A MOVING LEMMA FOR COHOMOLOGY WITH SUPPORT

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ABSTRACT. For a natural class of cohomology theories with support (including étale or pro-étale cohomology with suitable coefficients), we prove a moving lemma for cohomology classes with support on smooth quasi-projective k-varieties that admit a smooth projective compactification (e.g. char $k = 0$). This has the following consequences for such k-varieties and cohomology theories: a local and global generalization of the effacement theorem of Quillen, Bloch–Ogus, and Gabber, a finite level version of the Gersten conjecture in characteristic zero, and a generalization of the injectivity property and the codimension 1 purity theorem for étale cohomology. Our results imply that the new invariants from [Sch21b] are motivic.

CONTENTS

1. INTRODUCTION

On a smooth quasi-projective variety X, Chow's moving lemma [Chow56] allows to move an algebraic cycle modulo rational equivalence into good position with respect to a closed subset $S \subset X$.

The essential content of the Gersten conjecture for smooth varieties, proven for K-theory by Quillen [Qui73] and for étale cohomology by Bloch–Ogus [BO74] and Gabber [Gab94, CTHK97], is an effacement theorem, which is equivalent to a similar moving lemma for classes with support in the particular case where X is affine and S is a finite set of points. For instance, in the case of étale cohomology, the effacement theorem is equivalent to saying that for any smooth affine k-variety X, any finite set $S \subset X$, and any class $\alpha \in H^i_Z(X)$ whose support $Z \subset X$ is no-where dense, there is a class $\alpha' \in H^i_{Z'}(X)$ with $S \cap Z' = \emptyset$ such that α and α' have the same image in $H^i_W(X)$ for some closed $W \subset X$ with $Z, Z' \subset W$ and $\dim W = \dim Z + 1$. This is a fundamental result in algebraic geometry. We refer to the surveys [CTHK97, Mo16] for more details, applications, references, and historical remarks.

The above analogy leads naturally to the question whether the effacement theorems of Quillen, Bloch–Ogus, and Gabber are special instances of a more general moving lemma which allows to move a class with support $Z \subset X$ to one with support Z' such that Z' is in good position with respect to an arbitrary given closed subset $S \subset X$. This paper answers that question positively for a natural class of cohomology theories on smooth quasi-projective varieties in characteristic zero (or more generally, with a smooth projective compactification). In the classical case where X is affine and $\dim S = 0$, this yields a new proof of the Gersten conjecture for étale cohomology in characteristic zero. Our proof yields a stronger conclusion than what was known before, as it is well-behaved with respect to localization and hence gives rise to new Gersten type resolutions on finite levels. The general case where $\dim S > 0$ is new and has several applications that go beyond the original conjecture of Gersten.

1.1. **Main result.** We fix a field k and a twisted cohomology theory with support $(X, Z) \longmapsto H_Z^*(X, n)$, for smooth equi-dimensional algebraic k-schemes X with $Z \subset X$ closed. We assume that some natural axioms, as outlined in Section 3 below, are satisfied. Concrete examples include étale or pro-étale cohomology with suitable coefficients, see Proposition 3.2.

To simplify notation, we write $H^*(X, n) := H_X^*(X, n)$ and $H_Z^*(U, n) := H_{Z \cap U}^*(U, n)$ for $U \subset X$ open. Our main result is the following.

Theorem 1.1 (Moving lemma). *Let X be a smooth equi-dimensional k-scheme that admits a smooth projective compactification. Let $S, Z \subset X$ be closed subsets with $\dim Z < \dim X$. Then there are closed subsets $Z' \subset W \subset X$ with $Z \subset W$, $\dim Z' = \dim Z$, and $\dim W = \dim Z + 1$, such that Z' and $W \setminus Z$ meet S properly and for any $\alpha \in H_Z^*(X, n)$ there is a class $\alpha' \in H_{Z'}^*(X, n)$ such that α and α' have the same image in $H_W^*(X, n)$.*

In Theorem 5.1 below, we prove a stronger, but more technical version of the above theorem, where $S \subset X$ is replaced by a morphism $f : S \to X$ and the behaviour under localization on X is discussed. The main idea of our proof is to reduce the problem to a version of Chow's moving lemma due to Levine. In particular, even if X in Theorem 1.1 is affine, our proof is global and makes essential use of a smooth projective compactification. In contrast, the Gersten conjecture is a purely local statement. We explain several applications in the following two subsections.

1.2. **Effacement theorems and a finite level version of Gersten's conjecture.** By the long exact sequence of triples, the conclusion of Theorem 1.1 is equivalent to saying that the natural map $H_Z^*(X, n) \to H_W^*(X \setminus Z', n)$ is zero. We thus get the following.

Corollary 1.2 (Global effacement). *Let X be a smooth equi-dimensional k-scheme that admits a smooth projective compactification. Let $S, Z \subset X$ be closed subsets with $\dim S + \dim Z < \dim X$. Then there is a neighbourhood $U \subset X$ of S and a closed subset $W \subset X$ with $Z \subset W$ and $\dim W = \dim Z + 1$ such that the following composition is zero:*

$$H_Z^*(X, n) \longrightarrow H_W^*(X, n) \longrightarrow H_W^*(U, n).$$

The case $\dim S = 0$ implies formally by specialization that the result holds for any finite set of (possibly non-closed) points $S \subset X$ and any no-where dense closed subset $Z \subset X$. This is the aforementioned effacement theorem of Bloch–Ogus and Gabber in our context, cf. [CTHK97, Theorems 2.2.1]. A similar effacement theorem for K-theory had previously been proven by Quillen [Qui73, p. 125, Theorem 5.11].

Let now X_S be the (Zariski) localization of X along $S \subset X$, i.e. the pro-scheme given by the system of all open neighbourhoods $U \subset X$ of S in X. One defines

$$H_Z^*(X_S, n) := \varinjlim_{S \subset U \subset X} H_Z^*(U, n) \quad \text{and} \quad H^*(X_S \setminus Z, n) := \varinjlim_{S \subset U \subset X} H^*(U \setminus Z, n).$$

Corollary 1.3. *Let X be a smooth equi-dimensional k-scheme that admits a smooth projective compact-ification. Let $S, Z \subset X$ be a closed subsets with $\dim S + \dim Z < \dim X$. Then $H_Z^*(X_S, n) \to H^*(X_S, n)$ is zero and the long exact sequence of triples induces for all i a short exact sequence:*

$$0 \longrightarrow H^i(X_S, n) \longrightarrow H^i(X_S \setminus Z, n) \xrightarrow{\partial} H_Z^{i+1}(X_S, n) \longrightarrow 0.$$

An interesting special case is when $S = Z$ with $\dim Z < \frac{1}{2} \dim X$, where we get an algebro-geometric analogue of a well-known result in differential topology concerning the cohomology of the tubular neigh-bourhood of a submanifold with trivial Euler class. Our result applies also to singular subvarieties $Z \subset X$ and to the (comparatively coarse) Zariski localization X_S; the triviality of the Euler class is replaced by the condition $\dim Z < \frac{1}{2} \dim X$.

Contrary to Chow's moving lemma, the subset Z' in Theorem 1.1 cannot be chosen to be well-behaved with respect to localization: if we shrink X, then Z' typically has to be enlarged, see Remark 5.3. Surprisingly, the closed subset W in Theorem 1.1 is much better behaved. This yields the following.

Corollary 1.4 (Local effacement). *Let X be a smooth equi-dimensional k-scheme which admits a smooth projective compactification. Let $Z \subset X$ be a no-where dense closed subset. Let $S \subset X$ be either closed with $\dim S < \operatorname{codim} Z - 1$ or a finite set of points. Then there is a closed subset $W \subset X$ with $Z \subset W$ and $\dim W = \dim Z + 1$, such that the natural map $H_Z^*(X_S, n) \to H_W^*(X_S, n)$ is zero.*

The above corollary is new even in the classical case where S is a finite set of points. In this case, $X_S = \operatorname{Spec} \mathcal{O}_{X,S}$ is the spectrum of the semi-local ring $\mathcal{O}_{X,S}$ and it was known before that the above vanishing result holds if one passes to the direct limit over all such closed subsets W, see [BO74, Theorem 4.2.3]. The fact a single subscheme suffices is new (and may be somewhat surprising). In fact, Colliot-Thélène–Hoobler–Kahn write in [CTHK97, Remark 2.2.8] the following concerning this issue:

"*We would like to point out that (contrary to the definition of effaceability) the statement in* [CTHK97, *Theorem 2.2.7] is not local: the proof by no means implies that the map of Theorem 2.2.7 (2) remains 0 when U is replaced by a smaller open set. [...] This shows the subtlety of the situation and probably why Gersten's conjecture is so difficult for general regular local rings of dimension ≥ 2.*"

As a consequence, we obtain the following finite level version of the Gersten conjecture for the étale cohomology of varieties over fields of characteristic zero.

Corollary 1.5 (Gersten's conjecture on finite levels). *Let X be a smooth affine variety over a field k. Assume X admits a smooth projective compactification (e.g. $\operatorname{char} k = 0$). Let $x \in X^{(c)}$ with localization $X_x = \operatorname{Spec} \mathcal{O}_{X,x}$. Let $Z_c = \{x\} \subset Z_{c-1} \subset \cdots \subset Z_1 \subset Z_0 = X_x$ be a chain of closed subsets of X_x of increasing dimensions. Up to replacing the given chain $\{Z_j\}_j$ by one that is finer (i.e. by a chain $\{Z_j'\}$ as above with $Z_j \subset Z_j'$ for all j), the following complex is exact:*

$$0 \to H^i(X_x, n) \to H_{BM}^i(X_x \setminus Z_1) \xrightarrow{\partial} H_{BM}^{i-1}(Z_1 \setminus Z_2) \xrightarrow{\partial} \cdots \xrightarrow{\partial} H_{BM}^1(Z_{i-1} \setminus Z_i) \xrightarrow{\partial} H_{BM}^0(Z_i \setminus Z_{i+1}) \to 0,$$

where $Z_j = \emptyset$ for $j > c$,

$$H_{BM}^{i-j}(Z_j \setminus Z_{j+1}) := \varinjlim_{x \in U \subset X} H_{\bar{Z}_j \setminus \bar{Z}_{j+1}}^{i+j}(U \setminus \bar{Z}_{j+1}, n+j)$$

and $\bar{Z}_j \subset X$ denotes the closure of Z_j in X. If moreover $T \subset X_x$ is closed and each closed subset $Z_j \subset X_x$ of the given chain $\{Z_j\}$ meets $T \setminus \{x\}$ dimensionally transversely, then the above chosen refinement $\{Z_j'\}$ still satisfies this transversality condition with respect to $T \setminus \{x\}$,

If $H_Z^*(X, n) = H_Z^*(X_{\text{ét}}, \mu_{\ell r}^{\otimes n})$ is étale cohomology with support and coefficients in $\mu_{\ell r}^{\otimes n}$ for some prime ℓ invertible in k, then $H_{BM}^{i-j}(Z_j \setminus Z_{j+1})$ as defined above coincides up to a shift with Borel–Moore homology of $Z_j \setminus Z_{j+1}$ (cf. [BO74, §1]), hence the notation. If moreover $Z_j \setminus Z_{j+1}$ is regular and equi-dimensional, then $H_{BM}^{i-j}(Z_j \setminus Z_{j+1}) = H^{i-j}(Z_j \setminus Z_{j+1}, \mu_{\ell r}^{\otimes n+j})$ agrees by Gabber's purity theorem [Fuj02]

with ordinary étale cohomology of $Z_j \setminus Z_{j+1}$. The latter holds in particular in the limit where we run over all chains $\{Z_j\}_j$ as above.

The corollary says that we can compute the cohomology of the localization $X_x = \operatorname{Spec} \mathcal{O}_{X,x}$ in terms of arbitrarily fine stratifications of X_x. The original conjecture of Gersten for étale cohomology, proven in [BO74], asserts this only in the limit: the above complex is exact if we pass to the direct limit over all chains $\{Z_j\}_j$. The fact that exactness happens already on finite levels as well as the possibility of asking transversality conditions with respect to a closed subset $T \subset X_x$ are new.

1.3. **Codimension j+1 purity and new motivic invariants.** For a subset $S \subset X$, we denote by $F_j^S X$ the pro-scheme given by the inverse system of all open neighbourhoods $U \subset X$ of S with $\operatorname{codim}_X(X \setminus U) > j$. We further set $F_j X := F_j^\emptyset X$. As before, the cohomology $H^*(F_j^S X, n)$ is defined as direct limit over $H^*(U, n)$, where U runs through the given inverse system. Intuitively, one obtains $F_j^S X$ by successively removing from X all closed subsets $Z \subset X$ with $\operatorname{codim}_X Z \geq j+1$ that are disjoint of S. For instance, if X is irreducible, $F_0 X \simeq \operatorname{Spec} k(X)$, and if S is a finite set, $F_0^S X \simeq \operatorname{Spec} \mathcal{O}_{X,S}$.

Important consequences of [BO74], highlighted for instance in [CT95, §3.8], are the injectivity and codimension 1 purity property for étale cohomology, see [CT95, Theorems 3.8.1 and 3.8.2]. Theorem 1.1 implies the following generalization.

Corollary 1.6 (Injectivity and codimension j+1 purity). *Let X be a smooth equi-dimensional k-scheme that admits a smooth projective compactification and let $S \subset X$ be closed.*

    (1) The restriction map $H^*(F_j^S X, n) \to H^*(F_j X, n)$ is injective for $j \geq \dim S$;

    (2) A class in $H^*(F_j X, n)$ that lifts to $F_{j+1} X$ also lifts to $F_{j+1}^S X$ for $j \geq \dim S - 1$.

The $\dim S = 0$ case in the above corollary implies by specialization that the results hold for $j \geq 0$ and any finite set of points $S \subset X$. Hence, the case $j = \dim S = 0$ of the above corollary corresponds to the injectivity property and codimension 1 purity theorem in étale cohomology, see [CT95, Theorems 3.8.1 and 3.8.2]. We note that our purity result is new even for $j = 0$: we get that an unramified class on the generic point of X lifts to an open neighbourhood of any given closed curve in X, while this was previously only known for points in X.

Following [Sch21b], the refined unramified cohomology associated to the given twisted cohomology theory is given by
$$H_{j,nr}^*(X, n) := \operatorname{im}(H^*(F_{j+1} X, n) \to H^*(F_j X, n)).$$
In other words, an element $[\alpha] \in H_{j,nr}^*(X, n)$ is represented by a class $\alpha \in H^*(U, n)$ on some open $U \subset X$ whose complement has codimension $j + 2$ and any two such representatives yield the same element in $H_{j,nr}^*(X, n)$ if they coincide on some open subset $V \subset X$ whose complement has codimension $j + 1$.

This is a common generalization of traditional unramified cohomology and Kato homology, see [Sch21b, Section 1.3]. The latter are known to be motivic as a consequence of the Gersten conjecture [BO74]. The generalization of the Gersten conjecture proven in this paper implies that refined unramified cohomology $H_{j,nr}^i(X, n)$ is motivic for all i and j.

Corollary 1.7. *Let X and Y by smooth projective equi-dimensional schemes over a field k with $d_X = \dim X$. For all $c, i, j \geq 0$, there is a natural bi-additive pairing*
$$\operatorname{CH}^c(X \times Y) \times H_{j,nr}^i(X, n) \longrightarrow H_{j+c-d_X,nr}^{i+2c-2d_X}(Y, n + c - d_X), \quad ([\Gamma], [\alpha]) \longmapsto [\Gamma]_*([\alpha])$$
which is functorial with respect to the composition of correspondences.

The above theorem establishes in particular the existence of functorial pullbacks $f^* : H_{j,nr}^i(Y, n) \to H_{j,nr}^i(X, n)$ along morphisms $f : X \to Y$ between smooth projective varieties; cf. Corollary 6.9. The existence of pullbacks is nontrivial already in the case where $Y = \mathbb{P}^n$ and X is the blow-up in a smooth subvariety. The main problem being that a class on some open subset $U \subset Y$ can of course be pulled

back to a class on $f^{-1}(U)$, but if f is not flat, then the complement of $f^{-1}(U)$ in X may have the wrong codimension. The key ingredient that allows to overcome this issue is the codimension j purity property proven in Corollary 1.6, which allows to represent unramified classes by classes on particular open subsets U where this issue does not occur.

Despite the simple definition, the refined unramified cohomology groups of a variety X are rather subtle invariants that interpolate between cohomology and cycle theory of X, see [Sch21b, Sch21c]. For instance, $H^i_{0,nr}(X, n) \simeq H^i_{nr}(X, n)$ is classical unramified cohomology, while

$$H^i_{j,nr}(X, n) \simeq H^i(X, n) \quad \text{for} \quad \lceil i/2 \rceil \leq j,$$

see Lemma 6.10 below or [Sch21b, Corollary 5.10].

If $\lceil i/2 \rceil > j$, then the refined unramified cohomology groups differ in general from ordinary cohomology. In this (interesting) range we show that refined unramified cohomology satisfies the following basic properties that generalize the fact that traditional unramified cohomology is a stable birational invariant by [CTO89]. To state our result, recall the decreasing filtration F^* on $H^i_{j,nr}(X, n)$, given by $F^m H^i_{j,nr}(X, n) := \operatorname{im}(H^i(F_m X, n) \to H^i(F_j X, n))$ for $m \geq j + 1$.

Corollary 1.8. *Let X and Y be smooth projective equi-dimensional k-schemes and let $i, j, n, m \geq 0$. Then the following properties hold true:*

(1) *There is a natural isomorphism*

$$\sum_l f_l : \bigoplus_{l=0}^{\min(j,n)} H^{i-2l}_{j-l,nr}(Y, m-l) \xrightarrow{\simeq} H^i_{j,nr}(Y \times \mathbb{P}^n_k, m),$$

where f_l is the composition of the pullback along the projection $Y \times \mathbb{P}^{n-l}_k \to Y$ followed by the pushforward along the inclusion $Y \times \mathbb{P}^{n-l}_k \to Y \times \mathbb{P}^n_k$ induced by some linear embedding $\mathbb{P}^{n-l}_k \subset \mathbb{P}^n_k$.

(2) *If $f : X \dashrightarrow Y$ is a birational map that is an isomorphism in codimension c, then f induces for any $j \leq c$ an isomorphism $f^* : H^i_{j,nr}(Y, n) \xrightarrow{\simeq} H^i_{j,nr}(X, n)$.*

(3) *The isomorphisms in (1) and (2) respect the decreasing filtration F^* on both sides.*

The formula proven in (1) implies for instance

$$H^i_{j,nr}(\mathbb{P}^n_k, m) \simeq \bigoplus_{l=0}^{\min(j,n)} H^{i-2l}(\operatorname{Spec} k, m-l).$$

This is, thanks to Corollary 1.7, a consequence of the motivic decomposition of $\mathbb{P}^n_k$, while a direct computation of $H^i_{j,nr}(\mathbb{P}^n_k, m)$ appears to be a rather difficult task, because $H^i(F_j \mathbb{P}^n_k, m)$ is typically a huge group.

Item (3) is new even for $j = 0$, where it shows that the unramified cohomology groups from [CTO89] that are known to be stable birational invariants, carry a filtration that is still a stable birational invariant.

