by $\Gamma = \Gamma(X)$ and fix any orientation of Γ . Let

$$\tilde{X} = \coprod_{i=1}^n \tilde{X}_i \rightarrow X$$

be the normalization of X . The dual abelian variety of $J\tilde{X}$ is identified with itself via the principal polarization, and the character group of the torus $T = \text{Ker}(JX \rightarrow J\tilde{X})$ is canonically identified with $H_1(\Gamma, \mathbb{Z})$. Hence,

$$\text{Ext}(J\tilde{X}, T) = \text{Hom}(\text{Hom}(T, \mathbb{G}_m), J\tilde{X}) = \text{Hom}(H_1(\Gamma, \mathbb{Z}), J\tilde{X}). \quad (2.8)$$

Moreover, the homomorphism

$$c^t: H_1(\Gamma, \mathbb{Z}) \rightarrow J\tilde{X} \quad (2.9)$$

corresponding to $[JX] \in \text{Ext}(J\tilde{X}, T)$ via (2.8) is described explicitly in the following way. Every edge e of Γ corresponds to a node P of X , and the orientation defines an ordered pair of points (P^+, P^-) on \tilde{X} . Put

$$c^t(e) = P^+ - P^- \in \text{Pic}(\tilde{X}),$$

and extend this by linearity to the free module $C_1(\Gamma, \mathbb{Z})$ on the edges of Γ . Let $C_0(\Gamma, \mathbb{Z})$ be the free \mathbb{Z} -module on the set of vertices of Γ . For an edge e , let $\text{end}(e)$ and $\text{beg}(e)$ be the end-vertex and begin-vertex of e , as determined by the orientation of e , and define

$$\partial: C_1(\Gamma, \mathbb{Z}) \rightarrow C_0(\Gamma, \mathbb{Z}) \quad \text{by} \quad \partial(e) = \text{end}(e) - \text{beg}(e) \quad \text{for an edge } e \text{ of } \Gamma.$$

If $h \in H_1(\Gamma, \mathbb{Z}) = \text{Ker}(\partial: C_1(\Gamma, \mathbb{Z}) \rightarrow C_0(\Gamma, \mathbb{Z}))$, one has $c^t(h) \in J\tilde{X}$, and this construction defines the homomorphism (2.9), see [Ale04, Section 2.4].

2.5. Graph homology and extension classes. We continue with the notation of Section 2.4. Let v_1, \dots, v_m be a set of vertices and e_1, \dots, e_m a set of edges of Γ , such that for each $j \in \{1, \dots, m-1\}$, the edge e_j connects the vertices v_j and v_{j+1} , and e_m connects v_m and v_1 . In particular, if $m = 1$, then e_1 is a loop connecting v_1 to itself.

Let

$$\tilde{X}_{i_1}, \tilde{X}_{i_2}, \dots, \tilde{X}_{i_m}$$

be the connected components of the normalization \tilde{X} of X that correspond to the vertices v_1, \dots, v_m . For $j \in \{1, \dots, m-1\}$, the orientation of e_j defines an ordered pair of points (P_j^+, P_j^-) such that P_j^+ lies either on \tilde{X}_{i_j} or on $\tilde{X}_{i_{j+1}}$, and the opposite is true for P_j^- ; define

$\varepsilon_j \in \{1, -1\}$ by declaring that $\varepsilon_j = 1$ if P_j^- lies on \tilde{X}_{i_j} and $\varepsilon_j = -1$ otherwise. We obtain a cycle

$$\gamma = \sum_{j=1}^m \varepsilon_j \cdot e_j \in C_1(\Gamma, \mathbb{Z}), \quad (2.10)$$

and one readily observes that $\partial(\gamma) = 0$, so that γ defines an element of $H_1(\Gamma, \mathbb{Z})$.

Lemma 2.10. *Let X be a connected nodal curve with dual graph Γ . Then there exists a linearly independent subset $S = \{\gamma_1, \dots, \gamma_k\} \subset H_1(\Gamma, \mathbb{Z})$ consisting of homology classes of the form (2.10) such that S defines a basis of $H_1(\Gamma, \mathbb{Q})$.*

Proof. Recall the following fact. Let Y be a path-connected one-dimensional CW complex with basepoint y_0 , a 0-cell. Then every loop in Y is homotopic to a loop consisting of a finite sequence of edges traversed monotonically, see [Hat02, Section 1.1, Exercise 19]. Consequently, homology classes of the form (2.10) generate $H_1(\Gamma, \mathbb{Z})$, yielding the lemma. (We do not ask that S is an integral basis, because not any generating set of a free \mathbb{Z} -module contains a basis.) \square

Lemma 2.11. *Let X be a connected nodal curve with dual graph Γ . Let $\tilde{X}_1, \dots, \tilde{X}_n$ be the connected components of the normalization \tilde{X} of X . Let $\gamma \in H_1(\Gamma, \mathbb{Z})$ be a class of the form (2.10), and consider the homomorphism $c^t: H_1(\Gamma, \mathbb{Z}) \rightarrow J\tilde{X}$, see (2.8) and (2.9). There are points $p_i, q_i \in \tilde{X}_i$ for each $i \in \{1, \dots, n\}$, such that*

$$c^t(\gamma) = (p_1 - q_1, \dots, p_n - q_n) \in J\tilde{X}_1 \times \dots \times J\tilde{X}_n = J\tilde{X}. \quad (2.11)$$

Note that we do not require that all the points p_i and q_i are distinct.

Proof. In the notation used above equation (2.10), we have

$$c^t(\gamma) = \left(P_{i_1}^+ - P_{i_{m-1}}^-, P_{i_2}^+ - P_{i_1}^-, P_{i_3}^+ - P_{i_2}^-, \dots, P_{i_m}^+ - P_{i_{m-1}}^- \right) \in \prod_{j=1}^m J\tilde{X}_{i_j} \subset J\tilde{X}.$$

The lemma follows. \square

2.6. Extension classes of nodal hyperelliptic curves. We turn to the hyperelliptic case.

Lemma 2.12. *Let X be an irreducible one-nodal hyperelliptic curve of genus $g \geq 1$, with normalization $\tilde{X} \rightarrow X$. Consider the homomorphism $c^t: H_1(\Gamma, \mathbb{Z}) \rightarrow J\tilde{X}$, where Γ denotes the dual graph of X , see equation (2.9). If γ is a generator for $H_1(\Gamma, \mathbb{Z}) \simeq \mathbb{Z}$, there is a point $x \in \tilde{X}$ such that $c^t(\gamma) = x - \iota(x)$, where $\iota: \tilde{X} \rightarrow \tilde{X}$ is the hyperelliptic involution.*

Proof. By the description of the map c^t , we have $c^t(\gamma) = p - q$, where p, q denote the points on the normalization \tilde{X} of X that are glued to form the nodal curve X . As X is hyperelliptic, the hyperelliptic involution on \tilde{X} descends to an involution on X , which implies that $q = \iota(p)$. \square

Lemma 2.13. *Let X be a very general one-nodal hyperelliptic curve. There is no positive dimensional abelian subvariety of JX .*

Proof. Let $T = \text{Ker}(JX \rightarrow J\tilde{X})$ and consider the extension

$$0 \rightarrow T \rightarrow JX \rightarrow J\tilde{X} \rightarrow 0. \quad (2.12)$$

Since X is very general, $J\tilde{X}$ is simple. Hence any positive dimensional abelian subvariety of JX must be isogenous to $J\tilde{X}$. If such a subvariety exists, then the extension (2.12) splits up to isogeny, and so it suffices to exclude the latter. By Lemma 2.12, the isomorphism $\text{Ext}(J\tilde{X}, T) \simeq J\tilde{X}$ (cf. Section 2.3) identifies $[JX]$ with $x - \iota(x)$ for some $x \in \tilde{X}$, hence (2.12) splits up to isogeny if and only if $x - \iota(x) \in J\tilde{X}$ is torsion. As the one-nodal hyperelliptic curve X is very general, this is not the case. \square

2.7. Nodal degeneration and vanishing cycles. Recall the following result.

Proposition 2.14. *Let D be the open unit disc with origin $0 \in D$. Let X be a complex analytic space and $f: X \rightarrow D$ a proper map. Put $X_0 = f^{-1}(0)$. If X is smooth, and f is smooth over $D - \{0\}$, then the inclusion $X_0 \hookrightarrow X$ is a homotopy equivalence. If X is any complex analytic space and f any proper map $X \rightarrow D$, then the same is true up to shrinking D around 0.*

Proof. See for instance [PS08, Proposition C.11 and Remark C.12.ii]. \square

Lemma 2.15. *Let X be an analytic space, $(D, 0)$ the pointed unit disc, and $f: X \rightarrow D$ a family of nodal curves over D which is smooth over $D^* = D - \{0\}$. There exists a complex manifold \tilde{X} and a projective morphism $h: \tilde{X} \rightarrow X$ which is an isomorphism over the regular locus of X , such that $f \circ h$ defines a family of nodal curves $\tilde{X} \rightarrow D$ which is smooth over D^* .*

Proof. See [Jon96, Lemma 3.2]; the main point being that we can find a resolution \tilde{X} of X such that the fibres of $\tilde{X} \rightarrow D$ are reduced. \square

Consider a proper holomorphic map

$$f: \mathcal{X} \rightarrow D$$

from an n -dimensional complex manifold \mathcal{X} to a disc D . Assume that f is a submersion over the punctured disc D^* , and that over $0 \in D$, there are k critical points $x_1, \dots, x_k \in X_0 = f^{-1}(0)$ for some $k \in \mathbb{Z}_{\geq 1}$, and that these are non-degenerate. Assume that \mathcal{X} is regular (something we can always achieve by modifying \mathcal{X} , see Lemma 2.15).

Lemma 2.16. *Continue with the above notation, and let $t \in D^*$. There are k disjoint spheres $S_1^{n-1}, \dots, S_k^{n-1} \subset X_t$ and a deformation retraction of \mathcal{X} onto the union of X_t and k disjoint n -dimensional balls B_1^n, \dots, B_k^n , where the ball B_i^n is glued to X_t along the sphere $S_i^{n-1} \subset X_t$.*

Proof. This is a straightforward generalization of [Voi07, Theorem 2.16]. \square

Corollary 2.17. *Continue with the above notation. Let $t \in D^*$ and let $i: X_t \hookrightarrow \mathcal{X}$ be the inclusion. Then $i_*: H_m(X_t, \mathbb{Z}) \rightarrow H_m(\mathcal{X}, \mathbb{Z})$ is an isomorphism for $m < n - 1$. For $m = n - 1$, the map i_* is surjective, with kernel generated by the cohomology classes of the spheres $S_1^{n-1}, \dots, S_k^{n-1} \subset X_t$.*

Proof. This is a straightforward generalization of [Voi07, Corollary 2.17]. \square

Lemma 2.18. *Continue with the above notation and let $t \in D^*$. Let $\delta_1, \dots, \delta_n \in H^{n-1}(X_t, \mathbb{Z})$ be the Poincaré duals of the homology classes of the vanishing spheres $S_1^{n-1}, \dots, S_k^{n-1} \subset X_t$, see Lemma 2.16. Let $(-, -): H^{n-1}(X_t, \mathbb{Z}) \times H^{n-1}(X_t, \mathbb{Z}) \rightarrow \mathbb{Z}$ be the cup-product pairing. For some $\epsilon_n \in \{\pm 1\}$, depending only on n , the natural generator $T \in \text{Aut}(H^1(X_t, \mathbb{Z}))$ of the monodromy group satisfies*

$$T(\alpha) = \alpha + \epsilon_n \cdot \sum_{i=1}^k (\alpha, \delta_i) \delta_i \quad \forall \alpha \in H^{n-1}(X_t, \mathbb{Z}).$$

Proof. This is a straightforward generalization of [Voi07, Theorem 3.16]. \square

Next, we verify that, for a nodal degeneration of curves over a disc with smooth general fibre, the monodromy invariant part of the first integral cohomology group of the general fibre does not change after any finite base change.

Lemma 2.19. *Let $D \ni 0$ be the pointed unit disc. Let $\mathcal{X} \rightarrow D$ be a family of nodal curves, smooth over $D^* = D \setminus \{0\}$. Let $\tau: D' = D \rightarrow D$ be the map $z \mapsto z^m$, let $0' \in D'$ be the preimage of $0 \in D$, and consider the base change $\mathcal{X}' := \mathcal{X} \times_D D'$. Let $t \neq 0 \in D$ and fix a preimage $t' \in D'$ of $t \in D$. Let $T \in \text{Aut}(H^1(X_t, \mathbb{Z}))$ and $T' \in \text{Aut}(H^1(X'_{t'}, \mathbb{Z}))$ be the monodromy operators induced by the restrictions of the families to D^* and D'^* . Then the invariant subspaces of T and T' coincide, that is,*

$$H^1(X_t, \mathbb{Z})^T = H^1(X'_{t'}, \mathbb{Z})^{T'}.$$

Proof. To prove the lemma, we may assume that \mathcal{X} is regular, see Lemma 2.15. Suppose that the central fibre X_0 has k nodes, and let $\delta_1, \dots, \delta_k \in H^1(X_t, \mathbb{Z})$ be the cohomology classes attached to the vanishing spheres $S_1, \dots, S_k \subset X_t$, see Lemma 2.16. By Lemma 2.18, we have the following generalization of the Picard–Lefschetz formula: for each $\alpha \in H^1(X_t, \mathbb{Z})$, one has

$$T(\alpha) = \alpha + \sum_{i=1}^k (\alpha \cdot \delta_i) \delta_i \in H^1(X_t, \mathbb{Z}).$$

On the one hand, the monodromy operator T' on $H^1(X'_{t'}, \mathbb{Z}) = H^1(X_t, \mathbb{Z})$ satisfies $T' = T^m$. On the other hand, we have

$$T^m(\alpha) = \alpha + m \cdot \sum_{i=1}^k (\alpha \cdot \delta_i) \delta_i \in H^1(X_t, \mathbb{Z}), \quad \alpha \in H^1(X_t, \mathbb{Z}).$$

This implies that, for $\alpha \in H^1(X_t, \mathbb{Z})$, one has:

$$T'(\alpha) = \alpha \iff m \cdot \sum_{i=1}^k (\alpha \cdot \delta_i) \delta_i = 0 \iff \sum_{i=1}^k (\alpha \cdot \delta_i) \delta_i = 0 \iff T(\alpha) = \alpha.$$

This proves the lemma. \square

Remark 2.20. In the course of the proof of Theorem 1.1 we will be forced to perform various base changes. This is a priori a subtle issue for the following reasons. We plan to degenerate to different nodal fibres, which is equivalent to the degeneration to one fixed nodal fibre followed

by the application of a monodromy operator. However, the monodromy action on cohomology with finite coefficients may become trivial after a base change, while the basic criterion in Lemma 2.1 that we aim to exploit is in fact equivalent to the analogous assertion for (sufficiently divisible) finite coefficients. For this reason, the fact that even after an arbitrary base change one can deduce additional information from degeneration to various nodal fibres (a fact which was already exploited in [BP89; NP18]), seems somewhat surprising. The key reason which makes these arguments work is given in Lemma 2.19 above, which says that the monodromy invariant subspace is not affected by any finite base change.

We conclude the section with the following lemma.

Lemma 2.21. *Let \mathcal{C} be complex analytic space, $D \subset \mathbb{C}$ an open disc around $0 \in \mathbb{C}$, and $q: \mathcal{C} \rightarrow D$ a family of nodal curves over D . Suppose that for each $s \in D^* = D - \{0\}$, the curve $C_s = q^{-1}(s)$ is of compact type. Then up to shrinking D around 0, the following holds: for $t \in D^*$ and $T \in \text{Aut}(H^1(C_t, \mathbb{Z}))$ a generator of the monodromy group, the natural map*

$$H^1(\mathcal{C}, \mathbb{Z}) \longrightarrow H^1(C_t, \mathbb{Z})^T$$

is surjective.

Proof. Let \mathcal{C}_i ($i \in I$) denote the irreducible components of \mathcal{C} . By flatness of q , the induced morphism $q_i: \mathcal{C}_i \rightarrow D$ is surjective. We claim that the general fibre of q_i is irreducible. Clearly, it suffices to prove this after shrinking the disc D . The main point is then that $\mathcal{C} \rightarrow D$ is a family of nodal curves, hence the fibre above 0 is reduced. This in turn implies that for each irreducible component $C_{0,j}$ of the special fibre of q , there is up to shrinking a section of q that passes through a general point of $C_{0,j}$. This shows that the component \mathcal{C}_i of \mathcal{C} that contains $C_{0,j}$ admits a section and hence has irreducible general fibre. Running through all components $C_{0,j}$ of the special fibre of q , we get this way that each $q_i: \mathcal{C}_i \rightarrow D$ has irreducible general fibre.

Since the general fibre of $q: \mathcal{C} \rightarrow D$ is of compact type, the general fibre of q_i is smooth and the index set I forms the vertices of a tree that indicates which components of a general fibre of q meet. If $i, j \in I$ are joined by an edge $e_{i,j}$, or equivalently, if the general fibres of \mathcal{C}_i and \mathcal{C}_j are glued at a point, then we get a section $e_{i,j}: D^* \rightarrow \mathcal{C}$, which has to extend across the puncture by properness of q . Since the arithmetic genus of the fibres of each $q_i: \mathcal{C}_i \rightarrow D$ is constant, and because the same holds for $q: \mathcal{C} \rightarrow D$, we see that \mathcal{C} is given by the quotient

$$\mathcal{C} = \left(\coprod_{i \in I} \mathcal{C}_i \right) / \sim$$

where we glue for each edge $e_{i,j}$ between some indices $i, j \in I$ according to the section constructed above. Since the fibres of q are nodal, we see moreover that the points that are glued via \sim on the special fibre lie in the smooth locus of $\bigsqcup_{i \in I} \mathcal{C}_i \rightarrow D$.

A simple Mayer–Vietoris argument now reduces us to show that

$$H^1(\mathcal{C}_i, \mathbb{Z}) \longrightarrow H^1(C_{it}, \mathbb{Z})^T$$

is surjective for each $i \in I$. In other words, we have reduced the result to the case where q is smooth over the punctured disc D^* . In this case we apply Lemma 2.15 and get a modification $\tau: \tilde{\mathcal{C}} \rightarrow \mathcal{C}$ given by successive blow-ups of the singular points in the central fibre, such that the fibres of $\tilde{\mathcal{C}} \rightarrow D$ are reduced and hence nodal curves. By [Kol93, Theorem 7.8], $\tau_*: \pi_1(\tilde{\mathcal{C}}) \rightarrow \pi_1(\mathcal{C})$ is an isomorphism. Passing to the abelianization and applying $\text{Hom}(-, \mathbb{Z})$, we find that $\tau^*: H^1(\mathcal{C}, \mathbb{Z}) \rightarrow H^1(\tilde{\mathcal{C}}, \mathbb{Z})$ is an isomorphism. This reduces us to the case where \mathcal{C} is regular and q is smooth over D^* . By Corollary 2.17, $H_1(C_t, \mathbb{Z}) \rightarrow H_1(\mathcal{C}, \mathbb{Z})$ is surjective. It follows that $H^1(\mathcal{C}, \mathbb{Z}) \rightarrow H^1(C_t, \mathbb{Z})$ is injective with torsion-free cokernel. By the local invariant cycle theorem (cf. [Mor84]), the map $H^1(\mathcal{C}, \mathbb{Z}) \rightarrow H^1(C_t, \mathbb{Z})^T$ becomes surjective after tensoring with \mathbb{Q} ; as its cokernel is torsion-free, it is surjective. This concludes the proof of the lemma. \square

2.8. Degenerations of hyperelliptic curves. In this section, we construct a family of stable hyperelliptic curves satisfying suitable properties. The base will be higher dimensional with several divisors each of which giving rise to a family of one-nodal hyperelliptic curves, allowing us to degenerate a very general hyperelliptic curve in different directions. We will ultimately need these different degenerations in order to prove Theorem 1.1, see Section 8.

Most of the following lemma is certainly well-known; we include some details for convenience of the reader.

Lemma 2.22. *Let $n, g \geq 2$ be positive integers with $n \leq g$. Consider the affine space \mathbb{A}^{2g-1} with coordinates z_1, \dots, z_{2g-1} . There exists a non-empty Zariski open subset $U \subset \mathbb{A}^{2g-1}$, irreducible divisors $\Delta_i \subset U$ for $i = 1, \dots, n$, and a family of genus g stable hyperelliptic curves*

$$p: \mathcal{X} \rightarrow U \tag{2.13}$$

such that the following holds.

- (1) Let $\Delta := \cup_i \Delta_i$. The family (2.13) is smooth over $U - \Delta$, the fibre $X_t = p^{-1}(t)$ for very general $t \in U$ is a very general hyperelliptic curve of genus g , and for each $i \in \{1, \dots, n\}$, the fibre X_{0_i} above a very general point $0_i \in \Delta_i$ is a very general one-nodal hyperelliptic curve of arithmetic genus g .
- (2) For general $t \in U$ and $0_i \in \Delta_i$ ($i \in \{1, \dots, n\}$), there exists a one-dimensional disc $D_i \subset U$ intersecting Δ transversally in $0_i \in \Delta_i$ such that, for general $t_i \in D_i - \{0_i\}$, there is a path $\rho_i: [0, 1] \rightarrow U - \Delta$ from t to t_i with the following properties.
 - (a) The restriction of $p: \mathcal{X} \rightarrow U$ to the disc $D_i \subset U$ is a Lefschetz degeneration $p|_{D_i}: \mathcal{X}|_{D_i} \rightarrow D_i$ with nodal central fibre above $0_i \in D_i$.
 - (b) Let $\delta_i \in H^1(X_{t_i}, \mathbb{Z})$ be the corresponding vanishing cycle, and view δ_i as an element of $H^1(X_t, \mathbb{Z})$ via the canonical isomorphism $H^1(X_t, \mathbb{Z}) \simeq H^1(X_{t_i}, \mathbb{Z})$ induced by the path ρ_i . Then $\delta_1, \dots, \delta_n$ can be completed to a symplectic basis

$$H^1(X_t, \mathbb{Z}) = \langle \delta_1, \dots, \delta_g; \gamma_1, \dots, \gamma_g \rangle, \tag{2.14}$$

where symplectic means that $\delta_i \cdot \gamma_i = 1$ for all i , and all other products of bases elements vanish.

