# INVISIBLE SINGULARITIES IN COMPLEX ALGEBRAIC GEOMETRY

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Abstract. We construct morphisms between smooth complex projective varieties that have singular fibers, but look topologically smooth. We use this to give negative answers to the following four conjectures and questions: the smoothness conjecture of Fernández de Bobadilla and Kollár [FdBK12], a question of Kollár and Pardon on universal covers [KP12], Kotschick's conjecture on 1-forms [Kot22], and a conjecture of Schreieder on Aomoto complexes [Sch21].

## 1. Introduction

A central question of equisingularity theory is to understand the relationship between the algebraic, analytic and topological versions. After giving full answers for curves, Zariski outlined a series of higher-dimensional questions in [Zar71]. The general expectation was that the algebraic and topological concepts of equisingularity coincide.

After many positive results [Mil68, LR76, Tei80, Mas91, Sie01], Fernández de Bobadilla [FdB05] constructed families of cubic hypersurfaces $Y \to C$ over a curve $C$, for which the various notions introduced by Zariski and Massey are not equivalent. In these, and the later examples in [FdBK12], the total space $Y$ is always singular.

Note also that if $Y \to C$ is any proper morphism from a smooth variety $Y$ to a curve $C$, and $Y_c$ is a fiber with at worst isolated singularities, then the retraction map from a nearby fiber $Y_{c'}$ to $Y_c$ is an isomorphism on $\mathbb{Q}$-homology iff $Y_c$ is smooth, see [Mil68].

These led to the belief that, if $Y$ is smooth and $g\colon Y \to C$ is proper, then critical points of $g$ are always visible topologically; see [FdBK12, LMW21, FdB22, FdBP24] for further discussions and results.

Nonetheless, our main examples show that the topology of $Y$ does not always detect the presence of non-isolated critical points of a morphism $g\colon Y \to C$.

**Theorem 1.** *There are morphisms $g\colon Y \to \mathbb{P}^1$ from smooth complex projective varieties $Y$, where $g$ has singular fibers but looks topologically smooth. That is,*

*(i) the $R^i g_* \mathbb{Z}_Y$ are constant sheaves on $\mathbb{P}^1$,*
*(ii) $g$ is a homotopy fiber bundle with simply connected fibers (see Definition 5),*
*(iii) all the fibers of $g$ are PL-manifolds, but $g$ is not a $C^0$-fiber bundle, and*
*(iv) in the homotopy class of $g$ there is a locally trivial, $C^\infty$-fiber bundle $\tilde{g}\colon Y \to \mathbb{P}^1$ that agrees with $g$ outside a small neighborhood of the singular fibers of $g$.*

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Theorem 1 disproves the smoothness conjecture of Fernández de Bobadilla–Kollár [FdBK12, Conjecture 3]; see also Corollaries 8 and 16 below.

**Remark 2.** Our construction of these $g\colon Y \to \mathbb{P}^1$ is completely explicit and simple.

The smallest example we have has $\dim Y = 5$, the critical locus of $g$ has dimension 1, and, near any critical point, one can write $g$ as $(x_1, \dots, x_5) \mapsto x_1^2+x_2^2+x_3^2+x_4^3$ (in suitable local analytic coordinates), see Remark 24. Moreover, $g$ is birationally equivalent to the projection $\mathbb{P}^4 \times \mathbb{P}^1 \to \mathbb{P}^1$, see Example 23.

One can deduce from [HS21] that there are no such examples with $\dim Y \leq 3$, see Theorem 40 below. The question is open in dimension 4.

We construct the example in Theorem 1 by choosing the singularities of $g$ carefully. Then item (i) in Theorem 1 follows from a computation involving vanishing cycles. For simply connected fibers (i) implies (ii) by Whitehead's theorem. Item (iii) follows from the works of Brieskorn [Bri66] and Milnor [Mil68].

It turns out that (ii) implies (iv). Here we rely on Smale's h-cobordism theorem [Sm62] and Cerf's pseudo-isotopy theorem [Cer70]; see Proposition 20 below. However these provide very little information about $\tilde{g}$ beyond its existence.

We also get negative answers to [KP12, Question 27] and to Kotschick's conjecture, first asked around 2013, but published only later in [Kot22, Question 15] and [Sch21, HS21, SY25, Pie25a].

**Corollary 3.** *Let $E$ be an elliptic curve and $E \to \mathbb{P}^1$ a double cover unramified over the critical values of $g\colon Y \to \mathbb{P}^1$ from Theorem 1. Set $f\colon X := Y \times_{\mathbb{P}^1} E \to E$. Then*

*(i) the universal cover $\widetilde{X}$ has the homotopy type of a finite CW complex, but $f$ is not smooth;*

*(ii) every nonzero class $u \in H^1(X, \mathbb{R})$ can be represented by a real, closed one-form without zeros, yet the harmonic representative of every such class has zeros. In particular, $X$ admits no holomorphic one-form without zeros.*

Proofs are given in Corollaries 30 and 27.

A related construction disproves a conjecture proposed in [Sch21, Remark 1.7].

**Theorem 4.** *There is a smooth, complex, projective variety $X$ such that, for every finite connected étale cover $\pi\colon X' \to X$ and nonzero $\omega \in H^0(X, \Omega^1_X)$, the Aomoto complex $(H^*(X', \mathbb{C}), \wedge\pi^*\omega)$ is exact, but every real closed one-form on $X$ has a zero.*

**Outline of the paper.** We work throughout this paper over the field of complex numbers.

In Section 2, we construct flat, projective morphisms $f\colon X \to \Delta$ over the disc $\Delta$ with smooth total space $X$ such that $f$ is a $\mathbb{Z}$-homology fiber bundle but not a homotopy fiber bundle. There are examples whose general fibers are bielliptic surfaces and whose special fiber is non-normal. We also construct higher-dimensional examples with normal (and in fact terminal) fibers, cf. Example 14.

All the above examples have nontrivial fundamental group. In Section 3, we use these examples and a blow-up construction to produce examples of non-smooth morphisms $f\colon X \to \Delta$ with smooth total space that are $\mathbb{Z}$-homology fiber bundles and whose fibers

are simply connected. By Whitehead's theorem, these maps are then homotopy fiber bundles, and we observe that they are not submersions.

In Section 4, we prove a $C^\infty$-smoothing result up to homotopy for morphisms to Riemann surfaces, which shows in particular that item (ii) in Theorem 1 implies item (iv). In Section 5, we globalize the constructions from Sections 2–3 and prove Theorem 1. In Section 6, we discuss applications to the Kollár–Pardon question and Kotschick's conjecture, and prove items (i) and (ii) of Corollary 3. In Section 7 we prove a topological equisingularity theorem for proper morphisms with simply connected fibers to Riemann surfaces.

In Section 8 we apply constructions similar to those used in Theorem 1 to the rational cohomology torus of Debarre–Jiang–Lahoz [DJL17]. Using this construction together with the results in Section 7 proves Theorem 4. Finally, in Section 8.5 we use some of the positive results of [HS21] to prove Theorem 40, which shows that counterexamples—such as those produced in Theorem 1—do not occur in dimension three.

## 2. Homology fiber bundles that are not homotopy fiber bundles

In this section we discuss the local situation and provide homology fiber bundles over the complex disc that are not homotopy fiber bundles. These will be used in the proof of Theorem 1.

### 2.1. Homology and homotopy fiber bundles.

We start by recalling the following.

**Definition 5.** Let $f\colon M \to N$ be a proper morphism between complex spaces. It is a *$\mathbb{Z}$-homology fiber bundle* if there is an open covering $N = \bigcup U_i$ such that for each $y \in U_i$, the inclusion $f^{-1}(y) \hookrightarrow f^{-1}(U_i)$ induces an isomorphism on integral homology $H_*(f^{-1}(y), \mathbb{Z}) \xrightarrow{\cong} H_*(f^{-1}(U_i), \mathbb{Z})$, or, equivalently, on cohomology $H^*(f^{-1}(U_i), \mathbb{Z}) \xrightarrow{\cong} H^*(f^{-1}(y), \mathbb{Z})$. This condition is equivalent to requiring that $R^i f_* \mathbb{Z}$ be a local system for all $i$.

Similarly, $f$ is a *homotopy fiber bundle* if there is a covering as above such that the inclusion $f^{-1}(y) \hookrightarrow f^{-1}(U_i)$ is a homotopy equivalence for all $y \in U_i$.

We will use the following direct consequence of Whitehead's theorem.

**Corollary 6.** *Let $f\colon M \to N$ be a proper morphism between complex spaces. Assume that $f$ is a $\mathbb{Z}$-homology fiber bundle with simply connected fibers. Then*

*(i) $f$ is a homotopy fiber bundle;*
*(ii) if $N$ is contractible, then $M$ has the homotopy type of a finite CW complex.*

*Proof.* Let $y \in N$. Then there is an open neighborhood $U_y \subset N$ of $y$ such that the fiber $M_y := f^{-1}(y)$ is a deformation retract of $M_{U_y} := f^{-1}(U_y)$. Since the fibers of $f$ are simply connected, we can choose an open cover $N = \bigcup_{i\in I} U_i$ such that for all $i \in I$, $U_i$ is contractible and $f^{-1}(U_i)$ is simply connected. For all $y \in U_i$, the inclusion $M_y \hookrightarrow M_{U_i}$ is a map between simply connected and finite CW complexes which induces an isomorphism on homology by the Leray spectral sequence. Hence, $M_y \hookrightarrow M_{U_i}$ is a homotopy equivalence by Whitehead's theorem [Hat02, Corollary 4.33]. This concludes the proof of item (i).

In order to prove item (ii), assume that $N$ is contractible. By van Kampen's theorem, $M$ is then simply connected. Moreover, $M_y \to M$ induces an isomorphism on homology and hence is a homotopy equivalence by Whitehead's theorem [Hat02, Corollary 4.33]. In particular, $M$ has the homotopy type of a finite CW complex. □

2.2. **The quotient construction.** Let $Z$ be a complex manifold of dimension $n$, and let $\pi\colon Z \to \Delta$ be a projective morphism with finitely many critical points $p_i \in Z$, $i \in I$, all lying over the origin $0 \in \Delta$. Let $M_i$ denote the Milnor fiber of $\pi$ at $p_i$. We assume that there is at least one critical point, i.e. $I \neq \emptyset$.

Let $\gamma$ be an automorphism of $Z$ of order $1 < c < \infty$, that commutes with $\pi$ and fixes each $p_i$. Let $T_i$ denote the induced action on $H^{n-1}(M_i, \mathbb{Z})$ and let $\chi_i(x)$ be its characteristic polynomial. Let $A$ be an abelian variety and let $\tau\colon A \to A$ be a translation of order $c$. Set

$$\pi_Y\colon Y := (A \times Z)/\langle(\tau,\gamma)\rangle \longrightarrow \Delta, \tag{1}$$

where we take the quotient by the diagonal action. Note that $Y$ is smooth and $\pi_Y$ is projective.