2. Preliminaries

2.1. **Conventions.** An algebraic scheme is a separated scheme of finite type over a field; it is smooth if it is smooth over the ground field. A variety is an integral algebraic scheme. A closed subset of a scheme is implicitly identified with the corresponding reduced closed subscheme. A morphism $f : X \to Y$ of Noetherian schemes is of pure relative dimension d, if for each $x \in X$, $d = \dim_x(X) - \dim_{f(x)}(Y)$.

For a Noetherian scheme X, we denote by $X_{(j)}$ the set of points of dimension j, and put $X^{(j)} := X_{(d_X - j)}$, where $d_X = \dim X$. The free abelian group generated by $X^{(j)}$ is denoted by $Z^j(X)$. The

Chow group $\mathrm{CH}^j(X) := \mathrm{CH}_{d_X-j}(X)$ is the quotient of $Z^j(X)$ modulo rational equivalence, see [Ful98]. In this paper we use the above convention only in the case where X is pure-dimensional, so that points in $X^{(j)}$ have local codimension j in X.

The support of a cycle $\Gamma = \sum a_i Z_i \in Z^j(X)$ is the reduced subscheme $\mathrm{supp}(\Gamma) := \bigcup Z_i$, where the union runs through all i with $a_i \neq 0$. If X is equi-dimensional, then we say that two cycles $\Gamma_1 \in Z^{j_1}(X)$ and $\Gamma_2 \in Z^{j_2}(X)$ meet properly (or dimensionally transversely), if each point of $\mathrm{supp}(\Gamma_1) \cap \mathrm{supp}(\Gamma_2)$ has local codimension at least $j_1 + j_2$ in X. In this case, $\mathrm{supp}(\Gamma_1) \cap \mathrm{supp}(\Gamma_2)$ is either empty or it has pure codimension $j_1 + j_2$.

2.2. **Chows moving lemma.** We will need the following version of Chow's moving lemma due to Levine, see [Lev98, Part I, Chapter II, §3.5] and [Lev05, Theorem 2.13].

Theorem 2.1. *Let X be a smooth projective equi-dimensional scheme over a field k. Let S be a locally equi-dimensional algebraic k-scheme with a morphism $f : S \to X$. Then the following holds:*

(1) Any class $[\Gamma] \in \mathrm{CH}^c(X)$ can be represented by a cycle Γ such that each point of $f^{-1}(\mathrm{supp}(\Gamma))$ has locally codimension c on S (i.e. the expected codimension).

(2) Let $\Gamma \in Z^c(X)$ such that $f^{-1}(\mathrm{supp}(\Gamma))$ has locally codimension c at each point. If $\Gamma \sim_{\mathrm{rat}} 0$, then there is a closed subscheme $W \subset X$ of codimension $c-1$, such that:
- *$f^{-1}(W)$ has locally at each point codimension $c-1$ on S;*
- *$\mathrm{supp}\,\Gamma \subset W$ and Γ, viewed as a cycle on W, is rationally equivalent to zero on W.*

Occasionally, we will use the above result in conjunction with the following simple result.

Lemma 2.2. *Let $f : S \to X$ be a morphism between locally equi-dimensional algebraic k-schemes. Let $Z \subset X$ be closed so that $f^{-1}(Z)$ is locally on S of codimension at least $\dim(f(S))+1$. Then $f^{-1}(Z) = \emptyset$.*

Proof. It suffices to prove the result for each irreducible component of S, hence we may assume that S is irreducible. For a contradiction, assume that $f^{-1}(Z) \neq \emptyset$, or equivalently, $f(S) \cap Z \neq \emptyset$. Let $x \in f(S) \cap Z$ be a closed point. Since S is irreducible by the above reduction step, the theorem on the fibre dimensions of morphisms between algebraic schemes shows that $\dim S \leq \dim(f(S)) + \dim f^{-1}(x)$. Since $f^{-1}(x) \subset f^{-1}(Z)$, it follows that $f^{-1}(Z)$ is locally on S of codimension at most $\dim(f(S))$. This contradicts our assumptions, which concludes the proof. $\qquad\square$

3. Twisted cohomology theory with an action by cycles

In this section we list several natural properties of a twisted cohomology theory with support which admits an action by algebraic cycles. We will show that any theory that satisfies (some of) these properties satisfies the moving lemma for classes with support as in Theorem 1.1.

We fix a field k. A pair (X, Z) of algebraic k-schemes is an algebraic k-scheme X and a closed subset $Z \subset X$. A morphism of pairs $f : (X, Z_X) \to (Y, Z_Y)$ is a morphism of schemes $f : X \to Y$ with $f^{-1}(Z_Y) \subset Z_X$. The total space of a pair (X, Z) is the scheme X.

Let $\mathcal{V}_k$ be the category of pairs (X, Z) of algebraic k-schemes. A twisted cohomology theory with support on $\mathcal{V}_k$ is a family of contravariant functors

$$(3.1) \qquad\qquad \mathcal{V}_k \longrightarrow \{\text{graded abelian groups}\}, \quad (X, Z) \longmapsto H_Z^*(X, n)$$

for $n \in \mathbb{Z}$. The degree i part of $H_Z^*(X, n)$ is denoted by $H_Z^i(X, n)$. For any morphism $f : (X, Z_X) \to (Y, Z_Y)$, contravariance yields functorial pullback maps $f^* : H_{Z_Y}^i(Y, n) \to H_{Z_X}^i(X, n)$ for all i.

For $U \subset X$ open, we write $H_Z^*(U, n) := H_{Z \cap U}^*(U, n)$. (This is in line with the fact that $H_Z^*(-, n)$ is a Zariski presheaf on X.) Moreover, for $Z = X$, we write $H^*(X, n) := H_X^*(X, n)$.

We will need the following natural properties.

  C1 (Excision) Let $f : U \to X$ be an open immersion of smooth algebraic k-schemes and let $Z \subset X$ be closed with $Z \subset U$. Then the natural map $f^* : H_Z^i(X, n) \to H_Z^i(U, n)$ is an isomorphism.

C2 (Long exact sequence of triples) Let $Z \subset W \subset X$ be closed subsets. Then there is a long exact sequence

$$\ldots \longrightarrow H^i_Z(X,n) \xrightarrow{\iota_*} H^i_W(X,n) \xrightarrow{\text{restr}} H^i_{W \setminus Z}(X \setminus Z, n) \xrightarrow{\partial} H^{i+1}_Z(X,n) \longrightarrow \ldots,$$

where ι_* denotes the proper pushforward with respect to the identity on X and restr denotes the canonical restriction map. This sequence is functorial for pullbacks: for any morphism $f : X' \to X$ and closed subsets $Z' \subset W' \subset X'$ such that $f^{-1}(Z) \subset Z'$ and $f^{-1}(W) \subset W'$, pullback along f induces a commutative ladder between the long exact sequence of the triple (X', W', Z') and that of (X, W, Z).

C3 (Pushforwards) Let $f : X \to Y$ be a proper morphism between smooth equi-dimensional algebraic k-schemes. Let $Z_Y \subset Y$ and $Z_X \subset X$ be closed subsets with $f(Z_X) \subset Z_Y$. Then there are pushforward maps

$$f_* : H^{i-2c}_{Z_X}(X, n-c) \longrightarrow H^i_{Z_Y}(Y, n)$$

where $c := \dim Y - \dim X$. These are functorial, i.e. satisfy $f_* \circ g_* = (f \circ g)_*$.

Consider the following diagrams

$$
\begin{array}{ccc}
X' & \xrightarrow{g'} & X \\
{\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\
Y' & \xrightarrow{g} & Y,
\end{array}
\qquad
\begin{array}{ccc}
H^{i-2c}_{Z_{X'}}(X', n-c) & \xleftarrow{(g')^*} & H^{i-2c}_{Z_X}(X, n-c) \\
{\scriptstyle f'_*}\downarrow & & \downarrow{\scriptstyle f_*} \\
H^i_{Z_{Y'}}(Y', n) & \xleftarrow{g^*} & H^i_{Z_Y}(Y, n),
\end{array}
$$

where the diagram on the left is Cartesian, X' and Y' are smooth and equi-dimensional, f is proper, $Z_X \subset X$ and $Z_Y \subset Y$ are closed with $f(Z_X) \subset Z_Y$, $Z_{Y'} = g^{-1}(Z_Y)$, and $Z_{X'} = Z_{Y'} \times_Y Z_X \subset X'$.

  (a) If g is an open immersion, then the diagram on the right commutes.
  (b) If f and g are smooth of pure relative dimensions and if $Z_Y = Y$ and $Z_X = X$, then the diagram on the right commutes.

C4 (Action of cycles) For a cycle $\Gamma \in Z^c(X)$ on a smooth equi-dimensional algebraic k-scheme X and any closed subset $W \subset X$ with $\text{supp}(\Gamma) \subset W$, there is an additive action

$$\text{cl}^X_W(\Gamma) \cup : H^i_Z(X,n) \longrightarrow H^{i+2c}_{W \cap Z}(X, n+c), \quad \alpha \longmapsto \text{cl}^X_W(\Gamma) \cup \alpha$$

which is linear in Γ and such that the following holds:
  (a) If Γ, viewed as a cycle on W, is rationally equivalent to zero on W, then the above action is zero, i.e. $\text{cl}^X_W(\Gamma) \cup \alpha = 0$ for all α.
  (b) If $\Gamma = [W]$ is a prime cycle with smooth support W and inclusion $f : W \hookrightarrow X$, then

$$\text{cl}^X_W(\Gamma) \cup \alpha = f_*(f^*\alpha) \in H^{i+2c}_{W \cap Z}(X, n+c) \quad \text{for all } \alpha \in H^i_Z(X, n),$$

where $f^* : H^i_Z(X,n) \to H^i_{W \cap Z}(W,n)$ and $f_* : H^i_{W \cap Z}(W,n) \to H^{i+2c}_{W \cap Z}(X, n+c)$ denote the natural pullback and pushforward maps, respectively.
  (c) If $Z \subset Z' \subset X$ and $W' \subset X$ are closed with $W \subset W'$, then the following diagram commutes

$$
\begin{array}{ccc}
H^i_Z(X,n) & \xrightarrow{\iota_*} & H^i_{Z'}(X,n) \\
{\scriptstyle \text{cl}^X_W(\Gamma)\cup}\downarrow & & \downarrow{\scriptstyle \text{cl}^X_{W'}(\Gamma)\cup} \\
H^{i+2c}_{W \cap Z}(X, n+c) & \xrightarrow{\iota_*} & H^{i+2c}_{W' \cap Z'}(X, n+c),
\end{array}
$$

where ι_* denote the respective pushforwards with respect to the identity on X.

(d) For any open subset $U \subset X$, the following diagram commutes

$$
\begin{array}{ccc}
H_Z^i(X,n) & \xrightarrow{\mathrm{cl}_W^X(\Gamma)\cup} & H_{W\cap Z}^{i+2c}(X,n+c) \\
{\scriptstyle\mathrm{restr}}\downarrow & & \downarrow{\scriptstyle\mathrm{restr}} \\
H_Z^i(U,n) & \xrightarrow{\mathrm{cl}_{W\cap U}^U(\Gamma|_U)\cup} & H_{W\cap Z}^{i+2c}(U,n+c),
\end{array}
$$

where the vertical arrows are the canonical restriction maps and $\Gamma|_U \in Z^c(U)$ is the flat pullback of Γ.

(e) If $\Gamma \in Z^c(X)$ and $\Gamma' \in Z^{c'}(X)$ such that $W = \mathrm{supp}\,\Gamma$ meets $W' = \mathrm{supp}\,\Gamma'$ properly, then

$$\mathrm{cl}_W^X(\Gamma) \cup (\mathrm{cl}_{W'}^X(\Gamma') \cup \alpha) = \mathrm{cl}_{W\cap W'}^X(\Gamma\cdot\Gamma') \cup \alpha, \quad \text{for all } \alpha \in H_Z^i(X,n).$$

(f) Let $f : X' \to X$ be a morphism between smooth equi-dimensional algebraic k-schemes and let $\alpha \in H_Z^i(X,n)$.
- If f is flat, then $f^*(\mathrm{cl}_W^X(\Gamma) \cup \alpha) = \mathrm{cl}_{f^{-1}(W)}^{X'}(f^*\Gamma) \cup f^*\alpha$.
- If f is smooth and proper, then the following projection formulas hold true:
$$f_*(\mathrm{cl}_{W'}^{X'}(\Gamma) \cup f^*\alpha) = \mathrm{cl}_{f(W')}^X(f_*\Gamma) \cup \alpha \text{ and } f_*(f^*\,\mathrm{cl}_W^X(\Gamma) \cup \alpha) = \mathrm{cl}_W^X(\Gamma) \cup f_*\alpha.$$

C5 (Semi-purity) If X is smooth and equi-dimensional, then $H_Z^i(X,n) = 0$ for $i < 2\,\mathrm{codim}_Z(X)$.

Remark 3.1. *Items* C3b, C4d, C4e, C4f, *and* C5 *are not needed to prove the moving lemma (see Theorems* 1.1 *and* 5.1*) and its immediate applications (see Corollaries* 1.2, 1.4, 1.5, *and* 1.6*). We list these properties, because we will need items* C3b *and* C4d *to get a natural action on refined unramified cohomology, cf. Corollaries* 1.7 *and* 6.8*, while* C4e *and* C4f *are needed to guarantee that this action is functorial with respect to the composition of correspondences. Finally, semi-purity (i.e.* C5*) is only needed to guarantee some normalizations in Corollaries* 1.5 *and* 1.8*.*

3.1. Examples.

Proposition 3.2. *Let k be a field, let ℓ be a prime invertible in k and denote by $\pi_X : X \to \mathrm{Spec}\,k$ the structure map of a k-scheme X.*

(1) Let F be an ℓ^∞-torsion étale sheaf on $\mathrm{Spec}\,k$. The étale cohomology functor

$$(X,Z) \longmapsto H_Z^*(X,n) := H_Z^*(X_{\acute{e}t}, \mathcal{F}(n))$$

satisfies all properties C1–C5 *above, where $\mathcal{F}(n) = \mathrm{colim}_r(\pi_X^* F \otimes_{\mathbb{Z}} \mu_{\ell^r}^{\otimes n})$.*

(2) Let $F = (F_r)_r$ be an inverse system of étale sheaves on $\mathrm{Spec}\,k$ such that F_r is ℓ^r-torsion and the transition maps $F_{r+1} \to F_r$ are surjective. The continuous étale cohomology functor

$$(X,Z) \longmapsto H_Z^*(X,n) := H_{Z,cont}^*(X_{\acute{e}t}, \mathcal{F}(n))$$

satisfies all properties C1–C5 *above, where $\mathcal{F}(n) = (\pi_X^* F_r \otimes_{\mathbb{Z}/\ell^r} \mu_{\ell^r}^{\otimes n})_r$.*

(3) Let $K \in D_{cons}((\mathrm{Spec}\,k)_{pro\acute{e}t}, \widehat{\mathbb{Z}}_\ell)$ be a constructible complex of $\widehat{\mathbb{Z}}_\ell$-modules on the pro-étale site $(\mathrm{Spec}\,k)_{pro\acute{e}t}$. Then the pro-étale (hyper-)cohomology functor

$$(X,Z) \longmapsto H_Z^*(X,n) := \mathrm{R}^* \Gamma_Z(X_{pro\acute{e}t}, \mathcal{K}(n))$$

satisfies all properties C1–C4 *above, where $\mathcal{K}(n) = (\pi_X)_{comp}^* K \otimes_{\widehat{\mathbb{Z}}_\ell} \widehat{\mathbb{Z}}_\ell(n)$ is the n-th Tate twist of the completed pullback of K, cf. Appendix* A *below. If the complex K is concentrated in non-negative degrees, then property* C5 *holds true as well.*

The above result is certainly well-known to experts. For convenience of the reader, we include a detailed proof in the appendix of this paper. We limit ourselves here to sketch the argument briefly.

Using the compatibility of étale cohomology with direct limits in the coefficients from [Stacks21, Tag 09YQ], one reduces item (1) to the case where $\mathcal{F}$ is ℓ^r-torsion for some fixed r. Hence, [Jan88, (3.1)]

implies shows that (2) $\Rightarrow$ (1). Moreover, the arguments in [BS15, §5.6] imply that (3) $\Rightarrow$ (2) and so the last item in the above proposition implies the rest. Conversely, the way the six functor formalism of Bhatt–Scholze from [BS15, §6.7] works, allows one to essentially reduce item (3) to the case of étale cohomology. The proof of the above proposition then goes roughly as follows: Items *C1* and *C2* are straightforward, the pushforwards in item *C3* is a consequence of Poincaré duality (cf. [SGA4.3, p. 585, XVIII, 3.2.5]) and the action in item *C4* is given by cup products with a suitable cycle class (cf. [SGA4$\frac{1}{2}$, pp. 129]). Finally, semi-purity reduces by topological invariance to the case where k is perfect and so the result may be deduced from Poincaré duality.

Proposition 3.2 has an analogue for singular cohomology that applies to algebraic schemes over $k = \mathbb{C}$. Its proof follows the same lines as Proposition 3.2 but is simpler and we leave it to the reader.

Proposition 3.3. *Assume that $k = \mathbb{C}$ and let A be a $\mathbb{Z}$-module (i.e. an abelian group). Then the sheaf cohomology functor*

$$(X, Z) \longmapsto H_Z^*(X, n) := H_Z^*(X_{\mathrm{an}}, \underline{A}_X)$$

satisfies all properties C1–C5 *above, where X_{an} denotes the analytic space that underlies X and $\underline{A}_X$ denotes the constant sheaf on X_{an}, associated to A.*

4. ACTION OF CYCLES ON OPEN VARIETIES

In this section we fix a field k and a twisted cohomology theory as in (3.1) which satisfies the conditions *C1–C4* from Section 3. Moreover, X and Y denote smooth projective equi-dimensional algebraic schemes over k and we set $d_X := \dim(X)$. We will denote the natural projections by $p : X \times Y \to X$ and $q : X \times Y \to Y$, respectively.

Since X and Y are assumed to be smooth projective and equi-dimensional, any algebraic cycle $\Gamma \in Z^c(X \times Y)$ gives rise to an action

$$\Gamma_* : H^i(X, n) \to H^{i+c-d_X}(Y, n + c - d_X), \quad \alpha \longmapsto q_*(\mathrm{cl}_{X \times Y}^{X \times Y}(\Gamma) \cup p^* \alpha)$$

which by *C4* a depends only on the rational equivalence class of Γ. Roughly speaking, the purpose of this section is to generalize this result to smooth quasi-projective varieties that are not necessarily projective but which admit a smooth projective compactification. In other words, we aim to construct a similar action between the cohomology of suitable open subsets $U \subset X$ and $U' \subset Y$. Moreover, instead of ordinary cohomology groups, we will consider the more general situation of cohomology with support. A closely related discussion of the action of correspondences on Chow groups with support can be found in [CR11, Section 1].