Proof. Let $a_{n-1}, \dots, a_{2g-1} \in \mathbb{C}$ be general complex numbers, and consider the following equation:

$$y^2 = ((x - a_{n-1})^2 - z_1) \cdot ((x - a_n)^2 - z_2) \cdot \prod_{i=1}^{n-2} ((x - z_{n+i} - a_{n+i})^2 - z_{i+2}) \cdot \prod_{j=2n-1}^{2g-1} (x - z_j - a_j).$$

This defines a family of affine hyperelliptic curves over \mathbb{A}^{2g-1} , branched at the $2g + 2$ points

$$x = \infty, \quad x = \pm\sqrt{z_1} + a_{n-1}, \quad x = \pm\sqrt{z_2} + a_n, \quad x = \pm\sqrt{z_{i+2}} + z_{n+i} + a_{n+i}, \quad x = z_j + a_j$$

for $i = 1, \dots, n-2$ and $j = 2n-1, \dots, 2g-1$. This extends to a projective family of hyperelliptic curves and we denote by $U \subset \mathbb{A}^{2g-1}$ the open subset where the corresponding hyperelliptic curve attains at most one node. The corresponding projective family of hyperelliptic curves is denoted by $p: \mathcal{X} \rightarrow U$ and we note that \mathcal{X} is regular. By construction, for $u = (z_1, \dots, z_{2g-1}) \in U$, the fibre $X_u = p^{-1}(u)$ is either smooth or attains exactly one node; the latter happens if and only if $z_i = 0$ for some $i \in \{1, \dots, n\}$.

Let $\Delta_i := U \cap \{z_i = 0\}$ for $i \in \{1, \dots, n\}$. The above description of the ramification points of the hyperelliptic covering $X_u \rightarrow \mathbb{P}^1$ for $u \in U$ shows that the moduli map $\Delta_i \rightarrow \overline{\mathcal{M}}_g$ is generically finite onto its image (because the hyperelliptic locus in \mathcal{M}_g has dimension $2g - 1$). Altogether this proves item (1) in the lemma.

Fix a general base point $t \in U$. For $i \in \{1, \dots, n\}$, let $D_i \subset U$ be a disc that intersects Δ transversally in a general point $0_i \in \Delta_i$, and for a general point $t_i \in D_i - \{0_i\}$, define $\rho_i: [0, 1] \rightarrow \mathbb{C}^{2g-1}$ as the path of shortest length from t to t_i . We may and do assume that the path ρ_i is contained in $U - \Delta$.

The fibre X_t is a double covering $X_t \rightarrow \mathbb{P}^1$ branched along $2g + 2$ points $p_0, p_1, \dots, p_{2g+1}$. If we move along ρ_i , then precisely two of these branch points collide and the above description of the branch points together with the construction of the path ρ_i shows that there is no other monodromy. Up to ordering, we can assume that p_{2i-1} and p_{2i} collide along ρ_i for $i = 1, \dots, n$.

We pick a path of shortest distance between p_{2i-1} and p_{2i} on \mathbb{P}^1 for $i = 1, \dots, g$ and note that the preimage of this path in the hyperelliptic curve X_t gives rise to a homology class in $H_1(X_t, \mathbb{Z})$ (well-defined up to sign) whose Poincaré dual $\delta_i \in H^1(X_t, \mathbb{Z})$ is for $i = 1, \dots, n$ the vanishing cycle that corresponds to colliding p_{2i-1} and p_{2i} along the aforementioned path ρ_i of shortest distance. The classes $\delta_1, \dots, \delta_g$ are orthogonal to each other and can be completed to a symplectic bases, proving item (2b).

We note that the total space $\mathcal{X}|_{D_i}$ of the restriction $\mathcal{X}|_{D_i} \rightarrow D_i$ is regular: the only possible singularity is at the node of the central fibre, where analytically locally an equation of $\mathcal{X}|_{D_i}$ is given by $t = x^2 - y^2$, which yields a regular surface. Thus, $\mathcal{X}|_{D_i} \rightarrow D_i$ is a Lefschetz degeneration, proving item (2a) in the lemma. \square

Lemma 2.23. *Let $p: \mathcal{X} \rightarrow U$ be a family of hyperelliptic curves of genus g as in Lemma 2.22. For $i \in \{1, \dots, n\}$, consider the embedding $H^1(X_{0_i}, \mathbb{Z}) \hookrightarrow H^1(X_t, \mathbb{Z})$ defined as the composition of the inverse of the map $H^1(\mathcal{X}|_{D_i}, \mathbb{Z}) \rightarrow H^1(X_{0_i}, \mathbb{Z})$ (which is an isomorphism by Proposition*

(2.14) and the restriction $H^1(\mathcal{X}|_{D_i}, \mathbb{Z}) \rightarrow H^1(X_t, \mathbb{Z})$. Define

$$H'_i := \text{Im} (H^1(X_{0_i}, \mathbb{Z}) \hookrightarrow H^1(X_t, \mathbb{Z})) \quad \text{and} \quad V_i := \text{Im} (W_0 H^1(X_{0_i}, \mathbb{Z}) \hookrightarrow H^1(X_t, \mathbb{Z})),$$

where $W_0 H^1(X_{0_i}, \mathbb{Z}) = W_0 H^1(X_{0_i}, \mathbb{Q}) \cap H^1(X_{0_i}, \mathbb{Z})$ is the integral part of the zeroth piece of the weight filtration. For $i \in \{1, \dots, n\}$, let $T_i \in \text{Aut}(H^1(X_t, \mathbb{Z}))$ be the monodromy operator associated to the path ρ_i and the pointed disc $(D_i, 0_i)$. Then, with respect to the symplectic basis (2.14), we have

$$H'_i = H^1(X_t, \mathbb{Z})^{T_i} = \langle \delta_1, \dots, \delta_g; \gamma_1, \dots, \gamma_{i-1}, \widehat{\gamma}_i, \gamma_{i+1}, \dots, \gamma_g \rangle, \quad (2.15)$$

$$V_i = \text{Im} (T_i - \text{id}) = \langle \delta_i \rangle = \mathbb{Z} \cdot \delta_i. \quad (2.16)$$

Here, the module on the right in (2.15) denotes the submodule of $H^1(X_t, \mathbb{Z})$ obtained from $H^1(X_t, \mathbb{Z})$ by removing γ_i from the symplectic basis (2.14).

Proof. For each $i \in \{1, \dots, n\}$, consider the monodromy operator $T_i: H^1(X_t, \mathbb{Z}) \rightarrow H^1(X_t, \mathbb{Z})$. In view of item (2a) in Lemma 2.22, T_i is given by the formula

$$T_i(\alpha) = \alpha + (\alpha \cdot \delta_i) \delta_i, \quad \alpha \in H^1(X_t, \mathbb{Z}),$$

where the δ_i are the vanishing cycles $\delta_i \in H^1(X_t, \mathbb{Z})$ attached to the Lefschetz degenerations $\mathcal{X}|_{D_i} \rightarrow D_i$ and the path ρ_i (cf. item (2b) in Lemma 2.22). It is well-known that

$$V_i \otimes \mathbb{Q} = \text{Im}(T_i - \text{id}) \otimes \mathbb{Q} \subset \text{Ker}(T_i - \text{id}) \otimes \mathbb{Q} = H'_i \otimes \mathbb{Q} \subset H^1(X_t, \mathbb{Q}). \quad (2.17)$$

In fact, (2.17) holds integrally because the respective spaces are saturated in $H^1(X_t, \mathbb{Z})$. In particular, (2.15) follows. Note that $(T_i - \text{id})(\alpha) = (\alpha \cdot \delta_i) \delta_i$ for each $\alpha \in H^1(X_t, \mathbb{Z})$. As we have $\gamma_i \cdot \delta_i = -1$, this gives $(T_i - \text{id})(\gamma_i) = (\gamma_i \cdot \delta_i) \delta_i = -\delta_i$. Therefore, $V_i = \text{Im}(T_i - \text{id}) = \langle \delta_i \rangle = \mathbb{Z} \cdot \delta_i \subset H'_i$, proving (2.16), and thereby the lemma. \square

3. MOVING THE EXTENSION CLASS

Let $f: X \rightarrow Y$ be a flat morphism of complex analytic spaces with reduced fibres. Following [CHL06, Definition 5.3], a *simultaneous normalization* of f is a finite morphism of analytic spaces $\nu: Z \rightarrow X$ such that $\bar{f} := f \circ \nu$ is a flat morphism $Z \rightarrow Y$ whose non-empty fibres are normal, and such that for each $y \in f(X)$, the induced morphism of fibres $\nu_y: \bar{f}^{-1}(y) \rightarrow f^{-1}(y)$ is a normalization map. We call $f: X \rightarrow Y$ *equinormalizable* if a simultaneous normalization of f exists. If Y is normal, $f: X \rightarrow Y$ is flat with reduced fibres, and the connected components of X are equidimensional, then any simultaneous normalization $\nu: Z \rightarrow X$ of f is a normalization of X , see [CHL06, Proposition 5.4]; in particular, $\nu: Z \rightarrow X$ is then unique up to isomorphism.

Proposition 3.1. *Let X and Y be complex analytic spaces with Y normal. Let $f: X \rightarrow Y$ be a (proper) family of nodal curves with irreducible fibres (cf. Section 1.3). Assume the number of nodes of the curve $X_y = f^{-1}(y)$ is constant for $y \in Y$. Then f is equinormalizable.*

Proof. This can for instance be deduced from [CHL06, Theorem 5.6, Corollary 5.4.2, and Definition 5.1]; we include some details for convenience of the reader. Since f is a proper family of nodal curves, $\text{Sing}(f) \rightarrow Y$ is finite and unramified, see e.g. [Jon96, §2.21]. Since

the number of nodes is constant in the family, it follows that each component of $\text{Sing}(f)$ dominates Y . Using this we see that for each $x \in \text{Sing}(f)$ with image $y = f(x)$, there is a suitable neighbourhood $U \subset X$ such that $\text{Sing}(f) \cap U \cap f^{-1}(y) = \{x\}$. Up to shrinking U , we can assume that the fibres of $U \rightarrow Y$ have at most one node. Up to replacing U by the intersection with the preimage of a suitable neighbourhood of y in Y , we can moreover assume that $U \cap \text{Sing}(f) \rightarrow f(U)$ is finite and surjective. Moreover, since f is flat, so is $U \rightarrow f(U)$. We may then apply Theorem 5.6 in loc. cit. to conclude that f is equinormalizable at x (in the sense of the paragraph above Corollary 5.4.2 in loc. cit.). Since $x \in X$ was arbitrary, it follows from Corollary 5.4.2(ii) in loc. cit. that f is equinormalizable. \square

Proposition 3.2. *Let $p: \mathcal{X} \rightarrow H$ be a family of one-nodal hyperelliptic curves of arithmetic genus $g \geq 2$ over a connected normal complex analytic space H . Assume that:*

- (1) *For some $k \geq 1$ and some family of semi-stable curves $q: \mathcal{C} \rightarrow H$, there is an isogeny*

$$\psi: (J\mathcal{X})^k := J\mathcal{X} \times_H \cdots \times_H J\mathcal{X} \longrightarrow J\mathcal{C}$$

of semi-abelian varieties over H , where $J\mathcal{X}$ and $J\mathcal{C}$ are the relative Jacobians over H .

- (2) *If $\nu: \tilde{\mathcal{X}} \rightarrow \mathcal{X}$ is a simultaneous normalization (cf. Proposition 3.1), then*

- (i) *there is an isomorphism*

$$\tilde{\mathcal{X}} \simeq \tilde{X}_0 \times H$$

over H , where $\tilde{X}_0 \subset \tilde{\mathcal{X}}$ is the fibre above a base point $0 \in H$.

- (ii) *there is a non-constant morphism $H \rightarrow \tilde{X}_0$, $u \mapsto x_u$, such that for any $u \in H$, the fibre X_u is obtained from \tilde{X}_u by glueing the pair of points $(x_u, \iota(x_u))$, where ι denotes the hyperelliptic involution on \tilde{X}_0 .*

Let $\tilde{C}_{0,1}, \dots, \tilde{C}_{0,n}$ be the non-rational connected components of the normalization \tilde{C}_0 of the curve $C_0 = q^{-1}(0)$. Then there is an integer $N \geq 1$ and a matrix $\alpha = (a_{i,j})_{ij} \in M_k(\mathbb{Z})$ with non-zero determinant, such that for each $j \in \{1, \dots, k\}$, the image of the composition

$$g_j: \tilde{X}_0 \xrightarrow{f_j} (J\tilde{X}_0)^k \xrightarrow{\alpha} (J\tilde{X}_0)^k \xrightarrow{\tilde{\psi}_0} J\tilde{C}_0 = J\tilde{C}_{0,1} \times \cdots \times J\tilde{C}_{0,n}$$

is contained in

$$N \cdot (\tilde{C}_{0,1} - \tilde{C}_{0,1}) \times \cdots \times N \cdot (\tilde{C}_{0,n} - \tilde{C}_{0,n}) \subset J\tilde{C}_{0,1} \times \cdots \times J\tilde{C}_{0,n},$$

where $f_j(x) = (0, \dots, x - \iota(x), \dots, 0)$ with $x - \iota(x)$ placed on the j -th coordinate and where $\tilde{\psi}_0$ is the isogeny induced by ψ . Moreover, for each $i \in \{1, \dots, n\}$, there exists $j \in \{1, \dots, k\}$ such that the resulting morphism

$$\text{pr}_i \circ g_j: \tilde{X}_0 \longrightarrow N \cdot (\tilde{C}_{0,i} - \tilde{C}_{0,i})$$

is non-constant.

Proof. The image of g_j is analytic and the image of f_j is one-dimensional. Hence, in order to prove the proposition, we are allowed to perform a base change along an arbitrary morphism

$\tau: H' \rightarrow H$ of complex analytic spaces as long as $0 \in \text{im}(\tau)$ and the point $x_u \in \tilde{X}_0$ from item (ii) moves if u runs along $H \cap \text{im}(\tau)$. This easily reduces us to the situation where H is a disc.

Since the fibres of $p: \mathcal{X} \rightarrow H$ are one-nodal and $\psi: (J\mathcal{X})^k \rightarrow J\mathcal{C}$ is an isogeny, we have that $k = \text{rank}_{\mathbb{Z}}(H_1(\Gamma(C_0), \mathbb{Z}))$, where $\Gamma(C_0)$ denotes the dual graph on C_0 . By Lemma 2.10, there is a linearly independent subset $\{\gamma_1(0), \dots, \gamma_k(0)\} \subset H_1(\Gamma(C_0), \mathbb{Z})$ of homology classes of the form (2.10). Since $\mathcal{C}|_H \rightarrow H$ is equi-singular, the groups $H_1(\Gamma(C_u), \mathbb{Z})$ form for $u \in H$ a local system on H . Since H is a disc, the corresponding local system is trivial and so each $\gamma_j(0)$ extends to a section of classes $\gamma_j(u) \in H_1(\Gamma(C_u), \mathbb{Z})$. Hence, for each $u \in H$ we get a linearly independent subset

$$\{\gamma_1(u), \dots, \gamma_k(u)\} \subset H_1(\Gamma(C_u), \mathbb{Z}) \quad (3.1)$$

of homology classes of the form (2.10). Consider the homomorphism

$$c_t: H_1(\Gamma(C_u), \mathbb{Z}) \longrightarrow J\tilde{C}_u = J\tilde{C}_0 = \prod_{i=1}^n J\tilde{C}_{0,i},$$

see Section 2.4 and in particular equations (2.8) and (2.9). By Lemma 2.11, for each $u \in H$ there are points $p_{i,j}(u), q_{i,j}(u) \in \tilde{C}_{0,i}$ for $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, k\}$, such that

$$c^t(\gamma_j(u)) = (p_{1,j}(u) - q_{1,j}(u), \dots, p_{n,j}(u) - q_{n,j}(u)) \in J\tilde{C}_{0,1} \times \dots \times J\tilde{C}_{0,n}. \quad (3.2)$$

Up to a suitable base change we may assume that the points $p_{i,j}(u), q_{i,j}(u)$ depend holomorphically on u and give rise to sections of $\mathcal{C} \rightarrow H$.

By Lemma 2.8 and equation (2.8), for each $u \in H$, the isogeny of semi-abelian varieties $\psi_u: (JX_u)^k \rightarrow JC_u$ induces a canonical morphism $(\psi_u)_*: H_1(\Gamma(X_u), \mathbb{Z})^{\oplus k} \rightarrow H_1(\Gamma(C_u), \mathbb{Z})$ such that the following diagram commutes:

$$\begin{array}{ccc} H_1(\Gamma(X_u), \mathbb{Z})^{\oplus k} & \xrightarrow{(\psi_u)_*} & H_1(\Gamma(C_u), \mathbb{Z}) \\ \downarrow c_{(JX_u)^k}^t & & \downarrow c_{JC_u}^t \\ (J\tilde{X}_0)^k & \xrightarrow{\tilde{\psi}_0} & J\tilde{C}_0. \end{array} \quad (3.3)$$

By assumptions, the curve X_u is for $u \in H$ obtained from its normalization \tilde{X}_u by gluing x_u to $\iota(x_u) \in \tilde{X}_u$. In particular, the dual graph $\Gamma(X_u)$ consists of a single loop. We fix an orientation of this dual graph (see Definition 2.9) and obtain a canonical identification $H_1(\Gamma(X_u), \mathbb{Z}) = \mathbb{Z}$. Let

$$\mu(u)_r = (0, \dots, 0, 1, 0, \dots, 0) \in H_1(\Gamma(X_u), \mathbb{Z})^{\oplus k},$$

where 1 is placed on the r -th coordinate. Then

$$c_{(JX_u)^k}^t(\mu(u)_r) = (0, \dots, x_u - \iota(x_u), \dots, 0) \in (J\tilde{X}_u)^k = (J\tilde{X}_0)^k,$$

where $x_u - \iota(x_u)$ is placed on the r -th coordinate. Recall the set of classes $\gamma_j(u) \in H_1(\Gamma(C_u), \mathbb{Z})$ from (3.1). As the elements $\mu(u)_1, \dots, \mu(u)_k$ form a basis of $H_1(\Gamma(X_u), \mathbb{Z})^{\oplus k}$, and the cokernel of the embedding

$$(\psi_u)_*: H_1(\Gamma(X_u), \mathbb{Z})^{\oplus k} \longrightarrow H_1(\Gamma(C_u), \mathbb{Z})$$

is finite (because ψ_u is an isogeny), there exists an integer $N \in \mathbb{Z}_{\geq 1}$ and integers $a_{1,j}, \dots, a_{k,j} \in \mathbb{Z}$ for each $j \in \{1, \dots, k\}$, such that

$$N \cdot \gamma_j(u) = \sum_{r=1}^k a_{r,j} \cdot (\psi_u)_*(\mu(u)_r) \in H_1(\Gamma(C_u), \mathbb{Z}) \quad (3.4)$$

for each $j \in \{1, \dots, k\}$. By the commutativity of the diagram (3.3), together with (3.2), we obtain the following equalities for each $u \in H$ and each $j \in \{1, \dots, k\}$:

$$\begin{aligned} & \tilde{\psi}_0(a_{1,j} \cdot (x_u - \iota(x_u)), \dots, a_{k,j} \cdot (x_u - \iota(x_u))) \\ &= \sum_{r=1}^k a_{r,j} \cdot \tilde{\psi}_0(0, \dots, x_u - \iota(x_u), \dots, 0) = \sum_{r=1}^k a_{r,j} \cdot \tilde{\psi}_0\left(c_{(JX_u)^k}^t(\mu(u)_r)\right) \\ &= \sum_{r=1}^k a_{r,j} \cdot c_{JC_u}^t((\psi_u)_*(\mu(u)_r)) = c_{JC_u}^t\left(\sum_{r=1}^k a_{r,j} \cdot (\psi_u)_*(\mu(u)_r)\right) \\ &= c_{JC_u}^t(N \cdot \gamma_j) = N \cdot c_{JC_u}^t(\gamma_j) \\ &= N \cdot (p_{1,j}(u) - q_{1,j}(u), \dots, p_{n,j}(u) - q_{n,j}(u)) \in J\tilde{C}_{0,1} \times \dots \times J\tilde{C}_{0,n}. \end{aligned}$$

We may now consider the integral $k \times k$ matrix $\alpha := (a_{i,j})_{i,j}$ and we note that this matrix has non-zero determinant because $\gamma_1(u), \dots, \gamma_k(u) \in H_1(\Gamma(C_u), \mathbb{Z})$ from (3.1) form a rational basis. Then

$$\alpha \cdot f_j(x_u) = (a_{1,j} \cdot (x_u - \iota(x_u)), \dots, a_{k,j} \cdot (x_u - \iota(x_u))),$$

and so we conclude that

$$\tilde{\psi}_0(\alpha \cdot f_j(x_u)) = N \cdot (p_{1,j}(u) - q_{1,j}(u), \dots, p_{n,j}(u) - q_{n,j}(u)) \in J\tilde{C}_{0,1} \times \dots \times J\tilde{C}_{0,n}.$$

As $\tilde{\psi}_0$ is an isogeny, $\tilde{\psi}_0(\alpha \cdot f_j(x_u))$ moves with $u \in H$ because the map $H \rightarrow \tilde{X}_0$, $u \mapsto x_u$ is non-constant by assumption. This shows that the restriction of $g_j: \tilde{X}_0 \rightarrow J\tilde{C}_0$ to some analytically open non-empty subset of \tilde{X}_0 has the property claimed in the proposition, which suffices to conclude.

It remains to prove the last assertion. Note that the curves $f_j(\tilde{X}_0)$ for $j = 1, \dots, k$ generate $(J\tilde{X}_0)^k$. As $\tilde{\psi}_0$ and α are isogenies, the curves $g_j(\tilde{X}_0)$ for $j = 1, \dots, k$ generate $J\tilde{C}_0 = \prod_{i=1}^n J\tilde{C}_{0,i}$. Thus, for $i \in \{1, \dots, n\}$, the projection $\text{pr}_i: g_j(\tilde{X}_0) \rightarrow J\tilde{C}_{0,i}$, and hence the projection $\text{pr}_i: g_j(\tilde{X}_0) \rightarrow N \cdot (\tilde{C}_{0,i} - \tilde{C}_{0,i})$, is non-constant for some $j \in \{1, \dots, k\}$. This concludes the proof of the proposition. \square

4. HYPERELLIPTIC CURVES ON HYPERELLIPTIC JACOBIANS

The following result is due to Naranjo and Pirola, see [NP18, Theorem 1.1].