**Theorem 7.** *In the above notation, assume in addition that $\chi_i(1) = \pm 1$ for every $i$. Then $\pi_Y\colon Y \to \Delta$ is a $\mathbb{Z}$-homology fiber bundle but not a homotopy fiber bundle.*

The smallest such $Z$, to be constructed in Example 10, has dimension 2. Choosing $A$ suitably gives the following.

**Corollary 8.** *For every $m \geq 2$ there is a flat projective morphism $\pi_Y\colon Y \to \Delta$ of smooth analytic varieties of relative dimension $m$, such that $\pi_Y$ is a $\mathbb{Z}$-homology fiber bundle but not a homotopy fiber bundle.*

In all the examples constructed below, the fibers $Y_t$ have the property that their fundamental group is the extension of a finite group by $\mathbb{Z}^{2r}$ for some $r \geq 1$.

We first record the topological calculation used in the proof.

**Lemma 9.** *Let $n \geq 2$ and let $M$ be a topological space whose only nonzero cohomology groups are $H^0(M,\mathbb{Z}) = \mathbb{Z}$ and $H^{n-1}(M,\mathbb{Z}) \cong \mathbb{Z}^r$. Let $\gamma$ be an automorphism of $M$ of order $c$, let $T$ be its induced action on $H^{n-1}(M,\mathbb{Z})$, and let $\chi_T(x)$ be the characteristic polynomial of $T$. Let $A$ be an abelian variety and let $\tau\colon A \to A$ be a translation of order $c$. Then the projection*

$$(A \times M)/\langle(\tau,\gamma)\rangle \longrightarrow A/\langle\tau\rangle$$

*induces an isomorphism on integral cohomology if and only if $\chi_T(1) = \pm 1$.*

*Proof.* Topologically, an abelian variety of dimension $m$ is $(S^1)^{2m}$. We may assume that $\tau$ acts on the first factor only. Then

$$(A \times M)/\langle(\tau,\gamma)\rangle \simeq \big((S^1 \times M)/\langle(\tau,\gamma)\rangle\big) \times (S^1)^{2m-1}.$$

By the Künneth formula, it remains to consider the projection

$$q\colon (S^1 \times M)/\langle(\tau,\gamma)\rangle \longrightarrow S^1/\langle\tau\rangle.$$

By assumption, $R^i q_*\mathbb{Z}$ is nonzero only for $i = 0$ and $i = n-1$. The base $S^1/\langle\tau\rangle$ is homeomorphic to $S^1$. Thus the Leray spectral sequence degenerates at the $E_2$ page.

We have $q_*\mathbb{Z} \cong \mathbb{Z}$, and $R^{n-1}q_*\mathbb{Z}$ is the local system with fiber $H^{n-1}(M,\mathbb{Z})$ and monodromy $T$; denote it by $L_T$. A direct computation gives that

$$H^0(S^1, L_T) = \ker(T-I) \qquad \text{and} \qquad H^1(S^1, L_T) = \operatorname{coker}(T-I).$$

Thus both groups vanish if and only if $T-I \in \mathrm{GL}_r(\mathbb{Z})$, which is equivalent to $\det(T-I) = \pm 1$, and hence to $\chi_T(1) = \pm 1$. The assertion follows now from the Leray spectral sequence and the Künneth formula. □

*Proof of Theorem 7.* Let $p_i \in B_i \subset Z$ be disjoint, $\gamma$-invariant balls around the critical points $p_i$. Let $B$ be the union of the $B_i$, with interior $B^\circ$ and boundary $\partial B$. For $t \in \Delta$ near 0, $Z_t$ is the union of $B \cap Z_t$ and $Z_t \setminus B^\circ$, meeting along $\partial B \cap Z_t$. Correspondingly, $Y_t$ is the union of

$$\big(A \times (B \cap Z_t)\big)/\langle(\tau,\gamma)\rangle \quad \text{and} \quad \big(A \times (Z_t \setminus B^\circ)\big)/\langle(\tau,\gamma)\rangle, \tag{2}$$

meeting along

$$\big(A \times (\partial B \cap Z_t)\big)/\langle(\tau,\gamma)\rangle. \tag{3}$$

For all sufficiently small $t$, retraction to $Z_0$ gives homeomorphisms between the second spaces in (2), and the same holds for the intersection spaces in (3). Since $B_i \cap Z_0$ is contractible, each

$$\big(A \times (B_i \cap Z_0)\big)/\langle(\tau,\gamma)\rangle$$

retracts to $A/\langle\tau\rangle$.

Using the Mayer–Vietoris sequence, it remains to show that the natural maps

$$\big(A \times (B_i \cap Z_t)\big)/\langle(\tau,\gamma)\rangle \longrightarrow A/\langle\tau\rangle$$

induce isomorphisms on integral cohomology. For $t \neq 0$, $B_i \cap Z_t$ is the Milnor fiber $M_i$, and the assertion follows from Lemma 9 and the assumption $\chi_i(1) = \pm 1$. Hence $\pi_Y$ in (1) is a $\mathbb{Z}$-homology fiber bundle, as we want.

It remains to show that $\pi_Y$ is not a homotopy fiber bundle. For a contradiction, assume that $\pi_Y$ is a homotopy fiber bundle. Then the same holds for $A \times Z \to \Delta$, because $A \times Z \to Y$ is finite étale. This is a contradiction, because $Z \to \Delta$ has at least one isolated critical point in the central fiber and so $Z \to \Delta$ is not a $\mathbb{Q}$-homology fiber bundle; the same follows for $A \times Z \to \Delta$ by the Künneth formula. □

2.3. **A family of bielliptic surfaces.** We start with the simplest, two-dimensional example.

**Example 10.** Set

$$S \coloneqq \big\{zy^2 + x^3 + tz^3 = 0\big\} \subset \mathbb{P}^2_{x,y,z} \times \Delta, \tag{4}$$

where $t$ is a coordinate on the disc $\Delta$. Let $\xi$ be a primitive sixth root of unity and consider the order-six automorphism

$$\xi_1 \colon S \longrightarrow S, \qquad ([x:y:z],t) \longmapsto ([\xi^2 x : \xi^3 y : z], t).$$

Let $F$ be an elliptic curve and let $\xi_2 \in \mathrm{Aut}(F)$ be translation by a point of order six. Set

$$Y \coloneqq (S \times F)/\langle(\xi_1,\xi_2)\rangle,$$

and let $f \colon Y \to \Delta$ be the natural projection.

The Milnor fiber of $S \to \Delta$ can be identified with the affine elliptic curve

$$M = \{y^2 + x^3 + 1 = 0\} \subset \mathbb{A}^2.$$

Thus $H^1(M, \mathbb{Z}) \cong \mathbb{Z}^2$, and the induced action has characteristic polynomial

$$\chi(x) = x^2 - x + 1.$$

In particular, $\chi(1) = 1$, and Theorem 7 shows that $f$ is a $\mathbb{Z}$-homology fiber bundle. The special fiber $Y_0$ is a cuspidal curve bundle over $F/\langle \xi_2 \rangle$, whereas the other fibers are bielliptic surfaces. In particular, $f$ is neither smooth nor a $C^0$-fiber bundle. It is not a homotopy fiber bundle either: the finite étale cover $S \times F \to Y$ is not even a $\mathbb{Q}$-homology fiber bundle over $\Delta$.

The general fibers of $Y \to \Delta$ constructed above are surfaces with surprisingly simple cohomology, as we show next.

**Lemma 11.** *The general fiber $Y_t$ of $Y \to \Delta$ in Example 10 is a bielliptic surface. Its integral cohomology is torsion-free and is given by*

$$H^k(Y_t, \mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & k = 0, 4, \\ \mathbb{Z}^2, & k = 1, 2, 3. \end{cases}$$

*Proof.* For $t \neq 0$, the fiber $S_t$ of $S \to \Delta$ is an elliptic curve and

$$Y_t \simeq (S_t \times F)/\langle (\xi_1, \xi_2) \rangle$$

is a bielliptic surface of Bagnera–de Franchis type $\mathbb{Z}/6\mathbb{Z}$ (type 7 in the numbering of [Ser90, Proposition 1.2]). The assertion over $\mathbb{Q}$ follows by analyzing the invariant part of the cohomology of $S_t \times F$. Iitaka showed that its integral cohomology is torsion-free, see [Ser90, Remark 1.6]. This proves the lemma. □

2.4. **Further examples.** Fix $n \geq 2$ and a tuple $\vec{c} = (c_1, \dots, c_n) \in \mathbb{N}^n$ of pairwise coprime integers $c_i \geq 2$. Set $c := \prod_i c_i$, $a_i := c/c_i$ and

$$Z^\circ_{\vec{c}} := \{\textstyle\sum_{i=1}^n x_i^{c_i} = t\} \subset \mathbb{A}^n \times \mathbb{A}^1_t.$$

There is a $\mu_c$-action on $Z^\circ_{\vec{c}}$, given by

$$\gamma_{\vec{c}} \colon (x_1, \cdots, x_n, t) \longmapsto (\zeta^{a_1} x_1, \cdots, \zeta^{a_n} x_n, t),$$

where $\zeta$ denotes a primitive c-th root of unity.

We check that the assumptions of Lemma 9 are satisfied. The eigenvalues of the monodromy on the Milnor fiber of a sum of powers were determined in [Pha65], see also [Bri66]. We recall the calculation needed here.

**Lemma 12.** *Let $T_{\vec{c}}$ denote the monodromy on the reduced middle homology of the Milnor fiber $M_{\vec{c}}$ of $\sum_i x_i^{c_i} = t$, and let $\chi_{\vec{c}}$ be its characteristic polynomial. Then*

$$\chi_{\vec{c}}(1) = \pm 1.$$

*Proof.* For $x^r = t$, the $\mu_r$-action $x \mapsto \zeta_r x$ equals the monodromy action, whose eigenvalues on reduced homology are

$$\exp(2\pi i a) \qquad \text{with} \qquad a = \frac{1}{r}, \dots, \frac{r-1}{r}.$$

By the Thom–Sebastiani theorem, the $\mu_c$-action on the homology of $M_{\vec{c}}$ equals the monodromy action, and its eigenvalues are

$$\exp(2\pi i a) \qquad \text{with} \qquad a = \sum_{i=1}^{n} \frac{b_i}{c_i} \qquad \text{and} \qquad b_i = 1, \dots, c_i - 1.$$

Since the $c_i$ are pairwise coprime, these eigenvalues are pairwise distinct and none of these is a $c_i$-th root of unity for any $i$. Therefore $\chi_{\vec{c}}(\lambda)$ divides

$$\frac{\lambda^c - 1}{\lambda - 1} \prod_i \frac{\lambda - 1}{\lambda^{c_i} - 1}.$$