Lemma 4.1. *Let $\Gamma \in Z^c(X \times Y)$ be a cycle and let $W \subset X \times Y$ be a closed subset that contains the support of Γ: $\mathrm{supp}(\Gamma) \subset W$. Let $R, Z \subset X$ be closed and put $W_R := W \cap (R \times Y)$ and $W_Z := W \cap (Z \times Y)$. Let $R', Z' \subset Y$ be closed with $q(W_R) \subset R'$ and $q(W_Z) \subset Z'$. Let finally $U := X \setminus R$ and $U' := Y \setminus R'$ be the complements of R and R', respectively.*

Then there is an additive action (to be defined in (4.1) below):

$$\Gamma(W)_* : H_Z^i(U, n) \longrightarrow H_{Z'}^{i+2c-2d_X}(U', n + c - d_X).$$

If $\Gamma = \Gamma_1 + \Gamma_2$ with $\mathrm{supp}(\Gamma_i) \subset W$ for $i = 1, 2$, then $\Gamma(W)_* = \Gamma_1(W)_* + \Gamma_2(W)_*$.

Proof. We denote the flat pullback of Γ to $U \times Y$ by $\Gamma|_{U \times Y}$ and define $\Gamma(W)_*$ by the following composition

(4.1)
$$H^i_Z(U, n)$$
$$\downarrow{\scriptstyle p^*}$$
$$H^i_{Z \times Y}(U \times Y, n)$$
$$\downarrow{\scriptstyle \mathrm{cl}^{U \times Y}_{W \setminus W_R}(\Gamma|_{U \times Y}) \cup}$$
$$H^{i+2c}_{W_Z}(U \times Y, n + c)$$
$$\simeq \downarrow{\scriptstyle \mathrm{exc}}$$
$$H^{i+2c}_{W_Z}((X \times Y) \setminus W_R, n + c)$$
$$\downarrow{\scriptstyle \mathrm{restr.}}$$
$$H^{i+2c}_{W_Z}(X \times U', n + c)$$
$$\downarrow{\scriptstyle q_*}$$
$$H^{i+2c-2d_X}_{Z'}(U', n + c - d_X),$$

where p^* denotes the pullback map with respect to the projection $U \times Y \to U$, $\mathrm{cl}^{U \times Y}_{W \setminus W_R}(\Gamma|_{U \times Y}) \cup$ denotes the action from item *C4*, exc is the isomorphism given by excision (see item *C1*), restr denotes the restriction map given by pullback and q_* denotes the proper pushforward from item *C3*. Additivity of the above action in Γ follows from the fact that the action of Γ from item *C4* is additive in Γ. This concludes the lemma. $\qquad\square$

Lemma 4.2. *Assume in the notation of Lemma 4.1 that Γ, viewed as a cycle on W, is rationally equivalent to zero on W. Then the action $\Gamma(W)_*$ from Lemma 4.1 is zero.*

Proof. The assumption implies that $\Gamma|_{U \times Y}$ is rationally equivalent to zero on $W \cap (U \times Y) = W \setminus W_R$. The result thus follows from condition *C4* a, which implies that the second morphism in the composition (4.1) is zero. $\qquad\square$

Lemma 4.3. *Let $\Gamma \in Z^c(X \times Y)$ be a cycle and let $W_i \subset X \times Y$ for $i = 1, 2$ be closed subsets with $\mathrm{supp}(\Gamma) \subset W_i$ for $i = 1, 2$. Let $Z_1 \subset Z_2 \subset X$ and $Z'_1 \subset Z'_2 \subset X$ be closed with $q(W_i \cap (Z_i \times Y)) \subset Z'_i$ for $i = 1, 2$. Let $R_1 \subset R_2 \subset X$ and $R'_1 \subset R'_2 \subset Y$ be closed with $q(W_i \cap (R_i \times Y)) \subset R'_i$ and with complements $U_i := X \setminus R_i$ and $U'_i := Y \setminus R'_i$ for $i = 1, 2$.*
Then the following diagram commutes

(4.2)
$$\begin{array}{ccccc} H^i_{Z_1}(U_1, n) & \xrightarrow{\mathrm{restr}} & H^i_{Z_1}(U_2, n) & \xrightarrow{\iota_*} & H^i_{Z_2}(U_2, n) \\ \downarrow{\scriptstyle \Gamma(W_1)_*} & & \downarrow{\scriptstyle \Gamma(W_1)_*} & & \downarrow{\scriptstyle \Gamma(W_2)_*} \\ H^{i+2c-2d_X}_{Z'_1}(U'_1, n+c-d_X) & \xrightarrow{\mathrm{restr}} & H^{i+2c-2d_X}_{Z'_1}(U'_2, n+c-d_X) & \xrightarrow{\iota_*} & H^{i+2c-2d_X}_{Z'_2}(U'_2, n+c-d_X), \end{array}$$

where the vertical arrows are the respective actions from Lemma 4.1.

Proof. We assume first that $W_1 \subset W_2$. The action from Lemma 4.1 is defined via the diagram in (4.1). Since $W_1 \subset W_2$, the diagram in question may therefore be expanded to the following big diagram

$$\begin{array}{ccccc}
H^i_{Z_1}(U_1,n) & \xrightarrow{\mathrm{restr}} & H^i_{Z_1}(U_2,n) & \xrightarrow{\iota_*} & H^i_{Z_2}(U_2,n) \\
\downarrow{\scriptstyle p^*} & & \downarrow{\scriptstyle p^*} & & \downarrow{\scriptstyle p^*} \\
H^i_{Z_1 \times Y}(U_1 \times Y,n) & \xrightarrow{\mathrm{restr}} & H^i_{Z_1 \times Y}(U_2 \times Y,n) & \xrightarrow{\iota_*} & H^i_{Z_2 \times Y}(U_2 \times Y,n) \\
\downarrow{\scriptstyle \mathrm{cl}^{U_1 \times Y}_{W_1 \setminus W_1 R_1}(\Gamma|_{U_1 \times Y})\cup} & & \downarrow{\scriptstyle \mathrm{cl}^{U_2 \times Y}_{W_1 \setminus W_1 R_2}(\Gamma|_{U_2 \times Y})\cup} & & \downarrow{\scriptstyle \mathrm{cl}^{U_2 \times Y}_{W_2 \setminus W_2 R_2}(\Gamma|_{U_2 \times Y})\cup} \\
H^{i+2c}_{W_1 Z_1}(U_1 \times Y,n+c) & \xrightarrow{\mathrm{restr}} & H^{i+2c}_{W_1 Z_1}(U_2 \times Y,n+c) & \xrightarrow{\iota_*} & H^{i+2c}_{W_2 Z_2}(U_2 \times Y,n+c) \\
\simeq\downarrow{\scriptstyle \mathrm{exc}} & & \simeq\downarrow{\scriptstyle \mathrm{exc}} & & \simeq\downarrow{\scriptstyle \mathrm{exc}} \\
H^{i+2c}_{W_1 Z_1}((X \times Y) \setminus W_{1 R_1},n+c) & \xrightarrow{\mathrm{restr}} & H^{i+2c}_{W_1 Z_1}((X \times Y) \setminus W_{1 R_2},n+c) & \xrightarrow{\xi} & H^{i+2c}_{W_2 Z_2}((X \times Y) \setminus W_{2 R_2},n+c) \\
\downarrow{\scriptstyle \mathrm{restr}} & & \downarrow{\scriptstyle \mathrm{restr}} & & \downarrow{\scriptstyle \mathrm{restr}} \\
H^{i+2c}_{W_1 Z_1}(X \times U'_1,n+c) & \xrightarrow{\mathrm{restr}} & H^{i+2c}_{W_1 Z_1}(X \times U'_2,n+c) & \xrightarrow{\iota_*} & H^{i+2c}_{W_1 Z_1}(X \times U'_2,n+c) \\
\downarrow{\scriptstyle q_*} & & \downarrow{\scriptstyle q_*} & & \downarrow{\scriptstyle q_*} \\
H^{i+2c-2d_X}_{Z'_1}(U'_1,n+c-d_X) & \xrightarrow{\mathrm{restr}} & H^{i+2c-2d_X}_{Z'_1}(U'_2,n+c-d_X) & \xrightarrow{\iota_*} & H^{i+2c-2d_X}_{Z'_2}(U'_2,n+c-d_X), .
\end{array}$$

where ξ is given by the following composition

$$H^{i+2c}_{W_1 Z_1}((X \times Y) \setminus W_{1 R_2},n+c) \xrightarrow{\mathrm{restr}} H^{i+2c}_{W_1 Z_1}((X \times Y) \setminus W_{2 R_2},n+c) \xrightarrow{\iota_*} H^{i+2c}_{W_2 Z_2}((X \times Y) \setminus W_{2 R_2},n+c)$$

We show first that each square on the left is commutative. This follows for the first, third, and fourth square from functoriality of pullbacks. The second square on the left is commutative because of $C4$d, while the last square on the left column is commutative because of $C3$a.

Next, we show that the right squares of the above diagram are commutative. This follows for the first square from the compatibility of proper pushforwards and pullbacks via open immersions (see $C3$a), and for the third and fourth square by the same compatibility together with the functoriality of pullbacks. Commutativity of the second square on the right follows from $C4$c, while commutativity of the last square on the right follows from the functoriality of pushforwards, see $C3$. This concludes the proof in the case where $W_1 \subset W_2$. The general case follows from this by replacing W_2 by $W_1 \cup W_2$ and noting that this does not require to change R'_2, because $R'_1 \subset R'_2$ holds by assumption. $\square$

Lemma 4.4. *Assume in the notation of Lemma 4.1 that $X = Y$, $c = \dim X$ and $\Gamma = \Delta_X$ is the diagonal with $\Delta_X \subset W$. Then $U' \subset U$ and $Z \subset Z'$ and the action*

$$\Gamma(W)_* : H^i_Z(U,n) \longrightarrow H^i_{Z'}(U',n)$$

from Lemma 4.1 identifies to the natural composition

$$H^i_Z(U,n) \xrightarrow{\iota_*} H^i_{Z'}(U,n) \xrightarrow{\mathrm{restr}} H^i_{Z'}(U',n).$$

Proof. Assume first that $W = \mathrm{supp}\,\Delta_X$. Then the result is a straightforward consequence of item $C4$b together with the compatibility of pushforwards and restrictions to open subsets (see $C3$a) and the functoriality of pushforwards (see $C3$). The general case follows from this and Lemma 4.3. $\square$

Remark 4.5. *Item $C2$ together with the functoriality of pullbacks and pushforwards implies $H^i_\emptyset(X,n) = 0$ for any smooth equi-dimensional k-scheme X. Indeed, $C2$ applied to $Z = W = \emptyset$, shows that the*

sequence

$$H^i_\emptyset(X, n) \xrightarrow{\iota_*} H^i_\emptyset(X, n) \xrightarrow{\mathrm{restr}} H^i_\emptyset(X, n)$$

is exact, while both arrows are isomorphisms by functoriality of pullbacks and pushforwards, respectively.

Lemma 4.6. *Assume in the notation of Lemma 4.1 that* $(\mathrm{supp}\,\Gamma) \cap (Z \times Y) \subset X \times R'$. *Then* $\Gamma(W)_*$ *from Lemma 4.1 is zero.*

Proof. The assumption $(\mathrm{supp}\,\Gamma) \cap (Z \times Y) \subset X \times R'$ implies by Lemma 4.3 that $\Gamma(W)_*$ from Lemma 4.1 factors through the group $H^{i+2c-2d_X}_\emptyset(U', n+c-d_X)$, which vanishes by Remark 4.5. This concludes the proof. $\square$

The proof of the following functoriality property of the action from Lemma 4.1 is slightly tedious. The result is only needed to see that the action of cycles on refined unramified cohomology that we construct in this paper is functorial with respect to the composition of correspondences. In particular, the result is not needed in the proof of the moving lemma for classes with support (see Theorem 1.1 and 5.1) and its immediate consequences. Readers who are mainly interested in the latter may therefore skip the next result and move directly to Section 5 below.

In the following, we let $p_i : X_1 \times X_2 \times X_3 \to X_i$, $p_{ij} : X_1 \times X_2 \times X_3 \to X_i \times X_j$, $p_i^{ij} : X_i \times X_j \to X_i$, and $p_j^{ij} : X_i \times X_j \to X_j$ be the natural projections.

Proposition 4.7. *Let* X_i *for* $i = 1, 2, 3$ *be smooth equi-dimensional algebraic k-schemes of dimensions* $d_{X_i} = \dim X_i$. *Let* $\Gamma_1 \in Z^{c_1}(X_1 \times X_2)$ *and* $\Gamma_2 \in Z^{c_2}(X_2 \times X_3)$ *be cycles and let* $W_i := \mathrm{supp}\,\Gamma_i$ *for* $i = 1, 2$. *Let* $W_{12} := (W_1 \times X_3) \cap (X_1 \times W_2)$ *and assume that* W_{12} *has codimension at least* $c_1 + c_2$. *Let* $W_3 := p_{13}(W_{12}) \subset X_1 \times X_3$ *which is of codimension at least* $c_3 := c_1 + c_2 - d_{X_2}$. *Consider the (well-defined) cycle*

$$\Gamma_3 := (p_{13})_*(p_{12}^* \Gamma_1 \cdot p_{23}^* \Gamma_2) \in Z^{c_3}(X_1 \times X_3)$$

with $\mathrm{supp}\,\Gamma_3 \subset W_3$. *Let* $U_1 \subset X_1$ *be an open subset with complement* R_1. *Let*

$$(4.3) \qquad R_2 := p_2^{12}(W_1 \cap (R_1 \times X_2)) \subset X_2, \qquad R_3 := p_3(W_{12} \cap (R_1 \times X_2 \times X_3)) \subset X_3,$$

and put $U_i := X_i \setminus R_i$ *for* $i = 2, 3$. *Then the following diagram commutes*

$$\begin{array}{ccc} H^i(U_1, n) & \xrightarrow{\Gamma_3(W_3)_*} & \\ {\scriptstyle \Gamma_1(W_1)_*} \downarrow & & \searrow \\ H^{i+2c_1-2d_{X_1}}(U_2, n+c_1-d_{X_1}) & \xrightarrow{\Gamma_2(W_2)_*} & H^{i+2c_3-2d_{X_1}}(U_3, n+c_3-d_{X_1}). \end{array}$$

Before we turn to the proof of the proposition, we need the following result on R_3 from (4.3).

Lemma 4.8. *In the notation of Proposition 4.7, we have*

$$R_3 = p_3^{13}(W_3 \cap (R_1 \times X_3)) = p_3^{23}(W_2 \cap (R_2 \times X_3)).$$

Proof. We have

$$\begin{aligned} p_3^{13}(W_3 \cap (R_1 \times X_3)) &= p_3^{13}(p_{13}(W_{12}) \cap (R_1 \times X_3)) \\ &= p_3^{13}(p_{13}(W_{12} \cap (R_1 \times X_2 \times X_3))) \\ &= p_3(W_{12} \cap (R_1 \times X_2 \times X_3)), \end{aligned}$$

where the second equality follows from the projection formula. This proves the first equality claimed in the lemma.

Since $R_2 = p_2(W_1 \cap (R_1 \times X_2))$, we have

$$R_2 \times X_3 = p_2(W_1 \cap (R_1 \times X_2)) \times X_3 = p_{23}(p_{12}^{-1}(W_1) \cap (R_1 \times X_2 \times X_3))$$

and hence
$$W_2 \cap (R_2 \times X_3) = W_2 \cap p_{23}(p_{12}^{-1}(W_1) \cap (R_1 \times X_2 \times X_3)).$$

The projection formula with respect to p_{23} then gives

(4.4) $$W_2 \cap (R_2 \times X_3) = p_{23}(p_{23}^{-1}(W_2) \cap p_{12}^{-1}(W_1) \cap (R_1 \times X_2 \times X_3)).$$

Since $p_{23}^{-1}(W_2) = X_1 \times W_2$ and $p_{12}^{-1}(W_1) = W_1 \times X_3$, we conclude
$$p_3^{23}(W_2 \cap (R_2 \times X_3)) = p_3(W_{12} \cap (R_1 \times X_2 \times X_3)) = R_3,$$

which proves the second equality in the lemma. This concludes the proof. $\qquad\square$

Proof of Proposition 4.7. For $\alpha \in H^i(U_1, n)$, we have by construction in Lemma 4.1:

(4.5) $$\Gamma_3(W_3)_*(\alpha) = (p_3^{13})_* \left(\mathrm{exc}_1 \left(\mathrm{cl}^{U_1 \times X_3}_{W_3 \backslash (W_3)_{R_1}} ((p_{13})_*(p_{12}^* \Gamma_1 \cdot p_{23}^* \Gamma_2)) \cup p_1^* \alpha \right) |_{X_1 \times U_3} \right),$$

where
$$\mathrm{exc}_1 : H^*_{W_3 \backslash (W_3)_{R_1}}(U_1 \times X_3, -) \xrightarrow{\simeq} H^*_{W_3 \backslash (W_3)_{R_1}}((X_1 \times X_3) \backslash (W_3)_{R_1}, -)$$

denotes the isomorphism given by excision and $p_3^{13} : X_1 \times U_3 \to U_3$ denotes the projection. Here $(W_3)_{R_1} = W_3 \cap (R_1 \times X_3)$ and so $p_3^{13}((W_3)_{R_1}) = R_3$ by Lemma 4.8.