Theorem 4.1 (Naranjo–Pirola). *Let X be a very general hyperelliptic curve of genus $g \geq 3$. Suppose that the Jacobian JC of some hyperelliptic curve C is isogenous to JX . Then $C \simeq X$.*

Theorem 4.1 is the part of [NP18, Theorem 1.1] that is not affected by the gap that we point out in Remark 4.5 below. In the appendix to this paper, we include an important ingredient that goes into its proof, see Proposition A.1.

The goal of this section is to deduce from the Naranjo–Pirola theorem the following generalization of Theorem 4.1.

Theorem 4.2. *Let X be a very general hyperelliptic curve of genus ≥ 3 . Let C_1, \dots, C_n be hyperelliptic curves of genus ≥ 1 such that there exists an isogeny $JC_1 \times \dots \times JC_n \rightarrow (JX)^k$ for some $k \geq 1$. Then $n = k$, there is an isomorphism $C_i \simeq X$ for each $i \in \{1, \dots, n\}$, and the induced isogeny $(JX)^k \simeq JC_1 \times \dots \times JC_n \rightarrow (JX)^k$ is given by a matrix $\alpha \in M_k(\mathbb{Z})$.*

The proof of Theorem 4.2 relies on Theorem 4.1 and the following two results, the first of which is probably well-known, and the second of which is due to Lazarsfeld and Martin [LM23].

Lemma 4.3. *Let A an abelian variety with $\text{End}(A) = \mathbb{Z}$. The natural maps $M_k(\mathbb{Z}) \rightarrow \text{End}(A^k)$ and $\text{GL}_k(\mathbb{Z}) \rightarrow \text{Aut}(A^k)$ are isomorphisms. Moreover, if there exist abelian varieties B_1, \dots, B_n and an isomorphism $\varphi: \prod_{i=1}^n B_i \xrightarrow{\sim} A^k$, then there exists $\alpha \in \text{Aut}(A^k) = \text{GL}_k(\mathbb{Z})$ such that the composition $\alpha \circ \varphi: \prod_{i=1}^n B_i \rightarrow A^k$ respects the product structures on both sides. Thus, in that case, there is a partition $k = k_1 + \dots + k_n$ such that $B_i \simeq A^{k_i}$ for each i .*

Proof. The space of endomorphisms $\text{End}(A^k)$ is naturally given by $k \times k$ matrices whose entries are endomorphisms of A . Since $\text{End}(A) = \mathbb{Z}$, we find that $\text{End}(A^k) \simeq M_k(\mathbb{Z})$, which proves the first claim in the lemma. This also implies $\text{Aut}(A^k) \simeq \text{GL}_k(\mathbb{Z})$.

The i -th factor B_i yields a projector $p_i \in \text{End}(A^k)$. By what we have said above, p_i can be identified with a $k \times k$ matrix with $p_i \cdot p_i = p_i$ and $p_i \circ p_j = p_j \circ p_i$ for all i, j . By simultaneous diagonalization of permuting projectors, we find a change of coordinates, i.e. an automorphism $\alpha \in \text{Aut}(A^k) = \text{GL}_k(\mathbb{Z})$, such that $\alpha \circ \varphi$ has the property claimed in the lemma. \square

Proposition 4.4 (Lazarsfeld–Martin). *Let X be a very general hyperelliptic curve of genus $g \geq 3$ and let $Z \subset JX \times JX$ be an irreducible curve whose normalization is hyperelliptic. Then Z generates a proper subtorus of $JX \times JX$.*

Proof. See [LM23, Proposition 3.1]. In the statement of that proposition, there is the additional assumption that Z lifts to a curve on $X \times X$, but this assumption is not used in the proof. Indeed, the proof in *loc. cit.* immediately starts with the hyperelliptic curve Z inside $JX \times JX$ and spreads this out to a family of hyperelliptic curves $Z_s \subset JX_s \times JX_s$. Then the hyperelliptic Jacobian JX_s is specialized to $JX_s = B_s \times E$, where E is a fixed elliptic curve and B_s is the Jacobian of a very general hyperelliptic curve of genus $g - 1$ which varies with s . Under the assumption that Z generates $JX \times JX$, the same will be true for Z_s for all s . In particular, under this assumption the image of Z_s via the projection $JX_s \times JX_s \rightarrow E \times E$ is a curve in $E \times E$. It is then shown (see [LM23, Claim in Section 3]) that this curve varies with s . Since the normalization of Z_s is hyperelliptic, its image in $E \times E$ yields a rational curve on the Kummer surface associated to $E \times E$. The latter are rigid, because Kummer surfaces are not ruled. This contradiction concludes the argument. \square

Proof of Theorem 4.2. As JX is simple, we have $g(C_i) \geq g(X)$ for each i , and $n \leq k$. If $g(C_i) > g(X)$ for some i , then $k \geq 2$ and there exists a surjection

$$JC_i \twoheadrightarrow JX \times JX.$$

As C_i is hyperelliptic, and X is very general hyperelliptic, this contradicts Proposition 4.4. We conclude that $g(C_i) = g(X)$ for each i , and that $n = k$. In particular, for each i , there exists an isogeny $JC_i \rightarrow JX$. By Theorem 4.1, we have $C_i \simeq X$ for each i . As the hyperelliptic curve X is very general, the composition $(JX)^k \simeq \prod_{i=1}^n JC_i \rightarrow (JX)^k$ is given by a matrix in $M_k(\mathbb{Z})$, see Lemma 4.3. \square

Remark 4.5. The content of [NP18, Theorem 1.1] is a combination of Theorem 4.1 stated above with the $k = 1$ case of Theorem 1.1. The proof of the latter contains a gap, see [NP18, p. 903, line -5 of the proof of Theorem 1.1]. Here, it is claimed that the nodal curves C_0 and D_0 in *loc. cit.* are isomorphic, while the given arguments only suffice to conclude that their normalizations are isomorphic. This step could be fixed if the dimension count on the bottom of [NP18, p. 901] was correct, which is not the case.

As Theorem 1.1 is more general than the problematic part of [NP18, Theorem 1.1], proving this theorem will fix the gap in *loc. cit.*. To overcome the difficulties encountered in *loc. cit.* and make our proof work, the key point is that we use four degenerations (instead of two as in *loc. cit.*) together with various results on symplectic lattices that we prove in Section 7.

Naranjo and Pirola have informed us that it should be possible to find an alternative fix via a careful study of infinitesimal variations of Hodge structures due to Griffiths and Voisin.

5. POLARIZATIONS ON POWERS OF ABELIAN VARIETIES AND BILINEAR FORMS

This section has two goals. Consider a principally polarized abelian variety A with endomorphism ring \mathbb{Z} . Firstly, we classify isomorphism classes of principal polarizations on any power of A , see Section 5. Secondly, we investigate principal polarizations on any abelian variety B isogenous to a power of A , see Section 5.2.

5.1. Polarizations on powers of a very general abelian variety. Let A be an abelian variety. Recall that there is a natural injective map

$$\Phi: \text{NS}(A) \longrightarrow \text{Hom}(A, A^\vee), \quad D \longmapsto \varphi_D,$$

where A^\vee is the dual abelian variety of A . Here, $\varphi_D(x) = t_x^* D' - D'$, where $t_x: A \rightarrow A$ denotes the translation by x and D' denotes any divisor on A that represents D in $\text{NS}(A)$. The image of Φ is contained in the subset $\text{Hom}^{\text{sym}}(A, A^\vee) \subset \text{Hom}(A, A^\vee)$ of maps $\phi: A \rightarrow A^\vee$ that satisfy $\phi^\vee = \phi$ (viewed as maps $A^{\vee\vee} = A \rightarrow A^\vee$). A class $D \in \text{NS}(A)$ is ample if and only if $\varphi_D: A \rightarrow A^\vee$ is an isogeny, in which case D (resp. φ_D) is called a polarization. The polarization is principle if $h^0(A, \mathcal{O}_A(D)) = 1$, or equivalently, if φ_D is an isomorphism; see e.g. [Mil86] or [BL04] for more details.

Lemma 5.1. *Let $A = V/\Lambda$ be a complex torus. Let $k \in \mathbb{Z}_{\geq 1}$ and consider the natural embedding*

$$\iota_A: M_k(\mathbb{Z}) \longrightarrow \text{End}(A^k).$$

Let $\alpha = (a_{ij})_{ij} \in M_k(\mathbb{Z})$ with attached endomorphism $\iota_A(\alpha) \in \text{End}(A^k)$. We have an equality $\iota_A(\alpha)^\vee = \iota_{A^\vee}(\alpha^t)$ of maps $(A^\vee)^k \rightarrow (A^\vee)^k$, where $\alpha^t \in M_k(\mathbb{Z})$ is the transpose of the matrix α .

Proof. The lemma follows from the following linear algebra statement: if Λ is a free \mathbb{Z} -module of positive and finite rank, and if $\alpha = (a_{ij}) \in M_k(\mathbb{Z})$, then the endomorphism $f_\Lambda(\alpha): \Lambda^k \rightarrow \Lambda^k$ that α induces satisfies $f_\Lambda(\alpha)^\vee = f_{\Lambda^\vee}(\alpha^t)$ as morphisms $(\Lambda^\vee)^k \rightarrow (\Lambda^\vee)^k$. Here, $f_{\Lambda^\vee}(\alpha^t)$ is the endomorphism of $(\Lambda^\vee)^k$ attached to the transpose $\alpha^t \in M_k(\mathbb{Z})$ of α . We leave the proof of this statement to the reader. \square

Lemma 5.2. *Let $n \in \mathbb{Z}_{\geq 1}$ and let A be an abelian variety with $\text{End}(A) = \mathbb{Z}$, principally polarized by $\lambda: A \xrightarrow{\sim} A^\vee$. Consider the map*

$$M: \left\{ \text{polarizations on } A^k \right\} \longrightarrow M_k(\mathbb{Z})$$

that sends a polarization $\mu: A^k \rightarrow (A^k)^\vee$ to the matrix $M(\mu) \in M_k(\mathbb{Z})$ attached to $\text{diag}(\lambda^{-1}) \circ \mu$ (cf. Lemma 4.3), where $\text{diag}(\lambda^{-1})$ is the isomorphism $(A^k)^\vee \rightarrow A^k$ induced by λ^{-1} . The following holds.

(1) *The map M induces a bijection*

$$M: \left\{ \text{polarizations on } A^k \right\} \xrightarrow{\sim} P_k(\mathbb{Z}), \tag{5.1}$$

where $P_k(\mathbb{Z}) \subset M_k(\mathbb{Z})$ denotes the subset of positive definite and symmetric matrices.

- (2) *The map M is equivariant with respect to the evident $\text{Aut}(A^k) = \text{GL}_k(\mathbb{Z})$ -action on both sides (to be recalled in the proof). In particular, two polarizations μ_1 and μ_2 are $\text{Aut}(A^k)$ -conjugate if and only if the matrices $M(\mu_1)$ and $M(\mu_2)$ are $\text{GL}_k(\mathbb{Z})$ -equivalent.*
- (3) *The map M induces a bijection between the set of (isomorphism classes of) principal polarizations on A^k and the set of (isomorphism classes of) unimodular positive definite symmetric bilinear forms on \mathbb{Z}^k .*

Proof. Let us prove statement (1). By [BL04, Theorem 5.2.4], M yields a bijection between polarizations on A^k and totally positive symmetric endomorphisms of A^k . Here, *symmetric* is understood to be with respect to the Rosati involution $\dagger: \text{End}(A^k) \rightarrow \text{End}(A^k)$ defined by λ , and an endomorphism $\varphi: X \rightarrow X$ of an abelian variety X is said to be *totally positive* if the zeros of the characteristic polynomial of the analytic representation $\varphi^{an}: \text{Lie}(X) \rightarrow \text{Lie}(X)$ of φ are all positive. Consider the canonical embedding

$$\iota_A: M_k(\mathbb{Z}) \hookrightarrow \text{End}(A^k).$$

It remains to prove the following assertions.

- (a) The embedding ι_A identifies the symmetric matrices $(f_{ij}) \in M_k(\mathbb{Z})$ with the symmetric elements $f \in \text{End}(A^k)$.
- (b) The bilinear form attached to a matrix $(f_{ij}) \in M_k(\mathbb{Z})$ is positive definite if and only if the associated endomorphism $f \in \text{End}(A^k)$ is totally positive.

Item (a) follows from Lemma 5.1. Item (b) follows readily from the fact that for any complex vector space V , an \mathbb{R} -linear map $\mathbb{R}^k \rightarrow \mathbb{R}^k$ has positive eigenvalues if and only if the induced \mathbb{C} -linear map $V^k = \mathbb{R}^k \otimes_{\mathbb{R}} V \rightarrow \mathbb{R}^k \otimes_{\mathbb{R}} V = V^k$ has positive real eigenvalues.

Since $\text{End}(A) = \mathbb{Z}$, the canonical map $\iota_A: \text{GL}_k(\mathbb{Z}) \rightarrow \text{Aut}(A^k)$ is an isomorphism (see Lemma 4.3) and we will use ι_A to identify the two groups; thus $\text{Aut}(A^k) = \text{GL}_k(\mathbb{Z})$ in the

remaining part of the proof. Observe that an element $\alpha \in \text{Aut}(A^k)$ acts naturally on a polarization $\mu : A^k \rightarrow (A^k)^\vee$ via

$$\mu \mapsto \alpha^\vee \circ \mu \circ \alpha,$$

where $\alpha^\vee : (A^\vee)^k \rightarrow (A^\vee)^k$ is the automorphism of $(A^\vee)^k$ induced by α via duality. The map M associates to a polarization on A^k the corresponding matrix and using Lemma 5.1, we find

$$M(\alpha^\vee \circ \mu \circ \alpha) = M(\alpha^\vee) \circ M(\mu) \circ M(\alpha) = \alpha^t \cdot M(\mu) \cdot \alpha,$$

where $\alpha \in \text{Aut}(A^k) = \text{GL}_k(\mathbb{Z})$ is identified with a $k \times k$ matrix and α^t is the transpose of this matrix. This proves (2).

Finally, the polarization μ is a principal if and only if the positive definite matrix $M(\mu)$ is unimodular. Item (3) of the lemma follows then from item (2), and we are done. \square

Corollary 5.3. *Let (A, λ) a principally polarized abelian variety with $\text{End}(A) = \mathbb{Z}$. Let k be a positive integer, and μ a polarization on A^k . Let $\alpha = M(\mu) \in \text{M}_k(\mathbb{Z})$ be the attached matrix.*

- (1) *The map M from Lemma 5.2 induces a bijection between isomorphism classes of decompositions $(A^k, \mu) \simeq \prod_i (B_i, \lambda_i)$ for some polarized abelian varieties (B_i, λ_i) , and isomorphism classes of decompositions $(\mathbb{Z}^k, \alpha) \simeq \prod_i (\mathbb{Z}^{k_i}, \alpha_i)$ for some integral symmetric spaces $(\mathbb{Z}^{k_i}, \alpha_i)$.*
- (2) *Suppose that the polarization μ on A^k is principal. Let $(B, \mu|_B) \subset (A^k, \mu)$ be a principally polarized abelian subvariety. Then $B \simeq A^n$ for some $n \leq k$.*

Proof. Consider an isomorphism $\varphi : \prod_i (B_i, \lambda_i) \xrightarrow{\sim} (A^k, \mu)$ of polarized abelian varieties. Up to replacing φ by the composition $\alpha \circ \varphi$ and μ by the conjugate polarization $((\alpha^{-1})^\vee) \circ \mu \circ \alpha^{-1}$, we can assume that φ , viewed as an isomorphism $\prod B_i \rightarrow A^k$ respects the product decompositions on both sides and hence identifies each of the B_i 's with some power A^{k_i} . Item (1) is now a consequence of item (2) in Lemma 5.2.

Item (2) follows from item (1) together with the fact that the subvariety $(B, \mu|_B) \subset (A^k, \mu)$ admits a principal polarized complement, because $\mu|_B$ and μ are both principal polarizations and the category of principally polarized abelian varieties is semi-simple. \square

5.2. Polarizations on abelian varieties isogenous to a power of an abelian variety.

Lemma 5.4. *Let A be an abelian variety and let $\lambda : A \rightarrow A^\vee$ be a polarization. There is a canonical isomorphism $H_1(A^\vee, \mathbb{Z}) = H_1(A, \mathbb{Z})^\vee$. Moreover, if $E : H_1(A, \mathbb{Z}) \times H_1(A, \mathbb{Z}) \rightarrow \mathbb{Z}$ is the alternating form corresponding to λ , then the push-forward*

$$\lambda_* : H_1(A, \mathbb{Z}) \rightarrow H_1(A^\vee, \mathbb{Z}) = H_1(A, \mathbb{Z})^\vee$$

satisfies $\lambda_(x)(y) = E(x, y)$ for all $x, y \in H_1(A, \mathbb{Z})$.*

Proof. This is well-known and follows for instance from [BL04, Lemma 2.4.5]. \square

Lemma 5.5. *Let (A, λ_A) and (B, λ_B) be principally polarized abelian varieties. Let β be a positive definite symmetric integral $k \times k$ matrix and let λ_{A^k} be the natural product polarization*

on A^k . Let $A^k \rightarrow B$ be an isogeny for some integer $k \geq 1$, and suppose that the principal polarization λ_B of B pulls back to the polarization $\beta \cdot \lambda_{A^k}$ on A^k defined as

$$\beta \cdot \lambda_A: A^k \longrightarrow (A^\vee)^k, \quad x \longmapsto \beta \cdot \lambda_{A^k}(x).$$

Define $M = H_1(B, \mathbb{Z})$ and $H = H_1(A, \mathbb{Z})$, and let E_M and $E_H^{\oplus k}$ be the symplectic forms on M and $H^{\oplus k}$ associated to the respective principal polarizations. With respect to the natural embedding $M \subset H^{\oplus k}$ induced by the isogeny $A^k \rightarrow B$, we have

$$E_M(x, y) = E_H^{\oplus k}(x, \beta^{-1} \cdot y) = E_H^{\oplus k}(\beta^{-1} \cdot x, y) \quad \text{for each } x, y \in M.$$

Proof. Via the principal polarizations, the given isogeny $\phi: A^k \rightarrow B$ induces an isogeny $\psi: B \rightarrow A^k$. We claim that $\psi \circ \phi = \beta$ as isogenies $A^k \rightarrow A^k$. To see this, recall that the pull-back $\phi^*(\lambda_B)$ of the principal polarization λ_B of B is the isogeny $A^k \rightarrow (A^k)^\vee$ given by the composition

$$A^k \xrightarrow{\phi} B \xrightarrow{\lambda_B} B^\vee \xrightarrow{\phi^\vee} (A^k)^\vee.$$

As this is β times the natural principal polarization on A^k , the claim follows.

On the level of lattices, the maps ϕ and ψ induce embeddings

$$H^{\oplus k} \xleftarrow{f} M \xleftarrow{g} H^{\oplus k}.$$

The claim above implies that $g \circ f = \beta$ as linear maps $H^{\oplus k} \rightarrow H^{\oplus k}$. By assumption, we have $\phi^*(\lambda_B) = \beta \cdot \lambda_{A^k}$. Therefore, by Lemma 5.4, we have

$$E_M(f(x), f(y)) = (f^* E_M)(x, y) = E_H^{\oplus k}(\beta \cdot x, y) \quad \forall x, y \in H^{\oplus k}.$$

If we view M as a sublattice of $H^{\oplus k}$ via g , then the above equality implies that

$$E_M(\beta \cdot x, \beta \cdot y) = E_M(f(x), f(y)) = E_H^{\oplus k}(\beta \cdot x, y) \quad \forall x, y \in H^{\oplus k}.$$

In particular, $E_M(\beta \cdot x, \beta \cdot y) = E_H^{\oplus k}(\beta \cdot x, y)$ for all $x, y \in M$, hence $E_M(x, y) = E_H^{\oplus k}(x, \beta^{-1} \cdot y)$ for $x, y \in M$. As $E_H^{\oplus k}(x, \beta^{-1} \cdot y) = E_H^{\oplus k}(\beta^{-1} \cdot x, y)$ for each $x, y \in H^{\oplus k}$, the lemma follows. \square

6. POWERS OF ABELIAN VARIETIES ISOMORPHIC TO PRODUCTS OF JACOBIANS

The goal of this section is to prove the following theorem.

Theorem 6.1. *Let $g \in \mathbb{Z}_{\geq 1}$ and let $Z \subset \mathcal{A}_g$ be an irreducible subvariety of the moduli space of principally polarized abelian varieties of dimension g with the following properties:*

- *there is a point $[(A_0, \Theta_0)] \in Z$ such that $A_0 \simeq E_0 \times B_0$ (as unpolarized abelian varieties), where B_0 is an abelian variety of dimension $g - 1$ and E_0 is an elliptic curve with transcendental j -invariant;*
- *any very general point $[(A, \Theta)] \in Z$ satisfies $\text{End}(A) = \mathbb{Z}$.*

If for some very general point $[(A, \Theta)] \in Z$ and some integers $k, n \geq 1$, there are some smooth projective connected curves C_1, \dots, C_n of positive genus and an isomorphism $\prod_{i=1}^n JC_i \simeq A^k$ of unpolarized abelian varieties, then $k = n$ and for each i we have an isomorphism $(JC_i, \Theta_{C_i}) \simeq (A, \Theta)$ of polarized abelian varieties.

6.1. Applications of Theorem 6.1. Before we turn to the proof of Theorem 6.1, we show that it implies the following, which is a more general version of Theorem 1.2 stated in the introduction.