The value of this expression at $\lambda = 1$ is 1, which proves the claim. □

It is most economical to compactify $Z^\circ_{\vec{c}}$ in a weighted projective space. Let

$$Z_{\vec{c}} \coloneqq \{\textstyle\sum_{i=1}^n x_i^{c_i} = t w^c\} \subset \mathbb{P}(a_1, \dots, a_n, 1) \times \mathbb{A}^1_t,$$

where $x_i$ has weight $a_i$ and $w$ has weight 1. A $\mu_c$-action is given by

$$\gamma_{\vec{c}} \colon ([x_1 : \cdots : x_n : w], t) \longmapsto ([\zeta^{a_1} x_1 : \cdots : \zeta^{a_n} x_n : w], t).$$

In the chart $w \neq 0$, the central fiber has the unique singularity $\sum_i x_i^{c_i} = 0$, and the other fibers are smooth. Along $w = 0$ the fibers are singular, but they admit a $\mu_c$-equivariant simultaneous resolution. Denote the resulting family and action by

$$\pi_{\widetilde{Z}_{\vec{c}}} \colon \widetilde{Z}_{\vec{c}} \longrightarrow \mathbb{A}^1_t \qquad \text{and} \qquad \gamma_{\vec{c}} \colon \mu_c \times \widetilde{Z}_{\vec{c}} \longrightarrow \widetilde{Z}_{\vec{c}},$$

respectively. Thus $\pi_{\widetilde{Z}_{\vec{c}}}$ has a unique critical point, where

$$\pi_{\widetilde{Z}_{\vec{c}}}(x_1, \dots, x_n) = \textstyle\sum_i x_i^{c_i}, \qquad \gamma_{\vec{c}}(x_1, \dots, x_n) = (\zeta^{a_1} x_1, \dots, \zeta^{a_n} x_n).$$

By [Bri66, Satz 1], the link of such a singularity is a topological sphere for $n \geq 4$, so the central fiber of $\pi_{\widetilde{Z}_{\vec{c}}}$ is a PL-manifold. However, the link is usually not a standard $C^\infty$-sphere; see [Bri66, Satz 3] for the precise conditions.

**Example 13.** Take $\vec{c} = (2, 3, 5)$. Then $c = 30$, and in the weighted model

$$Z_{\vec{c}} = \{x^2 + y^3 + z^5 = t w^{30}\} \subset \mathbb{P}(15, 10, 6, 1) \times \mathbb{A}^1_t.$$

The chart $w \neq 0$ is the standard smoothing of the $E_8$ surface singularity $x^2 + y^3 + z^5 = 0$. The action is

$$\gamma_{\vec{c}}([x : y : z : w], t) = ([\zeta^{15} x : \zeta^{10} y : \zeta^6 z : w], t),$$

where $\zeta$ is a primitive thirtieth root of unity.

Let $A$ be an elliptic curve and let $\tau \in \mathrm{Aut}(A)$ denote translation by a point of order 30. The special fiber of the quotient family (1) is singular along $[0 : 0 : 0 : 1] \times A/\langle\tau\rangle$ and is locally the product of $A/\langle\tau\rangle$ with the $E_8$ surface singularity. For the $E_8$ singularity, the Milnor lattice is the $E_8$ lattice and the Milnor monodromy is the corresponding Coxeter transformation [Pha65, Bri66]. Since $T - I$ is unimodular, the resulting family is a $\mathbb{Z}$-homology fiber bundle but is not smooth.

The special fiber of the resulting family in Theorem 7 is a threefold with canonical singularities, locally analytically isomorphic to the product of an $E_8$ surface singularity with an elliptic curve.

**Example 14.** Adding an even number of squares does not change the monodromy action on the Milnor fiber, cf. [Bri66, Lemmas 1, 3, 4]. Thus for any $n \geq 1$, the Milnor fiber $M$ of

$$Z := \{x_1x_2 + \cdots + x_{2n-1}x_{2n} + y^2 + z^3 + t = 0\} \subset \mathbb{A}^{2n+2} \times \mathbb{A}^1_t$$

satisfies $H_{2n+1}(M, \mathbb{Z}) = \mathbb{Z}^2$, and the $\mathbb{Z}/6$-action

$$(x_1, \ldots, x_{2n}, y, z, t) \longmapsto (\xi^3 x_1 \ldots, \xi^3 x_{2n}, \xi^3 y, \xi^2 z, t)$$

has characteristic polynomial $x^2 - x + 1$.

The central fiber $Z_0$ has an isolated terminal singularity at the origin. Let $B_\epsilon \subset \mathbb{A}^{2n+2}$ be an analytic ball of radius $\epsilon > 0$ and consider the neighborhood $U_\epsilon = Z_0 \cap B_\epsilon$. For sufficiently small $\epsilon$, [Mil68, Theorem 2.10] implies that $U_\epsilon$ is homeomorphic to the cone $C(\partial U_\epsilon)$ over the link $\partial U_\epsilon$ of $0 \in Z_0$ and the homeomorphism $U_\epsilon \cong C(\partial U_\epsilon)$ is a diffeomorphism outside the singular point of $Z_0$. By [Bri66, Satz 1], $\partial U_\epsilon$ is homeomorphic to a sphere $S^{4n+1}$. Hence, $Z_0$ is a PL-manifold.

If $n = 1$, then the given differentiable structure on $\partial U_\epsilon$ must be the standard structure on $S^5$ by [KM63]. We can thus endow $U_\epsilon$ with the standard differentiable structure of the ball and get a $C^\infty$-manifold structure on $Z_0$ which agrees with the one induced by the algebraic structure of $Z_0$ outside the singularity.

## 3. Homotopy fiber bundles that are not submersions

In this section we analyze the topology of the blow-up of the examples from Section 2. To this end, let $Y \to \Delta$ be a projective morphism which is a $\mathbb{Z}$-homology fiber bundle but not a submersion (e.g. not a homotopy fiber bundle), with $Y$ smooth. Let $P \to \Delta$ be a smooth projective morphism with simply connected fibers which admits a relative projective embedding $Y \hookrightarrow P$ over $\Delta$ of codimension at least 2. Let $X := \mathrm{Bl}_Y P$.

**Theorem 15.** *In the above notation, the induced map $X \to \Delta$ is a homotopy fiber bundle which is not a submersion.*

Combining this with the examples in the previous section, we obtain

**Corollary 16.** *For every $n \geq 4$ there is a flat, projective morphism $f\colon X \to \Delta$ of smooth analytic varieties of relative dimension $n$, such that $f$ is a homotopy fiber bundle but not a submersion.*

We will prove the above theorem in the remainder of this section.

3.1. **Submersions and homology fiber bundles under blow-ups.** In this section, $f\colon M \to N$ denotes a proper submersion between complex manifolds and

$$\pi\colon M' := \mathrm{Bl}_Z M \longrightarrow M$$

denotes the blow-up of $M$ along a closed connected complex submanifold $Z \subset M$ of codimension $c \geq 2$. We denote the exceptional divisor of $\pi$ by

$$E := \mathbb{P}(N_{Z/M}) \subset M' \qquad \text{with projection} \qquad p\colon E \to Z.$$

The natural maps induced by $f$ are denoted by

$$f'\colon M' \longrightarrow N \qquad \text{and} \qquad f_Z\colon Z \longrightarrow N.$$

With this notation, we have the following simple but useful result.

**Proposition 17.** *In the above notation, the following holds:*

*(i) $f'$ is a submersion if and only if $f_Z$ is a submersion.*
*(ii) $f'$ is a $\mathbb{Z}$-homology fiber bundle if and only if $f_Z$ is a $\mathbb{Z}$-homology fiber bundle.*

*Proof.* Let us first prove item (i). If $f_Z$ is a submersion, then $f_Z$ is flat and the fibers of $f_Z$ are smooth. It follows that the fibers of $f'$ are smooth, hence $f'$ is a submersion. Conversely, assume that $f'$ is a submersion and let $\widetilde{z} = (z, [\ell]) \in E = \mathbb{P}(N_{Z/M})$. Then

$$\operatorname{im}(d\pi_{\widetilde{z}}) = T_z Z + \widetilde{\ell},$$

where $\widetilde{\ell} \subset T_z M$ is a lift of $\ell$. Suppose that $f_Z$ is not a submersion at $z$. Choose a hyperplane $H \subset T_{f(z)} N$ containing $df_z(T_z Z)$. Since $df_z$ is surjective and $\operatorname{codim}_M Z \geq 2$, there is a line $\widetilde{\ell} \subset df_z^{-1}(H)$ which is not contained in $T_z Z$. At the corresponding point $\widetilde{z} \in E$, the image of $df'_{\widetilde{z}}$ is contained in $H$. Hence $f'$ is not a submersion, proving (i).

Next, we prove item (ii). By the blow-up formula, we have

$$R\pi_* \mathbb{Z}_{M'} \simeq \mathbb{Z}_M \oplus \bigoplus_{r=1}^{c-1} \iota_* \mathbb{Z}_Z[-2r] \tag{5}$$

in the derived category of sheaves of abelian groups on $M$, where $\iota\colon Z \to M$ denotes the inclusion. This follows, for instance, from the projective-bundle formula for the exceptional divisor and the Gysin morphism, as may be checked stalkwise. For the analogous statement with $\ell$-adic coefficients on the étale site, see [SGA5, Exp. VII, Thm. 2.2.1 and Thm. 8.1]; see also [Voi02, Thm. 7.31] for the resulting decomposition of integral cohomology groups.

Applying $Rf_*$ to (5), taking cohomology sheaves (and omitting the subscripts from the constant sheaves with stalk $\mathbb{Z}$), we obtain, for every $k$,

$$R^k f'_* \mathbb{Z} \simeq R^k f_* \mathbb{Z} \oplus \bigoplus_{i=1}^{c-1} R^{k-2i} f_{Z*} \mathbb{Z}. \tag{6}$$

Since $f$ is a proper submersion, Ehresmann's lemma implies that $R^k f_* \mathbb{Z}$ is a local system for every $k$.

Suppose first that $f_Z$ is a $\mathbb{Z}$-homology fiber bundle. Then each $R^q f_{Z*} \mathbb{Z}$ is a local system. It then follows immediately from (6) that each $R^k f'_* \mathbb{Z}$ is a local system. Hence $f'\colon M' \to N$ is a $\mathbb{Z}$-homology fiber bundle.

Conversely, suppose that $f'\colon M' \to N$ is a $\mathbb{Z}$-homology fiber bundle. Then $R^k f'_* \mathbb{Z}$ is a local system for every $k$. Since $c \geq 2$, the summand corresponding to $i = 1$ occurs in (6). Thus, for every $q$, $R^q f_{Z*} \mathbb{Z}$ is a direct summand of $R^{q+2} f'_* \mathbb{Z}$. A direct summand of a local system is again a local system. Hence all $R^q f_{Z*} \mathbb{Z}$ are local systems, and therefore $f_Z$ is a $\mathbb{Z}$-homology fiber bundle. This concludes the proof. □

### 3.2. Proof of Theorem 15 and Corollary 16.

*Proof of Theorem 15.* Let $Y \to \Delta$ be a projective morphism which is a $\mathbb{Z}$-homology fiber bundle but not a submersion. Let $P \to \Delta$ be a smooth projective morphism and let $Y \hookrightarrow P$ be a relative projective embedding of codimension at least 2 over $\Delta$.