Since $W_3 = p_{13}(W_{12})$, the projection formula in C4f yields

(4.6) $$\mathrm{cl}^{U_1 \times X_3}_{W_3 \backslash (W_3)_{R_1}} ((p_{13})_*(p_{12}^* \Gamma_1 \cdot p_{23}^* \Gamma_2)) \cup p_1^* \alpha = (p_{13})_* \left(\mathrm{cl}^{U_1 \times X_2 \times X_3}_{W_{12} \backslash (W_{12})_{R_1}} (p_{12}^* \Gamma_1 \cdot p_{23}^* \Gamma_2) \cup p_{13}^*(p_1^* \alpha) \right),$$

where $p_1^* \alpha$ denotes the pullback of α via the projection $U_1 \times X_3 \to U_1$. Note that $p_{12}^* \Gamma_1 \cdot p_{23}^* \Gamma_2 = p_{23}^* \Gamma_2 \cdot p_{12}^* \Gamma_1$ by the commutativity of the intersection product of cycles. Since $W_{12} = (W_1 \times X_3) \cap (X_1 \times W_2)$ has codimension at least $c_1 + c_2$, the compatibility of the action of cycles with the intersection product (see C4e) thus shows

$$\mathrm{cl}^{U_1 \times X_2 \times X_3}_{W_{12} \backslash (W_{12})_{R_1}} (p_{12}^* \Gamma_1 \cdot p_{23}^* \Gamma_2) \cup p_{13}^*(p_1^* \alpha) = \mathrm{cl}^{U_1 \times X_2 \times X_3}_{U_1 \times W_2} (p_{23}^* \Gamma_2) \cup \left(\mathrm{cl}^{U_1 \times X_2 \times X_3}_{(W_1 \backslash (W_1)_{R_1}) \times X_3} (p_{12}^* \Gamma_1) \cup p_{13}^*(p_1^* \alpha) \right).$$

Note that $p_{13}^*(p_1^* \alpha) = p_{12}^* p_1'^* \alpha$, where $p_1'^* \alpha$ denotes the pullback of α via the projection $U_1 \times X_2 \to U_1$. Using the compatibility with pullbacks from C4f, we thus get

$$\mathrm{cl}^{U_1 \times X_2 \times X_3}_{W_{12} \backslash (W_{12})_{R_1}} (p_{12}^* \Gamma_1 \cdot p_{23}^* \Gamma_2) \cup p_{13}^*(p_1^* \alpha) = \mathrm{cl}^{U_1 \times X_2 \times X_3}_{U_1 \times W_2} (p_{23}^* \Gamma_2) \cup p_{12}^* \left(\mathrm{cl}^{U_1 \times X_2}_{W_1 \backslash (W_1)_{R_1}} (\Gamma_1) \cup (p_1'^* \alpha) \right).$$

Substituting this into (4.6) and plugging the result into (4.5), we get:

$$\Gamma_3(W_3)_*(\alpha) = (p_3^{13})_* \left(\mathrm{exc}_1 \left((p_{13})_* \left(\mathrm{cl}^{U_1 \times X_2 \times X_3}_{U_1 \times W_2} (p_{23}^* \Gamma_2) \cup p_{12}^* \left(\mathrm{cl}^{U_1 \times X_2}_{W_1 \backslash (W_1)_{R_1}} (\Gamma_1) \cup (p_1'^* \alpha) \right) \right) \right) |_{X_1 \times U_3} \right).$$

Compatibility of proper pushforward with pullbacks via open immersions (see C3a) and the fact that exc_1 is the inverse of a pullback via an open immersion, shows that

$$\Gamma_3(W_3)_*(\alpha) = (p_3^{13})_* \left((p_{13})_* \left(\mathrm{exc}_2 \left(\mathrm{cl}^{U_1 \times X_2 \times X_3}_{U_1 \times W_2} (p_{23}^* \Gamma_2) \cup p_{12}^* \left(\mathrm{cl}^{U_1 \times X_2}_{W_1 \backslash (W_1)_{R_1}} (\Gamma_1) \cup (p_1'^* \alpha) \right) \right) |_{X_1 \times X_2 \times U_3} \right) \right),$$

where
$$\mathrm{exc}_2 : H^*_{W_{12} \backslash (W_{12})_{R_1}}(U_1 \times X_2 \times X_3, -) \xrightarrow{\simeq} H^*_{W_{12} \backslash (W_{12})_{R_1}}((X_1 \times X_2 \times X_3) \backslash (W_{12})_{R_1}, -)$$

is the isomorphism given by excision. Here we use that $(W_{12})_{R_1} = W_{12} \cap (R_1 \times X_2 \times X_3)$ and $W_3 = p_{13}(W_{12})$, and so $p_3((W_{12})_{R_1}) = p_3^{13}(W_3 \cap R_1 \times X_3) = R_3 = X_3 \backslash U_3$ by Lemma 4.8.

Functoriality of pushforwards then shows:

$$\Gamma_3(W_3)_*(\alpha) = (p_3^{23})_* \left((p_{23})_* \left(\mathrm{exc}_2 \left(\mathrm{cl}^{U_1 \times X_2 \times X_3}_{U_1 \times W_2} (p_{23}^* \Gamma_2) \cup p_{12}^* \left(\mathrm{cl}^{U_1 \times X_2}_{W_1 \backslash (W_1)_{R_1}} (\Gamma_1) \cup (p_1'^* \alpha) \right) \right) |_{X_1 \times X_2 \times U_3} \right) \right).$$

Since the action of cycles is compatible with pullbacks along open immersions (hence with excision) by $C4$d, $\Gamma_3(W_3)_*(\alpha)$ is given by

$$(p_3^{23})_* \left((p_{23})_* \left(\mathrm{exc}_2 \left(\mathrm{cl}_{X_1 \times W_2}^{X_1 \times X_2 \times X_3}(p_{23}^* \Gamma_2) \cup p_{12}^* \circ \mathrm{exc}_3 \left(\mathrm{cl}_{W_1 \setminus (W_1)_{R_1}}^{U_1 \times X_2}(\Gamma_1) \cup (p_1'^* \alpha) \right) |_{X_1 \times U_2} \right) |_{X_1 \times X_2 \times U_3} \right) \right),$$

where

$$\mathrm{exc}_3 : H_{W_1 \setminus (W_1)_{R_1}}^*(U_1 \times X_2, -) \xrightarrow{\simeq} H_{W_1 \setminus (W_1)_{R_1}}^*((X_1 \times X_2) \setminus (W_1)_{R_1}, -)$$

is the isomorphism given by excision, and where we use that $R_2 = p_2^{12}((W_1)_{R_1})$, see (4.3).

Compatibility of proper pushforwards with pullbacks along open immersions (hence with excision) from $C3$a then shows that $\Gamma_3(W_3)_*(\alpha)$ is given by

$$(p_3^{23})_* \left(\mathrm{exc}_4 \left((p_{23})_* \left(\mathrm{cl}_{X_1 \times W_2}^{X_1 \times X_2 \times X_3}(p_{23}^* \Gamma_2) \cup p_{12}^* \circ \mathrm{exc}_3 \left(\mathrm{cl}_{W_1 \setminus (W_1)_{R_1}}^{U_1 \times X_2}(\Gamma_1) \cup (p_1'^* \alpha) \right) |_{X_1 \times U_2} \right) \right) |_{X_2 \times U_3} \right),$$

where

$$\mathrm{exc}_4 : H_{W_2}^*(U_2 \times X_3, -) \xrightarrow{\simeq} H_{W_1 \setminus (W_1)_{R_1}}^*((X_2 \times X_3) \setminus p_{23}((W_{12})_{R_1}), -)$$

is the isomorphism given by excision, where $p_{23}((W_{12})_{R_1}) = p_{23}(W_{12} \cap (R_1 \times X_2 \times X_3))$ and so $p_3(p_{23}((W_{12})_{R_1})) = R_3$, see (4.3).

Applying the projection formula from $C4$f to p_{23} then shows that $\Gamma_3(W_3)_*(\alpha)$ is given by

$$(p_3^{23})_* \left(\mathrm{exc}_4 \left(\mathrm{cl}_{W_2}^{X_2 \times X_3}(\Gamma_2) \cup (p_{23})_* \circ p_{12}^* \circ \mathrm{exc}_3 \left(\mathrm{cl}_{W_1 \setminus (W_1)_{R_1}}^{U_1 \times X_2}(\Gamma_1) \cup (p_1'^* \alpha) \right) |_{X_1 \times U_2} \right) |_{X_2 \times U_3} \right).$$

We then consider the natural pushforward map

$$\epsilon : H_{W_1 \setminus (W_1)_{R_1}}^*(X_1 \times U_2, -) \longrightarrow H^*(X_1 \times U_2, -)$$

and note that by functoriality of pullbacks and proper pushforwards, the above class identifies to

$$(p_3^{23})_* \left(\mathrm{exc}_4 \left(\mathrm{cl}_{W_2}^{X_2 \times X_3}(\Gamma_2) \cup (p_{23})_* \circ p_{12}^* \circ \epsilon \circ \mathrm{exc}_3 \left(\mathrm{cl}_{W_1 \setminus (W_1)_{R_1}}^{U_1 \times X_2}(\Gamma_1) \cup (p_1'^* \alpha) \right) |_{X_1 \times U_2} \right) |_{X_2 \times U_3} \right).$$

By item $C3$b, we have $(p_{23})_* \circ p_{12}^* \circ \epsilon = (p_2^{23})^* \circ (p_2^{12})_* \circ \epsilon$, and so $\Gamma_3(W_3)_*(\alpha)$ is given by

$$(p_3^{23})_* \left(\mathrm{exc}_4 \left(\mathrm{cl}_{W_2}^{X_2 \times X_3}(\Gamma_2) \cup (p_2^{23})^* \circ (p_2^{12})_* \circ \epsilon \circ \mathrm{exc}_3 \left(\mathrm{cl}_{W_1 \setminus (W_1)_{R_1}}^{U_1 \times X_2}(\Gamma_1) \cup (p_1'^* \alpha) \right) |_{X_1 \times U_2} \right) |_{X_2 \times U_3} \right).$$

This shows that

$$\Gamma_3(W_3)_*(\alpha) = (p_3^{23})_* \left(\mathrm{exc}_4 \left(\mathrm{cl}_{X_1 \times W_2}^{X_2 \times X_3}(\Gamma_2) \cup (p_2^{23})^* \left(\Gamma_1(W_1)_*(\alpha) \right) \right) |_{X_2 \times U_3} \right)$$
$$= \Gamma_2(W_2)_*(\Gamma_1(W_1)_*(\alpha)),$$

as we want. $\square$

5. The moving lemma

Theorem 1.1 stated in the introduction will be deduced from the following stronger (but more technical) result.

Theorem 5.1. *Let k be a field and fix a twisted cohomology theory as in (3.1) which satisfies the conditions C1–C4 from Section 3. Let U be a smooth equi-dimensional algebraic k-scheme that admits a smooth projective compactification and let $f : S \to U$ be a morphism from an algebraic k-scheme S that is locally of pure dimension. Then for any no-where dense closed subset $Z \subset U$, there are closed subsets $Z' \subset W \subset U$ with $Z \subset W$ and $\dim W \leq \dim Z + 1$, such that:*

(1) The subsets Z' and $W \setminus Z$ are in good position with respect to f in the following sense:
   (a) the codimension of $f^{-1}(Z')$ in S is locally at each point at least $\mathrm{codim}_U Z$;
   (b) the codimension of $f^{-1}(W \setminus Z)$ in S is locally at each point at least $\mathrm{codim}_U Z - 1$.

(2) *Classes with support on Z can be moved along W to classes with support on Z' in the following sense:*

$$\mathrm{im}(H_Z^*(U,n) \to H_W^*(U,n)) \subset \mathrm{im}(H_{Z'}^*(U,n) \to H_W^*(U,n)).$$

Moreover, the following additional results hold true:

(3) *Up to replacing S by some dense open subset $S^\circ \subset S$ of the form $S^\circ = S \setminus f^{-1}(\bar{A})$ for some closed $\bar{A} \subset U$, W is well-behaved under localization in the following sense: If we replace U by an open subset $V \subset U$, and S by $f^{-1}(V) \cap S^\circ$, then W can be replaced by $W \cap V$. Moreover, if $\dim f(S) < \mathrm{codim}_U Z - 1$, then $S^\circ = S$.*

(4) *Any component $Z'' \subset Z'$ with $\dim Z'' > \dim Z$ satisfies $f^{-1}(Z'') = \emptyset$. If we only ask items (1) and (2), but not (3), then we may assume that no such component exists.*

Remark 5.2. *Item (3) in the above theorem is equivalent to saying that there is an open subset $U^\circ \subset U$ with the following properties:*

- *$f^{-1}(U^\circ)$ is dense in S;*
- *If we replace U by an open subset $V \subset U^\circ$, and S by $f^{-1}(V)$, then the closed subset W in Theorem 5.1 can be replaced by $W \cap V$.*

Remark 5.3. *Even if $\dim S = 0$, the subset $Z' \subset U$ in Theorem 5.1 can in general not be chosen to be well-behaved with respect to localization. This can already be observed in the case where $U = \mathbb{P}^2$, $Z \subset \mathbb{P}^2$ is a line, $S \subset Z$ is a closed point, $k = \bar{k}$ is algebraically closed and $H^*(-,n)$ denotes étale cohomology with coefficients in $\mu_{\ell^r}^{\otimes n}$ for some prime ℓ invertible in k. Indeed, if in this case $Z' \subset \mathbb{P}^2$ is any closed subset which meets S properly, then $Z \not\subset Z'$ and so we can pick two points $\{p, q\} \subset Z \setminus Z'$. But then there is a class*

$$\alpha \in H_Z^3(\mathbb{P}^2 \setminus \{p, q\}, n) \simeq H^1(Z \setminus \{p, q\}, n-1) \simeq \mathbb{Z}/\ell^r$$

with nontrivial residue $\partial_p \alpha$ at p and so the pushforward of α to $H^3(\mathbb{P}^2 \setminus \{p, q\}, n)$ does not admit a lift to $H_{Z'}^3(\mathbb{P}^2 \setminus \{p, q\}, n)$, because the latter agrees with $H_{Z'}^3(\mathbb{P}^2, n)$ by excision and so the residue at p would need to be trivial.

Proof of Theorem 5.1. Let X be a smooth projective compactification of U and let $\bar{Z} \subset X$ be the closure of Z in X. Since U is equi-dimensional, so is X and we let $d_X := \dim X$. Let $R := \bar{Z} \setminus Z$. By excision (see item C1), the canonical restriction map $H_Z^*(X \setminus R, n) \to H_Z^*(U, n)$ is an isomorphism. Since the pushforward maps from C3 are compatible with respect to pullbacks along open immersions (see C3a), one easily concludes that it suffices to prove the theorem in the case where

(5.1) $$U = X \setminus R \quad \text{with} \quad R := \bar{Z} \setminus Z.$$

The idea is to apply the moving lemma for algebraic cycles to the diagonal $\Delta_X \subset X \times X$ and to exploit the action of cycles on open varieties from Section 4. The former yields the following precise statement.

Lemma 5.4. *There is a closed subset $W_{X \times X} \subset X \times X$ of codimension $d_X - 1$ and a cycle $\Delta_X' \in Z^{d_X}(X \times X)$ with $\mathrm{supp}(\Delta_X') \subset W_{X \times X}$ and $\Delta_X \subset W_{X \times X}$, such that Δ_X and Δ_X', viewed as cycles on $W_{X \times X}$, are rationally equivalent to each other on $W_{X \times X}$. Moreover, the following transversality conditions hold:*

(1) *The closed subset $W_{X \times X}$ meets $\bar{Z} \times X$ dimensionally transversely;*

(2) *Consider the morphism*

$$g := \tau_R \times f : R^\nu \times S \longrightarrow X \times X,$$

where R^ν denotes the disjoint union of the components of R and $\tau_R : R^\nu \to X$ denotes the natural map. Then the preimage $g^{-1}(W_{X \times X})$ (resp. $g^{-1}(\mathrm{supp}(\Delta_X'))$) has locally codimension at least $d_X - 1$ (resp. at least d_X) in $R^\nu \times S$.

(3) *Consider the morphism*

$$h := \tau_{\bar{Z}} \times f : \bar{Z}^\nu \times S \longrightarrow X \times X,$$

where $\bar{Z}^\nu$ *denotes the disjoint union of the irreducible components of* $\bar{Z}$ *and* $\tau_{\bar{Z}_1} : \bar{Z}^\nu \to X$ *denotes the natural map. Then the following holds:*
  (a) *the preimage* $h^{-1}(\mathrm{supp}(\Delta'_X))$ *has locally codimension at least* d_X *in* $\bar{Z}^\nu \times S$;
  (b) *the preimage* $h^{-1}(W_{X \times X} \setminus \Delta_X)$ *has locally codimension at least* $d_X - 1$.

Proof. The lemma follows from the moving lemma for algebraic cycles (see Theorem 2.1) applied to a suitable morphism from a locally equi-dimensional algebraic k-scheme to $X \times X$. We will indicate which morphism we take to arrange that each of the items (1)–(3) holds individually; the disjoint union of these morphisms will then ensure that we can arrange that all items hold simultaneously.

Note that Δ_X meets $\bar{Z} \times X$ dimensionally transversely. Item (1) follows therefore by applying Theorem 2.1 to the natural map $\bar{Z}^\nu \times X \to X \times X$.

Since $\mathrm{im}\, f \subset U$, while $R = X \setminus U$, we find that $g^{-1}(\Delta_X) = \emptyset$. Item (2) follows therefore by applying Theorem 2.1 to the morphism g above.

Applying Theorem 2.1 to h restricted to the complement of $h^{-1}(\Delta_X)$ in $\bar{Z}^\nu \times S$, we can arrange item (3b) to hold true. By the same argument, we can arrange that $h^{-1}(\mathrm{supp}(\Delta'_X)) \setminus h^{-1}(\Delta_X)$ has codimension d_X locally at each point. To prove (3a), it thus suffices to arrange that

$$h^{-1}(\mathrm{supp}(\Delta'_X)) \cap h^{-1}(\Delta_X) = \emptyset.$$

Since $h^{-1}(\Delta_X) = h^{-1}(\Delta_{\bar{Z}})$, where $\Delta_{\bar{Z}} \subset X \times X$ denotes the diagonal of $\bar{Z}$, it suffices to see that Δ'_X does not intersect $\Delta_{\bar{Z}} \subset X \times X$, which we can arrange by the moving lemma because $\bar{Z} \subset X$ is no-where dense by assumption and so $\dim \Delta_X = d_X < d_X + 1 \leq \mathrm{codim}_{X \times X} \Delta_{\bar{Z}}$. This concludes the proof of the lemma. $\qquad \square$

We define

$$(5.2) \qquad \bar{Z}' := q(W_{X \times X} \cap (R \times X)) \cup q(\mathrm{supp}(\Delta'_X) \cap (\bar{Z} \times X)), \qquad Z' := \bar{Z}' \cap U,$$

$$(5.3) \qquad \bar{W} := q(W_{X \times X} \cap (\bar{Z} \times X)), \quad \text{and} \quad W := \bar{W} \cap U.$$

(Here and in the following, $q : X \times X \to X$ denotes, as before, the projection onto the second factor.) Since $\Delta_X \subset W_{X \times X}$, we have $Z \subset W$. Since $R \subset \bar{Z}$ (by the definition of R, cf. (5.1)) and $\mathrm{supp}(\Delta'_X) \subset W_{X \times X}$, we also find that $Z' \subset W$. Moreover, since $W_{X \times X} \subset X \times X$ has codimension $d_X - 1$, item (1) in Lemma 5.4 implies $\dim W \leq \dim Z + 1$, as we want.

It remains to prove items (2)–(4) in the theorem.

Step 1. Proof of item (2).

Let $U' := U \setminus Z'$. By the long exact sequence of triples (see *C2*), it suffices to show that the following composition is zero:

$$(5.4) \qquad H^*_Z(U, n) \xrightarrow{\iota_*} H^*_W(U, n) \xrightarrow{\mathrm{restr}} H^*_W(U', n).$$

We view Δ_X as a cycle on $X \times X$. Its support is contained in $W_{X \times X}$. By (5.3), $W = q(W_{X \times X} \cap (\bar{Z} \times X)) \cap U$. Moreover, (5.2) implies that $q(W_{X \times X} \cap (R \times X)) \subset X \setminus U'$. Lemma 4.1 thus yields an action

$$\Delta_X(W_{X \times X})_* : H^*_Z(U, n) \longrightarrow H^*_W(U', n).$$

By Lemma 4.4, this action identifies to the composition in (5.4).

By Lemma 5.4, Δ_X is rationally equivalent to Δ'_X on $W_{X \times X}$. Lemma 4.2 and the additivity of the action thus imply that

$$\Delta_X(W_{X \times X})_* = \Delta'_X(W_{X \times X})_* : H^*_Z(U, n) \longrightarrow H^*_W(U', n).$$

By (5.2), $q(\mathrm{supp}(\Delta'_X) \cap (\bar{Z} \times X)) \subset \bar{Z}'$. Moreover, $U' \cap \bar{Z}' = \emptyset$, because $U' = U \setminus Z' = X \setminus \bar{Z}'$, where the last equality uses that $\bar{Z} \setminus Z = X \setminus U$ by (5.1). Hence, Lemma 4.6 implies that

$$\Delta'_X(W_{X \times X})_* : H^*_Z(U, n) \longrightarrow H^*_W(U', n)$$

is zero. This concludes the proof of step 1.