Corollary 6.2. *Let $g \in \mathbb{Z}_{\geq 1}$ and let $Z \subset \mathcal{M}_g$ be an irreducible subvariety which contains the hyperelliptic locus. Let X be a curve that corresponds to a very general point in Z . If for some $k, n \geq 1$, there is an isomorphism of unpolarized abelian varieties $\prod_{i=1}^n JC_i \simeq (JX)^k$ for some smooth projective connected curves C_i of positive genus, then $k = n$ and $C_i \simeq X$ for each i .*

Proof. Let $\bar{Z} \subset \mathcal{A}_g$ be the closure of Z in \mathcal{A}_g . By assumption \bar{Z} contains the hyperelliptic locus. Since the Jacobian JX of a very general hyperelliptic curve X satisfies $\text{End}(JX) = \mathbb{Z}$, we conclude via specialization that $\text{End}(A) = \mathbb{Z}$ for any very general point $[(A, \Theta)] \in \bar{Z}$. Moreover, there are hyperelliptic compact type curves X_0 with $JX_0 \simeq E_0 \times B_0$ for an elliptic curve E_0 with transcendental j -invariant. Since \bar{Z} is closed in \mathcal{A}_g and contains the hyperelliptic locus, the point $[JX_0, \Theta_{X_0}]$ is contained in \bar{Z} and so Theorem 6.1 applies to the subvariety \bar{Z} of \mathcal{A}_g . Hence, for each i we have $(JC_i, \Theta_{C_i}) \simeq (JX, \Theta_X)$ as polarized abelian varieties and this implies $C_i \simeq X$ by the Torelli theorem. This concludes the proof of the corollary. \square

Proof of Theorem 1.2. This follows directly from Corollary 6.2, by taking $Z = \mathcal{M}_g$ or $Z = \mathcal{H}_g$, the hyperelliptic locus in \mathcal{M}_g . \square

Another consequence of Theorem 6.1 is as follows.

Corollary 6.3. *Let Y be a very general cubic threefold and k a positive integer. There exist no smooth projective curves C_1, \dots, C_n such that $JC_1 \times \dots \times JC_n \simeq (J^3Y)^k$.*

Proof. By [Col82, Theorem (0.1)], we can degenerate Y into a singular cubic threefold Y_0 such that $J^3Y_0 = JX$ is the Jacobian of a very general hyperelliptic curve X of genus five. In particular, the closure of the locus of intermediate Jacobians of cubics inside \mathcal{A}_5 contains the locus of Jacobians of hyperelliptic curves. We can then argue as in Corollary 6.2 to deduce from Theorem 6.1 that (J^3Y, Θ_Y) is isomorphic to the Jacobian of a curve, which contradicts the main result of [CG72]. This concludes the proof of the corollary. \square

The remaining part of Section 6 will be devoted to a proof of Theorem 6.1.

6.2. Special subvarieties and powers of abelian varieties isomorphic to Jacobians.

Proof of Theorem 6.1. Recall that $Z \subset \mathcal{A}_g$ is an irreducible subvariety such that

- there is a point $[(A_0, \Theta_0)] \in Z$ such that $A_0 \simeq E_0 \times B_0$ and some elliptic curve E_0 with transcendental j -invariant;
- the very general point $[(A, \Theta)] \in Z$ satisfies $\text{End}(A) = \mathbb{Z}$.

We assume that for some $k, n \geq 1$, there are some smooth projective connected curves C_1, \dots, C_n of positive genus and an isomorphism $\prod_{i=1}^n JC_i \simeq A^k$ of unpolarized abelian varieties. By Lemma 4.3, we reduce to the case $n = 1$ and get an isomorphism

$$f: JC \xrightarrow{\simeq} A^k$$

for some $k \geq 1$, where $C := C_1$. We aim to prove that $k = 1$. Since $\text{End}(A) = \mathbb{Z}$, this already implies $(JC, \Theta_C) \simeq (A, \Theta)$ as polarized varieties, because A carries only one principal polarization, since $\text{NS}(A) = \mathbb{Z}$.

The above isomorphism $f: JC \rightarrow A^k$ provides A^k with an irreducible principal polarization, say λ . Since $\text{End}(A) = \mathbb{Z}$, λ corresponds to a positive definite symmetric unimodular bilinear form α on \mathbb{Z}^k , see Lemma 5.2. As λ is irreducible, (\mathbb{Z}^k, α) is indecomposable. Kneser's classification of indecomposable integral inner product spaces of rank at most 16 implies then that $k = 1$ or $k \geq 8$, see [MH73, p. 28, Remark 1] and [Kne57].

As explained above we only need to show $k = 1$ and so we assume for a contradiction that $k \geq 8$. By our assumptions there is a degeneration of A to $A_0 \simeq E_0 \times B_0$, where E_0 is an elliptic curve with transcendental j -invariant. This yields an isomorphism $(E_0 \times B_0)^k \simeq JC_0$ for some compact type degeneration C_0 of C . If $C_{0,1}, \dots, C_{0,n}$ are the non-rational irreducible components of C_0 , then we can write

$$(E_0 \times B_0)^k = E_0^k \times B_0^k \simeq JC_{0,1} \times \cdots \times JC_{0,n}.$$

This is an isomorphism of principally polarized abelian varieties, where the polarization on E_0^k and B_0^k is given by α . In particular, (E_0^k, α) is an irreducible principally polarized abelian variety. By uniqueness of the decomposition of any principally polarized abelian variety into a product of principally polarized abelian subvarieties [CG72; Deb96], it follows that for some i , there is an isomorphism of principally polarized abelian varieties $(JC_{0,i}, \Theta_{C_{0,i}}) \simeq (E_0^k, \alpha)$.

To simplify notation we write $E := E_0$ and $C := C_{0,i}$ and get an isomorphism $(JC, \Theta_C) \simeq (E^k, \alpha)$ of principally polarized abelian varieties. Since E has transcendental j -invariant, the above isomorphism spreads out to a one-dimensional family of isomorphisms which for any general elliptic curve E_t identifies (E_t^k, α) with the Jacobian of some smooth projective curve. We have a morphism of moduli stacks $\phi: \mathcal{A}_1 \rightarrow \mathcal{A}_k$ defined as $E_t \mapsto (E_t^k, \alpha)$.

We claim that the closure of the image of ϕ is a special subvariety of \mathcal{A}_k . Indeed, there is a universal polarized abelian scheme $\mathcal{E} \rightarrow \mathcal{A}_1$ which we can equip with the polarization α , as well as with the canonical principal polarization. The condition that the image of ϕ is a special subvariety of \mathcal{A}_k does not depend on the polarization of $\mathcal{E} \rightarrow \mathcal{A}_1$, see [MO13, Remark 3.13]. Thus, the claim follows from the well-known fact that for each integer $g \geq 1$, the image of the map $\mathcal{A}_g \rightarrow \mathcal{A}_{gk}$ defined as $A \mapsto A^k$ yields a special subvariety $Z \subset \mathcal{A}_{gk}$.

Let $Z_\phi \subset \mathcal{A}_k$ be the special subvariety given by the image of the map $\phi: \mathcal{A}_1 \rightarrow \mathcal{A}_k$ defined above. By assumption, the generic point $\eta \in Z_\phi$ is contained in the open Torelli locus $\mathcal{T}_k^0 \subset \mathcal{A}_k$, i.e. the locus of principally polarized abelian varieties of dimension k that are isomorphic to the Jacobian of a smooth projective curve. This implies that

$$Z_\phi \subset \mathcal{T}_k \subset \mathcal{A}_k,$$

where \mathcal{T}_k is the closure of \mathcal{T}_k^0 in \mathcal{A}_k , and $Z_\phi \cap \mathcal{T}_k^0 \neq \emptyset$.

It remains to show that since $k \geq 8$, this is impossible. For $k \geq 12$, this follows from [LZ19, Theorem A]. For $8 \leq k \leq 11$, the only indecomposable integral inner product space is, by Kneser's classification, given by the E_8 -lattice, see [MH73, p. 28, Remark 1] and [Kne57].

Hence we are reduced to the case $k = 8$ and we have $(E^8, \alpha) \simeq (JC, \Theta_C)$ for a smooth projective curve C of genus eight. The polarization α is induced by the E_8 -lattice. In particular, the automorphism group of (E^8, α) is isomorphic to $W(E_8)$, the Weyl group of type E_8 , hence $|\text{Aut}(E, \alpha)| = |W(E_8)| = 4! \cdot 6! \cdot 8! = 696\,729\,600$ by [Hum90, Section 2.12]. Since $(E^8, \alpha) \simeq (JC, \Theta_C)$ as principally polarized abelian varieties, the Torelli theorem implies $\text{Aut}(C) = W(E_8)$ or $\text{Aut}(C) \times \langle \pm 1 \rangle = W(E_8)$. This is impossible: the genus g of C is equal to eight, hence $|\text{Aut}(C)| \leq 84(g - 1) = 84 \cdot 7 = 588$. This contradiction concludes the proof of the theorem. \square

7. MODULES AND LATTICE THEORY

The goal of this section is to prove Lemmas 7.5 and 7.9 below. We consider unimodular symplectic lattices M and H such that $M \subset H^{\oplus k}$ as well as four matrices $\alpha_i \in M_k(\mathbb{Z})$ for $i = 1, 2, 3, 4$. In Lemma 7.5 we give sufficient conditions that guarantee the inclusions $\alpha_i H^{\oplus k} \subset M$, and in Lemma 7.9 we give sufficient conditions for the inclusions $\alpha_i H^{\oplus k} \subset M$ to be an equality. When combined, these lemmas provide a key technical step in the proof of our main theorem, which shall be provided in the next section, see Section 8.

7.1. Preliminary lemmas. We start by collecting three basic lemmas for future reference.

Lemma 7.1. *Let Λ be a free \mathbb{Z} -module of positive rank. Let $\alpha_1, \alpha_2 \in M_n(\mathbb{Z}) \cap \text{GL}_n(\mathbb{Q})$ for some $n \in \mathbb{Z}_{\geq 1}$. The following assertions are equivalent:*

- (1) *We have $\alpha_1 \cdot \Lambda^{\oplus n} = \alpha_2 \cdot \Lambda^{\oplus n}$, for the natural actions of the α_i on $\Lambda^{\oplus n}$.*
- (2) *There exists an invertible matrix $\gamma \in \text{GL}_n(\mathbb{Z})$ such that $\alpha_2 = \alpha_1 \gamma \in M_n(\mathbb{Z})$.*

Proof. Clearly, (2) implies (1). To prove the other implication, assume that (1) holds. Define $\gamma = \alpha_1^{-1} \alpha_2 \in \text{GL}_n(\mathbb{Q})$. One readily shows that $\gamma \in \text{GL}_n(\mathbb{Z})$. \square

Lemma 7.2. *Let M be a free \mathbb{Z} -module of finite rank and let $N \subset M$ be a saturated submodule. Let $\alpha \in M_k(\mathbb{Z}) \cap \text{GL}_k(\mathbb{Q})$ for some $k \in \mathbb{Z}_{\geq 1}$. Then $N^{\oplus k} \cap (\alpha \cdot M^{\oplus k}) = \alpha \cdot N^{\oplus k}$.*

Proof. We may assume that $N \neq M$. In particular, M/N is a non-zero free \mathbb{Z} -module. Moreover, the matrix $\alpha \in M_k(\mathbb{Z})$ induces an endomorphism

$$\alpha: (M/N)^{\oplus k} \rightarrow (M/N)^{\oplus k}. \quad (7.1)$$

As α has non-zero determinant, and $(M/N)^{\oplus k}$ is torsion-free, the endomorphism (7.1) is injective. Therefore, $N^{\oplus k} \cap (\alpha \cdot M^{\oplus k}) \subset \alpha \cdot N^{\oplus k}$. The other inclusion is clear. \square

Lemma 7.3. *Let Λ be a \mathbb{Z} -module. Let M_1, M_2 and N be submodules of Λ with $M_1 \subset M_2$. Suppose that $M_1 + N = M_2 + N$, and that $M_1 \cap N = M_2 \cap N$. Then $M_1 = M_2 \subset \Lambda$.*

Proof. It suffices to prove that $M_2 \subset M_1$. Let $x_2 \in M_2$. The hypotheses imply that there exists an element $z \in N$ such that $x_2 = x_1 + z$ for some $x_1 \in M_1$. As $M_1 \subset M_2$, we have $x_1 \in M_2$, hence $x_2 - x_1 = z \in M_2 \cap N = M_1 \cap N$. Therefore, $z \in M_1$, so that $x_2 = x_1 + z \in M_1$. \square

7.2. Matrices and saturated submodules. Consider finite free \mathbb{Z} -modules M and H such that $M \subset H^{\oplus k}$. For matrices $\alpha_i \in M_k(\mathbb{Z})$ ($i = 1, 2, 3, 4$), we would like to know whether $\alpha_i H^{\oplus k} \subset M$, assuming that this holds in certain subquotients of $H^{\oplus k}$. The goal of this section is to provide some sufficient conditions. The main result in this direction is Lemma 7.5 below.

Lemma 7.4. *Let H be a free \mathbb{Z} -module of finite rank. Let $W \subset H$ be a submodule, and let $V_i \subset H$ be saturated submodules for $i = 1, 2$, such that $V_1 \cap V_2 = 0$ and $V_1 \oplus V_2 \subsetneq W$. Let $M \subset H^{\oplus k}$ be a submodule such that for each $i \in \{1, 2\}$ and some $\alpha_i \in M_k(\mathbb{Z}) \cap \mathrm{GL}_k(\mathbb{Q})$, we have*

$$M \cap W^{\oplus k} \equiv \alpha_i \cdot W^{\oplus k} \pmod{V_i^{\oplus k}}. \quad (7.2)$$

Then the following holds:

- (1) There exists $\gamma \in \mathrm{GL}_k(\mathbb{Z})$ such that $\alpha_2 = \alpha_1 \gamma$.
- (2) We have

$$M \cap W^{\oplus k} = \alpha_1 \cdot W^{\oplus k} = \alpha_2 \cdot W^{\oplus k}. \quad (7.3)$$

Proof. Replacing M by $M \cap W^{\oplus k}$ and H by W , we may assume that $M \subset W^{\oplus k} = H^{\oplus k}$. Notice that

$$M \equiv \alpha_1 \cdot W^{\oplus k} \equiv \alpha_2 \cdot W^{\oplus k} \pmod{V_1^{\oplus k} \oplus V_2^{\oplus k}}. \quad (7.4)$$

As $(W/(V_1 \oplus V_2))^{\oplus k}$ is torsion-free, item (1) follows from Lemma 7.1 because $V_1 \oplus V_2 \neq W$.

Next, we claim that

$$M \subset \alpha \cdot W^{\oplus k}, \quad \text{where} \quad \alpha := \alpha_1. \quad (7.5)$$

To prove this, let $x \in M \subset W^{\oplus k}$. By (7.2), we can write $x = \alpha \cdot w_1 + v_1 = \alpha \cdot w_2 + v_2 \in W^{\oplus k}$ with $w_1, w_2 \in W^{\oplus k}$ and $v_i \in V_i^{\oplus k}$. Therefore, we have

$$\alpha \cdot (w_1 - w_2) = -v_1 + v_2 \in \alpha \cdot W^{\oplus k} \cap (V_1^{\oplus k} \oplus V_2^{\oplus k}) = \alpha (V_1^{\oplus k} \oplus V_2^{\oplus k}) = \alpha V_1^{\oplus k} \oplus \alpha V_2^{\oplus k},$$

where we used Lemma 7.2, which applies because $V_1^{\oplus k} \oplus V_2^{\oplus k}$ is saturated in $H^{\oplus k}$ by assumption. In particular, $v_1 \in \alpha V_1^{\oplus k}$ and $v_2 \in \alpha V_2^{\oplus k}$. Thus, we have $x = \alpha w_1 + v_1 \in \alpha W^{\oplus k}$. This proves the inclusion (7.5).

Furthermore, we claim that

$$\alpha \cdot V_1^{\oplus k} \subset M \cap V_1^{\oplus k}. \quad (7.6)$$

To prove this, notice that $\alpha \cdot V_1^{\oplus k} \subset \alpha \cdot W^{\oplus k} \equiv M \cap W^{\oplus k} \pmod{V_2^{\oplus k}}$, where we use the assumption $V_1 \subset W$ for the inclusion and (7.2) for the congruence. As $\alpha \cdot V_1^{\oplus k} \subset V_1^{\oplus k}$, we obtain

$$\alpha \cdot V_1^{\oplus k} \subset M \cap V_1^{\oplus k} \pmod{V_2^{\oplus k}}. \quad (7.7)$$

Notice that (7.7) implies (7.6), because $V_1 \cap V_2 = 0$. Our claim is proved. In a similar way (or by symmetry), one proves that $\alpha \cdot V_2^{\oplus k} \subset M \cap V_2^{\oplus k} \subset M$.

As $V_1 \cap V_2 = 0$, it follows that $\alpha \cdot V_1^{\oplus k} \oplus \alpha \cdot V_2^{\oplus k} \subset M$. Via Lemma 7.2, we thus obtain:

$$\alpha \cdot W^{\oplus k} \cap (V_1^{\oplus k} \oplus V_2^{\oplus k}) = \alpha \cdot (V_1^{\oplus k} \oplus V_2^{\oplus k}) \subset M \cap (V_1^{\oplus k} \oplus V_2^{\oplus k}). \quad (7.8)$$

By (7.5), we have $M \subset \alpha \cdot W^{\oplus k}$, so that

$$M \cap (V_1^{\oplus k} \oplus V_2^{\oplus k}) \subset \alpha \cdot W^{\oplus k} \cap (V_1^{\oplus k} \oplus V_2^{\oplus k}). \quad (7.9)$$

Combining (7.8) and (7.9), we see that

$$M \cap (V_1^{\oplus k} \oplus V_2^{\oplus k}) = \alpha \cdot W^{\oplus k} \cap (V_1^{\oplus k} \oplus V_2^{\oplus k}). \quad (7.10)$$

We are now in position to apply Lemma 7.3 to the following \mathbb{Z} -modules: let $\Lambda := W^{\oplus k}$, $M_1 := M \subset \Lambda$, $M_2 := \alpha \cdot W^{\oplus k} \subset \Lambda$, and $N := V_1^{\oplus k} \oplus V_2^{\oplus k}$. Observe that (7.4), (7.5) and (7.10) imply respectively that $M_1 + N = M_2 + N$, $M_1 \subset M_2$, and $M_1 \cap N = M_2 \cap N$. Therefore, by Lemma 7.3, we have $M_1 = M_2$. That is, $M = \alpha \cdot W^{\oplus k}$, and the lemma follows. \square

Lemma 7.5. *Let H be a free \mathbb{Z} -module of finite rank. Assume that, for each $i \in \{1, 2, 3, 4\}$, there exists a sequence of free submodules*

$$V_i \subset W_i \subset H,$$

such that the following properties are satisfied:

- (1) *The natural map $V_1 \oplus V_2 \oplus V_3 \oplus V_4 \rightarrow H$ is an embedding with torsion-free cokernel.*
- (2) *We have $(W_1 \cap W_2) + (W_3 \cap W_4) = H$.*
- (3) *The intersections*

$$W_1 \cap W_2, \quad W_3 \cap W_4, \quad \text{and} \quad W_1 \cap W_2 \cap W_3 \cap W_4$$

are non-zero and saturated in H , and $W_1 \cap W_2$ (resp. $W_3 \cap W_4$, resp. $W_1 \cap W_2 \cap W_3 \cap W_4$) strictly contains $V_1 \oplus V_2$ (resp. $V_3 \oplus V_4$, resp. $V_1 \oplus V_2 \oplus V_3 \oplus V_4$).

Let $M \subset H^{\oplus k}$ be a submodule of $H^{\oplus k}$ for some $k \geq 1$ that satisfies the following condition: for each $i \in \{1, 2, 3, 4\}$ there exists a matrix $\alpha_i \in M_k(\mathbb{Z})$ with non-zero determinant, such that

$$M \cap W_i^{\oplus k} \equiv \alpha_i \cdot W_i^{\oplus k} \pmod{V_i^{\oplus k}}. \quad (7.11)$$

Then for each $i, j \in \{1, 2, 3, 4\}$, we have:

$$\alpha_i \cdot H^{\oplus k} = \alpha_j \cdot H^{\oplus k} \subset M. \quad (7.12)$$

Proof. We claim that the following equality holds:

$$\alpha_1 \cdot ((W_1 \cap W_2)/(V_1 \oplus V_2))^{\oplus k} = \alpha_2 \cdot ((W_1 \cap W_2)/(V_1 \oplus V_2))^{\oplus k}. \quad (7.13)$$

To prove this, first observe that $(W_1 \cap W_2)/(V_1 \oplus V_2)$ is saturated in $W_i/(V_1 \oplus V_2)$ for $i = 1, 2$, and that both modules are non-zero and torsion-free. Indeed, by condition (1), $V_1 \oplus V_2$ is saturated in H and hence in any submodule that contains it; by condition (3), we have $V_1 \oplus V_2 \subsetneq W_1 \cap W_2 \subset W_i$ and hence the quotients $(W_1 \cap W_2)/(V_1 \oplus V_2)$ and $W_i/(V_1 \oplus V_2)$ are non-zero and torsion-free. The saturation of $(W_1 \cap W_2)/(V_1 \oplus V_2) \subset W_i/(V_1 \oplus V_2)$ follows from

the saturation of $W_1 \cap W_2 \subset W_i$, which holds because of condition (3). In view of Lemma 7.2, we deduce that

$$\alpha_1 \cdot (W_1^{\oplus k} \cap W_2^{\oplus k}) \equiv (\alpha_1 \cdot W_1^{\oplus k}) \cap W_2^{\oplus k} \pmod{V_1^{\oplus k} \oplus V_2^{\oplus k}}, \quad (7.14)$$

$$W_1^{\oplus k} \cap (\alpha_2 \cdot W_2^{\oplus k}) \equiv \alpha_2 \cdot (W_1^{\oplus k} \cap W_2^{\oplus k}) \pmod{V_1^{\oplus k} \oplus V_2^{\oplus k}}. \quad (7.15)$$

Moreover, because of (7.11), we have:

$$\begin{aligned} (\alpha_1 \cdot W_1^{\oplus k}) \cap W_2^{\oplus k} &\equiv (M \cap W_1^{\oplus k}) \cap W_2^{\oplus k} \equiv W_1^{\oplus k} \cap (M \cap W_2^{\oplus k}) \\ &\equiv W_1^{\oplus k} \cap (\alpha_2 \cdot W_2^{\oplus k}) \pmod{V_1^{\oplus k} \oplus V_2^{\oplus k}}. \end{aligned} \quad (7.16)$$

Taken together, (7.14), (7.15) and (7.16) imply (7.13), proving the claim.