Let $X := \mathrm{Bl}_Y P$ be the blow-up of $P$ along $Y$. By Proposition 17, the induced map $f\colon X \to \Delta$ is a $\mathbb{Z}$-homology fiber bundle which is not a submersion.

Since the fibers of $P \to \Delta$ are simply connected, the same holds for the fibers of $f$. Indeed, let $y \in \Delta$ and let $U_y \subset \Delta$ be a sufficiently small disc centered at $y$. Then $X_{U_y} = f^{-1}(U_y)$ is the blow-up of $P_{U_y}$ along a closed submanifold, hence it is simply connected because $P_{U_y}$ is.

It then follows from Whitehead's theorem (see Corollary 6) that $f$ is a homotopy fiber bundle, as we want. □

*Proof of Corollary 16.* Example 10 and Theorem 7 show that there is a flat projective morphism $Y \to \Delta$ of relative dimension two which is a $\mathbb{Z}$-homology fiber bundle but not a submersion. If $n \geq 6$, then we can choose a closed embedding $Y \hookrightarrow P := \mathbb{P}^n \times \Delta$ (see Lemma 18 below) and define $X := \mathrm{Bl}_Y P$. By Theorem 15, $X \to \Delta$ is a homotopy fiber bundle which is not a submersion. This proves the corollary for $n \geq 6$. The cases $n = 4, 5$ follow similarly by embedding $Y$ into a bundle of rationally connected fourfolds over $\Delta$, cf. Example 23 below. □

**Lemma 18.** *Let $B$ be a smooth curve and $q\colon Y \to B$ a nonconstant morphism from a smooth projective variety of dimension $m$. Then there is a closed embedding*

$$\iota\colon Y \hookrightarrow \mathbb{P}^{2m} \times B \qquad \textit{such that} \quad \mathrm{pr}_2 \circ \iota = q.$$

*Proof.* Choose a closed embedding $j\colon Y \hookrightarrow \mathbb{P}^n$, with $n$ sufficiently large. Let $\Sigma \subset \mathbb{P}^n$ be the closure of the union of all lines $\overline{j(x)j(y)}$, where $x \neq y$ and $q(x) = q(y)$, together with all tangent lines to $j(Y)$ corresponding to nonzero vectors in $\ker(dq_y)$ for some $y \in Y$. One checks that $\dim \Sigma \leq 2m$. Hence, if $p\colon \mathbb{P}^n \to \mathbb{P}^{2m}$ is a general linear projection, then $\iota := (p \circ j, q)$ has the properties we want. □

**Remark 19.** Theorem 15 can in particular be applied to $P = \mathbb{P}^N \times \Delta$ and $Y \to \Delta$ as in Corollary 8, together with an embedding $Y \hookrightarrow P$ over $\Delta$ which exists for $N \geq 6$, cf. Lemma 18. The resulting family has rationally connected fibers.

To get fibers of general type one can start with the blow-up $\hat{H} = \mathrm{Bl}_p H$ of any simply connected smooth projective general type variety $H$ of dimension at least 7 in a point $p$ with exceptional divisor $E = \mathbb{P}^N$, $N = \dim H - 1 \geq 6$, and use the embedding $Y \hookrightarrow E \times \Delta \hookrightarrow \hat{H} \times \Delta$.

## 4. Fibrations over Riemann surfaces

In this section we prove the following result of independent interest, which will be used in the proof of Theorem 1.

**Proposition 20.** *Let $g\colon X \to \Sigma$ be a proper holomorphic map between connected complex manifolds with the following properties:*

*(i)* $\dim_{\mathbb{C}} \Sigma = 1$ *and* $\dim_{\mathbb{C}} X \geq 4$*;*
*(ii)* $R^i g_* \mathbb{Z}$ *is a local system for all* $i$*;*
*(iii) the fibers of $g$ are simply connected.*

*Then, for any analytic neighborhood $U \subset \Sigma$ of the critical values $g(\mathrm{Crit}(g))$ of $g$, there is a proper $C^\infty$-submersion $\tilde{g}\colon X \to \Sigma$, which agrees with $g$ outside of $g^{-1}(U)$ and which is homotopic to $g$.*

Before we prove the proposition, we formulate the following version of Smale's h-cobordism theorem for manifolds with boundary, that we will use.

**Theorem 21** (Relative h-cobordism Theorem)**.** *Let $I_x, I_y$ denote intervals $[0,1]$ with coordinates $x$ and $y$. Let $C$ be a compact simply connected smooth manifold with corners of dimension at least six and with a proper surjective map $g\colon C \to I_x \times I_y$. Assume that the boundary of $C$ has four faces $M_0$, $M_1$, $V_0$, $V_1$ corresponding to $y=0$, $y=1$, $x=0$, $x=1$, respectively. Suppose that $g$ is a submersion over $y=0$, $x=0$, and $x=1$, with smooth fiber $F$ and that there are compatible bundle trivializations*

$$F \times I_x \xrightarrow{\sim} M_0 \qquad \text{and} \qquad F \times I_y \xrightarrow{\sim} V_j \quad \text{for} \quad j=0,1. \tag{7}$$

*If the inclusions $M_i \hookrightarrow C$ are homotopy equivalences for $i=0,1$, then the boundary identifications (7) extend to a diffeomorphism*

$$\Psi\colon F \times I_x \times I_y \xrightarrow{\sim} C.$$

*Proof.* The result follows from the more general version of the s-cobordism theorem for manifolds with boundary, see [KS77, Essay I, §1, p. 4].

Alternatively, glue $\partial M_0 \times I_y \times \mathbb{R}_{\geq 0}$ to the vertical boundary of $C$. The prescribed product structure in (7) gives the embedding at infinity required in [Sm62, Theorem 3.1]; that theorem yields a product diffeomorphism extending this embedding, whose restriction to $C$ gives the required map $\Psi$. □

*Proof of Proposition 20.* The critical values of $g$ form a closed discrete subset of $\Sigma$. It thus suffices to treat the case when $g$ has only one critical value and $U \subset \Sigma$ is a small disc containing the critical value. Let $I_x = [0,1]$ and $I_y = [0,1]$ be intervals with coordinates $x$ and $y$, respectively. Choose a closed embedding

$$Q \coloneqq I_x \times I_y \hookrightarrow U \subset \Sigma$$

of a square which contains the critical value of $g$ in its interior. In particular, $g$ is a submersion over a neighborhood of the boundary $\partial Q$.

Put

$$C = g^{-1}(Q) \qquad \text{and} \qquad M_i = g^{-1}(I_x \times \{i\}) \quad \text{for } i=0,1, \tag{8}$$

and

$$V = g^{-1}\big((\{0\} \times I_y) \cup (\{1\} \times I_y)\big). \tag{9}$$

Since $g$ is a submersion over a neighborhood of $\partial Q$, Ehresmann's lemma allows us to fix compatible bundle trivializations

$$M_0 \simeq F \times I_x \qquad \text{and} \qquad V \simeq (F \times I_y) \sqcup (F \times I_y), \tag{10}$$

where $F$ is a fixed smooth fiber of $g$.

Note that $C$ is a manifold with corners. It is a cobordism between the manifolds with boundary $M_0$ and $M_1$. The idea of the proof is to apply the h-cobordism theorem (see Theorem 21). This trivializes $C$ but the trivialization $\Psi$ might not be compatible with the natural projection to $I_x$ at the boundary component $y=1$. To fix this, we will apply Cerf's theorem [Cer70]. Once $\Psi$ is compatible with the projection to $I_x \times I_y$ over

(a neighborhood of) the boundary of $I_x \times I_y$, we can define $\widetilde{g}$ via the diffeomorphism $\Psi$ and the projection $F \times I_x \times I_y \to I_x \times I_y$, which is clearly a submersion.

**Step 1.** The inclusion $M_i \hookrightarrow C$ is a homotopy equivalence for $i = 0, 1$. Moreover, $C$ is simply connected.

*Proof.* The argument is similar to the proof of Corollary 6. Indeed, since $g$ is a $\mathbb{Z}$-homology fiber bundle and $I_x$ and $Q$ are contractible, $M_i \hookrightarrow C$ induces an isomorphism on homology, hence is a homotopy equivalence by Whitehead's theorem [Hat02, Corollary 4.33]. Here we used that $M_i$ and $C$ are simply connected, because the fibers of $g$ are simply connected, see [Sm57, Main Theorem]. □

**Step 2.** There is a diffeomorphism

$$\Psi \colon (F \times I_x) \times I_y \longrightarrow C$$

which agrees with the chosen trivialization (10) on

$$\big((F \times I_x) \times \{0\}\big) \ \cup \ \big((F \times \{0,1\}) \times I_y\big).$$

*Proof.* By Step 1, $C$ is a h-cobordism between the manifolds with boundary $M_0$ and $M_1$. Since $\dim_{\mathbb{R}} C = 2 \cdot \dim_{\mathbb{C}} X \geq 8$, this h-cobordism is trivial by Smale's h-cobordism theorem with boundary, see Theorem 21. This concludes Step 2. □

Since $g$ is a submersion over the top edge $I_x \times \{1\}$ as well, we can fix by Ehresmann's lemma a bundle trivialization

$$\tau \colon F \times I_x \longrightarrow M_1 \qquad \text{with} \qquad g(\tau(z, x)) = (x, 1),$$

which agrees with $\Psi$ from Step 2 at $x = 0$. We then consider the composition

$$\Phi \coloneqq \tau^{-1} \circ \Psi|_{(F \times I_x) \times \{1\}} \colon F \times I_x \longrightarrow F \times I_x,$$

which is the identity at $x = 0$.

**Step 3.** There is a smooth path of pseudo-isotopies

$$\Phi_t \colon F \times I_x \longrightarrow F \times I_x$$

with $\Phi_0 = \Phi$ and $\Phi_1 = \mathrm{id}$.

*Proof.* Cerf [Cer70, p. 7, Définition 2] considers the group

$$P(F) \coloneqq \{\varphi \in \mathrm{Diff}(F \times I_x) \mid \varphi|_{F \times \{0\}} = \mathrm{id}\}$$

of pseudo-isotopies, equipped with the $C^\infty$-topology. Note that $F$ is a simply connected compact manifold without boundary of real dimension at least 6. Hence, [Cer70, p. 8, Théorème 0] implies that $P(F)$ is connected and there is a smooth path with the required properties. This concludes Step 3. □

**Step 4.** There is a diffeomorphism

$$\Psi' \colon (F \times I_x) \times I_y \longrightarrow C$$

which agrees with the chosen trivializations (10) on

$$\big((F \times I_x) \times \{0\}\big) \cup \big((F \times \{0\}) \times I_y\big),$$

is fiber-preserving on the right vertical face, in the sense that

$$g\big(\Psi'(z,1,y)\big) = (1,y) \qquad \text{for all } (z,y) \in F \times I_y,$$

and satisfies $\Psi'|_{(F\times I_x)\times\{1\}} = \tau$.