Step 2. Proof of item (1).

Since $\mathrm{im}\, f \subset U \subset X$, the definition of Z' in (5.2) implies that

$$f^{-1}(Z') = f^{-1}(q(W_{X \times X} \cap (R \times X))) \cup f^{-1}(q(\mathrm{supp}(\Delta'_X) \cap (\bar{Z} \times X))).$$

This splits into a union of two closed subsets and it suffices to bound the codimension of each of these two subsets separately.

Consider the following commutative diagram

$$
\begin{array}{ccccc}
R^\nu \times S & \xrightarrow{\mathrm{id} \times f} & R^\nu \times X & \xrightarrow{\tau_R \times \mathrm{id}} & X \times X \\
\downarrow{\scriptstyle \mathrm{pr}_2} & & \downarrow{\scriptstyle \mathrm{pr}_2} & & \downarrow{\scriptstyle q} \\
S & \xrightarrow{\quad f \quad} & X & \xrightarrow{\quad \mathrm{id} \quad} & X
\end{array}
$$

where each vertical arrow is the respective projection onto the second factor, the square on the left is Cartesian, and the composition of the two horizontal arrows on the top coincides with the morphism g from item (2) from Lemma 5.4. It is elementary to check that we have an equality of sets

$$\mathrm{pr}_2((\mathrm{id} \times f)^{-1}(A)) = f^{-1}(\mathrm{pr}_2(A))$$

for any subset $A \subset R^\nu \times X$. Applying this to $A = (\tau_R \times \mathrm{id})^{-1}(W_{X \times X})$, we get

$$\mathrm{pr}_2(g^{-1}(W_{X \times X})) = f^{-1}(\mathrm{pr}_2((\tau_R \times \mathrm{id})^{-1}(W_{X \times X}))).$$

The above right hand side agrees with $f^{-1}(q(W_{X \times X} \cap (R \times X)))$, and so

$$\mathrm{pr}_2(g^{-1}(W_{X \times X})) = f^{-1}(q(W_{X \times X} \cap (R \times X)))$$

The codimension of $\mathrm{pr}_2(g^{-1}(W_{X \times X}))$ in S is bounded from below by

$$\mathrm{codim}_{R^\nu \times S}(g^{-1}(W_{X \times X})) - \dim R^\nu.$$

By item (2) in Lemma 5.4, the above number is bounded from below by $d_X - 1 - \dim R$, where we use $\dim R = \dim R^\nu$. Since $R = \bar{Z} \setminus Z$ is no-where dense in $\bar{Z}$, we finally conclude that the above number is bounded from below by $d_X - \dim Z$. Hence, $f^{-1}(q(W_{X \times X} \cap (R \times X)))$ has codimension at least $\mathrm{codim}_X \bar{Z} = \mathrm{codim}_U Z$ in S, as we want.

Similarly, consider the commutative diagram

$$(5.5)
\begin{array}{ccccc}
\bar{Z}^\nu \times S & \xrightarrow{\mathrm{id} \times f} & \bar{Z}^\nu \times X & \xrightarrow{\tau_{\bar{Z}} \times \mathrm{id}} & X \times X \\
\downarrow{\scriptstyle \mathrm{pr}_2} & & \downarrow{\scriptstyle \mathrm{pr}_2} & & \downarrow{\scriptstyle q} \\
S & \xrightarrow{\quad f \quad} & X & \xrightarrow{\quad \mathrm{id} \quad} & X,
\end{array}
$$

where the composition of the morphisms in the top row is h. As before, on checks that there is an equality of sets

$$(5.6) \qquad\qquad \mathrm{pr}_2((\mathrm{id} \times f)^{-1}(A)) = f^{-1}(\mathrm{pr}_2(A))$$

for any subset $A \subset \bar{Z}^\nu \times X$. Applying this to $A = (\tau_{\bar{Z}} \times \mathrm{id})^{-1}(\mathrm{supp}\,\Delta'_X)$, we conclude in a similar way as above that the codimension of $f^{-1}(q(\mathrm{supp}(\Delta'_X) \cap (\bar{Z} \times X)))$ in S is bounded from below by

$$\mathrm{codim}_{\bar{Z}^\nu \times S}(h^{-1}(\mathrm{supp}(\Delta'_X))) - \dim \bar{Z}.$$

By item (3a) in Lemma 5.4, this number is bounded from below by $d_X - \dim \bar{Z} = \operatorname{codim}_X \bar{Z}$, as we want. This concludes item (1a).

Next, we aim to prove (1b). Using (5.5) and applying (5.6) to $(\tau_{\bar{Z}} \times \operatorname{id})^{-1}(W_{X \times X})$, we find that

$$f^{-1}(\operatorname{pr}_2((\tau_{\bar{Z}} \times \operatorname{id})^{-1}(W_{X \times X}))) = \operatorname{pr}_2(h^{-1}(W_{X \times X})).$$

Note that $\operatorname{pr}_2((\tau_{\bar{Z}} \times \operatorname{id})^{-1}(W_{X \times X})) = q(W_{X \times X} \cap (\bar{Z} \times X)) = \bar{W}$, where the first equality uses that $\bar{Z}^\nu$ is the disjoint union of the irreducible components of $\bar{Z}$ and the last equality uses (5.3). Hence,

$$f^{-1}(W) = f^{-1}(\bar{W}) = \operatorname{pr}_2(h^{-1}(W_{X \times X})),$$

where the first equality uses $\operatorname{im} f \subset U$. Similarly, $f^{-1}(Z) = \operatorname{pr}_2(h^{-1}(\Delta_X))$ and so

$$f^{-1}(W \setminus Z) \subset \operatorname{pr}_2(h^{-1}(W_{X \times X} \setminus \Delta_X)).$$

Hence, the codimension of $f^{-1}(W \setminus Z)$ in S is bounded from below by

$$\operatorname{codim}_{\bar{Z}^\nu \times S}(h^{-1}(W_{X \times X} \setminus \Delta_X)) - \dim \bar{Z}.$$

By item (3b) in Lemma 5.4, the above number is bounded from below by

$$d_X - 1 - \dim Z = \operatorname{codim}_U Z - 1.$$

This proves (1b) in Theorem 5.1 and hence concludes step 2.

Step 3. Proof of item (3).

If $\operatorname{codim}_U Z = 1$, then we may take $W = U$ and so the claim is clear. Since $Z \subset X$ is no-where dense, it remains to consider the case $\operatorname{codim}_U Z \geq 2$.

In this case we consider $h^{-1}(W_{X \times X} \setminus \Delta_X) \subset \bar{Z}^\nu \times S$ and define

$$S^\circ := S \setminus S^{\mathrm{cl}} \quad \text{with} \quad S^{\mathrm{cl}} := \overline{\operatorname{pr}_2(h^{-1}(W_{X \times X} \setminus \Delta_X))} \subset S.$$

By (5.6),

$$S^{\mathrm{cl}} = \overline{f^{-1}(A)}, \quad \text{where} \quad A := \operatorname{pr}_2((\tau \times \operatorname{id})^{-1}(W_{X \times X} \setminus \Delta_X)).$$

Hence, $S^\circ = S \setminus f^{-1}(\bar{A})$, for the closed subset $\bar{A} \subset X$, as we want in item (3) of the theorem.

By item (3) in Lemma 5.4, $h^{-1}(W_{X \times X} \setminus \Delta_X)$ has locally codimension at least $d_X - 1$. Hence, S^{cl} has locally codimension at least $\dim U - \dim Z - 1 = \operatorname{codim}_U Z - 1$ in S. In particular $S^\circ \subset S$ is dense because $\operatorname{codim}_U Z \geq 2$. Note further that $\operatorname{im} h = \bar{Z} \times f(S)$. Therefore, if $\dim f(S) < \operatorname{codim}_U Z - 1$, then $\dim(\operatorname{im} h) < d_X - 1$. The aforementioned observation that $h^{-1}(W_{X \times X} \setminus \Delta_X)$ has locally codimension at least $d_X - 1$ thus implies by Lemma 2.2 that $S = S^\circ$, as we want.

It remains to prove that up to replacing S by S°, the closed subset W is well-behaved under localization. By definition of S°, this replacement has the effect that we may from now on assume that

(5.7) $$h^{-1}(W_{X \times X} \setminus \Delta_X) = \emptyset.$$

Let now $V \subset U$ be an open subset and let $R_V := X \setminus V$. By excision, we reduce to the case where $R_V \subset \bar{Z}$. Then $R \subset R_V \subset \bar{Z}$ and replacing U by V amounts to replacing R by R_V and S by $S_V := f^{-1}(V)$. We aim to show that under these replacements, W can be replaced by $W \cap V$. In view of the definition in (5.3), it will be enough to show that $W_{X \times X}$ from Lemma 5.4 does not need to be changed if we replace R by R_V and S by S_V.

It is clear that the conclusion of Lemma 5.4 is not affected if we replace S by S_V and so we may from now on assume that $f(S) \subset V$. It remains to analyse the effect of replacing R by the possibly larger subset $R_V \subset \bar{Z}$. Items (1) and (3) are not affected by this replacement. It thus suffices to show that item (2) in Lemma 5.4 remains true after replacing R by R_V. To this end we consider the map

$$g_V := \tau_{R_V} \times f : R_V^\nu \times S \longrightarrow X \times X$$

that we obtain by replacing R by R_V in item (2) of Lemma 5.4. Since $R_V \subset \bar{Z}$ (by the above reduction step), $h^{-1}(W_{X \times X} \setminus \Delta_X) = \emptyset$ (see (5.7)) implies $g_V^{-1}(W_{X \times X} \setminus \Delta_X) = \emptyset$. Since $f(S) \subset V$ by the above

reduction step, we have $R_V \cap f(S) = \emptyset$ and so $g_V^{-1}(\Delta_X) = \emptyset$. We thus conclude $g_V^{-1}(W_{X \times X}) = \emptyset$. Since $\mathrm{supp}(\Delta'_X) \subset W_{X \times X}$, this also implies that $g_V^{-1}(\mathrm{supp}(\Delta'_X)) = \emptyset$. Hence, the conclusion in item (2) of Lemma 5.4 remains true if we replace R by R_V, as claimed. This concludes the proof of item (3) in the theorem.

Step 4. Proof of item (4).

We first note that $Z' \subset W$. Since $\dim W \leq \dim Z + 1$, we thus get $\dim Z' \leq \dim Z + 1$. By item (1a), $f^{-1}(Z')$ has locally at each point codimension at least $\mathrm{codim}_U Z$. But if $Z'' \subset Z'$ is a component of dimension $\dim Z + 1$, then $f^{-1}(Z'')$, if non-empty, has locally codimension at most $\mathrm{codim}_U Z - 1$. Hence, $f^{-1}(Z'') = \emptyset$, as claimed.

Finally, we only ask that items (1) and (2) hold true (but not (3)) and we aim to show that we can arrange $\dim Z \geq \dim Z'$. To this end we use the moving lemma for cycles (see Theorem 2.1) to ask in Lemma 5.4 that $W_{X \times X}$ meets $R \times X$ dimensionally transversally, and that $\mathrm{supp}(\Delta'_X)$ meets $\bar{Z} \times X$ dimensionally transversely. (We can do this because the respective transversality conditions are obviously satisfied for the diagonal Δ_X.) Then the definition of Z' in (5.2) implies $\dim Z \geq \dim Z'$, as we want. (Here we emphasize that we cannot ensure any longer that (3) holds true: unless every component of $W_{X \times X} \cap (\bar{Z} \times X)$ dominates a component of $\bar{Z}$, we cannot ensure for varying R that the intersection of $W_{X \times X}$ with $R \times X$ is dimensionally transversely, hence we may loose the localization property from Step 3 above.)

This concludes the proof of Step 4, and hence the proof of the theorem. $\square$

6. Applications: Proof of Theorem 1.1 and Corollaries 1.2–1.8

6.1. **Proof of Theorem 1.1.**

Proof of Theorem 1.1. Let X be a smooth equi-dimensional k-scheme that admits a smooth projective compactification and let $S, Z \subset X$ be closed subsets. Let S^ν be the disjoint union of the irreducible components of S and let $f : S^\nu \to X$ be the natural map. We apply Theorem 5.1 to $U := X$. We only ask that items (1) and (2) hold true, but not (3). We may thus by item (4) assume that $\dim Z' \leq \dim Z$. By item (2),

$$(6.1) \qquad \mathrm{im}(H_Z^*(X, n) \to H_W^*(X, n)) \subset \mathrm{im}(H_{Z'}^*(X, n) \to H_W^*(X, n)).$$

In other words, for any $\alpha \in H_Z^*(X, n)$ there is a class $\alpha' \in H_{Z'}^*(X, n)$ such that α and α' have the same image in $H_W^*(X, n)$. We further know that $\dim W \leq \dim Z + 1$ and by item (1) we have $\mathrm{codim}_X(S \cap Z') \geq \mathrm{codim}_X S + \mathrm{codim}_X Z$ and $\mathrm{codim}_X(S \cap (W \setminus Z)) \geq \mathrm{codim}_X S + \mathrm{codim}_X Z - 1$. So if we knew that equalities in the inequalities $\dim Z' \leq \dim Z$ and $\dim W \leq \dim Z + 1$ hold true, we would be done. Note however that passing from inequalities to equalities is trivial: if some of the inequalities are strict, we add to Z' (resp. W) closed subsets of the desired dimensions, in general position with respect to S, in such a way that $Z' \subset W$ remains true. Condition (6.1) then remains true by functoriality of pushforwards (see *C3*) and the conditions $\mathrm{codim}_X(S \cap Z') \geq \mathrm{codim}_X S + \mathrm{codim}_X Z$ and $\mathrm{codim}_X(S \cap (W \setminus Z)) \geq \mathrm{codim}_X S + \mathrm{codim}_X Z - 1$ remain true by the fact that we added subsets in general position with respect to S. Since $\dim W = \dim Z + 1$ and $\dim Z' = \dim Z$, it follows that Z' and $W \setminus Z$ meet S dimensionally transversely. This concludes the proof of the theorem. $\square$

6.2. **Global and local effacement theorems.** Corollary 1.2 stated in the introduction follows from the following more general statement.

Corollary 6.1. *Let X be a smooth equi-dimensional k-scheme that admits a smooth projective compactification. Let $Z \subset X$ be a closed subset and let $f : S \to X$ be a morphism from an algebraic k-scheme S that is locally of pure dimension. Then there is a closed subset $W \subset X$ with $Z \subset W$ and $\dim W = \dim Z + 1$, and there is an open subset $U \subset X$ with $\mathrm{codim}_X(X \setminus U) \geq \mathrm{codim}_X(Z)$, such that*

$f^{-1}(X \setminus U) \subset S$ *has locally codimension at least* $\mathrm{codim}_X(Z)$ *and such that the following composition is zero:*

$$H_Z^*(X, n) \longrightarrow H_W^*(X, n) \longrightarrow H_W^*(U, n).$$

Moreover, if $\dim f(S) + \dim Z < \dim X$, *then* U *is an open neighbourhood of* $f(S)$ *in* X.

Proof. We apply Theorem 5.1 to X and ask that items (1), (2), and (4) hold true, but we do not require the localization property from item (3). We thus get closed subsets $Z' \subset W \subset X$ with $Z \subset W$, $\dim Z' = \dim Z$ and $\dim W = \dim Z + 1$, such that $f^{-1}(Z') \subset S$ has locally codimension at least $\mathrm{codim}_X(Z)$ (see item (1a)). We then define $U := X \setminus Z'$. By the long exact sequence of triples (see C2), (6.1) implies that the composition $H_Z^*(X, n) \to H_W^*(X, n) \to H_W^*(U, n)$ is zero as claimed. Moreover, if $\dim f(S) + \dim Z < \dim X$, then Lemma 2.2 together with the fact that $f^{-1}(Z') \subset S$ has locally codimension at least $\mathrm{codim}_X(Z)$ implies $f^{-1}(X \setminus U) = f^{-1}(Z') = \emptyset$ and so U is an open neighbourhood of $f(S)$, as we want. This concludes the corollary. $\qquad\square$

The following result proves Corollary 1.3, stated in the introduction.

Corollary 6.2. *Let X be a smooth equi-dimensional k-scheme that admits a smooth projective compactification. Let $Z, S \subset X$ be closed subsets with $\dim S + \dim Z > \dim X$. Then the natural map $H_Z^*(X_S, n) \to H^*(X_S, n)$ is zero and there is a natural short exact sequence:*

$$0 \longrightarrow H^i(X_S, n) \longrightarrow H^i(X_S \setminus Z, n) \xrightarrow{\ \partial\ } H_Z^{i+1}(X_S, n) \longrightarrow 0.$$

Proof. Recall that the proper pushforwards from C3 are functorial and compatible with respect to pullbacks along open immersions (see C3a). Using this, we see that Corollary 1.2 implies that the natural map $H_Z^*(X_S, n) \to H^*(X_S, n)$ is zero. The given short exact sequence is then a direct consequence of the long exact sequence of triples C2 which is functorial with respect to pullbacks and hence with respect to restrictions to the open neighbourhoods of S in X that define the pro-scheme X_S. This concludes the proof. $\qquad\square$

The following corollary implies Corollary 1.4 stated in the introduction.

Corollary 6.3. *Let X be a smooth equi-dimensional k-scheme which admits a smooth projective compactification. Let $Z \subset X$ be a no-where dense closed subset. Let $S \subset X$ be either closed with $\dim S < \mathrm{codim}\, Z - 1$ or a finite set of points. Then there is a closed subset $W \subset X$ with $Z \subset W$ and $\dim W = \dim Z + 1$, such that the natural map $H_Z^*(X_S, n) \to H_W^*(X_S, n)$ is zero. Moreover, if S is a finite set of points and $T \subset X_S$ is closed, then we may assume that the restriction of $W \setminus Z$ to X_S meets T dimensionally transversally.*

Proof. Let $Z \subset X$ be closed and no-where dense. We first deal with the case where $S \subset X$ is closed with $\dim S + \dim Z < \dim X - 1$. Let S^ν be the disjoint union of the irreducible components of S and let $f : S^\nu \to X$ be the natural map. We apply Theorem 5.1 to $U := X$ and f to get closed subsets $Z', W \subset X$ with $Z, Z' \subset W$ and $\dim W \leq \dim Z + 1$. In contrast to the proof of Theorem 1.1, we require that in addition to items (1) and (2), also item (3) holds true and so our assumptions on S ensure that W is well-behaved with respect to localization. The disadvantage is that Z' may have components of dimension $\dim Z + 1$, but item (4) ensures that any such component is disjoint from S. Hence, $U = X \setminus Z'$ is a neighbourhood of S so that the composition $H_Z^*(X, n) \to H_W^*(X, n) \to H_W^*(U, n)$ is zero. The fact that W is well-behaved under localization then implies that $H_Z^*(X_S, n) \to H_W^*(X_S, n)$ is zero, as claimed.