From Lemma 7.1 and equation (7.13), we conclude that there exists $\gamma_{12} \in \mathrm{GL}_k(\mathbb{Z})$ such that $\alpha_2 = \alpha_1 \gamma_{12} \in M_k(\mathbb{Z})$. By symmetry, there exists a matrix $\gamma_{34} \in \mathrm{GL}_k(\mathbb{Z})$ such that $\alpha_4 = \alpha_3 \gamma_{34}$. Replacing α_2 by $\alpha_2 \gamma_{12}^{-1} = \alpha_1$ and α_4 by $\alpha_4 \gamma_{34}^{-1} = \alpha_3$, we may (and will) assume that $\alpha_1 = \alpha_2$ and $\alpha_3 = \alpha_4$. Consider, for $i = 1, 2, 3, 4$, the inclusions

$$V_1 \oplus V_2 \oplus V_3 \oplus V_4 \subset W_1 \cap W_2 \cap W_3 \cap W_4 \subset W_i.$$

Both inclusions are saturated by conditions (1) and (3). Consequently, for each i , the inclusion

$$(W_1 \cap W_2 \cap W_3 \cap W_4) / (V_1 \oplus V_2 \oplus V_3 \oplus V_4) \subset W_i / (V_1 \oplus V_2 \oplus V_3 \oplus V_4)$$

is a saturated embedding of free \mathbb{Z} -modules, which are non-zero in view of condition (3).

We obtain the following congruences:

$$\begin{aligned} \alpha_1 \cdot (W_1 \cap W_2 \cap W_3 \cap W_4)^{\oplus k} &\equiv (\alpha_1 \cdot W_1^{\oplus k}) \cap (W_1 \cap W_2 \cap W_3 \cap W_4)^{\oplus k} \\ &\equiv (M \cap W_1^{\oplus k}) \cap W_2^{\oplus k} \cap W_3^{\oplus k} \cap W_4^{\oplus k} \\ &\equiv W_1^{\oplus k} \cap W_2^{\oplus k} \cap (M \cap W_3^{\oplus k}) \cap W_4^{\oplus k} \\ &\equiv W_1^{\oplus k} \cap W_2^{\oplus k} \cap (\alpha_3 \cdot W_3^{\oplus k}) \cap W_4^{\oplus k} \\ &\equiv \alpha_3 \cdot (W_1 \cap W_2 \cap W_3 \cap W_4)^{\oplus k} \pmod{V_1^{\oplus k} \oplus V_2^{\oplus k} \oplus V_3^{\oplus k} \oplus V_4^{\oplus k}}. \end{aligned}$$

Here, the first congruence follows from Lemma 7.2, the second from equation (7.11), the third congruence is clear, the fourth congruence follows from (7.11) again and the final congruence from Lemma 7.2 again.

Therefore, by Lemma 7.1 using that $(W_1 \cap W_2 \cap W_3 \cap W_4) / (V_1 \oplus V_2 \oplus V_3 \oplus V_4)$ is a non-zero free \mathbb{Z} -module, there exists an invertible matrix $\gamma \in \mathrm{GL}_k(\mathbb{Z})$ such that $\alpha_3 = \alpha_1 \gamma$. Let $\alpha = \alpha_1$. Replacing α_3 by $\alpha_3 \gamma^{-1} = \alpha_1$, we may and do assume that

$$\alpha = \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4. \quad (7.17)$$

By (7.11) and (7.17), we get:

$$M \cap W_i^{\oplus k} \equiv \alpha \cdot W_i^{\oplus k} \pmod{V_i^{\oplus k}} \quad \forall i \in \{1, 2\}. \quad (7.18)$$

Combining (7.18) with Lemma 7.2, using that, for $i = 1, 2$, $(W_1 \cap W_2)/(V_1 \oplus V_2) \subset W_i/(V_1 \oplus V_2)$ is a saturated inclusion of free \mathbb{Z} -modules (see conditions (1) and (3)), we obtain:

$$M \cap (W_1 \cap W_2)^{\oplus k} \equiv \alpha \cdot (W_1 \cap W_2)^{\oplus k} \pmod{V_i^{\oplus k}} \quad \forall i \in \{1, 2\}. \quad (7.19)$$

By Lemma 7.4 and the fact that $V_1 \cap V_2 = 0$, it follows from (7.19) that

$$M \cap (W_1 \cap W_2)^{\oplus k} = \alpha \cdot (W_1 \cap W_2)^{\oplus k}.$$

By symmetry, we obtain $M \cap (W_3 \cap W_4)^{\oplus k} = \alpha \cdot (W_3 \cap W_4)^{\oplus k}$.

Finally, as we have an equality $H^{\oplus k} = (W_1 \cap W_2)^{\oplus k} + (W_3 \cap W_4)^{\oplus k}$, see condition (2), we conclude that

$$\alpha \cdot H^{\oplus k} = \alpha \cdot (W_1 \cap W_2)^{\oplus k} + \alpha \cdot (W_3 \cap W_4)^{\oplus k} = M \cap (W_1 \cap W_2)^{\oplus k} + M \cap (W_3 \cap W_4)^{\oplus k} \subset M.$$

In other words, the inclusion (7.12) is proved, and we are done. \square

7.3. Sublattices of powers of unimodular symplectic lattices. We continue by investigating sublattices $M \subset H^{\oplus k}$ of some power of a unimodular symplectic lattice H . This section is independent of Section 7.2. The main result of this section is Lemma 7.9.

Lemma 7.6. *Let (H, E_H) be a unimodular symplectic lattice. Let $U \subset H$ be a saturated isotropic subspace. Then there is a saturated subspace $U' \subset H$ such that $U \oplus U' \subset H$ is a unimodular subspace.*

Proof. We do induction on the rank of U . If U has rank one, then $U = \langle u \rangle$. The map $E_H(u, -) : H \rightarrow \mathbb{Z}$ is surjective because u is indivisible and E_H is unimodular. Hence there is a class $u' \in H$ with $E_H(u, u') = 1$, as we want.

If U has rank $r \geq 2$, then we pick a saturated subspace $U_1 \subset U$ of rank $r - 1$ and apply the induction hypothesis to U_1 to get a unimodular subspace $U_1 \oplus U'_1 \subset H$. Any unimodular subspace of a unimodular lattice admits a unimodular complement. Hence, we can extend a basis of U_1 to a symplectic basis of H . It follows that there is a symplectic basis

$$\{e_1, \dots, e_g, f_1, \dots, f_g\}$$

of H with $U = \langle e_1, \dots, e_{r-1} \rangle \oplus \langle u \rangle$ for some primitive element $u \in \langle e_r, \dots, e_g, f_r, \dots, f_g \rangle$. Applying the $r = 1$ case of the lemma to the subspace $\langle u \rangle \subset \langle e_r, \dots, e_g, f_r, \dots, f_g \rangle$, we get an element $u' \in \langle e_r, \dots, e_g, f_r, \dots, f_g \rangle$ such that

$$\langle u, u' \rangle \subset \langle e_r, \dots, e_g, f_r, \dots, f_g \rangle$$

is a unimodular subspace. This subspace has a complement and hence up to change of basis we can assume $u = e_r$ and $u' = f_r$. At this point the lemma is clear. \square

Lemma 7.7. *Let (H, E_H) be a unimodular symplectic lattice of rank $2g$. Let $k \geq 1$ and consider the induced unimodular symplectic lattice $(H^{\oplus k}, E_H^{\oplus k})$. Let $M \subset H^{\oplus k}$ be a sublattice such that there is a matrix $\beta \in M_k(\mathbb{Z})$ with positive determinant such that the intersection form*

$$E_M(-, -) := E_H^{\oplus k}(\beta^{-1} -, -)$$

is unimodular and integral on M . Assume that there is a matrix $\alpha \in M_k(\mathbb{Z})$ with nonzero determinant such that $\alpha H^{\oplus k} \subset M$. Let $[H^{\oplus k}: M]$ be the index of M in $H^{\oplus k}$. Then

$$[H^{\oplus k}: M] = \det(\beta)^g \mid \det(\alpha)^{2g}.$$

Proof. The inclusions $\alpha H^{\oplus k} \subset M \subset H^{\oplus k}$, together with the fact that the index of $\alpha H^{\oplus k}$ in $H^{\oplus k}$ equals $\det(\alpha)^{2g}$, show that the index of M in $H^{\oplus k}$ divides $\det(\alpha)^{2g}$. Thus, it suffices to prove that $[H^{\oplus k}: M] = \det(\beta)^g$.

To prove this, note that E_M and ψ induce isomorphisms $M \simeq M^\vee$ and $H \simeq H^\vee$. In particular, we can dualize the inclusion $M \subset H^{\oplus k}$ to obtain an inclusion $H^{\oplus k} \subset M$, and, with respect to this embedding, the index of M in $H^{\oplus k}$ equals the index of $H^{\oplus k}$ in M . The fact that $E_H^{\oplus k}(\beta^{-1} \cdot, -)$ restricts to the unimodular pairing E_M on M implies that E_M restricts to $E_H^{\oplus k}(\beta \cdot, -)$ on $H^{\oplus k} \subset M$. This means precisely that the composition

$$H^{\oplus k} \hookrightarrow M \xrightarrow{\simeq} M^\vee \hookrightarrow (H^{\oplus k})^\vee = (H^\vee)^{\oplus k} \quad (7.20)$$

is given by the map

$$x \mapsto \left(y \mapsto E_H^{\oplus k}(\beta \cdot x, y) \right).$$

Consequently, by identifying M and $H^{\oplus k}$ with their respective duals M^\vee and $(H^{\oplus k})^\vee$ (via E_M and $E_H^{\oplus k}$) in (7.20), it follows that the composition

$$H^{\oplus k} \hookrightarrow M \hookrightarrow H^{\oplus k}$$

is given by multiplication by the matrix β . In particular, if $[H^{\oplus k}: M]$ denotes the index of M in $H^{\oplus k}$ and $[H^{\oplus k}: \beta \cdot H^{\oplus k}]$ denotes the index of $\beta \cdot H^{\oplus k}$ in $H^{\oplus k}$, then

$$[H^{\oplus k}: M]^2 = [H^{\oplus k}: \beta \cdot H^{\oplus k}] = \det(\beta)^{2g}.$$

As $\det(\beta)$ is positive, we get $[H^{\oplus k}: M] = \det(\beta)^g$ as desired. \square

Lemma 7.8. *Let Λ be a free \mathbb{Z} -module and put $\Lambda_{\mathbb{Q}} = \Lambda \otimes \mathbb{Q}$. Let $\psi: \Lambda_{\mathbb{Q}} \times \Lambda_{\mathbb{Q}} \rightarrow \mathbb{Q}$ be a bilinear form that takes integral values on $\Lambda \times \Lambda$. Let $\alpha_i, \beta_i \in M_k(\mathbb{Z})$ ($i = 1, 2$) be matrices with non-zero determinant. Let $e_1, \dots, e_k, f_1, \dots, f_k \in \Lambda$. For $\gamma = (\gamma_{ij}) \in M_k(\mathbb{Q})$, define $\gamma e_i = \sum_j \gamma_{ji} e_j$ and $\gamma f_i = \sum_j \gamma_{ji} f_j$, and assume $\psi(\beta_1^{-1} \alpha_1 e_i, \beta_2^{-1} \alpha_2 f_j) = \delta_{ij}$ for all $i, j \in \{1, \dots, k\}$.*

Then $\det(\alpha_1) \det(\alpha_2) \mid \det(\beta_1) \det(\beta_2)$.

Proof. Let $\{h_1, \dots, h_g\}$ be the canonical basis of $\mathbb{Z}^{\oplus k}$, and define a bilinear form

$$\Psi: \mathbb{Q}^{\oplus k} \times \mathbb{Q}^{\oplus k} \rightarrow \mathbb{Q} \quad \text{by} \quad \Psi(h_i, h_j) = \psi(e_i, f_j).$$

Then Ψ takes integral values on $\mathbb{Z}^{\oplus k} \times \mathbb{Z}^{\oplus k}$. Let $Q = (Q_{ij})$ be the $k \times k$ -matrix with entries $Q_{ij} = \Psi(h_i, h_j)$. Then $Q \in M_k(\mathbb{Z})$, and for $x, y \in \mathbb{Q}^{\oplus k}$, we have

$$\Psi(x, y) = x^\top Q y.$$

Moreover, $\Psi(\beta_1^{-1} \alpha_1 h_i, \beta_2^{-1} \alpha_2 h_j) = \psi(\beta_1^{-1} \alpha_1 e_i, \beta_2^{-1} \alpha_2 f_j) = \delta_{ij}$. Therefore,

$$\delta_{ij} = \Psi(\beta_1^{-1} \alpha_1 h_i, \beta_2^{-1} \alpha_2 h_j) = (\beta_1^{-1} \alpha_1 h_i)^\top \cdot Q \cdot (\beta_2^{-1} \alpha_2 h_j) = h_i^\top \left(\alpha_1^\top \beta_1^{-\top} Q \beta_2^{-1} \alpha_2 \right) h_j.$$

Hence, the matrix $\alpha_1^\top \beta_1^{-\top} Q \beta_2^{-1} \alpha_2 \in M_k(\mathbb{Q})$ is the identity matrix. In particular,

$$\det(\alpha_1^\top \beta_1^{-\top} Q \beta_2^{-1} \alpha_2) = \det(\alpha_1) \det(\alpha_2) \det(\beta_1)^{-1} \det(\beta_2)^{-1} \det(Q) = 1,$$

which implies that $\det(\alpha_1) \det(\alpha_2) \det(Q) = \det(\beta_1) \det(\beta_2)$. We have $\det(Q) \in \mathbb{Z}$ because $Q \in M_k(\mathbb{Z})$, and the lemma follows. \square

We come to the main result of Section 7.3, which is the following lemma. A proof of a simpler version of this lemma is contained in the appendix, see Proposition A.1.

Lemma 7.9. *Let (H, E_H) be a unimodular symplectic lattice of rank $2g$. Let $k \geq 1$ and consider the unimodular lattice $(H, E_H)^{\oplus k} = (H^{\oplus k}, E_H^{\oplus k})$ with symplectic basis*

$$\{e_{1i}, \dots, e_{gi}, f_{1i}, \dots, f_{gi}\}, \quad i = 1, \dots, k.$$

Let

$$H'_1 := \bigoplus_{i=1}^k \langle e_{1i}, \dots, e_{gi}, f_{2i}, f_{3i}, \dots, f_{gi} \rangle, \quad H'_2 := \bigoplus_{i=1}^k \langle e_{1i}, \dots, e_{gi}, f_{1i}, f_{3i}, \dots, f_{gi} \rangle,$$

and define

$$V_1 := \bigoplus_{i=1}^k \langle e_{1i} \rangle, \quad V_2 := \bigoplus_{i=1}^k \langle e_{2i} \rangle.$$

Let $M \subset H^{\oplus k}$ be a sublattice such that there is a matrix $\beta \in M_k(\mathbb{Z})$ with positive determinant such that the intersection form

$$E_M(-, -) := E_H^{\oplus k}(\beta^{-1} -, -)$$

is unimodular and integral on M . Assume that there are matrices $\alpha_1, \alpha_2 \in M_k(\mathbb{Z})$ with nonzero determinant such that $\alpha_i H^{\oplus k} \subset M$ for $i = 1, 2$, and such that

$$M \cap H'_i \equiv \alpha_i H'_i \pmod{V_i}, \quad i = 1, 2. \quad (7.21)$$

Then there exists $\gamma \in \mathrm{GL}_k(\mathbb{Z})$ such that $\alpha_2 = \alpha_1 \gamma$, and we have $(\det \alpha_i)^2 = \det \beta$ for $i = 1, 2$. Furthermore, we have $M = \alpha_i H^{\oplus k}$ for $i = 1, 2$.

Proof. As a first step, we aim to show that there exists $\gamma \in \mathrm{GL}_k(\mathbb{Z})$ such that $\alpha_2 = \alpha_1 \gamma$. To prove this, notice that by Lemma 7.2, we have, for $i = 1, 2$:

$$(\alpha_i \cdot H'_1) \cap H'_2 = \alpha_i \cdot (H'_1 \cap H'_2) = H'_1 \cap (\alpha_i \cdot H'_2). \quad (7.22)$$

Moreover, $V_1 \oplus V_2 \subset H'_1 \cap H'_2$. Modulo $V_1 \oplus V_2$, the equality (7.22) combined with (7.21) gives

$$\begin{aligned} \alpha_1 \cdot (H'_1 \cap H'_2) &\equiv (\alpha_1 \cdot H'_1) \cap H'_2 \equiv (M \cap H'_1) \cap H'_2 \equiv H'_1 \cap (M \cap H'_2) \\ &\equiv H'_1 \cap (\alpha_2 \cdot H'_2) \equiv \alpha_2 \cdot (H'_1 \cap H'_2) \pmod{V_1 \oplus V_2}. \end{aligned} \quad (7.23)$$

Remark that $V_1 \oplus V_2 \subset H'_1 \cap H'_2$ is saturated in $H'_1 \cap H'_2$. Thus, by Lemma 7.1, equation (7.23) implies that there exists $\gamma \in \mathrm{GL}_k(\mathbb{Z})$ such that $\alpha_2 = \alpha_1 \gamma$, as we want.

As a second step, we let

$$U := \alpha \cdot \langle e_1, e_2 \rangle^{\oplus k} \subset M, \quad \alpha := \alpha_1 \in M_k(\mathbb{Z}),$$

and claim that U is saturated in M . To prove this, let $x \in M$ and assume that $rx \in U$ for some $r \in \mathbb{Z}_{\geq 1}$. We need to show that $x \in U$. As $rx \in U$, we have $x \in \langle e_1, e_2 \rangle^{\oplus k}$. In particular, $x \in H'_i$ for $i = 1, 2$, hence $x \in M \cap H'_1 \cap H'_2$. By (7.21), this means that we can write

$$\begin{aligned} x &= \alpha h'_1 + v_1, & \text{for some } h'_1 \in H'_1 & \text{ and } v_1 \in V_1, \\ x &= \alpha h'_2 + v_2, & \text{for some } h'_2 \in H'_2 & \text{ and } v_2 \in V_2. \end{aligned}$$

As $x \in \langle e_1, e_2 \rangle^{\oplus k}$, we get that $h'_i \in \langle e_1, e_2 \rangle^{\oplus k}$ and hence

$$\alpha(h'_1 - h'_2) = -v_1 + v_2 \in \langle e_1, e_2 \rangle^{\oplus k} = V_1 \oplus V_2.$$

As the action of α respects the decomposition $V_1 \oplus V_2$, this implies $v_1 \in \alpha V_1$ and $v_2 \in \alpha V_2$. Therefore, $x = \alpha h'_1 + v_1 \in \alpha \langle e_1, e_2 \rangle^{\oplus k} = U$, proving our claim that U is saturated in M .

Next, observe that the subspace U of M is isotropic. As we have just proved that $U \subset M$ is saturated in M , we conclude from Lemma 7.6 that there are classes $g_{1i}, g_{2i} \in M$ such that

$$U \oplus \bigoplus_{i=1}^k \langle g_{1i}, g_{2i} \rangle$$

is a unimodular sublattice of M . More precisely, we can assume that the g_{ai} ($a \in \{1, 2\}$) are chosen such that $E_M(\alpha e_{ai}, g_{bj}) = \delta_{ab} \delta_{ij}$. Hence

$$E_H^{\oplus k}(\beta^{-1} \alpha e_{ai}, g_{bj}) = \delta_{ab} \delta_{ij}. \quad (7.24)$$

Since the e_{rs}, f_{rs} form a basis of $H^{\oplus k}$, we can write

$$g_{ij} = \sum_{r,s} (a_{ijrs} e_{rs} + b_{ijrs} f_{rs})$$

for uniquely determined $a_{ijrs}, b_{ijrs} \in \mathbb{Z}$. By (7.24) we have

$$E_H^{\oplus k}(\beta^{-1} \alpha e_{1i}, g_{2j}) = 0 \quad \text{for all } i, j.$$

Since α and β have nonzero determinants, the classes $\beta^{-1} \alpha e_{1i}$ with $i = 1, \dots, k$ span rationally the space $\bigoplus_{i=1}^k \langle e_{1i} \rangle$ and so we find that g_{2j} contains the basis element f_{1i} trivially, that is, we have $b_{2j1s} = 0$ for all $j, s = 1, \dots, k$. This implies $g_{2j} \in M \cap H'_1$ for all $j = 1, \dots, k$. Hence, by (7.21), there exist classes $g'_{2j} \in H'_1$ such that, for all j , we have

$$g_{2j} = \alpha g'_{2j} + e'_j, \quad \text{for some } e'_j \in V_1. \quad (7.25)$$

By (7.24), we have

$$E_H^{\oplus k}(\beta^{-1} \alpha e_{2i}, g_{2j}) = \delta_{ij}. \quad (7.26)$$

Combining (7.25) and (7.26), and noticing that $E_H^{\oplus k}(\beta^{-1} \alpha e_{2i}, e'_j) = 0$, we get

$$\delta_{ij} = E_H^{\oplus k}(\beta^{-1} \alpha e_{2i}, g_{2j}) = E_H^{\oplus k}(\beta^{-1} \alpha e_{2i}, g_{2j} - e'_j) = E_H^{\oplus k}(\beta^{-1} \alpha e_{2i}, \alpha g'_{2j})$$

for all $i, j \in \{1, \dots, k\}$. By Lemma 7.8, this implies that $\det(\alpha)^2 \mid \det(\beta)$.