*Proof.* Define

$$\Xi\colon (F \times I_x) \times I_y \longrightarrow (F \times I_x) \times I_y, \qquad \big((z,x),y\big) \longmapsto \big(\Phi_{1-y}^{-1}(z,x),y\big),$$

and use it to modify $\Psi$ from Step 2 as follows:

$$\Psi' = \Psi \circ \Xi.$$

Since $\Phi_1 = \mathrm{id}$, the map $\Psi'$ agrees with $\Psi$ on the bottom face $I_x \times \{0\}$. Since every $\Phi_t$ fixes $F \times \{0\}$ pointwise, it also agrees with $\Psi$ on the left vertical face. Moreover, every $\Phi_t$ preserves $F \times \{1\}$, and hence $\Psi'$ is fiber-preserving on the right vertical face $\{1\} \times I_y$. Finally, using $\Phi_0 = \Phi$, we obtain

$$\Psi'|_{(F\times I_x)\times\{1\}} = \Psi|_{(F\times I_x)\times\{1\}} \circ \Phi^{-1} = \tau.$$

This proves Step 4. □

By Step 4, we have

$$g\big(\Psi'(z,x,y)\big) = (x,y)$$

for all $(x,y) \in \partial Q$ and $z \in F$. After modifying $\Psi'$ by a diffeomorphism supported in a collar of the boundary and equal to the identity on the boundary, we may assume that the same holds in some neighborhood of $\partial Q$. We may then define

$$\tilde{g}_Q\colon C \longrightarrow Q \qquad \text{by} \qquad \tilde{g}_Q\big(\Psi'(z,x,y)\big) = (x,y).$$

This is a smooth submersion and agrees with $g$ on a neighborhood of $\partial C$.

Since $Q = [0,1]^2$ is convex,

$$H\colon C \times [0,1] \longrightarrow Q, \qquad (c,t) \mapsto H_t(c) \coloneqq \big((1-t)\cdot g(c) + t \cdot \tilde{g}_Q(c)\big)$$

defines a homotopy from $g|_C$ to $\tilde{g}_Q$. On the neighborhood of $\partial C$ where $g = \tilde{g}_Q$, this homotopy is constant. Thus $g|_C$ and $\tilde{g}_Q$ are homotopic relative to that neighborhood. The map $g$, restricted to a small neighborhood of $X \setminus C$, thus glues with $\tilde{g}_Q$ to produce a $C^\infty$-submersion $\tilde{g}\colon X \to \Sigma$ which is homotopic to $g$ and agrees with $g$ in a neighborhood of $X \setminus g^{-1}(Q) \supset X \setminus g^{-1}(U)$. Since $g^{-1}(Q)$ and $Q$ are compact, $\tilde{g}$ is proper. This concludes the proof. □

## 5. Proof of Theorem 1

5.1. **Promoting homology fiber bundles to homotopy fiber bundles.** Let $h\colon V \to \mathbb{P}^1$ be a morphism of smooth projective varieties which is a $\mathbb{Z}$-homology fiber bundle but not a submersion. Let $P \to \mathbb{P}^1$ be a smooth projective morphism with simply connected fibers and with an embedding $V \hookrightarrow P$ which lifts $h$. Assume $\operatorname{codim}_P V \geq 2$, and put

$$Y \coloneqq \mathrm{Bl}_V P \qquad \text{with natural map} \qquad g\colon Y \longrightarrow \mathbb{P}^1. \tag{11}$$

**Proposition 22.** *In the above notation, the following holds.*

*(i) All fibers of $g\colon Y \to \mathbb{P}^1$ are simply connected.*

*(ii) For all $i$, $R^i g_*\mathbb{Z}$ is a trivial local system on $\mathbb{P}^1$.*

*(iii) $g$ is a homotopy fiber bundle.*
*(iv) $g$ is not a submersion.*

*Proof.* Fix $y \in \mathbb{P}^1$ and choose a sufficiently small neighborhood $U \subset \mathbb{P}^1$ of $y$ such that $g^{-1}(y) \hookrightarrow g^{-1}(U)$ and $P_y \hookrightarrow P_U = P \times_{\mathbb{P}^1} U$ are deformation retracts. Since $P_y$ is simply connected, the same holds for $P_U$ and hence for the blow-up $g^{-1}(U)$. Thus, $g^{-1}(y)$ is also simply connected. This proves item (i).

By item (ii) in Proposition 17, $g$ is a $\mathbb{Z}$-homology fiber bundle. Since $\mathbb{P}^1$ is simply connected, each $R^i g_* \mathbb{Z}$ is trivial, which proves item (ii).

Since $g$ is a $\mathbb{Z}$-homology fiber bundle with simply connected fibers, Corollary 6 implies that $g$ is a homotopy fiber bundle. This proves (iii).

Finally, $g$ is not a submersion by Proposition 17. This concludes the proof. □

5.2. **Explicit examples and proof of Theorem 1.** Let $L = \mathcal{O}_{\mathbb{P}^1}(1)$, and let $s$ be a general section of $L^{\otimes 6}$. Define the surface

$$S := \{zy^2 + x^3 + sz^3 = 0\} \subset \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus L^{\otimes 2} \oplus L^{\otimes 3}),$$

where the coordinates $x, y, z$ correspond to the summands $L^{\otimes 2}$, $L^{\otimes 3}$, and $\mathcal{O}_{\mathbb{P}^1}$, respectively. This is a rational elliptic surface with six singular fibers of Kodaira type II; rational elliptic surfaces with this configuration, including their various projective models, are studied in detail in [CGMVZ26].

Around each zero of $s$, the projection $\pi\colon S \to \mathbb{P}^1$ is analytically locally isomorphic to the local model from (4). We have an order-six fiberwise action on $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus L^{\otimes 2} \oplus L^{\otimes 3})$:

$$\xi_6\colon [x : y : z] \mapsto [\xi^2 x : \xi^3 y : z],$$

where $\xi$ denotes a primitive sixth root of unity. Let $A$ be an abelian variety and $\tau_6 \in \mathrm{Aut}(A)$ be given by translation by a point of order six. We define

$$V := (S \times A)/\langle(\xi_6, \tau_6)\rangle,$$

which admits a finite étale cover $S \times A \to V$ of degree 6. Consider the natural morphism

$$h\colon V \longrightarrow \mathbb{P}^1$$

that is induced by $S \to \mathbb{P}^1$. By Theorem 7 and Example 10, this map is a $\mathbb{Z}$-homology fiber bundle, but not a submersion.

From now on, we specialize to the case where $A$ is an elliptic curve. Our aim is to lift $h\colon V \to \mathbb{P}^1$ to an embedding $V \hookrightarrow P$, for some smooth projective morphism $P \to \mathbb{P}^1$ with simply connected fibers. By Lemma 18 we can choose $P := \mathbb{P}^N \times \mathbb{P}^1$ for any $N \geq 6$, but with more work we can get such a $P \to \mathbb{P}^1$ of relative dimension four, as we show next.

**Example 23.** Let $T$ be the degree 6 del Pezzo surface obtained by blowing up the three coordinate vertices of $\mathbb{P}^2$. The standard Cremona transformation $[x_0 : x_1 : x_2] \mapsto [x_0^{-1} : x_1^{-1} : x_2^{-1}]$ lifts to an order 2 automorphism $\tau_2$ of $T$, and $[x_0 : x_1 : x_2] \mapsto [x_1 : x_2 : x_0]$ lifts to an order 3 automorphism $\tau_3$ of $T$. Since $\tau_2$ and $\tau_3$ commute, $\tau_6 := \tau_2 \tau_3^{-1}$ is an order 6 automorphism of $T$.

The action is free away from the fixed points of $\tau_2$ and of $\tau_3$. These are $[\pm 1 : \pm 1 : \pm 1]$ and $[1 : \epsilon : \epsilon^2]$ where $\epsilon$ is a third root of unity.

Let $A \subset T$ be the curve $\{\sum_{i\neq j} x_i x_j^2 = 0\}$. Then $A$ is a $\tau_6$-invariant, smooth, elliptic curve which does not contain any of the $\tau_2$ or $\tau_3$ fixed points. Thus $\tau_6$ restricts to a translation of order six on $A$.

Consider

$$V := (S \times A)/\langle(\xi_6, \tau_6)\rangle \subset \big(\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus L^{\otimes 2} \oplus L^{\otimes 3}) \times T\big)/\langle(\xi_6, \tau_6)\rangle =: P'.$$

Here $P' \to \mathbb{P}^1$ is a fiber bundle with fiber

$$R' := \big(\mathbb{P}^2 \times T\big)/\langle(\xi_6, \tau_6)\rangle.$$

Let $R \to R'$ be a functorial resolution of singularities. It extends to a fiber bundle $P \to \mathbb{P}^1$ with fiber $R$, such that the projection $P \to P'$ is an isomorphism over the smooth locus of $P'$. Since $R$ is unirational, $P \to \mathbb{P}^1$ has simply connected fibers.

In fact, $R$ is rational because $R \to \mathbb{P}^2/\langle\xi_6\rangle$ admits a rational section, given by $[1:1:1]$ and so its generic fiber is a del Pezzo surface of degree 6 with a rational point, hence is rational, see [Poo17, Theorem 9.4.8(iii)]. Moreover, $\mathbb{P}^2/\langle\xi_6\rangle$ is rational and so is $R$. The same argument shows that the generic fiber of $P \to \mathbb{P}^1$ is rational. In particular, $P \to \mathbb{P}^1$ is birational to the projection $\mathbb{P}^4 \times \mathbb{P}^1 \to \mathbb{P}^1$.

Note that $P'$ is singular exactly at the images of the fixed points of the square or cube of $(\xi_6, \tau_6)$, hence $P'$ is smooth along the image of $V$. Thus the embedding $V \hookrightarrow P'$ lifts to an embedding $V \hookrightarrow P$. The resulting $g\colon Y \to \mathbb{P}^1$ from (11) has then $\dim Y = 5$.

With the above example at hand, we are finally able to prove Theorem 1 stated in the introduction.

*Proof of Theorem 1.* We define

$$Y := \mathrm{Bl}_V P \qquad \text{with natural map} \qquad g\colon Y \longrightarrow \mathbb{P}^1, \tag{12}$$

where $V$ and $P$ are as in Example 23; in particular, $\dim Y = 5$. Items (i) and (ii) of Theorem 1 follow directly from Proposition 22; item (iv) is established in Proposition 20.

It remains to show that the fibers of $g$ are PL-manifolds, but $g$ is not a $C^0$-fiber bundle. In local analytic coordinates, the singular fibers of $g$ are obtained by blowing up

$$\{x^2 + y^3 = u = 0\} \subset \mathbb{A}^4_{xyuv}.$$

After blow-up the chart

$$\{x^2 + y^3 - u'w = 0\} \subset \mathbb{A}^5_{xyu'vw} \tag{13}$$

given by $u = u'w$ contains the singularities.