It remains to deal with the case where $S \subset X$ is a finite set of possibly non-closed points. Let $\bar{S}^\nu$ be the disjoint union of the closures in X of the points in S. (In particular, there S corresponds to the generic points of the irreducible components of $\bar{S}^\nu$.) We apply Theorem 5.1 to $U = X$ and the natural map $f : \bar{S}^\nu \to X$. By item (3) in Theorem 5.1, there is a dense open subset $\bar{S}^\circ \subset \bar{S}^\nu$ such that the

closed subset W in Theorem 5.1 with respect to $f|_{\bar{S}^\circ} : \bar{S}^\circ \to X$ is well-behaved under localization. The result follows, because the open subset $\bar{S}^\circ \subset \bar{S}^\nu$ is dense and so S corresponds by the construction of $\bar{S}^\nu$ to the generic points of the components of $\bar{S}^\circ$.

If moreover $T \subset X_S$ is closed, we denote its closure in X by $\bar{T} \subset X$. Let further $\bar{T}^\nu$ denote the disjoint union of the irreducible components of $\bar{T}$. We then replace in the above argument the morphism f by the natural morphism $\tilde{f} : \bar{S} \sqcup \bar{T}^\nu \to X$. By item (3) in Theorem 5.1, we get dense open subsets $\bar{S}^\circ \subset \bar{S}$ and $\bar{T}^\circ \subset \bar{T}^\nu$ such that the closed subset $W \subset X$ produced this way is well-behaved under localization. We claim that $W \setminus Z$ meets locally at any point of S each component of $\bar{T}$ dimensionally transversally. This will follow from item (1b) in Theorem 5.1, if we can show that

$$(6.2) \qquad S \cap \bar{T} = S \cap \bar{T}^\circ,$$

as this implies that $\bar{T}$ and $\bar{T}^\circ$ have the same restriction to X_S and so $\bar{T}^\circ|_{X_S} = T$. To prove (6.2), we note that by item (3) in Theorem 5.1, $\bar{S}^\circ \cup \bar{T}^\circ$ is the preimage of some open subset via the natural map $\bar{S} \cup \bar{T} \to X$. Since $S \subset \bar{S}^\circ$, we get that $S \cap \bar{T} \subset \bar{T}^\circ$ and so (6.2) holds, as we want. This concludes the proof of the corollary. $\qquad\square$

6.3. Finite level version of the Gersten conjecture. The following result implies Corollary 1.5, stated in the introduction.

Corollary 6.4. *Assume that the given cohomology theory with support satisfies* C1–C5. *Let X be a smooth affine variety over a field k. Assume that X admits a smooth projective compactification (e.g. $\operatorname{char} k = 0$). Let $S \subset X^{(c)}$ be a finite set of codimension c points and let $X_S = \operatorname{Spec}\mathcal{O}_{X,S}$ be the localization of X at S. Let $T \subset X_S$ be closed and let $Z_c = S \subset Z_{c-1} \subset \cdots \subset Z_1 \subset Z_0 = X_S$ be a chain of closed reduced subschemes of X_S of increasing dimensions such that each Z_j meets $T \setminus S$ properly. Up to replacing the given chain $\{Z_j\}_j$ by one that is finer, i.e. by a chain $\{Z_j'\}$ as above with $Z_j \subset Z_j'$ for all j (in particular, each Z_j' meets $T \setminus S$ properly), the following complex is exact:*

$$0 \to H^i(X_S, n) \to H^i_{BM}(X_S \setminus Z_1) \xrightarrow{\partial} H^{i-1}_{BM}(Z_1 \setminus Z_2) \xrightarrow{\partial} \cdots \xrightarrow{\partial} H^1_{BM}(Z_{i-1} \setminus Z_i) \xrightarrow{\partial} H^0_{BM}(Z_i \setminus Z_{i+1}) \to 0,$$

where

$$H^{i-j}_{BM}(Z_j \setminus Z_{j+1}) := \varinjlim_{S \subset U \subset X} H^{i+j}_{\bar{Z}_j \setminus \bar{Z}_{j+1}}(U \setminus \bar{Z}_{j+1}, n + j),$$

$\bar{Z}_j \subset X$ *denotes the closure of Z_j in X, and ∂ is induced by the respective residue maps from the long exact sequence of triples in* C2.

Proof. This follows from Corollary 6.3 by standard arguments (cf. [CTHK97, §1]). We give some details for convenience of the reader.

Taking the closure of the given chain $\{Z_j\}_j$ of closed subsets of the localization X_S, we get a chain

$$\bar{Z}_c = \bar{S} \subset \bar{Z}_{c-1} \subset \cdots \subset \bar{Z}_1 \subset \bar{Z}_0 = X$$

of closed subsets of X with $\dim \bar{Z}_j = \dim \bar{Z}_{j+1} + 1$. We replace this chain inductively by a finer chain $\{\bar{Z}_j'\}$ of closed subsets of X, as follows. We let $\bar{Z}_c' := \bar{Z}_c = S$ and if $\bar{Z}_{j+1}'$ is defined, then we define $\bar{Z}_j'$ to be the closed subset from Corollary 6.3 which satisfies:

- $\bar{Z}_{j+1}' \subset \bar{Z}_j'$ and $\dim \bar{Z}_j' = \dim \bar{Z}_{j+1}' + 1$;
- the closure $\bar{T} \subset X$ of T in X meets locally at any point of S the subset $\bar{Z}_j' \setminus \bar{Z}_{j+1}'$ dimensionally transversely;
- the map

$$(6.3) \qquad H^i_{\bar{Z}_{j+1}'}(X_S, n) \longrightarrow H^i_{\bar{Z}_j'}(X_S, n)$$

  is zero.

Up to enlarging $\bar{Z}'_j$, we may in addition assume that $\bar{Z}_j \subset \bar{Z}'_j$, so that the inductively defined chain $\{\bar{Z}'_j\}$ is a refinement of $\{\bar{Z}_j\}$. Since $\bar{Z}_j$ meets $\bar{T} \setminus \bar{S}$ properly, $\bar{T}$ meets $\bar{Z}'_j \setminus \bar{Z}'_{j+1}$ properly for all $j \geq c - 1$. Since $\bar{Z}'_c = \bar{Z}_c = \bar{S}$, it follows inductively that $\bar{T}$ meets $\bar{Z}'_j \setminus \bar{S}$ properly. This way we constructed a refinement $\{\bar{Z}'_j\}$ of $\{\bar{Z}_j\}$. We then let $Z'_j \subset X_S$ be the closed subset given by restriction of $\bar{Z}'_j \subset X$. This way we get a chain $\{Z'_j\}$ of closed subsets of X_S such that each Z'_j meets $T \setminus S$ properly and (6.3) is zero. Hence, up to replacing the chain $\{Z_j\}$ of closed subsets of X_S by the refinement $\{Z'_j\}$ induced by $\{\bar{Z}'_j\}$, we may assume that each Z_j meets $T \setminus S$ properly and the natural map

$$(6.4) \qquad\qquad H^i_{\bar{Z}_{j+1}}(X_S, n) \longrightarrow H^i_{\bar{Z}_j}(X_S, n)$$

is zero for all i, j.

For each open neighbourhood $U \subset X$ of S, the long exact sequence of triples from $C2$ applied to $\bar{Z}_{j+1} \subset \bar{Z}_j \subset U$ yields

$$\cdots \to H^{*-1}_{\bar{Z}_j}(U \setminus \bar{Z}_{j+1}, n) \xrightarrow{\partial} H^*_{\bar{Z}_{j+1}}(U, n) \xrightarrow{\iota_*} H^*_{\bar{Z}_j}(U, n) \xrightarrow{r} H^*_{\bar{Z}_j}(U \setminus \bar{Z}_{j+1}, n) \xrightarrow{\partial} H^{*+1}_{\bar{Z}_{j+1}}(U, n) \to \cdots .$$

These sequences are compatible with respect to restrictions to finer open subsets $U' \subset U$. This gives rise to compatible exact couples $D_1(U) \xrightarrow{\iota_*} D_1(U) \xrightarrow{r} E_1(U) \xrightarrow{\partial}$ with

$$D_1^{p,q}(U) := H^{p+q}_{\bar{Z}_p}(U, n) \quad \text{and} \quad E_1^{p,q}(U) := H^{p+q}_{\bar{Z}_p}(U \setminus \bar{Z}_{p+1}, n),$$

where ι_*, r, and ∂ are of bidegrees $(-1, 1)$, $(0, 0)$, and $(0, 1)$, respectively, see [CTHK97, §1.1]. Passing to the direct limit over all open neighbourhoods $U \subset X$ of S in X, we arrive at an exact couple $D_1 \xrightarrow{\iota_*} D_1 \xrightarrow{r} E_1 \xrightarrow{\partial}$, where

$$D_1^{p,q} := \varinjlim_{S \subset U \subset X} H^{p+q}_{\bar{Z}_p}(U, n) \quad \text{and} \quad E_1^{p,q} := \varinjlim_{S \subset U \subset X} H^{p+q}_{\bar{Z}_p}(U \setminus \bar{Z}_{p+1}, n).$$

The map $\iota_* : D_1^{p,q} \to D_1^{p-1,q+1}$ is zero by (6.4) for all p, q. It follows that the spectral sequence associated to the couple $D_1 \xrightarrow{\iota_*} D_1 \xrightarrow{r} E_1 \xrightarrow{\partial}$ degenerates at E_1. That is, we get a long exact exact sequence

$$\cdots \to E_1^{p-1,q} \to E_1^{p,q} \to E_1^{p+1,q} \to \cdots$$

where the differential is given by $r \circ \partial$. This is precisely the exact complex stated in the corollary. The corollary follows, because each Z_j meets $T \setminus S$ properly and semi-purity (see $C5$) implies that

$$H^{i-j}_{BM}(Z_j \setminus Z_{j+1}) = \varinjlim_{S \subset U \subset X} H^{i+j}_{\bar{Z}_j \setminus \bar{Z}_{j+1}}(U \setminus \bar{Z}_{j+1}, n + j)$$

vanishes for $j > i$; i.e. Borel–Moore cohomology vanishes in negative degrees. $\qquad\square$

6.4. Injectivity and codimension j+1 purity theorems. Corollary 1.6 follows from the following more general statement.

Corollary 6.5. *Let X be a smooth variety over k that admits a smooth projective compactification. Let $f : S \to X$ be a morphism from a locally equi-dimensional algebraic k-scheme S. Then the following holds for any twisted cohomology theory as in (3.1) that satisfies conditions C1–C4.*

(1) Any class in $H^*(X, n)$ that vanishes on the complement of a closed subset $Z \subset X$ already vanishes on the complement of another closed subset $Z' \subset X$ with $\dim Z' = \dim Z$ such that $f^{-1}(Z')$ has locally codimension at least $\operatorname{codim}_X Z$ on S. In particular, if $\dim f(S) < \operatorname{codim}_X Z$, then $f^{-1}(Z') = \emptyset$.

(2) Let $U \subset X$ be open with $\operatorname{codim}(X \setminus U) = j + 2$. There are open subsets $V \subset U$ and $U' \subset X$ with $\operatorname{codim}(X \setminus V) = j + 1$, $\operatorname{codim}(X \setminus U') = j + 2$, such that $f^{-1}(X \setminus U')$ has locally codimension at least $j + 2$ on S and

$$\operatorname{im}(H^*(U, n) \to H^*(V, n)) \subset \operatorname{im}(H^*(U', n) \to H^*(V, n)).$$

In particular, if $\dim f(S) < j + 2$, *then* $f^{-1}(X \setminus U') = \emptyset$ *and so* $f(S) \subset U'$.

Proof. To prove item (1), we apply Corollary 6.1 to the closed subset $Z \subset X$ and the morphism $f : S \to X$. We then get a closed subset $W \subset X$ with $\mathrm{codim}_X(W) \geq \mathrm{codim}_X(Z) - 1$ and $Z \subset W$, and an open subset $U \subset X$ with complement $Z' := X \setminus U$ such that $\mathrm{codim}_X(Z') \geq \mathrm{codim}_X(Z)$ and $f^{-1}(Z') \subset S$ has locally codimension at least $\mathrm{codim}_X(Z)$ and the composition

$$H_Z^*(X, n) \longrightarrow H_W^*(X, n) \longrightarrow H_W^*(U, n)$$

is zero. If $\dim f(S) < \mathrm{codim}_X(Z)$, then Lemma 2.2, together with the fact that $f^{-1}(Z') \subset S$ has codimension at least $\mathrm{codim}_X(Z)$, implies $f^{-1}(Z') = \emptyset$, as claimed.

Let now $\alpha \in H^*(X, n)$ be a class that vanishes on $X \setminus Z$. By the long exact sequence of triples from item *C2*, α lifts to a class $\alpha' \in H_Z^*(X, n)$. The fact that the image of α' in $H_W^*(U, n)$ vanishes, implies by the functoriality of the long exact sequence of triples in *C2*, that the image of α in $H^*(U, n)$ vanishes. The claim in item (1) of the corollary thus follows from the fact that $f^{-1}(Z') \subset S$ has locally codimension at least $\mathrm{codim}_X(Z)$.

To prove item (2), we apply Corollary 6.1 to the closed subset $Z := X \setminus U$ and to the morphism $f : S \to X$. Any class $\alpha \in H^*(U, n)$ gives by the long exact sequence of triples rise to a residue

$$\partial \alpha \in H_Z^*(X, n).$$

By Corollary 6.1, there is a closed subset $W \subset X$ with $Z \subset W$ and $\mathrm{codim}_X W \geq j + 1$, such that the following holds: There is an open subset $U' \subset X$ with $\mathrm{codim}(X \setminus U') \geq j + 2$ such that $f^{-1}(X \setminus U')$ has locally codimension at least $j + 2$ and the image of $\partial \alpha$ in $H_W^*(U', n)$ vanishes. By the functoriality of the long exact sequence of triples in *C2*, this implies that the restriction of α to $V := X \setminus W$ admits a lift to $H^*(U', n)$. Finally, we note that U' and W in the above argument do not depend on the class α. This proves item (2). $\square$

6.5. Refined unramified cohomology is motivic.

Definition 6.6. *Let X be a smooth equi-dimensional algebraic k-scheme. For $j \geq 0$, let $F_j X$ be the pro-scheme given by all open subsets $U \subset X$ with $X^{(j)} \subset U$. Let further*

$$H^*(F_j X, n) := \varinjlim_{U \subset X} H^*(U, n),$$

where U runs through all open subsets of X with $X^{(j)} \subset U$, i.e. with $\mathrm{codim}_X(X \setminus U) > j$.

Definition 6.7. *Let X be a smooth equi-dimensional algebraic k-scheme. The j-th refined unramified cohomology of X with respect to the cohomology theory (3.1) is defined by*

$$H_{j,nr}^*(X, n) := \mathrm{im}(H^*(F_{j+1} X, n) \to H^*(F_j X, n)).$$

Note that $H_{j,nr}^*(X, n) = H^*(X, n)$ for $j \geq \dim X$ (because $F_j X = X$ in this case); see Lemma 6.10 below for a more general statement. Moreover, $H_{0,nr}^*(X, n) = H_{nr}^*(X, n)$ agrees with traditional unramified cohomology (cf. [CTO89, CT95, Sch21a]), see [CT95, Theorem 4.1.1(a)] and [Sch21b, Lemma 5.8].

For $j' \leq j$, a class on $F_j X$ may be restricted to $F_{j'} X$. This implies in particular that for $j' \leq j$, there are canonical restriction maps

$$H_{j,nr}^*(X, n) \longrightarrow H_{j',nr}^*(X, n).$$

The main result of this section is the following application of the moving lemma (Theorem 5.1), which shows that refined unramified cohomology is a motivic invariant naturally attached to any smooth projective variety and to any cohomology functor satisfying *C1–C4*.

Corollary 6.8. *Assume that the functor from (3.1) satisfies* C1–C4 *and let X and Y be smooth projective equi-dimensional k-schemes. Then for any $c, i, j \geq 0$, there is a bi-additive pairing*

$$\mathrm{CH}^c(X \times Y) \times H^i_{j,nr}(X, n) \longrightarrow H^{i+2c-2d_X}_{j+c-d_X,nr}(Y, n + c - d_X), \quad ([\Gamma], [\alpha]) \longmapsto [\Gamma]_*[\alpha]$$

with the following properties:

(1) *Let $\Gamma \in Z^c(X \times Y)$ be a representative of $[\Gamma] \in \mathrm{CH}^c(X \times Y)$ and let $\alpha \in H^i(U, n)$ be a representative of $[\alpha] \in H^i_{j,nr}(X, n)$ for some open $U \subset X$ whose complement $R = X \setminus U$ has codimension at least $j + 2$. Let $W := \mathrm{supp}\,\Gamma$, $R' := q(W \cap (R \times Y))$, and $U' := Y \setminus R'$.*

   If $W \cap (R \times Y)$ has codimension at least $j + c + 2$ in $X \times Y$, then

$$[\Gamma]_*[\alpha] = [\Gamma(W)_*(\alpha)] \in H^{i+2c-2d_X}_{j+c-d_X,nr}(Y, n + c - d_X)$$

   is represented by $\Gamma(W)_*(\alpha) \in H^{i+2c-2d_X}(U', n + c - d_X)$ constructed in Lemma 4.1.

(2) *If $j' \leq j$, then the following diagram commutes*

$$\begin{array}{ccc}
\mathrm{CH}^c(X \times Y) \times H^i_{j,nr}(X, n) & \longrightarrow & H^{i+2c-2d_X}_{j+c-d_X,nr}(Y, n + c - d_X) \\
\downarrow & & \downarrow \\
\mathrm{CH}^c(X \times Y) \times H^i_{j',nr}(X, n) & \longrightarrow & H^{i+2c-2d_X}_{j'+c-d_X,nr}(Y, n + c - d_X),
\end{array}$$

   where the horizontal maps are the given pairings and the vertical maps are induced by the canonical restriction maps.

(3) *Let Z be a smooth projective equi-dimensional k-scheme and let $[\Gamma'] \in \mathrm{CH}^{c'}(Y \times Z)$. Then for all $[\alpha] \in H^i_{j,nr}(X, n)$,*

$$[\Gamma']_*([\Gamma]_*[\alpha]) = ([\Gamma'] \circ [\Gamma])_*[\alpha] \in H^{i+2c+2c'-2d_X-2d_Y}_{j+c+c'-d_X-d_Y,nr}(Z, n + c + c' - d_X - d_Y),$$

   where $[\Gamma'] \circ [\Gamma]$ is the composition of correspondences and $d_Y = \dim Y$.