Conversely, by Lemma 7.7, we have that $\det(\beta)^g \mid \det(\alpha)^{2g}$. We conclude that $\det(\alpha)^{2g} = \det(\beta)^g$. As $\det(\beta)$ is positive, it follows that $\det(\beta) = \det(\alpha)^2$. Finally, by Lemma 7.7 again,

we have that $[H^{\oplus k}: M] = \det(\beta)^g$. Consequently, $[H^{\oplus k}: M] = \det(\alpha)^{2g} = [H^{\oplus k}: \alpha H^{\oplus k}]$. As $\alpha H^{\oplus k} \subset M$, we must have $M = \alpha H^{\oplus k}$. This finishes the proof of the lemma. \square

8. PROOF OF THE MAIN THEOREM

The goal of this section is to prove Theorem 1.1, the main result of this paper. The idea is to degenerate the given isogeny $J\mathcal{C} \rightarrow (J\mathcal{X})^k$ in four different directions; these are provided by Lemma 2.22. For each such a degeneration $J\mathcal{C}_{0i} \rightarrow (J\mathcal{X}_{0i})^k$ ($i \in \{1, 2, 3, 4\}$), we consider the induced map on the compact quotients, or equivalently the map $H^1(\tilde{C}_{0i}, \mathbb{Z}) \rightarrow H^1(\tilde{X}_{0i}, \mathbb{Z})^{\oplus k}$ between the cohomology groups of the normalizations \tilde{C}_{0i} and \tilde{X}_{0i} of \mathcal{C}_{0i} and \mathcal{X}_{0i} . We want to show that, for each i , the image of this map is $\alpha_i \cdot H^1(\tilde{X}_{0i}, \mathbb{Z})^{\oplus k} \subset H^1(\tilde{X}_{0i}, \mathbb{Z})^{\oplus k}$ for some $\alpha_i \in M_k(\mathbb{Z})$. We then combine these pieces of information obtained in the different degenerations to conclude that $\alpha_j = \alpha_i \gamma_{ij}$ for some $\gamma_{ij} \in \mathrm{GL}_k(\mathbb{Z})$, and that the image of $H^1(\mathcal{C}, \mathbb{Z}) \rightarrow H^1(\mathcal{X}, \mathbb{Z})^{\oplus k}$ is given by $\alpha_i \cdot H^1(\mathcal{X}, \mathbb{Z})^{\oplus k}$. In this last step, we use the linear algebra and lattice theory worked out in Section 7. More precisely, this is where Lemmas 7.5 and 7.9 enter the picture.

8.1. Degeneration of the isogeny: moving the extension class. In this section, we prove:

Proposition 8.1. *Let Δ be a connected normal complex analytic space, and let $p: \mathcal{X} \rightarrow \Delta$ be a family of semi-stable curves of arithmetic genus $g \geq 2$. Let $k \geq 1$ be an integer, consider a family of semi-stable curves $q: \mathcal{C} \rightarrow \Delta$, and assume there is an isogeny*

$$\psi: (J\mathcal{X})^k := J\mathcal{X} \times_{\Delta} \cdots \times_{\Delta} J\mathcal{X} \longrightarrow J\mathcal{C}$$

of semi-abelian schemes over Δ , where $J\mathcal{X}$ and $J\mathcal{C}$ denote the relative Jacobians over Δ . Assume that the following conditions are satisfied.

- (I) For all $s \in \Delta$, $X_s = p^{-1}(s)$ is an irreducible one-nodal hyperelliptic curve.
- (II) The induced moduli map $\Delta \rightarrow \overline{\mathcal{M}}_g$ is generically finite onto its image, where $\overline{\mathcal{M}}_g$ is the moduli stack of stable curves of genus g .
- (III) For any simultaneous normalization $\nu: \tilde{\mathcal{X}} \rightarrow \mathcal{X}$ of p , see Proposition 3.1, the map from Δ to the moduli stack of smooth hyperelliptic curves of genus $g-1$ induced by the family of hyperelliptic curves $p \circ \nu: \tilde{\mathcal{X}} \rightarrow \Delta$ is dominant with positive dimensional generic fibre.

Then for any general point $0 \in \Delta$, there is an isomorphism of abelian varieties

$$J\tilde{\mathcal{C}}_0 \simeq (J\tilde{X}_0)^k,$$

and the composition $(J\tilde{X}_0)^k \xrightarrow{\tilde{\psi}_0} J\tilde{\mathcal{C}}_0 \simeq (J\tilde{X}_0)^k$ is given by a matrix $\alpha \in M_k(\mathbb{Z})$. Here, $\tilde{\mathcal{C}}_0$ and \tilde{X}_0 are the normalizations of \mathcal{C}_0 and \mathcal{X}_0 , and $\tilde{\psi}_0: (J\tilde{X}_0)^k \rightarrow J\tilde{\mathcal{C}}_0$ is the isogeny induced by ψ .

Proof. Let \mathcal{H}_{g-1} be the moduli stack of smooth hyperelliptic curves of genus $g-1$. Let

$$f: \Delta \longrightarrow \mathcal{H}_{g-1}$$

be the map induced by a simultaneous normalization of $p: \mathcal{X} \rightarrow \Delta$, see Proposition 3.1. Then for 0 in a dense open subset of Δ , the closed analytic subset

$$H'_0 := f^{-1}(f(0)) \subset \Delta$$

is positive dimensional by condition (III). For each $u \in H'_0$, the normalization \widetilde{X}_u of X_u is isomorphic to \widetilde{X}_0 . Moreover, as the map $\Delta \rightarrow \overline{\mathcal{M}}_g$ is generically finite onto its image (see condition (II)), the same holds for the composition $H'_0 \hookrightarrow \Delta \rightarrow \overline{\mathcal{M}}_g$. In particular, for general $0 \in \Delta$ as above, there exists a connected normal complex analytic space H_0 and a dominant generically finite morphism

$$H_0 \longrightarrow H'_0 \quad (8.1)$$

with the following properties, where $p|_{H_0}: \mathcal{X}|_{H_0} \rightarrow H_0$ denotes the pull-back of p along (8.1), and $\widetilde{\mathcal{X}}|_{H_0} \rightarrow \mathcal{X}|_{H_0}$ the simultaneous normalization of $p|_{H_0}$ (see Proposition 3.1): there is an isomorphism

$$\widetilde{\mathcal{X}}|_{H_0} \simeq \widetilde{X}_0 \times H_0$$

of families of curves over H_0 , and a non-constant morphism $H_0 \rightarrow \widetilde{X}_0$, $u \mapsto x_u$, such that for any $u \in H_0$, the fibre X_u is obtained from \widetilde{X}_u by glueing the pair of points $(x_u, \iota(x_u))$, where ι denotes the hyperelliptic involution on \widetilde{X}_0 .

Let $\widetilde{C}_{0,i}$ for $i = 1, \dots, n$ be the non-rational irreducible components of the normalization \widetilde{C}_0 of the fibre $C_0 = q^{-1}(0)$. We apply Proposition 3.2 to the family $p|_{H_0}: \mathcal{X}|_{H_0} \rightarrow H_0$. This yields an integer $N \in \mathbb{Z}_{\geq 1}$ such that for each $i \in \{1, \dots, n\}$, there is a non-constant morphism

$$\widetilde{X}_0 \longrightarrow N \cdot (\widetilde{C}_{0,i} - \widetilde{C}_{0,i}).$$

By Proposition 2.6, this implies that the curve $\widetilde{C}_{0,i}$ is hyperelliptic for each i . Moreover, the assumption that $g = g(X_t) \geq 4$ implies that $g(\widetilde{X}_0) \geq 3$ for the genus $g(\widetilde{X}_0)$ of the \widetilde{X}_0 .

By condition (III), the curve \widetilde{X}_0 is a very general hyperelliptic curve of genus $g - 1 \geq 3$. Thus, by Theorem 4.2, the fact that there exists an isogeny

$$\widetilde{\psi}_0: (J\widetilde{X}_0)^k \longrightarrow J\widetilde{C}_{0,1} \times \cdots \times J\widetilde{C}_{0,n}$$

implies that for each $i \in \{1, \dots, n\}$ we have an isomorphism $\widetilde{C}_{0,i} \simeq \widetilde{X}_0$, and that the composition $(J\widetilde{X}_0)^k \rightarrow J\widetilde{C}_{0,1} \times \cdots \times J\widetilde{C}_{0,n} \simeq (J\widetilde{X}_0)^k$ is given by a matrix in $M_k(\mathbb{Z})$. \square

8.2. Degeneration of the isogeny: one-dimensional base. Let $D \subset \mathbb{C}$ be a sufficiently small open disc around $0 \in \mathbb{C}$. Let

$$p: \mathcal{X} \longrightarrow D \quad \text{and} \quad q: \mathcal{C} \longrightarrow D$$

be families of nodal curves over D , cf. Section 1.3. Let ψ be an isogeny

$$\psi: (J\mathcal{X})^k := J\mathcal{X} \times_D \cdots \times_D J\mathcal{X} \longrightarrow J\mathcal{C} \quad (8.2)$$

of semi-abelian schemes over D , where $J\mathcal{X}$ and $J\mathcal{C}$ denote the relative Jacobians over D . Assume that:

- (1) for each $s \in D^* = D - \{0\}$, the fibre $X_s = p^{-1}(s)$ is a curve of compact type;
- (2) there exists an isomorphism $J\widetilde{C}_0 \simeq (J\widetilde{X}_0)^k$ and a matrix $\alpha \in M_k(\mathbb{Z})$ such that the composition

$$(J\widetilde{X}_0)^k \xrightarrow{\widetilde{\psi}_0} J\widetilde{C}_0 \simeq (J\widetilde{X}_0)^k$$

is given by the multiplication by α .

Proposition 8.2. *Under the above assumptions, up to possibly shrinking D around 0, the following holds. The natural map $H^1(\mathcal{X}, \mathbb{Z}) \rightarrow H^1(X_0, \mathbb{Z})$ is an isomorphism. Let $t \in D^*$, and consider the natural embeddings*

$$H^1(X_0, \mathbb{Z}) \subset H^1(X_t, \mathbb{Z}) \quad \text{and} \quad H^1(C_t, \mathbb{Z}) \subset H^1(X_t, \mathbb{Z})^{\oplus k} \quad (8.3)$$

induced by the specialization map

$$sp_{\mathcal{X}}: H^1(X_0, \mathbb{Z}) \xleftarrow{\sim} H^1(\mathcal{X}, \mathbb{Z}) \longrightarrow H^1(X_t, \mathbb{Z}) \quad (8.4)$$

and the isogeny $\psi_t: (JX_t)^k \rightarrow JC_t$. With respect to these embeddings, we have the following identity in $H^1(X_t, \mathbb{Z})^{\oplus k}$ modulo $W_0H^1(X_0, \mathbb{Z})^{\oplus k}$:

$$H^1(C_t, \mathbb{Z}) \cap H^1(X_0, \mathbb{Z})^{\oplus k} \equiv \alpha \cdot H^1(X_0, \mathbb{Z})^{\oplus k} \pmod{W_0H^1(X_0, \mathbb{Z})^{\oplus k}}, \quad (8.5)$$

where $W_0H^1(X_0, \mathbb{Z}) = W_0H^1(X_0, \mathbb{Q}) \cap H^1(X_0, \mathbb{Z})$ and α is the matrix from (2) above.

Before we prove Proposition 8.2, which is the main result of this section, we need the following lemma. To state it, for $s \in D$, let

$$\psi_s^*: H^1(C_s, \mathbb{Z}) \longrightarrow H^1(X_s, \mathbb{Z})^{\oplus k}$$

denote the morphism on cohomology induced by the isogeny $\psi_s: (JX_s)^k \rightarrow JC_s$.

Lemma 8.3. *Up to possibly shrinking D around 0, we have the following equality:*

$$\begin{aligned} & \text{Im} \left(H^1(C_t, \mathbb{Z}) \xrightarrow{\psi_t^*} H^1(X_t, \mathbb{Z})^{\oplus k} \right) \cap \text{Im} \left(H^1(X_0, \mathbb{Z})^{\oplus k} \xrightarrow{sp_{\mathcal{X}}^{\oplus k}} H^1(X_t, \mathbb{Z})^{\oplus k} \right) \\ &= \text{Im} \left(H^1(C_0, \mathbb{Z}) \xrightarrow{\psi_0^*} H^1(X_0, \mathbb{Z})^{\oplus k} \xrightarrow{sp_{\mathcal{X}}^{\oplus k}} H^1(X_t, \mathbb{Z})^{\oplus k} \right). \end{aligned} \quad (8.6)$$

Here, $sp_{\mathcal{X}}$ is the specialization map defined in (8.4).

Assuming Lemma 8.3, we can prove Proposition 8.2 as follows.

Proof of Proposition 8.2. By Proposition 2.14, we may shrink D around 0 so that the natural map $H^1(\mathcal{X}, \mathbb{Z}) \rightarrow H^1(X_0, \mathbb{Z})$ is an isomorphism. By assumption (2), there is an isomorphism $H^1(\tilde{C}_0, \mathbb{Z}) \simeq H^1(\tilde{X}_0, \mathbb{Z})^{\oplus k}$ such that the following diagram commutes, and its rows are exact:

$$\begin{array}{ccccccc} 0 & \longrightarrow & W_0H^1(C_0, \mathbb{Z}) & \longrightarrow & H^1(C_0, \mathbb{Z}) & \longrightarrow & H^1(\tilde{C}_0, \mathbb{Z}) \simeq H^1(\tilde{X}_0, \mathbb{Z})^{\oplus k} \longrightarrow 0 \\ & & \downarrow \psi_0^* & & \downarrow \psi_0^* & & \downarrow \alpha \\ 0 & \longrightarrow & W_0H^1(X_0, \mathbb{Z})^{\oplus k} & \longrightarrow & H^1(X_0, \mathbb{Z})^{\oplus k} & \longrightarrow & H^1(\tilde{X}_0, \mathbb{Z})^{\oplus k} \longrightarrow 0. \end{array}$$

Here, $W_0H^1(C_0, \mathbb{Z}) = W_0H^1(C_0, \mathbb{Q}) \cap H^1(C_0, \mathbb{Z})$. By the above commutative diagram with exact rows, we get

$$\text{Im} \left(H^1(C_0, \mathbb{Z}) \xrightarrow{\psi_0^*} H^1(X_0, \mathbb{Z})^{\oplus k} \longrightarrow H^1(\tilde{X}_0, \mathbb{Z})^{\oplus k} \right) = \alpha \cdot H^1(\tilde{X}_0, \mathbb{Z})^{\oplus k}.$$

Consequently, we have

$$\text{Im} \left(H^1(C_0, \mathbb{Z}) \xrightarrow{\psi_0^*} H^1(X_0, \mathbb{Z})^{\oplus k} \right) \equiv \alpha \cdot H^1(X_0, \mathbb{Z})^{\oplus k} \pmod{W_0H^1(X_0, \mathbb{Z})^{\oplus k}}. \quad (8.7)$$

Combining (8.6) with (8.7), we obtain:

$$\begin{aligned} & \operatorname{Im} \left(H^1(C_t, \mathbb{Z}) \xrightarrow{\psi_t^*} H^1(X_t, \mathbb{Z})^{\oplus k} \right) \cap \operatorname{Im} \left(H^1(X_0, \mathbb{Z})^{\oplus k} \xrightarrow{sp_{\mathcal{X}}^{\oplus k}} H^1(X_t, \mathbb{Z})^{\oplus k} \right) \\ &= \operatorname{Im} \left(H^1(C_0, \mathbb{Z}) \xrightarrow{\psi_0^*} H^1(X_0, \mathbb{Z})^{\oplus k} \xrightarrow{sp_{\mathcal{X}}^{\oplus k}} H^1(X_t, \mathbb{Z})^{\oplus k} \right) \\ &\equiv \alpha \cdot \operatorname{Im} \left(H^1(X_0, \mathbb{Z}) \xrightarrow{sp_{\mathcal{X}}} H^1(X_t, \mathbb{Z}) \right)^{\oplus k} \quad \text{mod } W_0 H^1(X_0, \mathbb{Z})^{\oplus k}. \end{aligned}$$

Thus, (8.5) holds, and hence the proposition is proved. \square

It remains to prove Lemma 8.3.

Proof of Lemma 8.3. By the existence of the isogeny (8.2), the restriction $J\mathcal{C}|_{D^*} \rightarrow D^*$ is an abelian scheme over D^* . In particular, $q|_{D^*}: \mathcal{C}|_{D^*} \rightarrow D^*$ is a family of compact type curves. Thus, $R^1(q|_{D^*})_*\mathbb{Z}$ is a local system on D^* , and in fact a sub-local system of $(R^1(p|_{D^*})\mathbb{Z})^{\oplus k}$. Let $S \in \operatorname{Aut}(H^1(C_t, \mathbb{Z}))$ and $T \in \operatorname{Aut}(H^1(X_t, \mathbb{Z}))$ be generators of the monodromy groups attached to $R^1(q|_{D^*})_*\mathbb{Z}$ and $R^1(p|_{D^*})_*\mathbb{Z}$, such that

$$T^{\oplus k}|_{H^1(C_t, \mathbb{Z})} = S \in \operatorname{Aut}(H^1(C_t, \mathbb{Z}))$$

with respect to the inclusion $H^1(C_t, \mathbb{Z}) \subset H^1(X_t, \mathbb{Z})^{\oplus k}$ given by ψ_t^* . Then, we have:

$$\begin{aligned} & \operatorname{Im} \left(H^1(C_t, \mathbb{Z}) \xrightarrow{\psi_t^*} H^1(X_t, \mathbb{Z})^{\oplus k} \right) \cap \left(H^1(X_t, \mathbb{Z})^{\oplus k} \right)^{T^{\oplus k}} \\ &= \operatorname{Im} \left(H^1(C_t, \mathbb{Z})^S \xrightarrow{\psi_t^*} H^1(X_t, \mathbb{Z})^{\oplus k} \right). \end{aligned} \tag{8.8}$$

By Proposition 2.14, we may shrink D around 0 so that pulling back along the inclusions $X_0 \hookrightarrow \mathcal{X}$ and $C_0 \hookrightarrow \mathcal{C}$ yields isomorphisms $H^1(\mathcal{X}, \mathbb{Z}) \simeq H^1(X_0, \mathbb{Z})$ and $H^1(\mathcal{C}, \mathbb{Z}) \simeq H^1(C_0, \mathbb{Z})$. Define $sp_{\mathcal{X}}: H^1(X_0, \mathbb{Z}) \rightarrow H^1(\mathcal{X}, \mathbb{Z})$ as in (8.4), and define in a similar way

$$sp_{\mathcal{C}}: H^1(C_0, \mathbb{Z}) \rightarrow H^1(\mathcal{C}, \mathbb{Z}).$$

Thus, $sp_{\mathcal{C}}$ is the composition of the inverse of the restriction map $H^1(\mathcal{C}, \mathbb{Z}) \xrightarrow{\sim} H^1(C_0, \mathbb{Z})$ with the restriction map $H^1(\mathcal{C}, \mathbb{Z}) \rightarrow H^1(C_t, \mathbb{Z})$. Then Lemma 2.21 applied to $p: \mathcal{X} \rightarrow D$ and to $q: \mathcal{C} \rightarrow D$ implies that, possibly after further shrinking D around 0, we have:

$$\begin{aligned} & \operatorname{Im} \left(sp_{\mathcal{X}}^{\oplus k}: H^1(X_0, \mathbb{Z})^{\oplus k} \rightarrow H^1(X_t, \mathbb{Z})^{\oplus k} \right) = (H^1(X_t, \mathbb{Z})^{\oplus k})^{T^{\oplus k}} \subset H^1(X_t, \mathbb{Z})^{\oplus k}, \\ & \operatorname{Im} (sp_{\mathcal{C}}: H^1(C_0, \mathbb{Z}) \rightarrow H^1(C_t, \mathbb{Z})) = H^1(C_t, \mathbb{Z})^S \subset H^1(C_t, \mathbb{Z}). \end{aligned} \tag{8.9}$$

Claim 8.2.1. We have $\psi_t^* \circ sp_{\mathcal{C}} = sp_{\mathcal{X}}^{\oplus k} \circ \psi_0^*$ as maps $H^1(C_0, \mathbb{Z}) \rightarrow H^1(X_t, \mathbb{Z})^{\oplus k}$.

Proof of Claim 8.2.1. The Leray spectral sequence provides canonical morphisms $H^1(\mathcal{X}, \mathbb{Z}) \rightarrow H^0(D, R^1p_*\mathbb{Z})$ and $H^1(\mathcal{C}, \mathbb{Z}) \rightarrow H^0(D, R^1q_*\mathbb{Z})$. These make the following diagram commute:

$$\begin{array}{ccccccc} H^1(C_0, \mathbb{Z}) & \xleftarrow{\sim} & H^1(\mathcal{C}, \mathbb{Z}) & \longrightarrow & H^0(D, R^1g_*\mathbb{Z}) & \longrightarrow & H^1(C_t, \mathbb{Z}) \\ \downarrow \psi_0^* & & & & \downarrow & & \downarrow \psi_t^* \\ H^1(X_0, \mathbb{Z})^{\oplus k} & \xleftarrow{\sim} & H^1(\mathcal{X}, \mathbb{Z})^{\oplus k} & \longrightarrow & H^0(D, R^1f_*\mathbb{Z})^{\oplus k} & \longrightarrow & H^1(X_t, \mathbb{Z})^{\oplus k}. \end{array}$$

As the specialization maps sp_C and $sp_{\mathcal{X}}^{\oplus k}$ are obtained by following the horizontal arrows in this diagram from left to right, it follows that $\psi_t^* \circ sp_C = sp_{\mathcal{X}}^{\oplus k} \circ \psi_0^*$ as desired. \square

We can finish the proof of Lemma 8.3. It suffices to prove the following sequence of equalities:

$$\begin{aligned}
& \operatorname{Im} \left(H^1(C_t, \mathbb{Z}) \xrightarrow{\psi_t^*} H^1(X_t, \mathbb{Z})^{\oplus k} \right) \cap \operatorname{Im} \left(H^1(X_0, \mathbb{Z})^{\oplus k} \xrightarrow{sp_{\mathcal{X}}^{\oplus k}} H^1(X_t, \mathbb{Z})^{\oplus k} \right) \\
&= \operatorname{Im} \left(H^1(C_t, \mathbb{Z}) \xrightarrow{\psi_t^*} H^1(X_t, \mathbb{Z})^{\oplus k} \right) \cap \operatorname{Im} \left(\left(H^1(X_t, \mathbb{Z})^{\oplus k} \right)^{T^{\oplus k}} \longrightarrow H^1(X_t, \mathbb{Z})^{\oplus k} \right) \\
&= \operatorname{Im} \left(H^1(C_t, \mathbb{Z})^S \longrightarrow H^1(C_t, \mathbb{Z}) \xrightarrow{\psi_t^*} H^1(X_t, \mathbb{Z})^{\oplus k} \right) \\
&= \operatorname{Im} \left(H^1(C_0, \mathbb{Z}) \xrightarrow{sp_C} H^1(C_t, \mathbb{Z}) \xrightarrow{\psi_t^*} H^1(X_t, \mathbb{Z})^{\oplus k} \right) \\
&= \operatorname{Im} \left(H^1(C_0, \mathbb{Z}) \xrightarrow{\psi_0^*} H^1(X_0, \mathbb{Z})^{\oplus k} \xrightarrow{sp_{\mathcal{X}}^{\oplus k}} H^1(X_t, \mathbb{Z})^{\oplus k} \right).
\end{aligned}$$

The first equality follows from (8.9), the second equality from (8.8), the third equality from (8.9) again, and the last equality from Claim 8.2.1. \square

8.3. Symplectic bases adapted to paths. Let U be a complex analytic space. Let n and g be positive integers with $n \leq g$, and for $i \in \{1, \dots, n\}$, let $\Delta_i \subset U$ be an effective divisor. For each i , consider a sufficiently small disc $D_i \subset U$ that intersects the divisor $\Delta := \cup_i \Delta_i$ transversally in a point $0_i \in \Delta_i$.