Let $Y_0$ be a singular fiber of $g$. Since $Y_0$ is algebraic, it admits a subanalytic triangulation, unique up to PL homeomorphism by [SY84, Corollary 4.3]. As we noted in Example 14, the link of every point in (13) is a standard PL sphere by [Bri66, Satz 1]. This shows that $Y_0$ is locally around each of its singular points a PL manifold; uniqueness up to PL homeomorphisms then shows that these local structures glue to a global PL-manifold structure on $Y_0$.

Note finally that $g$ has six singular fibers, so is not a submersion. To see that it is not a $C^0$-fiber bundle, note that the Milnor fiber $M$ of (13) satisfies $H_3(M, \mathbb{Z}) \cong \mathbb{Z}^2$ (see Example 14) and so there are local vanishing cycles. This implies that $g$ is not a $C^0$-fiber bundle, thus completing the proof of the theorem. □

**Remark 24.** The local computation leading to (13) also shows that, locally at each critical point, $g\colon Y \to \mathbb{P}^1$ can be written as $(x_1, \dots, x_5) \mapsto x_1^2 + x_2^2 + x_3^2 + x_4^3$ in suitable analytic local charts on $Y$. The singularities of the fibers are locally given by $\{x_1^2 + x_2^2 + x_3^2 + x_4^3 = 0\} \subset \mathbb{A}^5$. These are terminal, because they are given by the product of a three-dimensional compound $A_1$-singularity with $\mathbb{A}^1$, see [KM98, Corollary 5.38].

**Remark 25.** One can show that the singular fibers of $g\colon Y \to \mathbb{P}^1$ produced in the proof of Theorem 1 admit the structure of a $C^\infty$-manifold that agrees with the one induced by the algebraic structure outside small neighborhoods of the singular locus.

The surface $S$ used in the proof of Theorem 1 (cf. Example 23) can be replaced by any of the other examples from Section 2.4. In those examples, the singularities of $Z_{\vec{c}}^\circ$ are completely determined by [Bri66], but even if the singularities admit $C^\infty$-structures, it is not clear that there is a natural one.

## 6. Kotschick's conjecture and a question of Kollár–Pardon

Let $X$ be a compact Kähler manifold. In 2013, Kotschick asked whether the following two conditions are equivalent, cf. [Kot22, Question 15]:

(A) $X$ admits a holomorphic one-form without zeros;
(B) $X$ admits a real closed one-form without zeros.

Kotschick observed that (A) implies (B) and asked if the converse holds. By Tischler's theorem [Ti70], Property (B) is equivalent to

(B') $X$ is a smooth fiber bundle over $S^1$.

In the literature, the equivalence of (A) and (B) became known as Kotschick's conjecture. It has been verified in dimensions 2 and 3 in [Kot22, Sch21] and [HS21, Pie25a], respectively. For closely related work on holomorphic versions of Tischler's theorem, sparked by the above conjecture, see e.g. [DHL24, SY25, Hao24, CCH25, Chu24, HWZ26, Chu26, Pie25b, DHLW26].

As a consequence of Theorem 1, we prove the following.

**Theorem 26.** *There is a smooth complex projective variety $X$ of dimension 5 and with $b_1(X) = 2$, such that the Albanese morphism $f\colon X \to \mathrm{Alb}(X)$ has the following properties:*

*(i) $f$ is a homotopy fiber bundle;*
*(ii) there is a $C^\infty$-fiber bundle structure $\tilde{f}\colon X \to \mathrm{Alb}(X)$ which is homotopic to $f$;*
*(iii) $f$ is not a submersion and any holomorphic one-form on $X$ has a zero.*

*Proof.* Let $E$ be an elliptic curve and $E \to \mathbb{P}^1$ a surjective morphism whose critical values are disjoint from the critical values of $g\colon Y \to \mathbb{P}^1$ from (12). We then define

$$X := Y \times_{\mathbb{P}^1} E \qquad \text{with natural map} \qquad f\colon X \longrightarrow E. \tag{14}$$

Note that $X$ is smooth because $g$ is smooth in a neighborhood of the critical values of $E \to \mathbb{P}^1$. Moreover, $f$ is a homotopy fiber bundle with simply connected fibers since $g$ is. Similarly, $f$ is not a submersion because $g$ is not.

Since $f$ has simply connected fibers, $f$ induces an isomorphism on fundamental groups. Hence $\mathrm{Alb}(X) \cong E$ and $f$ is the Albanese morphism. Since $f$ is not a submersion and $\mathrm{Alb}(X)$ is an elliptic curve, each holomorphic one-form on $X$ has a zero.

By Proposition 20, there is a $C^\infty$-fiber bundle structure $\tilde{f}\colon X \to E$ which is homotopic to $f$. This concludes the proof of the theorem. □

Theorem 26 yields the following strong counterexample to Kotschick's conjecture.

**Corollary 27.** *Condition (B) does not imply condition (A). In fact, there is a projective manifold $X$ of dimension 5 with $b_1(X) = 2$ such that each nonzero cohomology class $u \in H^1(X, \mathbb{R})$ can be represented by a closed real one-form without zeros, while the harmonic representative of $u$ always has zeros.*

*Proof.* Let $X$ be as in (14). Then $b_1(X) = 2$, $f\colon X \to E$ is the Albanese morphism of $X$ and every holomorphic one-form on $X$ has a zero.

By Theorem 26, there is a $C^\infty$-fibration $\tilde{f}\colon X \to \mathrm{Alb}(X)$ which is homotopic to the Albanese morphism $f$. Hence, $\tilde{f}^*\colon H^1(\mathrm{Alb}(X), \mathbb{R}) \to H^1(X, \mathbb{R})$ is an isomorphism. Since $\tilde{f}$ is a submersion, it follows that every nonzero class in $H^1(X, \mathbb{R})$ can be represented by a closed real one-form without zeros.

Now let $\alpha_u \in A^1(X, \mathbb{R})$ be the harmonic representative of a nonzero class $u \in H^1(X, \mathbb{R})$ and set $\omega_u = (\alpha_u)^{1,0}$. Since $X$ is Kähler and $u \neq 0$, $\omega_u$ is a nontrivial holomorphic one-form. As we have seen above, $\omega_u$ vanishes at some point $x \in X$. Since $\alpha_u = \omega_u + \overline{\omega_u}$, it follows that $\alpha_u(x) = 0$. Hence, the harmonic representative $\alpha_u$ of $u$ has a zero, as we want. This concludes the proof of Corollary 27. □

By a theorem of Popa and Schnell [PS14], every holomorphic one-form on a smooth complex projective variety of general type has a zero. The variety in Theorem 26 has Kodaira dimension $-\infty$, since the general fiber of its Albanese map is rationally connected. This raises the following natural question.

**Question 28.** Can a smooth complex projective variety of general type admit a real closed one-form without zeros? Equivalently, can its underlying differentiable manifold fiber smoothly over $S^1$?

Let us also mention the following related question.

**Question 29.** Are conditions (A) and (B) from Kotschick's conjecture equivalent for smooth complex projective varieties with nef canonical class?

As another consequence of Theorem 1, we obtain a negative answer to a question of Kollár and Pardon [KP12, Question 27] (see also [LMW21, Question 1.3]).

**Corollary 30.** *Let $f\colon X \to E$ be as in (14). Then $f_*\colon \pi_1(X) \to \pi_1(E)$ is an isomorphism, the universal cover $\widetilde{X}$ of $X$ has the homotopy type of a finite CW complex, but $f$ is not a $C^\infty$-fiber bundle.*

*Proof.* The universal cover of $E$ is $\widetilde{E} = \mathbb{C}$. For a regular value $y \in \widetilde{E}$, the inclusion $\widetilde{X}_y \hookrightarrow \widetilde{X}$ is an integral homology equivalence by the Leray spectral sequence, since $\widetilde{X} \to \widetilde{E}$ is a $\mathbb{Z}$-homology fiber bundle and $\widetilde{E}$ is contractible. Both spaces are simply connected: this follows for $\widetilde{X}$ from the isomorphism $\pi_1(X) \simeq \pi_1(E)$ and for $\widetilde{X}_y$ from Proposition 22. Corollary 6 therefore shows that $\widetilde{X}_y \hookrightarrow \widetilde{X}$ is a homotopy equivalence. As $\widetilde{X}_y$ is compact, $\widetilde{X}$ has the homotopy type of a finite CW complex. Finally, $f$ is not a $C^\infty$-fiber bundle, since it is not a submersion by Proposition 22. □

## 7. A topological equisingularity theorem

While [KP12, Question 27] has a negative answer in general (see Corollary 30 above), we prove here the following positive answer to one of its topological versions.

**Theorem 31.** *Let $\Sigma$ be a (possibly non-compact) Riemann surface with $\pi_1(\Sigma) \neq 0$. Let $g\colon X \to \Sigma$ be a proper morphism between complex spaces. Assume that $g$ has simply connected fibers. Then the following are equivalent.*

*(i) $g$ is a $\mathbb{Z}$-homology fiber bundle.*
*(ii) $g$ is a homotopy fiber bundle.*
*(iii) The universal cover $\widetilde{X}$ is homotopy equivalent to a finite CW complex.*

*If, in addition, $\dim X \geq 4$ and $X$ is smooth, then these are also equivalent to*

*(iv) There exists a proper $C^\infty$-fiber bundle $g'\colon X \to \Sigma$ which is homotopic to $g$ and agrees with $g$ outside a small neighborhood of the singular fibers of $g$.*

We will need the following proposition.

**Proposition 32.** *Let $X$ be a complex space and let $f\colon X \to \Sigma$ be a proper morphism to a Riemann surface $\Sigma$ such that $b_1(\Sigma) < \infty$. Then the groups $H^i(X, \mathbb{Z})$ are finitely generated if and only if there is a finite subset $D \subseteq \Sigma$ such that $f$ is a $\mathbb{Z}$-homology fiber bundle over $\Sigma \setminus D$.*

*Proof.* There is a smallest discrete subset $D \subseteq \Sigma$ such that the sheaves $R^i f_* \mathbb{Z}_X$ are local systems on $\Sigma \setminus D$. If $D$ is finite, then the groups

$$H^j\big(\Sigma \setminus D, R^i f_* \mathbb{Z}_X\big)$$

are finitely generated, and hence so are the groups

$$H^j\big(X \setminus f^{-1}(D), \mathbb{Z}\big)$$

by the Leray spectral sequence. The Mayer–Vietoris sequence now gives finite generation of the groups $H^j(X, \mathbb{Z})$.