(4) *If $\Gamma \in Z^c(X \times Y)$ such that $q(\mathrm{supp}\,\Gamma) \subset Y$ has codimension at least $j + c - d_X + 1$, where $q : X \times Y \to Y$ denotes the second projection, then $[\Gamma]_*[\alpha] = 0$ for all $[\alpha] \in H^i_{j,nr}(X, n)$.*

Proof. We first construct an action as required. For this let $[\alpha] \in H^i_{j,nr}(X, n)$ and let $\Gamma \in Z^c(X \times Y)$. Let $W \subset X \times Y$ be a closed subset with $\mathrm{supp}\,\Gamma \subset W$ and let $S := W^\nu$ be the disjoint union of the irreducible components of W together with the natural map $f : S \to X$ given by the composition $S \to W \to X \times Y \xrightarrow{p} X$. Applying item (2) from Corollary 6.5, we find that $[\alpha]$ can be represented by a class $\alpha \in H^i(U, n)$, such that $R = X \setminus U$ satisfies $\mathrm{codim}_X(R) = j + 2$ and $f^{-1}(R) \subset S$ has locally codimension at least $j + 2$. Hence, if $c' := \mathrm{codim}_{X \times Y}(W)$, then $W \cap (R \times Y)$ has codimension at least $j + 2 + c'$ and so $R' := q(W \cap (R \times Y)) \subset Y$ has codimension at least $j + c' - d_X + 2$. We then get a class

$$\Gamma(W)_*(\alpha) \in H^{i+2c-2d_X}(U', n + c - d_X),$$

constructed in Lemma 4.1, where $U' := Y \setminus R'$.

From now on we assume that $c' \in \{c, c - 1\}$. Then $F_{j+c-d_X}X \subset U'$ and we get a class

$$\Gamma(W)_*(\alpha) \in H^{i+2c-2d_X}(F_{j+c-d_X}X, n + c - d_X).$$

It follows from the commutativity of the first square in (4.2) from Lemma 4.3 that the above class does not depend on the chosen representative α of $[\alpha] \in H^i_{j,nr}(X, n)$; that is, the above class does not change if we replace α by another representative $\tilde{\alpha} \in H^i(\tilde{U}, n)$ such that $\tilde{R} = X \setminus \tilde{R}$ satisfies $\mathrm{codim}_X(\tilde{R}) = j + 2$ and $f^{-1}(\tilde{R}) \subset S$ has locally codimension at least $j + 2$. Using this, we can apply the above construction to the union of two given closed subsets to deduce from the commutativity of the second square in (4.2) from Lemma 4.3 that the above class does not depend on the chosen closed subset $W \subset X \times Y$ with $\mathrm{supp}\,\Gamma \subset W$ and $\mathrm{codim}_{X \times X}(W) \in \{c, c - 1\}$. Applying this to the particular case where $W = \mathrm{supp}\,\Gamma$

has codimension c, the closed subset $R' \subset Y$ has codimension at least $j + c - d_X + 2$ and so we find that the above class is unramified:

$$\Gamma(W)_*(\alpha) \in H^{i+2c-2d_X}_{j+c-d_X, nr}(X, n + c - d_X).$$

Moreover, the additivity in Lemma 4.1 implies that the above class depends additively on Γ. Finally, if Γ is rationally equivalent to zero, then applying the above construction to some W of codimension $c - 1$ such that Γ is rationally equivalent to zero on W, Lemma 4.2 implies $\Gamma(W)_*(\alpha) = 0$. Since we already know additivity of the above construction in Γ, we conclude that the above class depends not on Γ, but only on the rational equivalence class of Γ. Altogether, we conclude that

$$[\Gamma]_*[\alpha] := [\Gamma(W)_*(\alpha)] \in H^{i+2c-2d_X}_{j+c-d_X, nr}(Y, n + c - d_X),$$

for some $W \subset X \times Y$ of codimension c or $c - 1$ and with $\operatorname{supp}\Gamma \subset W$, yields a well-defined pairing as we want.

Item (1) in the corollary is now a direct consequence of the construction.

Item (2) follows from the well-definedness of the construction together with commutativity of the first square of (4.2) in Lemma 4.3.

To prove item (3), let $[\Gamma'] \in \mathrm{CH}^{c'}(Y \times Z)$. Let further $\alpha \in H^i(U, n)$ be a representative of $[\alpha] \in H^i_{j, nr}(X, n)$, for some open subset $U \subset X$ with complement $R_1 \subset X$ of codimension at least $j + 2$. By the moving lemma for algebraic cycles (see Theorem 2.1), we may choose a representative Γ of $[\Gamma] \in \mathrm{CH}^c(X \times Y)$ such that $W_1 = \operatorname{supp}\Gamma$ meets $R_1 \times Y$ dimensionally transversely. Hence, $U_2 := Y \setminus p_2^{12}(W_1 \cap (R_1 \times Y))$ contains $F_{c-d_X+j+1}Y$. Applying Theorem 2.1 once again, we may choose a representative Γ' of $[\Gamma']$ with support $W_2 = \operatorname{supp}\Gamma'$ such that

$$R_3 = p_3((W_1 \times Z) \cap (X \times W_2) \cap (R_1 \times Y \times Z)) \subset Z$$

has codimension at least $c + c' - d_X - d_Y + j + 2$. Hence, the open subset $U_3 := Z \setminus R_3$ contains $F_{j+c+c'-d_X-d_Y+1}Z$ and we have by Proposition 4.7

$$(\Gamma \circ \Gamma')(W_{12})(\alpha) = \Gamma'(W_2)_*(\Gamma(W_1)_*(\alpha))$$

The compatibility stated in (3) therefore follows from the construction (cf. item (1) proven above), together with Proposition 4.7.

Finally, item (4) follows from item (1) and Lemmas 4.3 and 4.6.

This concludes the proof of the corollary. $\square$

Corollary 6.8 implies the existence of pullback maps along morphisms between smooth projective equi-dimensional k-schemes. This can be generalized to open varieties as follows.

Corollary 6.9. *Let* $f : X \to Y$ *be a morphism between smooth quasi-projective equi-dimensional* k-*schemes that admit a smooth projective compactification. Then there are functorial pullback maps* $f^* : H^i_{j, nr}(Y, n) \to H^i_{j, nr}(X, n)$ *with the following property: if* $[\alpha] \in H^i_{j, nr}(Y, n)$ *is represented by a class* $\alpha \in H^i(U, n)$ *with* $F_{j+1}Y \subset U$ *such that* $F_{j+1}X \subset f^{-1}(U)$, *then* $f^*[\alpha]$ *is represented by* $f^*(\alpha) \in H^i(f^{-1}(U), n)$.

Proof. By item (2) in Corollary 6.5 applied to the morphism $f : X \to Y$, we find that any class $[\alpha] \in H^i_{j, nr}(Y, n)$ is represented by a class $\alpha \in H^i(U, n)$ with $F_{j+1}Y \subset U$ such that $F_{j+1}X \subset f^{-1}(U)$. We may use such a representative to define $f^*[\alpha] := [f^*(\alpha)]$, where $f^*(\alpha) \in H^i(f^{-1}(U), n)$. Lemma 4.3 implies that this is well-defined; functoriality follows from that together with the functoriality of pullbacks for the functor in Section 3. This concludes the proof. $\square$

6.6. **Behaviour under products with projective space and birational properties of refined unramified cohomology.** The following lemma is an analogue of [Sch21b, Corollary 5.10].

Lemma 6.10. *Assume that the functor from (3.1) satisfies* C2 *and* C5. *Then for any smooth equi-dimensional quasi-projective k-scheme, the natural map $H^i(X, m) \xrightarrow{\cong} H^i_{j,nr}(X, m)$ is an isomorphism for $j \geq \lceil i/2 \rceil$.*

Proof. It suffices to prove that the natural map $H^i(X, m) \to H^i(F_j X, m)$ is an isomorphism for $j \geq \lceil i/2 \rceil$. To this end, let $U \subset X$ be open with complement $Z = X \setminus U$ of codimension at least $j + 1$. Then C2 yields a long exact sequence

$$\cdots \to H^i_Z(X, m) \to H^i(X, m) \to H^i(U, m) \to H^{i+1}_Z(X, m) \to \cdots.$$

By C5, $H^{i+1}_Z(X, m) = 0$ for $i + 1 < 2 \operatorname{codim}_X(Z)$. Since $\operatorname{codim}_X(Z) \geq j + 1$, we get $H^{i+1}_Z(X, m) = 0$ for $j + 1 > (i+1)/2$. The latter condition is equivalent to any of the following conditions: $j > (i-1)/2$, $j \geq i/2$, or $j \geq \lceil i/2 \rceil$. The same argument shows that $H^i_Z(X, m) = 0$ for $j + 1 > i/2$, which holds if $j \geq \lceil i/2 \rceil$. The above sequence thus proves the lemma. $\square$

Corollary 6.11. *For any twisted cohomology theory as in Section 3 which satisfies* C1–C5, *and any smooth projective equi-dimensional k-scheme Y, there is a canonical isomorphism*

$$\sum f_l : \bigoplus_{l=0}^{\min(j,n)} H^{i-2l}_{j-l,nr}(Y, m - l) \xrightarrow{\cong} H^i_{j,nr}(Y \times \mathbb{P}^n_k, m),$$

where $f_l : H^{i-2l}_{j-l,nr}(Y, m-l) \to H^i_{j,nr}(Y \times \mathbb{P}^n_k, m)$ is given by the natural pullback map along the projection $Y \times \mathbb{P}^{n-l}_k \to Y$ followed by the pushforward along the inclusion $Y \times \mathbb{P}^{n-l}_k \to Y \times \mathbb{P}^n_k$ induced by some linear embedding $\mathbb{P}^{n-l}_k \subset \mathbb{P}^n_k$.

Proof. By Lemma 6.10, the statement reduces for $j \geq \lceil i/2 \rceil$ to the well-known decomposition

$$H^i(Y \times \mathbb{P}^n_k, m) \simeq \bigoplus_{l=0}^{\min(\lfloor i/2 \rfloor, n)} H^{i-2l}(Y, m - l),$$

which may be deduced from the axioms in Section 3 and the motivic decomposition of $\mathbb{P}^n_k$. We will see now that a suitable variant of that argument proves the more general claim in the corollary for any j.

We first show that $\sum_l f_l$ is injective. To this end we consider the map $g_l : H^i_{j,nr}(Y \times \mathbb{P}^n_k, m) \to H^{i-2l}_{j-l,nr}(Y, m - l)$ given by the composition of the pullback along a linear embedding $Y \times \mathbb{P}^l_k \to Y \times \mathbb{P}^n_k$, followed by the pushforward along the projection $Y \times \mathbb{P}^l_k \to Y$. The composition $g_{l'} \circ f_l$ factors by definition as follows:

$$H^{i-2l}_{j-l,nr}(Y, m-l) \to H^{i-2l}_{j-l,nr}(Y \times \mathbb{P}^{n-l}_k, m-l) \to H^i_{j,nr}(Y \times \mathbb{P}^n_k, m) \to H^i_{j,nr}(Y \times \mathbb{P}^{l'}_k, m) \to H^{i-2l'}_{j-l',nr}(Y, m-l').$$

By C4b and C4f and the functoriality of pushforwards from C3, $g_{l'} \circ f_l$ thus identifies to the composition

$$H^{i-2l}_{j-l,nr}(Y, m-l) \xrightarrow{\operatorname{pr}_1^*} H^{i-2l}_{j-l,nr}(Y \times \mathbb{P}^{l'-l}_k, m-l) \xrightarrow{(\operatorname{pr}_1)_*} H^{i-2l'}_{j-l',nr}(Y, m-l').$$

This is the identity if $l = l'$ and it is zero if $l > l'$, because $\mathbb{P}^{l'-l}_k = \emptyset$ in this case. To prove injectivity of $\sum f_l$, we assume that $\sum f_l(\alpha_l) = 0$ such that not all α_l are zero. Then we can pick $l' \geq 0$ minimal with $\alpha_{l'} \neq 0$ and the above computations show that

$$0 = g_{l'}\left(\sum_l f_l(\alpha_l)\right) = \sum_l g_{l'}\left(f_l(\alpha_l)\right) = \alpha_{l'},$$

a contradiction. This shows that $\sum f_l$ is injective as claimed.

Note that $H^{i-2l}_{j-l,nr}(Y, m-l) = 0$ for $l > j$, becasue $F_j X = \emptyset$ for $j < 0$. To prove surjectivity, it thus suffices to show that

$$(6.5) \qquad \operatorname{coker}\left(\sum f_l : \bigoplus_{l=0}^{n} H^{i-2l}_{j-l,nr}(Y, m-l) \to H^i_{j,nr}(Y \times \mathbb{P}^n_k, m)\right) = 0.$$

We prove (6.5) by induction on n. For $n = 0$ the statement is trivially true. Assume that $n > 0$ and that we know (6.5) for products with projective spaces of dimension less than n. Note that there is a natural identification of the diagonal $\Delta_{Y \times \mathbb{P}^n}$ of $Y \times \mathbb{P}^n$ with the product $\Delta_Y \times \Delta_{\mathbb{P}^n}$. The standard decomposition of $\Delta_{\mathbb{P}^n}$ then yields a decomposition

$$\Delta_Y \times \Delta_{\mathbb{P}^n} = \sum_{l=0}^{n} \Delta_Y \times h^l \times h^{n-l} \in \mathrm{CH}^{d_Y+n}(Y \times Y \times \mathbb{P}^n \times \mathbb{P}^n),$$

where $d_Y = \dim Y$ and $h \in \mathrm{CH}^1(\mathbb{P}^n)$ denotes the hyperplane class. By Corollary 6.8 and Lemma 4.4, we have

$$\sum_{l=0}^{n} (\Delta_Y \times h^l \times h^{n-l})_* = \mathrm{id} : H^i_{j,nr}(Y \times \mathbb{P}^n_k, m) \longrightarrow H^i_{j,nr}(Y \times \mathbb{P}^n_k, m).$$

On the other hand, $(\Delta_Y \times h^l \times h^{n-l})_*$ coincides by functoriality of the action with the composition

$$H^i_{j,nr}(Y \times \mathbb{P}^n_k, m) \xrightarrow{\mathrm{restr}} H^i_{j,nr}(Y \times \mathbb{P}^l_k, m) \xrightarrow{(\mathrm{id}_Y \times p)^*} H^i_{j,nr}(Y \times \mathbb{P}^l_k \times \mathbb{P}^{n-l}_k, m) \xrightarrow{(\mathrm{id}_Y \times q)_*}$$
$$H^{i-2l}_{j-l,nr}(Y \times \mathbb{P}^{n-l}_k, m-l) \xrightarrow{\iota_*} H^i_{j,nr}(Y \times \mathbb{P}^n_k, m),$$

where the first map is induced by the restriction with respect to a linear embedding $\mathbb{P}^l_k \hookrightarrow \mathbb{P}^n_k$, p and q denote the two evident projections and ι_* denotes the pushforward induced by a linear inclusion $\mathbb{P}^{n-l}_k \hookrightarrow \mathbb{P}^n_k$.

If $l \geq 1$, then (6.5) holds for $H^{i-2l}_{j-l,nr}(Y \times \mathbb{P}^{n-l}_k, m)$ and so the image of the above composition is contained in the image of $\sum f_l$, as we want. If $l = 0$, then the above composition factors through f_0 and so its image is also in this case contained in the image of $\sum f_l$. Altogether we conclude that (6.5) holds, which concludes the proof of the corollary. $\qquad \square$

The groups $H^i_{j,nr}(X, n)$ carry a natural descending filtration F^* with $F^{j+1} H^i_{j,nr}(X, n) = H^i_{j,nr}(X, n)$, which for $m \geq j + 1$ is given by (see [Sch21b, Definition 5.3]):

$$F^m H^i_{j,nr}(X, n) := \mathrm{im}(H^i(F_m X, n) \to H^i(F_j X, n)).$$

Corollary 6.12. *Let X and Y be smooth projective equi-dimensional k-schemes and let $f : X \dashrightarrow Y$ be a birational map which is an isomorphism in codimension c, i.e. there are closed subsets $Z_X \subset X$ and $Z_Y \subset Y$ of codimensions $> c$, such that $f : X \setminus Z_X \to Y \setminus Z_Y$ is an isomorphism. Then for all $j \leq c$, there is a natural isomorphism*

$$f^* : H^i_{j,nr}(Y, n) \xrightarrow{\sim} H^i_{j,nr}(X, n),$$

given by the action $[\Gamma_{f^{-1}}]_*$ of the closure of the graph of f^{-1}. Moreover, f^* respects the filtration F^* on both sides.

Proof. Since f is birational, $d := \dim X = \dim Y$. Let $\Gamma := \Gamma_f \in Z^d(X \times Y)$ and $\Gamma' := \Gamma_{f^{-1}} \in Z^d(Y \times X)$ be the closures of the graphs of f and f^{-1}, respectively. Since f is an isomorphism in codimension c, we get

$$[\Gamma] \circ [\Gamma'] = [\Delta_Y] + [\Omega_Y] \in \mathrm{CH}^d(Y \times Y) \quad \text{and} \quad [\Gamma'] \circ [\Gamma] = [\Delta_X] + [\Omega_X] \in \mathrm{CH}^d(X \times X),$$

for some cycles Ω_Y, Ω_X with the property that the respective images via the second projection $\mathrm{pr}_2(\mathrm{supp}(\Omega_Y)) \subset Y$ and $\mathrm{pr}_2(\mathrm{supp}(\Omega_X)) \subset X$ have codimension at least $c + 1$. Hence, item (4) in Corollary 6.8 implies that

$$[\Omega_X]_* : H^i_{j,nr}(X, n) \longrightarrow H^i_{j,nr}(X, n) \quad \text{and} \quad [\Omega_Y]_* : H^i_{j,nr}(Y, n) \longrightarrow H^i_{j,nr}(Y, n)$$

are zero for $j \leq c$. The above decompositions together with item (3) in Corollary 6.8 thus implies that

$$[\Gamma]_* : H^i_{j,nr}(X, n) \longrightarrow H^i_{j,nr}(Y, n) \quad \text{and} \quad [\Gamma']_* : H^i_{j,nr}(Y, n) \longrightarrow H^i_{j,nr}(X, n)$$

are inverse to each other for $j \leq c$. Moreover, (2) in Corollary 6.8 implies that the above isomorphisms respect the filtrations F^* on both sides. This concludes the proof of the corollary. $\qquad \square$

Proof of Corollary 1.8. Corollary 1.8 follows from Corollaries 6.11 and 6.12. $\qquad \square$

APPENDIX A. TWISTED PRO-ÉTALE COHOMOLOGY SATISFIES THE PROPERTIES FROM SECTION 3

The purpose of this appendix is to prove Proposition 3.2 from Section 3.1. We will see that item (3) of Proposition 3.2 implies items (1) and (2), and so it suffices to treat pro-étale cohomology. Most results and arguments are certainly well-known to experts; we include them here for convenience of the reader.

A.1. **Pro-étale and continuous étale cohomology.** The pro-étale site $X_{\mathrm{proét}}$ of a scheme X is formed by weakly étale maps of schemes $U \to X$, see [BS15, Definition 4.1.1 and Remark 4.1.2]. Since any étale map is also weakly étale, there is a natural map of topoi $\nu : \mathrm{Shv}(X_{\mathrm{proét}}) \to \mathrm{Shv}(X_{\mathrm{ét}})$, see [BS15, §5]. In particular, we get functors $\nu^* : \mathrm{Ab}(X_{\mathrm{ét}}) \to \mathrm{Ab}(X_{\mathrm{proét}})$ and $\nu_* : \mathrm{Ab}(X_{\mathrm{proét}}) \to \mathrm{Ab}(X_{\mathrm{ét}})$ on the corresponding categories of sheaves of abelian groups. Here, ν^* is exact and right adjoint to the left exact functor ν_*, see [Stacks21, Tag 00X9]. Since ν^* is (left) exact, ν_* maps injective objects to injective objects.