Let $p: \mathcal{X} \rightarrow U$ be a family of nodal curves of arithmetic genus g over U . Assume that p is smooth over $U - \Delta$. For each i , shrink the disc D_i around 0_i so that the natural map

$$H^1(\mathcal{X}|_{D_i}, \mathbb{Z}) \longrightarrow H^1(X_{0_i}, \mathbb{Z})$$

is an isomorphism (see Proposition 2.14). Consider a point $t_i \in D_i - \{0_i\}$, and let

$$\rho_i: [0, 1] \longrightarrow U - \Delta \tag{8.10}$$

be a path from t to t_i . The path ρ_i together with a single counter-clockwise loop on D_i induces a loop on $U - \Delta$ and we let $T_i \in \operatorname{Aut}(H^1(X_t, \mathbb{Z}))$ denote the associated monodromy operator. The path ρ_i induces a canonical isomorphism $H^1(X_{t_i}, \mathbb{Z}) \xrightarrow{\sim} H^1(X_t, \mathbb{Z})$, and we let $sp_{\mathcal{X}}^i: H^1(X_{0_i}, \mathbb{Z}) \rightarrow H^1(X_t, \mathbb{Z})$ denote the composition

$$sp_{\mathcal{X}}^i: H^1(X_{0_i}, \mathbb{Z}) \xleftarrow{\sim} H^1(\mathcal{X}|_{D_i}, \mathbb{Z}) \longrightarrow H^1(X_{t_i}, \mathbb{Z}) \xrightarrow{\sim} H^1(X_t, \mathbb{Z}). \tag{8.11}$$

Define $W_0 H^1(X_{0_i}, \mathbb{Z}) = W_0 H^1(X_{0_i}, \mathbb{Q}) \cap H^1(X_{0_i}, \mathbb{Z})$.

Definition 8.4. Consider the above notation. Let

$$\{\delta_1, \dots, \delta_g; \gamma_1, \dots, \gamma_g\} \subset H^1(X_t, \mathbb{Z}) \tag{8.12}$$

be a symplectic basis for $H^1(X_t, \mathbb{Z})$. We say that the symplectic basis (8.12) is adapted to the paths $\rho_1, \dots, \rho_n: [0, 1] \rightarrow U - \Delta$ defined in (8.10) if, for each $i \in \{1, \dots, n\}$, we have:

$$\begin{aligned} \operatorname{Im} \left(H^1(X_{0_i}, \mathbb{Z}) \xrightarrow{sp_i^X} H^1(X_t, \mathbb{Z}) \right) &= H^1(X_t, \mathbb{Z})^{T_i} = \langle \delta_1, \dots, \delta_g; \gamma_1, \dots, \gamma_{i-1}, \widehat{\gamma}_i, \gamma_{i+1}, \dots, \gamma_g \rangle, \\ \operatorname{Im} \left(W_0 H^1(X_{0_i}, \mathbb{Z}) \xrightarrow{sp_i^X} H^1(X_t, \mathbb{Z}) \right) &= \mathbb{Z} \cdot \delta_i = \langle \delta_i \rangle. \end{aligned}$$

Lemma 8.5. *Consider the above notation. Let U' be a complex analytic space, let $t', t'_i, 0'_i \in U'$ be points on U' ($i = 1, \dots, n$), and let*

$$\pi: U' \longrightarrow U$$

be a surjective generically finite morphism of complex analytic spaces which is étale over $U - \Delta$, such that $\pi(t') = t, \pi(t'_i) = t_i$ and $\pi(0'_i) = 0_i$ for $i = 1, \dots, n$. Define $\Delta'_i = \pi^{-1}(\Delta_i)$ and $\Delta' = \cup_i \Delta'_i = \pi^{-1}(\Delta)$. Let $D'_i \subset U'$ be the unique connected component of $\pi^{-1}(D_i)$ that contains t'_i . Then D'_i is a disc and $D'_i \rightarrow D_i$ is a finite cover of discs, totally ramified at $0'_i$. Let

$$\rho'_i: [0, 1] \longrightarrow U' - \Delta' \quad (i = 1, \dots, n)$$

be the unique path in $U' - \Delta'$ from t' to t'_i that lifts ρ_i , and let $T'_i \in \operatorname{Aut}(H^1(X'_{t'_i}, \mathbb{Z}))$ be the monodromy operator induced by the path ρ'_i and the pointed disc $(D'_i, 0'_i)$. Suppose that $\{\delta_1, \dots, \delta_g; \gamma_1, \dots, \gamma_g\} \subset H^1(X_t, \mathbb{Z})$ is a symplectic basis adapted to the paths ρ_1, \dots, ρ_n , see Definition 8.4. Then the image of $\{\delta_1, \dots, \delta_g; \gamma_1, \dots, \gamma_g\}$ under the canonical isomorphism

$$H^1(X_t, \mathbb{Z}) \simeq H^1(X'_{t'}, \mathbb{Z}),$$

is a symplectic basis of $H^1(X'_{t'}, \mathbb{Z})$ adapted to the paths ρ'_1, \dots, ρ'_n .

Proof. Notice that $D'_i \rightarrow D_i$ is a finite cover of discs, of the form $z \mapsto z^{m_i}$ for some $m_i \in \mathbb{Z}_{\geq 1}$. If we identify $H^1(X'_{t'}, \mathbb{Z})$ with $H^1(X_t, \mathbb{Z})$, then the monodromy operator T'_i satisfies $T'_i = T_i^{m_i}$, as elements of $\operatorname{Aut}(H^1(X_t, \mathbb{Z}))$. The first thing to show is that

$$H^1(X_t, \mathbb{Z})^{T_i^p} = \langle \delta_1, \dots, \delta_g; \gamma_1, \dots, \gamma_{i-1}, \widehat{\gamma}_i, \gamma_{i+1}, \dots, \gamma_g \rangle \quad (8.13)$$

for $p = m_i$, knowing that it holds for $p = 1$. To prove this, let $x \in H^1(X_t, \mathbb{Z})$, and write

$$x = a \cdot \gamma_i + y, \quad a \in \mathbb{Z}, \quad y \in \langle \delta_1, \dots, \delta_g; \gamma_1, \dots, \gamma_{i-1}, \widehat{\gamma}_i, \gamma_{i+1}, \dots, \gamma_g \rangle = H^1(X_t, \mathbb{Z})^{T_i}.$$

We must show that $T_i^{m_i}(x) = x$ if and only if $a = 0$. In other words, we must prove that $T_i^p(\gamma_i) \neq \gamma_i$ for $p = m_i$, knowing that it holds for $p = 1$. But this is clear: if $T_i^{m_i}(\gamma_i) = \gamma_i$, then $T_i^{m_i}$ acts trivially on $H^1(X_t, \mathbb{Z})$, which is absurd since T_i does not act trivially on $H^1(X_t, \mathbb{Z})$. We conclude that, for each i , (8.13) holds for $p = m_i$.

Finally, if $f: H^1(X_t, \mathbb{Z}) \xrightarrow{\sim} H^1(X'_{t'}, \mathbb{Z})$ is the isomorphism induced by the canonical isomorphism $X'_{t'} \simeq X_t$, then f identifies the images of $W_0 H^1(X_{0_i}, \mathbb{Z})$ and $W_0 H^1(X'_{0'_i}, \mathbb{Z})$. \square

8.4. Extending the isogeny. To prove Theorem 1.1, we would like to apply Proposition 8.1. To do so, we will need the following lemma.

Lemma 8.6. *Let n and g be integers with $g \geq 2$ and $1 \leq n \leq g$. Consider a normal algebraic variety U of dimension $2g - 1$, irreducible divisors $\Delta_i \subset U$ for $i \in \{1, \dots, n\}$, and a family of stable genus g hyperelliptic curves*

$$p: \mathcal{X} \longrightarrow U \tag{8.14}$$

satisfying the conditions of Lemma 2.22. Assume that, for the very general fibre X_t of (8.14), there is a smooth projective curve C and an isogeny $\varphi: (JX_t)^k \rightarrow JC$. Let $t \in U$ be such a very general point. Then up to replacing U by a normal variety with surjective generically finite morphism $\pi: U' \rightarrow U$, p by its pull-back along π , t by a point in U' that lies over it, and Δ_i by an irreducible divisor in U' that dominates it, there exists a family of stable curves

$$q: \mathcal{C} \longrightarrow U' \quad \text{such that} \quad C_t = q^{-1}(t) = C,$$

together with an isogeny

$$\psi: (J\mathcal{X})^k := J\mathcal{X} \times_{U'} \cdots \times_{U'} J\mathcal{X} \longrightarrow JC \tag{8.15}$$

of semi-abelian schemes over U' that extends the given isogeny $\varphi: (JX_t)^k \rightarrow JC$, and such that the following conditions are satisfied:

- (i) *For each $i \in \{1, \dots, n\}$, the family $p: \mathcal{X}|_{\tilde{\Delta}_i} \rightarrow \tilde{\Delta}_i$ obtained by pulling back $p: \mathcal{X} \rightarrow U$ along the normalization $\tilde{\Delta}_i \rightarrow \Delta_i \subset U$ satisfies conditions (I)–(III) in Proposition 8.1.*
- (ii) *For general $t \in U'$ and $0_i \in \Delta_i$ ($i \in \{1, \dots, n\}$), the following holds. For each i , there is a disc $D_i \subset U'$ that intersects Δ transversally in $0_i \in \Delta_i$, so that for general $t_i \in D_i - \{0_i\}$, there is a path $\rho_i: [0, 1] \rightarrow U' - \Delta$ from t to t_i and a symplectic basis $\{\delta_1, \dots, \delta_g; \gamma_1, \dots, \gamma_g\} \subset H^1(X_t, \mathbb{Z})$ which is adapted to the paths ρ_i , see Definition 8.4.*

Proof. We need to spread out the curve C and the isogeny $\varphi: (JX_t)^k \rightarrow JC$, and our plan is to do this after a suitable base change $U' \rightarrow U$.

Claim 8.4.1. There is a normal variety U' , a map $\pi: U' \rightarrow U$, and a family of stable curves

$$q': \mathcal{C}' \longrightarrow U' \tag{8.16}$$

such that if $p': \mathcal{X}' \rightarrow U'$ is the pull-back of the family (8.14) along π , then the following properties hold:

- (1) The morphism $\pi: U' \rightarrow U$ is surjective, generically finite, and étale over $U - \Delta$.
- (2) The fibre $\mathcal{C}'_{t'} = (q')^{-1}(t')$ is isomorphic to C .
- (3) There is a dense open subset $V' \subset U' - \pi^{-1}(\Delta)$ such that the family of curves (8.16) is smooth over V' , and such that the isogeny $\varphi: (JX)^k \rightarrow JC$ extends to an isogeny of abelian schemes

$$(J\mathcal{X}')^k|_{V'} \longrightarrow J\mathcal{C}'|_{V'}. \tag{8.17}$$

Proof of Claim 8.4.1. As the very general fibre X_t of (8.14) satisfies the property that there exists an integer $k \geq 1$, a smooth curve C and an isogeny $\varphi: (JX_t)^k \rightarrow JC$, there exists a non-empty open subset $V \subset U$, a finite étale cover $V_1 \rightarrow V$, a family of smooth genus kg curves

$$q_{V_1}: \mathcal{C}_{V_1} \rightarrow V_1, \quad C_s = q_{V_1}^{-1}(s) = C, \quad (8.18)$$

such that $\varphi: (JX)^k \rightarrow JC$ extends to an isogeny of abelian schemes $\psi_{V_1}: (J\mathcal{X})^k|_{V_1} \rightarrow J\mathcal{C}_{V_1}$ over V_1 . The family of smooth curves (8.18) corresponds to a morphism

$$V_1 \rightarrow \mathcal{M}_g,$$

where \mathcal{M}_g denotes the moduli stack of smooth curves of genus g . Let ℓ be a prime number at least three, and define moduli spaces with level structures ${}_\ell M_g$ and ${}_\ell \overline{M}_g$ as in [Jon96, Section 2.4]. We have ${}_\ell M_g \subset {}_\ell \overline{M}_g$ as a dense open subscheme, and there is a natural finite étale cover ${}_\ell M_g \rightarrow \mathcal{M}_g$. In particular, there is a finite étale cover $V_2 \rightarrow V_1$ and a morphism

$$V_2 \rightarrow {}_\ell M_g$$

that commutes with ${}_\ell M_g \rightarrow \mathcal{M}_g$ and the map $V \rightarrow \mathcal{M}_g$. Define U_2 as the normalization of U in the function field of V_2 . The natural map $U_2 \rightarrow U$ is a finite cover that extends the finite étale cover $V_2 \rightarrow V$. Since ${}_\ell M_g$ is a dense open subscheme of ${}_\ell \overline{M}_g$, we obtain a rational map

$$U_2 \dashrightarrow {}_\ell \overline{M}_g \quad (8.19)$$

Since ${}_\ell \overline{M}_g$ is a proper variety, there exists an algebraic variety U_3 , a surjective birational morphism $U_3 \rightarrow U_2$, and a morphism

$$U_3 \rightarrow {}_\ell \overline{M}_g$$

that extends the rational map (8.19). Let U' be the normalization of U_3 . As U is normal and $U' \rightarrow U$ is étale over $U - \Delta$, the composition $U' \rightarrow U$ is étale over $U - \Delta$. Moreover, the natural map $U' \rightarrow U$ is surjective, generically finite, and étale over $U - \Delta$, and the map

$$U' \rightarrow {}_\ell \overline{M}_g$$

defines a family of stable curves

$$q': \mathcal{C}' \rightarrow U'$$

that has all the required properties. \square

Claim 8.4.2. Let U' , $q': \mathcal{C}' \rightarrow U'$ and $p': \mathcal{X}' \rightarrow U'$ be as in Claim 8.4.1. The isogeny (8.17), which is an isogeny of abelian schemes over the open subset $V' \subset U'$, extends to an isogeny

$$\psi': (J\mathcal{X}')^k = J\mathcal{X}' \times_{U'} \cdots \times_{U'} J\mathcal{X}' \rightarrow J\mathcal{C}'$$

of semi-abelian schemes over U' .

Proof of Claim 8.4.2. As U' is normal, this follows from [FC90, Chapter I, Proposition 2.7]. \square

Finally, to finish the proof of Lemma 8.6, it remains to prove that properties (i) and (ii) hold. As for property (i), this holds by Lemma 2.22 and by the fact that it is stable under base change. Property (ii) follows from Lemmas 2.22, 2.23 and 8.5. \square

8.5. Proof of the main theorem. We are now in position to prove our main theorem.

Proof of Theorem 1.1. To prove the theorem, we begin with the following reduction step.

Claim 8.5.1. Theorem 1.1 is implied by the following statement:

(*) *Let k be a positive integer. If, for a very general hyperelliptic curve X of genus $g \geq 4$, there exists a smooth projective curve C and an isogeny $(JX)^k \rightarrow JC$, then $k = 1$ and $C \simeq X$.*

Proof of Claim 8.5.1. Let $g \geq 4$ and let $Z \subset \mathcal{M}_g$ be an irreducible closed subvariety that contains the hyperelliptic locus. Let $[X] \in Z$ be a very general point, corresponding to a smooth curve X of genus g . As mentioned in the introduction, because JX is simple, Theorem 1.1 readily reduces to the case $n = 1$: there is an isogeny $(JX)^k \rightarrow JC$ between $(JX)^k$ and the Jacobian JC of a smooth projective connected curve C . We need to show, under the assumption that (*) holds, that $k = 1$ and $C \simeq X$. We specialize X to a very general hyperelliptic curve Y . This yields a specialization of C into a compact type curve D , and an isogeny $(JY)^k \rightarrow JD = JD_1 \times \cdots \times JD_n$, where the D_i are the non-rational irreducible components of D . As JY is simple, there is, for each i , an integer $k_i \leq k$ and an isogeny $\varphi_i: (JY)^{k_i} \rightarrow JD_i$. Then (*) implies $k_i = 1$ and $D_i \simeq Y$ for each i , hence $JD \simeq (JY)^k$. Lemma 2.2 implies $JC \simeq (JX)^k$, and then Corollary 6.2 implies $k = 1$ and $C \simeq X$ as wanted. \square

Our goal is to prove (*). Thus, let $k \geq 1$ be an integer, and assume that for a very general hyperelliptic curve X of genus $g \geq 4$, there exists a smooth projective curve C and an isogeny $\varphi: (JX)^k \rightarrow JC$. We aim to show that $k = 1$ and $C \simeq X$. The strategy is to spread out the isogeny φ to an isogeny of families. To this end we apply Lemma 8.6 and we get a normal algebraic variety U with irreducible divisors $\Delta_i \subset U$ ($i = 1, 2, 3, 4$), families of stable curves

$$p: \mathcal{X} \rightarrow U \quad \text{and} \quad q: \mathcal{C} \rightarrow U$$

with smooth general fibres, and an isogeny

$$\psi: (J\mathcal{X})^k \rightarrow J\mathcal{C}$$

of semi-abelian schemes over U , such that for some $t \in U$, we have $X_t = X, C_t = C$ and ψ restricts to the given isogeny $\varphi: (JX)^k \rightarrow JC$, and such that all the conditions in Lemma 8.6 are satisfied. In particular, $p: \mathcal{X} \rightarrow U$ is smooth over the complement $U - \Delta$ of the divisor $\Delta = \cup_i \Delta_i$, and the morphism $U - \Delta \rightarrow \mathcal{H}_g$ induced by p is dominant. At this point, in order to prove Theorem 1.1, it suffices to show that $k = 1$ and $C_t \simeq X_t$ (see Claim 8.5.1).

Claim 8.5.2. For $i \in \{1, 2, 3, 4\}$ and general $0_i \in \Delta_i$, there is an isomorphism of abelian varieties

$$J\tilde{C}_{0_i} \simeq (J\tilde{X}_{0_i})^k,$$

and the composition $(J\tilde{X}_{0_i})^k \xrightarrow{\tilde{\psi}_{0_i}} J\tilde{C}_{0_i} \simeq (J\tilde{X}_{0_i})^k$ is given by a matrix

$$\alpha_i \in M_k(\mathbb{Z}). \tag{8.20}$$

Here, the curves \tilde{C}_{0_i} and \tilde{X}_{0_i} are the normalizations of C_{0_i} and X_{0_i} respectively, and the morphism $\tilde{\psi}_{0_i}: (J\tilde{X}_{0_i})^k \rightarrow J\tilde{C}_{0_i}$ is the isogeny induced by ψ .

Proof of Claim 8.5.2. For $i \in \{1, \dots, 4\}$, the family $p: \mathcal{X}|_{\tilde{\Delta}_i} \rightarrow \tilde{\Delta}_i$ obtained by pulling back $p: \mathcal{X} \rightarrow U$ along the normalization $\tilde{\Delta}_i \rightarrow \Delta_i \subset U$ of the divisor Δ_i satisfies conditions (I)–(III) in Proposition 8.1, see Lemma 8.6. Therefore, the claim follows from Proposition 8.1. \square

By construction (see Lemma 8.6), for each $i \in \{1, 2, 3, 4\}$ and general $0_i \in \Delta_i$, the fibre X_{0_i} is an irreducible one-nodal hyperelliptic curve, and there exists a disc $D_i \subset U$ intersecting Δ transversally in $0_i \in \Delta_i$, a path ρ_i from $t \in U$ to a point $t_i \in D_i - \{0_i\}$, and a symplectic basis

$$\{\delta_1, \dots, \delta_g; \gamma_1, \dots, \gamma_g\} \subset H^1(X_t, \mathbb{Z})$$

which is adapted to the paths ρ_1, \dots, ρ_4 in the sense of Definition 8.4. For $i \in \{1, 2, 3, 4\}$, shrink D around 0 so that the path ρ_i induces a well-defined specialization map

$$sp_{\mathcal{X}}^i: H^1(X_{t_i}, \mathbb{Z}) \rightarrow H^1(X_t, \mathbb{Z}),$$

see (8.11). Consider the group $H^1(X_{0_i}, \mathbb{Z})$ as a submodule $H^1(X_{0_i}, \mathbb{Z}) \subset H^1(X_t, \mathbb{Z})$ via $sp_{\mathcal{X}}^i$ and consider $H^1(C_t, \mathbb{Z})$ as a submodule $H^1(C_t, \mathbb{Z}) \subset H^1(X_t, \mathbb{Z})^{\oplus k}$ of $H^1(X_t, \mathbb{Z})^{\oplus k}$ via the map $\psi_t^*: H^1(C_t, \mathbb{Z}) \rightarrow H^1(X_t, \mathbb{Z})^{\oplus k}$ induced by the isogeny $\psi_t: (JX_t)^k \rightarrow JC_t$. Define

$$W_0H^1(X_{0_i}, \mathbb{Z}) = W_0H^1(X_{0_i}, \mathbb{Q}) \cap H^1(X_{0_i}, \mathbb{Z}).$$

Thus, $W_0H^1(X_{0_i}, \mathbb{Z})$ is the integral part of the zeroth piece of the weight filtration on $H^1(X_{0_i}, \mathbb{Q})$.