If $D$ is infinite, then $\Sigma$ is noncompact, hence Stein. Artin's vanishing theorem for constructible sheaves on Stein manifolds therefore gives

$$H^j\big(\Sigma, R^i f_* \mathbb{Z}_X\big) = 0 \qquad \text{for } j > 1,$$

see [KS90, Theorem 10.3.8]. Thus all differentials in the Leray spectral sequence vanish. Therefore, the groups $H^i(X, \mathbb{Z})$ are finitely generated if and only if each group

$$H^j\big(\Sigma, R^i f_* \mathbb{Z}_X\big)$$

is finitely generated. The proposition therefore follows from Lemma 33 below. □

**Lemma 33.** *Let $\Sigma$ be a connected orientable topological surface with $b_1(\Sigma) < \infty$. Let $\mathcal{F}$ be a sheaf of abelian groups on $\Sigma$ whose stalks are finitely generated. Let*

$$\Sigma^\circ = \{x \in \Sigma \mid \mathcal{F} \text{ is a local system on a neighborhood of } x\}$$

*and put $D = \Sigma \setminus \Sigma^\circ$. Assume that $D$ is discrete. Then the groups $H^i(\Sigma, \mathcal{F})$ are finitely generated for every $i$ if and only if $D$ is finite.*

*Proof.* If $D$ is finite, then the claim is clear. Conversely, assume that $D$ is infinite. Let $\mathcal{F}_{sky} \subset \mathcal{F}$ be the maximal skyscraper subsheaf. If $\mathcal{F}_{sky}$ has infinitely many stalks, $H^0(\Sigma, \mathcal{F})$ is not finitely generated. We may thus assume that this is not the case. Then we replace $\mathcal{F}$ by $\mathcal{F}/\mathcal{F}_{sky}$ and assume that $\mathcal{F}$ does not contain skyscraper subsheaves.

Note that $D$ is closed and discrete, hence is countable and has no accumulation points in $\Sigma$. In particular, we can write

$$D = \{0_i \mid i = 1, 2, \ldots\}.$$

Since $\Sigma$ is orientable and $b_1(\Sigma) < \infty$, it is given by attaching to a compact orientable surface (called compact core) with boundary an infinite annulus to each of the finitely many boundary components. We can therefore choose a closed embedded tree $\Gamma \subset \Sigma$ which contains $D$ and is obtained from a finite tree contained in the compact core by attaching finitely many proper rays, one for each of the attached ends.

Let $W_1$ be a sufficiently small open thickening of $\Gamma$, chosen so that $\Gamma$ is a deformation retract of $W_1$, and put $W_2 \coloneqq \Sigma \backslash \Gamma$. Then $W_1$ is connected and contractible, while $W_2$ and $W_1 \cap W_2$ have the homotopy type of finite CW complexes and do not contain any points of $D$. The Mayer–Vietoris sequence therefore reduces us to the case where $\Sigma = W_1$ is a connected contractible surface without boundary.

We then choose an open covering $\Sigma = V \cup \bigsqcup_{i=1}^{\infty} U_i$, where the $U_i$ are pairwise disjoint open discs with centers $0_i \in U_i$, $V \cap U_i$ is nonempty and contractible, and $V$ is connected and disjoint from $D$. (For instance, we can take $V = \{(x, y) \in \mathbb{R}^2 \mid y > 1/8\}$ and $U_i$ a ball of radius $1/4$ around $0_i = (i, 0) \in \mathbb{R}^2$.) By the definition of $\Sigma^\circ$ and $D$, the sheaf $\mathcal{F}$ is locally constant outside of $D$, but $\mathcal{F}$ is not locally constant in the neighborhood of any of the points $0_i$.

**Claim.** The map $H^0(U_i, \mathcal{F}) \to H^0(V \cap U_i, \mathcal{F})$ is not surjective.

*Proof.* Set

$$j \colon U_i \setminus \{0_i\} \hookrightarrow U_i \qquad \text{and} \qquad \mathcal{L} \coloneqq \mathcal{F}|_{U_i \setminus \{0_i\}}.$$

There is a natural morphism

$$\mathcal{F}|_{U_i} \longrightarrow j_* \mathcal{L}. \tag{15}$$

Its kernel is supported at $0_i$, hence is a skyscraper subsheaf of $\mathcal{F}$. By our reduction, this kernel is zero.

Suppose, for contradiction, that

$$H^0(U_i, \mathcal{F}) \longrightarrow H^0(V \cap U_i, \mathcal{F}) \tag{16}$$

is surjective. Since $V \cap U_i$ is nonempty and contractible and $\mathcal{F}$ is locally constant there, its space of sections identifies with any stalk of $\mathcal{L}$ on $V \cap U_i$. Surjectivity of (16) thus implies that $\mathcal{L}$ is a trivial local system on $U_i \setminus \{0_i\}$. But then $j_* \mathcal{L}$ is a trivial local system on $U_i$ and the injective map (15) is surjective by surjectivity of (16). Hence, (15) is an isomorphism, which contradicts our assumptions. □

Let $U = \bigsqcup_{i \geq 1} U_i$ and consider the Mayer–Vietoris sequence

$$H^0(\Sigma, \mathcal{F}) \to \prod_{i=1}^{\infty} H^0(U_i, \mathcal{F}) \oplus H^0(V, \mathcal{F}) \to \prod_{i=1}^{\infty} H^0(U_i \cap V, \mathcal{F}) \to H^1(\Sigma, \mathcal{F}).$$

By the above claim, the cokernel of $\prod_{i=1}^{\infty} H^0(U_i, \mathcal{F}) \to \prod_{i=1}^{\infty} H^0(U_i \cap V, \mathcal{F})$ is not finitely generated. Since $\mathcal{F}$ has no skyscraper subsheaf and the stalks of $\mathcal{F}$ are finitely generated, $H^0(V, \mathcal{F})$ is finitely generated. Hence, the above exact sequence shows that $H^1(\Sigma, \mathcal{F})$ is not finitely generated. This concludes the proof of the lemma. □

*Proof of Theorem 31.* Item (i) implies (ii) by Corollary 6(i). Since $\pi_1(\Sigma) \neq 0$, the universal cover of $\Sigma$ is contractible and so (ii) implies (iii) by Corollary 6(ii).

By a theorem of Smale [Sm57, Main Theorem], $g\colon X \to \Sigma$ induces an isomorphism on fundamental groups, because it has simply connected fibers. Hence, $\widetilde{X} = X \times_{\Sigma} \widetilde{\Sigma}$, where $\widetilde{X}$ and $\widetilde{\Sigma}$ denote the universal covers of $X$ and $\Sigma$, respectively. Since $\pi_1(\Sigma) \neq 0$, the classification of Riemann surfaces implies that $\widetilde{\Sigma} \to \Sigma$ has infinite degree. Moreover, $b_1(\widetilde{\Sigma}) = 0$. Therefore, item (iii) implies (i) by Proposition 32, applied to the induced map $\widetilde{X} \to \widetilde{\Sigma}$.

Assume now that $X$ is smooth. By Proposition 20, item (i) implies item (iv) if $\dim X \geq 4$. Conversely, if (iv) holds, then the universal cover $\widetilde{X}$ of $X$ is a $C^\infty$-fiber bundle over the universal cover $\widetilde{\Sigma}$ of $\Sigma$. Since $\Sigma$ is not simply connected, $\widetilde{\Sigma}$ is either $\mathbb{C}$ or the disc $\Delta$. It follows that $\widetilde{X}$ is homotopy equivalent to a fiber of $\widetilde{X} \to \widetilde{\Sigma}$ and so (iii) holds. □

**Remark 34.** When $\Sigma$ is compact, then the equivalence of (i), (ii) and (iii) in Theorem 31 can be deduced from [LMW21, Propositions 5.5 and 5.6, and Lemma 5.9]; see also [LMW21, Corollary 1.6] for the case of elliptic curves.

## 8. One-forms without zeros and exactness of the Aomoto complex

We were led to some of the constructions used in this paper while analyzing the approach to Kotschick's conjecture developed in [Sch21], which plays a central role in the known low-dimensional cases, see [Sch21, HS21, Pie25a]. The approach relies on the following property:

(C) There exists a holomorphic one-form $\omega \in H^0(X, \Omega^1_X)$, such that for every finite étale cover $\pi\colon X' \to X$, the Aomoto complex[1]

$$H^0(X', \mathbb{C}) \xrightarrow{\wedge \pi^*\omega} H^1(X', \mathbb{C}) \xrightarrow{\wedge \pi^*\omega} H^2(X', \mathbb{C}) \xrightarrow{\wedge \pi^*\omega} \cdots$$

has vanishing cohomology in every degree.

It was shown in [Sch21] that Property (B) implies Property (C). It was further conjectured in *loc. cit.* that for any compact Kähler manifold $X$, Property (C) implies (A). In this section we will prove Theorem 4, stated in the introduction, which implies that in fact (C) implies neither (A) nor (B).

### 8.1. A rational cohomology torus.

We recall the Iitaka torus tower from [DJL17].

Let $E$ and $F$ be two elliptic curves, and let

$$p_C\colon C \longrightarrow E$$

[1] Introduced in [AK11] in a different context.

be a branched double cover of $E$. Let $\sigma_1 \colon F \to F$ be an order-two automorphism defined by translating by a nonzero 2-torsion point, and denote the deck involution of $p_C$ by $\sigma_2$. Let $\sigma = (\sigma_1, \sigma_2)$ be the induced diagonal action on $F \times C$, and define

$$S = (F \times C)/\langle \sigma \rangle.$$

Since $\sigma$ is free, $S$ is a smooth projective surface. Let

$$p \colon S \longrightarrow C/\langle \sigma_2 \rangle = E \tag{17}$$

be the natural projection. Then, away from the branch locus of $p_C$, the fibers of $p$ are isomorphic to $F$. Over the branch locus of $p_C$, $p$ has double fibers, and these fibers (with induced reduced scheme structure) are isomorphic to $F/\langle \sigma_1 \rangle$. The important feature of $S$ is that $R^i p_* \mathbb{Q}_S$ is a trivial local system on $E$ of rank 1, 2, 1, when $i = 0, 1, 2$, respectively, and for $i > 2$, $R^i p_* \mathbb{Q}_S = 0$, cf. Lemma 36 below. This implies that the Albanese map of $S$ induces an isomorphism of rational cohomology groups. In particular, the rational cohomology ring of $S$ is isomorphic to the rational cohomology ring of a 2-dimensional complex torus. However, since $p$ has double fibers at the branch locus of $p_C$, $R^2 p_* \mathbb{Z}_S$ is not a local system near those points.

Geometrically, $p$ is a locally trivial fibration with fiber $F$ away from the branch locus of $p_C$. Since locally in $S$, the map $p$ factors through the ramified cover $p_C \colon C \to E$, $p$ fails to be a submersion precisely over the branch locus of $p_C$.

8.2. **Construction of the example.** Let $N \geq 3$ be such that there is a closed embedding $\iota \colon S \to \mathbb{P}^N \times E$ which lifts $p \colon S \to E$ from (17). Via this map we consider $S$ as a subvariety of $\mathbb{P}^N \times E$. Let

$$X \coloneqq \mathrm{Bl}_S(\mathbb{P}^N \times E), \tag{18}$$

which is a smooth projective variety of dimension $N + 1$. We further consider the projection

$$h \colon X \longrightarrow E$$

which agrees with the Albanese map of $X$ and induces an isomorphism on fundamental groups.

In the remainder of this section, $X$ will always denote the blow-up in (18) and we will show that it satisfies the conclusions from Theorem 4.