Claim 8.5.3. For each $i \in \{1, 2, 3, 4\}$, consider the matrix $\alpha_i \in M_k(\mathbb{Z})$ of Claim 8.5.2, see equation (8.20). We have:

$$H^1(C_t, \mathbb{Z}) \cap H^1(X_{0_i}, \mathbb{Z})^{\oplus k} \equiv \alpha_i \cdot H^1(X_{0_i}, \mathbb{Z})^{\oplus k} \pmod{W_0H^1(X_{0_i}, \mathbb{Z})^{\oplus k}}. \quad (8.21)$$

Proof of Claim 8.5.3. By Proposition 8.2, the claim follows from Claim 8.5.2. \square

We can finish the proof of Theorem 1.1. By Claim 8.5.3, we know that (8.21) holds for the matrices $\alpha_i \in M_k(\mathbb{Z})$ of Claim 8.5.2. Moreover, by Lemma 8.6, the monodromy operators $T_1, \dots, T_4 \in \text{Aut}(H^1(X_t, \mathbb{Z}))$ induced by the paths ρ_i and the discs D_i satisfy the property that

$$\begin{aligned} \text{Im} \left(H^1(X_{0_i}, \mathbb{Z}) \xrightarrow{sp_{\mathcal{X}}^i} H^1(X_t, \mathbb{Z}) \right) &= H^1(X_t, \mathbb{Z})^{T_i} = \langle \delta_1, \dots, \delta_g; \gamma_1, \dots, \gamma_{i-1}, \hat{\gamma}_i, \gamma_{i+1}, \dots, \gamma_g \rangle, \\ \text{Im} \left(W_0H^1(X_{0_i}, \mathbb{Z}) \xrightarrow{sp_{\mathcal{X}}^i} H^1(X_t, \mathbb{Z}) \right) &= \mathbb{Z} \cdot \delta_i = \langle \delta_i \rangle. \end{aligned}$$

Thus, by Lemma 7.5, we have $\alpha_i \cdot H^1(X_t, \mathbb{Z})^{\oplus k} = \alpha_j \cdot H^1(X_t, \mathbb{Z})^{\oplus k} \subset H^1(C_t, \mathbb{Z}) \subset H^1(X_t, \mathbb{Z})^{\oplus k}$. In view of Lemma 7.1, there exist invertible matrices $\gamma_{ij} \in \text{GL}_k(\mathbb{Z})$ for $i, j \in \{1, 2, 3, 4\}$, such that $\alpha_j = \alpha_i \gamma_{ij}$ for each i, j . Moreover, as $\alpha_j \cdot H^1(X_t, \mathbb{Z})^{\oplus k} \subset H^1(C_t, \mathbb{Z})$, Lemma 7.9 implies

$$H^1(C_t, \mathbb{Z}) = \alpha_i \cdot H^1(X_t, \mathbb{Z})^{\oplus k} \subset H^1(X_t, \mathbb{Z})^{\oplus k} \quad \forall i = 1, 2, 3, 4.$$

Consequently, by Lemma 2.1, there exists an isomorphism of abelian schemes

$$JC \simeq (J\mathcal{X})^k$$

over U . By Corollary 6.2, it follows that $k = 1$ and $C_t \simeq X_t$.

We have proven that (*) holds. By Claim 8.5.1, we are done. \square

9. CURVES ON POWERS OF ABELIAN VARIETIES AND INTERMEDIATE JACOBIANS

The goal of this section is to prove Corollaries 1.3, 1.4, 1.5 and 1.6 stated in the introduction.

Proof of Corollary 1.3. First, we deal with the case where $A = J^3Y$ is the intermediate Jacobian of a very general cubic threefold Y . We claim that there is no integer $k \geq 1$ for which there exists an isogeny between $(J^3Y)^k$ and a product of Jacobians. To prove this, assume that such an integer and such an isogeny exist. As Y is very general, J^3Y is simple; in particular, we may assume that there exists a curve C and an isogeny $\phi: JC \rightarrow (J^3Y)^k$, with dual $\psi: (J^3Y)^k \rightarrow JC$. Degenerate Y into a singular cubic Y_0 such that $J^3Y_0 = JX$ is the Jacobian of a very general hyperelliptic curve X , cf. [Col82]. This leads to a degeneration of C into a compact type curve D and an isogeny $\phi_0: JD \rightarrow (JX)^k$. By Theorem 1.1, this implies $JD \simeq (JX)^k$, hence by Lemma 2.2, we get $JC \simeq (J^3Y)^k$, which contradicts Corollary 6.3.

It remains to show that if A is a very general principally polarized abelian variety of dimension $g \geq 4$, then there is no integer $k \geq 1$ for which there exists an isogeny between A^k and a product of Jacobians. To arrive at a contradiction, we may assume that, for some $k \in \mathbb{Z}_{\geq 1}$, there exists an isogeny $\phi: JC \rightarrow A^k$ for some smooth projective curve C . Specialize A to the Jacobian $A_0 = JX$ of a very general hyperelliptic curve of genus g . The curve C specializes to a compact type curve D , hence we obtain an isogeny $\phi_0: JD \rightarrow (JX)^k$. By Theorem 1.1, we have $JD \simeq (JX)^k$, hence there exists an isomorphism of abelian varieties $\psi: JC \simeq A^k$ by Lemma 2.2. This implies by Theorem 6.1 that (A, Θ) is isomorphic to the Jacobian of a curve, which is absurd by dimension reasons, because $g \geq 4$. The corollary follows. \square

Proof of Corollary 1.4. We fix an integer $g \geq 8$ which is divisible by some integer h with $4 \leq h < g$, and let $\mathcal{S} \subset \mathcal{A}_g$ be the locus of abelian varieties that are isogenous to a power of a lower dimensional principally polarized abelian variety of dimension at least four. Then \mathcal{S} is a countable union of its irreducible components, which are all special subvarieties of \mathcal{A}_g (this can for instance be deduced from [MO13, Remark 3.13]). By Corollary 1.3, none of these special subvarieties is contained in the closure of the Torelli locus of \mathcal{A}_g . Hence, to prove Corollary 1.4, it remains to show that \mathcal{S} is dense in \mathcal{A}_g .

Consider the genus g Siegel space \mathbb{H}_g and the quotient map $\pi: \mathbb{H}_g \rightarrow \mathcal{A}_g$. Let (A, λ_A) be any principally polarized abelian variety that defines a point $x \in \mathcal{S}$, and let $\tilde{x} \in \mathbb{H}_g$ such that $\pi(\tilde{x}) = x$. If B is an abelian variety associated to an element y in the orbit $\mathrm{Sp}_{2g}(\mathbb{Q}) \cdot \tilde{x} \subset \mathbb{H}_g$, then there exists an isomorphism of rational Hodge structures $H_1(A, \mathbb{Q}) \simeq H_1(B, \mathbb{Q})$. Consequently, we have $\pi(\mathrm{Sp}_{2g}(\mathbb{Q}) \cdot \tilde{x}) \subset \mathcal{S}$. Moreover, the orbit $\mathrm{Sp}_{2g}(\mathbb{Q}) \cdot \tilde{x}$ is analytically dense in \mathbb{H}_g because $\mathrm{Sp}_{2g}(\mathbb{Q})$ is dense in $\mathrm{Sp}_{2g}(\mathbb{R})$ by strong approximation and $\mathrm{Sp}_{2g}(\mathbb{R})$ acts transitively on \mathbb{H}_g . Therefore, the set $\pi(\mathrm{Sp}_{2g}(\mathbb{Q}) \cdot \tilde{x}) \subset \mathcal{A}_g$ is analytically dense in \mathcal{A}_g , and hence so is $\mathcal{S} \subset \mathcal{A}_g$. \square

Proof of Corollary 1.5. Let A be either the intermediate jacobian of a very general cubic threefold or a very general principally polarized abelian variety of dimension at least 4. Let A_1 be an abelian variety isogenous to a power of A and let A_2 be an abelian variety with $\mathrm{Hom}(A, A_2) = 0$.

Suppose that there are smooth projective curves C_1, \dots, C_n and an isomorphism

$$A_1 \times A_2 \simeq JC_1 \times \cdots \times JC_n. \quad (9.1)$$

We claim that there exists a non-empty subset $I \subset \{1, \dots, n\}$ such that $A_1 \simeq \prod_{i \in I} JC_i$. Indeed, the product polarization on $\prod_{i=1}^n JC_i$ and the isomorphism (9.1) equip $A_1 \times A_2$ with a principal polarization, call it λ . We have $\text{NS}(A_1 \times A_2) = \text{NS}(A_1) \times \text{NS}(A_2)$ because $\text{Hom}(A_1, A_2) = 0$. Hence $\lambda = \lambda_1 \times \lambda_2$ for principal polarizations λ_i on A_i . By [CG72, Corollary 3.23] (see also [Deb96]), the decomposition of a principally polarized abelian variety into a product of principally polarized abelian subvarieties is unique. Therefore, (A_1, λ_1) is isomorphic to $\prod_{i=1}^n JC_i$ for some non-empty subset $I \subset \{1, \dots, n\}$, proving the claim.

Since A_1 is isogenous to a power of A , we find that $\prod_{i=1}^n JC_i$ is isogenous to such a power, which contradicts Corollary 1.3. This concludes the proof of Corollary 1.5. \square

Proof of Corollary 1.6. This is a direct consequence of Corollary 1.5. \square

APPENDIX A. REMARK ON THE DEGENERATION METHOD

In an influential work, Bardelli and Pirola [BP89] proved that for a very general curve X of genus $g \geq 4$, the Jacobian JX is not isogenous to a Jacobian of any smooth curve C with $C \not\cong X$. Their argument is based on the following idea, which also played an important role in consecutive papers (including this paper). If $f: JC \rightarrow JX$ is an isogeny, then the goal is to show that there is an integer n such that the image of $f^*: H^1(JX, \mathbb{Z}) \rightarrow H^1(JC, \mathbb{Z})$ satisfies

$$\text{im}(f^*) = n \cdot H^1(JC, \mathbb{Z}) \subset H^1(JC, \mathbb{Z}). \quad (\text{A.1})$$

Indeed, (A.1) implies the existence of an isomorphism $JX \simeq JC$ with respect to which the isogeny f is given by multiplication by n . As X is very general, this isomorphism has to respect the polarizations, and thereby $C \simeq X$ because of the Torelli theorem for curves.

To prove (A.1), a degeneration argument to nodal curves is used to show that there are suitable monodromy operators T_1, T_2 corresponding to two different Picard–Lefschetz degenerations of X , such that (A.1) holds when intersected with the T_i -invariant subspaces for $i = 1, 2$. To get (A.1) from this, it is then used without proof in [BP89, Proposition 4.1.3] that

$$\text{im}(f^*) = \text{im}(f^*)^{T_1} + \text{im}(f^*)^{T_2}. \quad (\text{A.2})$$

The argument is formalized in [Mar13, Proposition 3.6], but also here, the identity (A.2) (which is the identity $H_1 + H_2 = \mathcal{H}_x$ in the notation of *loc. cit.*) is assumed implicitly in the proof. The same identity is used without proof in the proof of Theorem 4.1 in [NP18].

It turns out that the version in [Mar13, Proposition 3.6] is incorrect, see Proposition A.7 below. Nonetheless, it is possible to prove (A.2) under the additional assumption that $f^*\Theta_{JX}$ is a multiple of the theta divisor Θ_{JC} of JC (which is the case in [BP89; NP18]). For this, one can use the lattice theoretic results provided in Section 7, see in particular Lemmas 7.5 and 7.9. These statements and their proofs greatly simplify if one is only interested in the $k = 1$ case. For convenience of the reader, we state and prove the precise statement that one needs to prove (A.2) in the aforementioned applications in [BP89; NP18] in Proposition A.1 below.

Proposition A.1. *Let $H \subset G$ be free \mathbb{Z} -modules of the same finite rank and let E_G and E_H be unimodular symplectic forms on G and H , respectively. Let $\{\delta_1, \dots, \delta_g; \gamma_1, \dots, \gamma_g\} \subset G$ be a symplectic basis of G . Let $T_i \in \text{Aut}(G)$ for $i = 1, 2$ be linear automorphisms with $T_i(H) \subset H$ for all $i = 1, 2$. Assume the following conditions.*

- (i) *For each i , there is a positive integer n_i such that $H^{T_i} = n_i G^{T_i}$.*
- (ii) *The T_i -invariant subspaces of G are given by*

$$G^{T_1} = \langle \delta_1, \dots, \delta_g, \gamma_2, \dots, \gamma_g \rangle \quad \text{and} \quad G^{T_2} = \langle \delta_1, \dots, \delta_g, \gamma_1, \gamma_3, \dots, \gamma_g \rangle.$$

- (iii) *The form E_G on G restricts to a multiple of the form E_H on $H \subset G$.*

Then $n := n_1 = n_2$ and $H = nG \subset G$.

Before we prove Proposition A.1, we consider the following result, due to Marcucci.

Proposition A.2 (Marcucci). *Let $H \subset G$ be free \mathbb{Z} -modules of the same finite rank. Let $T_i \in \text{Aut}(G)$ for $i = 1, 2$ be linear automorphisms with $T_i(H) \subset H$ for all $i = 1, 2$. Assume that*

- (1) $G^{T_1} + G^{T_2} = G$;
- (2) $G^{T_1} \cap G^{T_2} \neq 0$;
- (3) *for each i , there is a positive integer n_i such that $H^{T_i} = n_i G^{T_i}$.*

Then $n := n_1 = n_2$ and $nG \subset H \subset G$.

Proof. See the proof of [Mar13, Proposition 3.6] (or see the more general Lemma 7.5). \square

Next, we prove Proposition A.1.

Proof of Proposition A.1. Note that $n_1 = n_2$ and $n_1 G \subset H$ by Proposition A.2. By Lemma 7.9, this implies $nG_1 = H$ as we want. For convenience of the reader, let us sketch the proof in this particular situation. Let $n := n_1$ and $U := \langle ne_1, ne_2 \rangle \subset G$. Then U is a saturated, isotropic subspace of H ; by Lemma 7.6, there are $g_1, g_2 \in G$ such that $U \oplus \langle g_1, g_2 \rangle \subset H$ is a unimodular sublattice of H . By condition (iii), there exists $m \in \mathbb{Z}_{\geq 1}$ such that $E_H(x, y) = E_G(m^{-1}x, y)$ for $x, y \in H$. Thus, $[G : H] = m^g$ (cf. Lemma 7.7), and $g_2 \in \langle \delta_1, \dots, \delta_g; \gamma_2, \dots, \gamma_g \rangle = G^{T_1}$.

In view of condition (i), there exists $g'_2 \in H_1$ such that $g_2 = ng'_2$. As we have an equality $E_H(ne_2, g_2) = E_G(m^{-1}ne_2, ng'_2) = 1$, we must have $n^2 \mid m$. Moreover, the inclusion $nG \subset H$ shows that $m^g = [G : H] \mid [G : nG] = n^{2g}$. Thus, $m = n^2$, and the equality $H = nG$ follows. \square

A.1. Counterexample. In this section we show that, in contrast to what is claimed in [Mar13, Proposition 3.6], the equality $nG = H$ does not follow in general from the conditions (1)–(3) of Proposition A.2. We try to keep our notation as compatible as possible with *loc. cit.*

Notation A.3. *Let G be a free \mathbb{Z} -module of rank four, with basis $\{e_1, e_2, f_1, f_2\} \subset G$. We equip G with the symplectic form*

$$(-, -) : G \times G \longrightarrow \mathbb{Z}$$

that has $\{e_1, e_2, f_1, f_2\}$ as symplectic basis. In other words, for $i, j \in \{1, 2\}$, we have

$$(e_i, f_j) = \delta_{ij}, \quad (e_i, e_j) = 0 = (f_i, f_j), \quad (y, x) = -(x, y) \quad \forall x, y \in G.$$

Let k and n be positive integers such that

$$k \mid n, \quad 1 < k \leq n. \quad (\text{A.3})$$

We define a submodule $H \subset G$ as follows:

$$H = \left\langle ne_1, ne_2, nf_1, \frac{n}{k}f_2 + \frac{n}{k}f_1 \right\rangle \subset \langle e_1, e_2, f_1, f_2 \rangle = G.$$

Notice that

$$nG = \langle ne_1, ne_2, nf_1, nf_2 \rangle \subset \left\langle ne_1, ne_2, nf_1, \frac{n}{k}f_2 + \frac{n}{k}f_1 \right\rangle = H.$$

Lemma A.4. *The inequality $nG \subset H$ is strict. More precisely, the quotient H/nG is a finite cyclic group of order $k > 1$.*

Proof. This follows from (A.3). □

Consider the free abelian group $\mathbb{Z}^{\oplus 2}$. We are going to define an action of $\mathbb{Z}^{\oplus 2}$ on G . For $i = 1, 2$, define an operator $T_i: G \rightarrow G$ by $T_i(x) = x + (x, ke_i)ke_i$. Then T_1 and T_2 are automorphisms of G . As $(e_1, e_2) = 0$, we obtain an action of $\mathbb{Z}^{\oplus 2}$ on G :

$$\mathbb{Z}^{\oplus 2} \longrightarrow \text{Aut}(G), \quad b_i \longmapsto T_i, \quad (\text{A.4})$$

where $b_1 = (1, 0) \in \mathbb{Z}^{\oplus 2}$ and $b_2 = (0, 1) \in \mathbb{Z}^{\oplus 2}$.

Lemma A.5. *The action of $\mathbb{Z}^{\oplus 2}$ on G preserves the submodule $H \subset G$.*

Proof. We have $T_1(ne_1) = ne_1$ and $T_1(ne_2) = ne_2$. Moreover, we have $T_1(nf_1) = nf_1 - k^2ne_1$ and $T_1\left(\frac{n}{k}f_2 + \frac{n}{k}f_1\right) = \frac{n}{k}f_2 + \frac{n}{k}f_1 - kne_1$. In a similar way, $T_2(ne_1) = ne_1$, $T_2(ne_2) = ne_2$, $T_2(nf_1) = nf_1$, and $T_2\left(\frac{n}{k}f_2 + \frac{n}{k}f_1\right) = \frac{n}{k}f_2 + \frac{n}{k}f_1 - kne_2$. These are all elements of H . □

Next, we would like to calculate $H^{T_i} = H \cap G^{T_i}$ for $i = 1, 2$.

Lemma A.6. *We have $H^{T_i} = n \cdot G^{T_i}$ for $i = 1, 2$.*

Proof. Notice that $G^{T_i} = \{x \in G \mid T_i(x) = x\} = \{x \in G \mid (x, e_i) = 0\}$. Hence,

$$G^{T_1} = \langle e_1, e_2, f_2 \rangle \quad \text{and} \quad G^{T_2} = \langle e_1, e_2, f_1 \rangle. \quad (\text{A.5})$$

Therefore, $H^{T_1} = H \cap G^{T_1} = H \cap \langle e_1, e_2, f_2 \rangle = \langle ne_1, ne_2, nf_2 \rangle = nG^{T_1}$. Similarly, we have $H^{T_2} = H \cap G^{T_2} = H \cap \langle e_1, e_2, f_1 \rangle = \langle ne_1, ne_2, nf_1 \rangle = nG^{T_2}$. □

From the previous results, we deduce the following result, which shows that [Mar13, Proposition 3.6] fails in the generality stated. For a corrected version with stronger hypotheses, which seems to suffice for most of the applications, see Proposition A.1.

Proposition A.7. *There exists a connected and path-connected pointed topological space (X, x) , two local systems of free abelian groups of rank four \mathcal{H} and \mathcal{G} on X , an injective morphism of local systems $\mathcal{H} \hookrightarrow \mathcal{G}$, and two elements $\gamma_i \in \pi_1(X, x)$ ($i = 1, 2$), such that the following holds. If the monodromy representations attached to \mathcal{H} and \mathcal{G} are denoted by*

$$\rho: \pi_1(X, x) \longrightarrow \text{Aut}(\mathcal{H}_x) \quad \text{and} \quad \sigma: \pi_1(X, x) \longrightarrow \text{Aut}(\mathcal{G}_x),$$

and if $G_i \subset \mathcal{G}_x$ and $H_i \subset \mathcal{H}_x$ are defined as

$$\begin{aligned} G_i &:= \text{Inv}(\sigma(\gamma_i)) = \{a \in \mathcal{G}_x \mid \sigma(\gamma_i)(a) = a\}, \\ H_i &:= \text{Inv}(\rho(\gamma_i)) = \{a \in \mathcal{H}_x \mid \rho(\gamma_i)(a) = a\}, \end{aligned} \tag{A.6}$$

then the following conditions are satisfied:

- (1) $G_1 + G_2 = \mathcal{G}_x$;
- (2) $G_1 \cap G_2 \neq 0$;
- (3) there exists an integer $n \in \mathbb{Z}_{\geq 1}$ such that $H_i = nG_i$ for $i = 1, 2$;
- (4) with respect to the integer n in condition (3), we have

$$\mathcal{H}_x \neq n\mathcal{G}_x$$

as submodules of \mathcal{G}_x .

Proof. Let $D^* = \{z \in \mathbb{C} \mid 0 < |z| < 1\}$, and define $X = D^* \times D^*$. Let $x \in X$ be any point. Then $\pi_1(X, x) = \mathbb{Z}^{\oplus 2}$.

Define two free \mathbb{Z} -modules of rank four $H \subset G$ as in Notation A.3. Let $\mathbb{Z}^{\oplus 2}$ act on G as in (A.4). In particular, the action of $\mathbb{Z}^{\oplus 2}$ on G restricts to an action of $\mathbb{Z}^{\oplus 2}$ on $H \subset G$, see Lemma A.5. This yields two representations $\rho: \pi_1(X, x) \rightarrow \text{Aut}(H)$ and $\sigma: \pi_1(X, x) \rightarrow \text{Aut}(G)$.

Let \mathcal{H} and \mathcal{G} be the local systems on X attached to the representation $\rho: \pi_1(X, x) \rightarrow \text{Aut}(H)$ and $\sigma: \pi_1(X, x) \rightarrow \text{Aut}(G)$, respectively. Define G_i and H_i as in (A.6).

We claim that conditions (1)–(4) are verified. Conditions (1) and (2) follow from (A.5). Condition (3) follows from Lemma A.6. Condition (4) follows from Lemma A.4. \square

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