8.3. **Cohomology of the Aomoto complexes.** In this section, we prove the following proposition, which shows that $X$ satisfies an even stronger condition than Property (C).

**Proposition 35.** *Let $\pi \colon X' \to X$ be any connected finite étale cover of $X$ and $\omega$ be any nonzero holomorphic one-form on $X'$. Then the complex*

$$H^0(X', \mathbb{C}) \xrightarrow{\wedge\omega} H^1(X', \mathbb{C}) \xrightarrow{\wedge\omega} H^2(X', \mathbb{C}) \xrightarrow{\wedge\omega} \cdots$$

*has vanishing cohomology in every degree.*

We begin with the following lemma.

**Lemma 36.** *The map $p \colon S \to E$ satisfies:*

$$Rp_* \mathbb{Q}_S \cong \mathbb{Q}_E \oplus \mathbb{Q}_E[-1]^{\oplus 2} \oplus \mathbb{Q}_E[-2]. \tag{19}$$

*Proof.* First, we prove that $p\colon S \to E$ is a $\mathbb{Q}$-homology fiber bundle. In fact, away from the branch locus of $p_C$, $p$ is an $F$-fiber bundle. Let $x \in E$ be a branch point of $p_C$, and let $x' \in E$ be a point different from but very close to $x$. Then there is a specialization map between the fibers of $p$, $\mathrm{sp}\colon S_{x'} \to S_x$, well-defined up to homotopy. This map can be chosen to be the finite étale double cover $F \to F/\langle\sigma_1\rangle$. Hence, it induces an isomorphism on rational (co-)homology. Therefore, $p\colon S \to E$ is a $\mathbb{Q}$-homology fiber bundle. In particular, $R^i p_*\mathbb{Q}_S$ is a local system for every $i$.

Away from the branch locus of $p_C$, $p$ is an $F$-fiber bundle, and the monodromy action is given by translation. Thus, the restriction of $R^i p_*\mathbb{Q}_S$ to $E^\circ := E \setminus \mathrm{Br}(p_C)$ is the trivial local system. Since the inclusion $E^\circ \to E$ induces a surjective map on fundamental groups, $R^i p_*\mathbb{Q}_S$ is a trivial local system for all $i$. Thus, the isomorphism (19) follows from the decomposition theorem [BBD82] and the fact that a general fiber of $p$ is an elliptic curve. □

**Corollary 37.** *There are $a_i \in \mathbb{Z}_{\geq 0}$ with*

$$Rh_*\mathbb{Q}_X \cong \bigoplus_{0\leq i\leq 2N} \mathbb{Q}_E[-i]^{\oplus a_i}.$$

*Proof.* Recall that $h\colon X \to E$ is equal to the composition of the blow-up map $X \to \mathbb{P}^N \times E$ and the projection $\mathbb{P}^N \times E \to E$. Taking the derived pushforward of $\mathbb{Q}_X$ to $\mathbb{P}^N \times E$, we obtain a direct sum of the constant sheaf $\mathbb{Q}_{\mathbb{P}^N\times E}$ with a direct sum of shifted trivial local systems on $S$, cf. (5). Note further that the derived pushforwards of $\mathbb{Q}_{\mathbb{P}^N\times E}$ and $\mathbb{Q}_S$ to $E$ are direct sums of shifted trivial local systems on $E$. Hence, $Rh_*(\mathbb{Q}_X)$ is isomorphic to a direct sum of shifted trivial local systems on $E$. The precise version in the corollary follows because $h$ has relative dimension $N$. □

The pullback map $h^*\colon H^*(E,\mathbb{Q}) \to H^*(X,\mathbb{Q})$ gives $H^*(X,\mathbb{Q})$ a graded $H^*(E,\mathbb{Q})$-module structure.

**Corollary 38.** *As a graded $H^*(E,\mathbb{Q})$-module, $H^*(X,\mathbb{Q})$ is free.*

*Proof.* In the notation of Corollary 37, we have the following isomorphism of graded $H^*(E,\mathbb{Q})$-modules,

$$H^*(X,\mathbb{Q}) \cong \bigoplus_{0\leq i\leq 2N} \left(H^*(E,\mathbb{Q})[-i]\right)^{\oplus a_i}.$$

Therefore, $H^*(X,\mathbb{Q})$ is a free $H^*(E,\mathbb{Q})$-module. □

*Proof of Proposition 35.* First, we consider the case when $\pi\colon X' \to X$ is the identity map. Since $h\colon X \to E$ coincides with the Albanese map of $X$, every nonzero holomorphic one-form $\omega$ on $X$ is of the form $h^*(\omega_E)$ for some nonzero holomorphic one-form $\omega_E$ on $E$. By Corollary 38, $H^*(X,\mathbb{C})$ is a free graded $H^*(E,\mathbb{C})$-module. Therefore, the Aomoto complex

$$(H^*(X,\mathbb{C}), \wedge\omega) \tag{20}$$

is a direct sum of shifted copies of

$$(H^*(E,\mathbb{C}), \wedge\omega_E).$$

The latter is exact because $H^*(E,\mathbb{C})$ is an exterior algebra on a two-dimensional vector space and wedge by a nonzero degree-one element gives the exact Koszul complex. Therefore, the complex (20) also has trivial cohomology.

In general, since $h\colon X \to E$ induces an isomorphism on fundamental groups, every finite étale cover $\pi\colon X' \to X$ is induced by a finite étale cover $E' \to E$. More precisely, $X' = X \times_E E'$ and under this identification, $\pi$ is the projection onto the first factor. Recall that $X$ is obtained as the blow-up of $\mathbb{P}^N \times E$ along $S$, and $S$ is a quotient of $F \times C$. Let $C' = C \times_E E'$, which is a smooth projective curve equipped with a ramified double cover $p_{C'}\colon C' \to E'$. Let $\sigma_2'$ denote the involution on $C'$ induced by $\sigma_2$. Repeating the construction, we obtain

$$S' = (F \times C')/\langle(\sigma_1, \sigma_2')\rangle.$$

We further get maps $S' \to \mathbb{P}^N \times E' \to E'$ which can be identified with the pullback of $S \to \mathbb{P}^N \times E \to E$ along $E' \to E$. Consequently,

$$X' \cong \mathrm{Bl}_{S'}(\mathbb{P}^N \times E').$$

Therefore, the conclusion established in the first paragraph applies equally to the morphism $X' \to E'$, proving the general case of Proposition 35. □

8.4. **Proof of Theorem 4.**

*Proof of Theorem 4.* Let $X$ be as in (18). By Lemma 18, we can choose $N = 4$ and so $\dim X = 5$. Let $X' \to X$ be a finite étale cover. By Proposition 35, for any nonzero $\omega' \in H^0(X', \Omega^1_{X'})$, the Aomoto complex $(H^*(X',\mathbb{C}), \wedge\omega')$ is exact. It remains to show that any real closed one-form on $X$ has a zero. Assume, for contradiction, that $X$ admits a real closed one-form without zeros. Then $X$ fibers smoothly over $S^1$, see [Ti70]. We can pick such a $C^\infty$-fibration $X \to S^1$ such that $\pi_1(X) \to \pi_1(S^1)$ is surjective. Let $\widetilde{X} \to X$ be the infinite cyclic cover corresponding to the kernel of $\pi_1(X) \to \pi_1(S^1)$. Then $\widetilde{X}$ is homotopy equivalent to a fiber of $X \to S^1$, hence has the homotopy type of a finite CW complex.

Recall that $h\colon X \to E$ induces an isomorphism on $\pi_1$ and let $\widetilde{E} \to E$ be the infinite cyclic covering, induced by the kernel of $\pi_1(E) \cong \pi_1(X) \to \pi_1(S^1)$. Then $h$ induces a natural map $\tilde{h}\colon \widetilde{X} \to \widetilde{E}$. This map is not a $\mathbb{Z}$-homology fiber bundle. Indeed, $R^4\widetilde{h}_*\mathbb{Z}$ fails to be locally constant at every point of $\widetilde{E}$ lying above the branch locus of $p_C\colon C \to E$, and this is an infinite set. Proposition 32 therefore implies that $H^i(\widetilde{X},\mathbb{Z})$ is not finitely generated for some $i$. In particular, $\widetilde{X}$ does not have the homotopy type of a finite CW complex. This contradiction completes the proof of the theorem. □

**Remark 39.** The above proof shows that an a priori slightly stronger version of Theorem 4 holds true: $b_1(X) \neq 0$ and for any finite connected étale cover $X' \to X$ and any nonzero $\omega' \in H^0(X', \Omega^1_{X'})$, the Aomoto complex $(H^*(X',\mathbb{C}), \wedge\omega')$ is exact, but neither $X$ nor $X'$ carry real closed one-forms without zeros.

8.5. **Three-dimensional smoothness conjecture for simply connected fibers.** Combining arguments used in this paper with [HS21, Pie25a], we obtain the following, which shows that counterexamples to the smoothness conjecture in [FdBK12] with simply connected fibers, constructed in Theorem 1 in dimension at least 5, do not exist in dimension three.

**Theorem 40.** *Let $Y$ be a smooth, complex, projective threefold and $g\colon Y \to \mathbb{P}^1$ a morphism with simply connected fibers. Assume that $g$ is a $\mathbb{Q}$-homology fiber bundle. Then $g$ is smooth.*

*Proof.* Let $E$ be an elliptic curve and $E \to \mathbb{P}^1$ a double cover whose critical values are disjoint from the critical values of $g$. Let $X := Y \times_{\mathbb{P}^1} E$ with natural map $f\colon X \to E$. Since $g$ and hence $f$ are $\mathbb{Q}$-homology fiber bundles, the decomposition theorem (see [BBD82]) implies $Rf_*\mathbb{Q}_X = \oplus R^i f_*\mathbb{Q}_X[-i]$. Since $f$ is a base change of $g$ and $\mathbb{P}^1$ is simply connected, $R^i f_*\mathbb{Q}_X \cong \mathbb{Q}_E^{\oplus r_i}$ is a trivial local system for all $i$, cf. Corollary 37. It follows that $H^*(X,\mathbb{Q})$ is a free $H^*(E,\mathbb{Q})$-module, cf. Corollary 38. Hence the Aomoto complex $(H^*(X,\mathbb{C}), \wedge f^*\omega)$ is exact for every nonzero $\omega \in H^0(E,\Omega^1_E)$. Since $g$ (and hence $f$) has simply connected fibers, $f$ induces an isomorphism $\pi_1(X) \cong \pi_1(E)$. In particular, any finite étale cover $\pi\colon X' \to X$ is induced by a finite étale cover $E' \to E$. Repeating the above argument for $E'$ in place of $E$ then shows that $(H^*(X',\mathbb{C}), \wedge \pi^* f^*\omega)$ is exact. At this point [HS21, Theorem 1.4] (see also [Pie25a, Theorem 1.4(iv)]) implies that $f^*\omega$ has no zeros, and so $f$ is smooth. This forces $g$ to be smooth as well. □

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