For a closed subset $Z \subset X$, we have the functor $\Gamma_Z : \mathrm{Ab}(X_{\mathrm{proét}}) \to \mathrm{Ab}$, given by taking global sections with support on Z. For a bounded complex K of sheaves of abelian groups on $X_{\mathrm{proét}}$, we let

$$(A.1) \qquad\qquad H_Z^i(X_{\mathrm{proét}}, K) := \mathrm{R}^i \Gamma_Z(X_{\mathrm{proét}}, K).$$

In the special case where K is a sheaf F concentrated in degree zero, we also write $H_Z^i(X_{\mathrm{proét}}, F) := \mathrm{R}^i \Gamma_Z(X_{\mathrm{proét}}, F)$. (Note that by slight abuse of notation, we do not distinguish between hypercohomology of a complex as in (A.1) and cohomology of a sheaf.)

Lemma A.1 ([BS15]). *Let X be a scheme and let (F_r) be an inductive system of sheaves on the small étale site $X_{\text{ét}}$ and with surjective transition maps. Let $Z \subset X$ be a closed subset. Then there is a canonical identification*

$$H_{Z,cont}^i(X_{\text{ét}}, (F_r)) \simeq H_Z^i(X_{\text{proét}}, \varprojlim_r \nu^* F_r),$$

where the left hand side denotes Jannsen's continuous étale cohomology groups with support [Jan88].

Proof. This follows by the same argument as in [BS15, Proposition 5.6.2]; we give the details for convenience of the reader. By definition, $H_{Z,cont}^i(X_{\text{ét}}, (F_r))$, resp. $H_Z^i(X_{\mathrm{proét}}, \lim \nu^* F_r)$, is the i-th cohomology of the complex

$$\mathrm{R}\,\Gamma_Z(X_{\text{ét}}, \mathrm{R}\lim(F_r)) \in D(\mathrm{Ab}) \quad \text{resp.} \quad \mathrm{R}\,\Gamma_Z(X_{\mathrm{proét}}, \lim \nu^* F_r) \in D(\mathrm{Ab}).$$

Since $\lim$ commutes with Γ_Z, we have $\mathrm{R}\,\Gamma_Z \circ \mathrm{R}\lim \simeq \mathrm{R}\lim \circ \mathrm{R}\,\Gamma_Z$ and so

$$(A.2) \qquad \mathrm{R}\,\Gamma_Z(X_{\text{ét}}, \mathrm{R}\lim(F_r)) \simeq \mathrm{R}\lim \circ \mathrm{R}\,\Gamma_Z(X_{\text{ét}}, F_r) \simeq \mathrm{R}\lim \circ \mathrm{R}\,\Gamma_Z(X_{\mathrm{proét}}, \nu^* F_r)$$
$$\simeq \mathrm{R}\,\Gamma_Z(X_{\mathrm{proét}}, \mathrm{R}\lim(\nu^* F_r)),$$

where the second isomorphism uses the fact that $\nu_* : \mathrm{Ab}(X_{\mathrm{proét}}) \to \mathrm{Ab}(X_{\text{ét}})$ maps injectives to injectives and $\mathrm{id} \to \mathrm{R}\,\nu_* \nu^*$ is an equivalence on bounded below complexes (see [BS15, Proposition 5.2.6(2)]), while the last equality uses $\mathrm{R}\,\Gamma_Z \circ \mathrm{R}\lim \simeq \mathrm{R}\lim \circ \mathrm{R}\,\Gamma_Z$. By [BS15, Proposition 3.2.3 and Proposition 4.2.8], the topos $\mathrm{Shv}(X_{\mathrm{proét}})$ is locally weakly contractible and so [BS15, Proposition 3.1.10] implies

$$(A.3) \qquad\qquad \mathrm{R}\lim(\nu^* F_r) \simeq \lim(\nu^* F_r),$$

where we use that the transition maps in (F_r) are surjective. Hence, the above isomorphism simplifies to $\mathrm{R}\,\Gamma_Z(X_{\text{ét}}, \mathrm{R}\lim(F_r)) \simeq \mathrm{R}\,\Gamma_Z(X_{\mathrm{proét}}, \lim(\nu^* F_r))$, which proves the proposition. $\qquad \square$

A.2. **Constructible complexes and six functor formalism.** For simplicity, we recall the six functor formalism in the pro-étale topology that we need from [BS15] only in the special case of algebraic schemes, which suffices for the purpose of this paper.

Let k be a field and let ℓ be a prime that is invertible in k. Let X be an algebraic scheme over k (i.e. a separated scheme of finite type over k). For any integer n, we consider the sheaf

$$\widehat{\mathbb{Z}}_\ell(n) := \varprojlim_r \mu_{\ell^r}^{\otimes n} \in \mathrm{Ab}(X_{\mathrm{proét}}),$$

and write $\widehat{\mathbb{Z}}_\ell := \widehat{\mathbb{Z}}_\ell(0)$, which is a sheaf of rings. Let $D(X_{\mathrm{proét}}, \widehat{\mathbb{Z}}_\ell)$ denote the derived category of the abelian category of $\widehat{\mathbb{Z}}_\ell$-modules on $X_{\mathrm{proét}}$.

A complex $K \in D(X_{\mathrm{proét}}, \widehat{\mathbb{Z}}_\ell)$ is constructible if it is complete, i.e. the canonical map

$$K \longrightarrow \mathrm{R}\lim(K \otimes_{\widehat{\mathbb{Z}}_\ell}^{\mathbb{L}} \mathbb{Z}/\ell^r)$$

is a quasi-isomorphism (see [BS15, Definition 3.5.2]), and if for each r, $K \otimes_{\widehat{\mathbb{Z}}_\ell}^{\mathbb{L}} \mathbb{Z}/\ell^r \simeq \nu^* K_r$ for some constructible complex $K_r \in D(X_{\mathrm{ét}}, \mathbb{Z}/\ell^r)$ of sheaves of $\mathbb{Z}/\ell^r$-modules on $X_{\mathrm{ét}}$, see [BS15, Definition 6.5.1]. The full triangulated subcategory spanned by constructible complexes is denoted by

$$D_{cons}(X_{\mathrm{proét}}, \widehat{\mathbb{Z}}_\ell) \subset D(X_{\mathrm{proét}}, \widehat{\mathbb{Z}}_\ell).$$

By [BS15, Lemma 6.5.3], each $K \in D_{cons}(X_{\mathrm{proét}}, \widehat{\mathbb{Z}}_\ell)$ is bounded. Since $\mu_{\ell^{r+1}}^{\otimes n} \to \mu_{\ell^r}^{\otimes n}$ is surjective in the étale topology, (A.3) implies that $\widehat{\mathbb{Z}}_\ell(n)$ is complete and so for any integer m, the complex $\widehat{\mathbb{Z}}_\ell(n)[m] \in D(X_{\mathrm{proét}}, \widehat{\mathbb{Z}}_\ell)$ is constructible.

For a morphism $f : X \to Y$ between algebraic schemes over k, there are functors

$$\mathrm{R}\,f_*, \mathrm{R}\,f_! : D_{cons}(X_{\mathrm{proét}}, \widehat{\mathbb{Z}}_\ell) \longrightarrow D_{cons}(Y_{\mathrm{proét}}, \widehat{\mathbb{Z}}_\ell), \quad f_{comp}^*, f^! : D_{cons}(Y_{\mathrm{proét}}, \widehat{\mathbb{Z}}_\ell) \longrightarrow D_{cons}(X_{\mathrm{proét}}, \widehat{\mathbb{Z}}_\ell)$$

where f_{comp}^* is defined as the usual pullback followed by derived completion, see [BS15, §6.7].

We aim to recall the construction of $f^!$. To this end, note first that

$$(\mathrm{A}.4) \qquad\qquad \nu^* : D_{cons}(X_{\mathrm{ét}}, \mathbb{Z}/\ell^r) \xrightarrow{\simeq} D_{cons}(X_{\mathrm{proét}}, \mathbb{Z}/\ell^r)$$

is an equivalence for all $r \geq 1$, see [BS15, paragraph after Definition 6.5.1], and we will freely use this in the following to identify complexes in the two categories with each other. For $K \in D_{cons}(Y_{\mathrm{proét}}, \widehat{\mathbb{Z}}_\ell)$, we set $K_r := K \otimes_{\widehat{\mathbb{Z}}_\ell}^{\mathbb{L}} \mathbb{Z}/\ell^r$. Note that any constructible complex of sheaves of $\mathbb{Z}/\ell^r$-modules on $X_{\mathrm{proét}}$ is also constructible when viewed as a complex of $\widehat{\mathbb{Z}}_\ell$-modules. Using the equivalence in (A.4), $f_r^! K_r$ may therefore be viewed as object in $D_{cons}(X_{\mathrm{proét}}, \widehat{\mathbb{Z}}_\ell)$, where $f_r^! : D_{cons}(Y_{\mathrm{ét}}, \mathbb{Z}/\ell^r) \longrightarrow D_{cons}(X_{\mathrm{ét}}, \mathbb{Z}/\ell^r)$ denotes the exceptional pullback along f in the étale site, cf. [SGA4.3, Éxpose XVIII]. By [BS15, Lemma 6.7.18], $f_r^! K_r \in D_{cons}(X_{\mathrm{proét}}, \widehat{\mathbb{Z}}_\ell)$ form a projective system. Bhatt–Scholze then define

$$(\mathrm{A}.5) \qquad\qquad f^! K := \mathrm{R}\lim f_r^! K_r \in D_{cons}(X_{\mathrm{proét}}, \widehat{\mathbb{Z}}_\ell),$$

see [BS15, Lemma 6.7.19].

Using the above construction of $f^!$, it follows from [SGA4.3, Exposé XVIII, Proposition 3.1.8] that for any closed immersion $i : Z \to X$,

$$(\mathrm{A}.6) \qquad\qquad \mathrm{R}\,\Gamma_Z(X_{\mathrm{proét}}, -) \simeq \mathrm{R}\,\Gamma(Z_{\mathrm{proét}}, i^!(-)).$$

By adjunction (see [BS15, Lemmas 6.7.2 and 6.7.19]), there are morphisms of functors $\mathrm{Tr}_f : \mathrm{R}\,f_! f^! \to \mathrm{id}$ and $\theta_f : \mathrm{id} \to \mathrm{R}\,f_* f_{comp}^*$. In particular, for any $K \in D(Y_{\mathrm{proét}}, \widehat{\mathbb{Z}}_\ell)$, θ_f yields functorial pullback maps

$$(\mathrm{A}.7) \qquad\qquad f^* : \mathrm{R}\,\Gamma^i(Y_{\mathrm{proét}}, K) \longrightarrow \mathrm{R}\,\Gamma^i(X_{\mathrm{proét}}, f_{comp}^* K).$$

If f is proper, then $\mathrm{R}\, f_* = \mathrm{R}\, f_!$ (see [BS15, Definition 6.7.6]) and so Tr_f yields a functorial pushforward map

$$(A.8) \qquad f_* : \mathrm{R}\,\Gamma^i(X_{\mathrm{proét}}, f^! K) \longrightarrow \mathrm{R}\,\Gamma^i(Y_{\mathrm{proét}}, K).$$

A.3. **Notation.** We aim to prove Proposition 3.2. To this end we use from now on the notation

$$H_Z^*(X, n) := H_Z^*(X_{\mathrm{proét}}, \pi_{comp}^* K \otimes_{\widehat{\mathbb{Z}}_\ell} \widehat{\mathbb{Z}}_\ell(n)),$$

where $K \in D_{cons}((\mathrm{Spec}\, k)_{\mathrm{proét}}, \widehat{\mathbb{Z}}_\ell)$ is a fixed constructible complexes of $\widehat{\mathbb{Z}}_\ell$-modules on $(\mathrm{Spec}\, k)_{\mathrm{proét}}$. Existence of functorial pullback maps follows from (A.7) together with the fact that the tensor product of two complete complexes is complete, see [BS15, Lemma 6.5.5].

A.4. **Poincaré duality.**

Lemma A.2. *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of algebraic schemes. Then there is a canonical isomorphism of functors $f^! g^! \xrightarrow{\simeq} (g \circ f)^!$ on $D_{cons}(Z_{proét}, \widehat{\mathbb{Z}}_\ell)$.*

Proof. This follows from (A.5) together with the analogous result on the étale site, see [Sch21b, Lemma 6.1(3)] for more details. $\square$

Lemma A.3 (Poincaré duality)**.** *Let $f : X \to Y$ be a smooth morphism of algebraic k-schemes of pure dimension d. Then there is a canonical isomorphism $f_{comp}^*(d)[2d] \xrightarrow{\simeq} f^!$ of functors on $D_{cons}(Y_{proét}, \widehat{\mathbb{Z}}_\ell)$, where $f_{comp}^*(d) := f_{comp}^*(- \otimes_{\widehat{\mathbb{Z}}_\ell} \widehat{\mathbb{Z}}_\ell(d))$.*

Proof. This follows from Poincaré duality on the étale site [SGA4.3, XVIII, Théorème 3.2.5] by applying $\mathrm{R}\lim$; the details are given in [Sch21b, Lemma 6.1(4)]. $\square$

For open immersions $j : U \to X$ we have a canonical isomorphism $j_{comp}^* \simeq j^!$, see e.g. [Sch21b, Lemma 6.1(2)].

Lemma A.4. *In the notation of Lemma A.3, the following holds. For any open immersion $j : U \hookrightarrow X$, and $f|_U := f \circ j$, the diagram*

$$
\begin{array}{ccc}
(f|_U)_{comp}^*(d)[2d] & \xrightarrow{\ \simeq\ } & f|_U^! \\
\uparrow & & \uparrow \\
j_{comp}^* f_{comp}^*(d)[2d] & \xrightarrow{\ \simeq\ } & j_{comp}^* f^! \simeq j^! f^!
\end{array}
$$

commutes, where the horizontal isomorphisms are given by Lemma A.3 and the vertical maps are the canonical maps, given by functoriality of f_{comp}^* and $f^!$, respectively.

Proof. This is [Sch21b, Lemma 6.1(5)]. $\square$

Lemma A.5 (Purity)**.** *Let $i : Z \hookrightarrow X$ be a closed immersion between smooth equi-dimensional algebraic schemes over k. Assume that Z has codimension c in X. Then there is a canonical isomorphism of functors*

$$i_{comp}^*(-c)[-2c] \xrightarrow{\ \simeq\ } i^!.$$

Proof. Let π_X and π_Z denote the structure morphisms of X and Z, respectively. There is a canonical isomorphism $\pi_X^! i^! \xrightarrow{\simeq} \pi_Z^!$ (see Lemma A.2). Since X and Z are smooth and equi-dimensional, $\pi_X^!$ and $\pi_Z^!$ are expressed by Lemma A.3 and we get a canonical isomorphism $i_{comp}^*(-c)[-2c] \xrightarrow{\simeq} i^!$. This proves the lemma. $\square$

A.5. **Pullbacks and proper pushforwards.**

Lemma A.6. *Let $f : X \to Y$ be a morphism between algebraic schemes and let $Z_Y \subset Y$ and $Z_X \subset X$ be closed subsets.*

(1) If $f^{-1}(Z_Y) \subset Z_X$, then there are functorial pullback maps

(A.9)
$$f^* : H^i_{Z_Y}(Y, n) \longrightarrow H^i_{Z_X}(X, n).$$

(2) Assume that X and Y are smooth. If f is proper of pure relative codimension c and $f(Z_X) \subset Z_Y$, then there are functorial pushforward maps

(A.10)
$$f_* : H^{i-2c}_{Z_X}(X, n - c) \longrightarrow H^i_{Z_Y}(Y, n).$$

Proof. The pullback maps are induced by the natural map of functors $\Gamma_{Z_Y}(Y_{\text{proét}}, -) \to \Gamma_{Z_X}(X_{\text{proét}}, f^*_{comp}(-))$ on $\text{Mod}(Y_{\text{proét}}, \widehat{\mathbb{Z}}_\ell)$. Functoriality is clear from this description. This proves (1).

Let $\pi_X : X \to \text{Spec}\,k$ and $\pi_Y : Y \to \text{Spec}\,k$ denote the structure morphisms, and let $i : Z_Y \to Y$ and $i' : Z_X \to X$ denote the respective closed immersions. Applying Lemmas A.2 and A.3, we get canonical isomorphisms of functors

(A.11)
$$(i')^!(\pi_X)^*_{comp}(n) \simeq (i')^! \pi^!_X \simeq (f|_{Z_X})^! i^! \pi^!_Y \simeq (f|_{Z_X})^! i^! (\pi_Y)^*_{comp}(n - c)[-2c],$$

where $c = \dim Y - \dim X$ and $f|_{Z_X} : Z_X \to Z_Y$ is the restriction of f. The trace map in (A.8) thus induces by (A.6) pushforward maps as claimed in item (2). Functoriality in f follows from the functoriality of the trace map. This concludes the proof of the lemma. $\square$

Remark A.7. *Let X be a smooth equi-dimensional algebraic scheme and let $\iota_{Z'} : Z' \hookrightarrow X$ and $\iota_Z : Z \to Z'$ be closed subschemes. By (A.6), $H^i_{Z'}(X_{proét}, -) = H^i(Z'_{proét}, \iota^!_{Z'}-)$ and*

$$H^i_Z(X_{proét}, -) = H^i(Z_{proét}, \iota^!_Z \iota^!_{Z'}-) = H^i(Z'_{proét}, \mathrm{R}\,\iota_{Z*} \iota^!_Z \iota^!_{Z'}-).$$

Hence, Tr_{ι_Z} induces a pushforward map

$$(\iota_Z)_* : H^i_Z(X, n) \longrightarrow H^i_{Z'}(X, n),$$

which agrees with the respective pushforward map along the identity in (A.10). By construction of the adjunction map $\mathrm{Tr}_{\iota_Z} : \mathrm{R}\,\iota_{Z*}\iota_Z^! \to \mathrm{id}$, we find that $(\iota_Z)_*$ agrees with the map induced by the natural map $\mathrm{R}\,\Gamma_Z(X_{proét}, -) \to \mathrm{R}\,\Gamma_{Z'}(X_{proét}, -)$ of functors, induced by the inclusion $\Gamma_Z(X_{proét}, -) \hookrightarrow \Gamma_{Z'}(X_{proét}, -)$.

Lemma A.8 (Topological invariance)**.** *Let $f : X' \to X$ be a morphism of algebraic k-schemes that is a universal homeomorphism (e.g. $X' = X^{\text{red}}$ or $X' = X \times_k k'$ for a purely inseparable base extension k'/k). Let $Z \hookrightarrow X$ be a closed immersion with base change $Z' \hookrightarrow X'$. Then $f^* : H^i_Z(X, n) \to H^i_{Z'}(X', n)$ is an isomorphism.*

Proof. The claim follows from [BS15, Lemma 5.4.2]. $\square$

A.6. **Comparison to étale cohomology and excision.** Since $K \in D_{cons}((\text{Spec}\,k)_{\text{proét}}, \widehat{\mathbb{Z}}_\ell)$, is constructible $K \otimes_{\widehat{\mathbb{Z}}_\ell} \mathbb{Z}/\ell^r = \nu^* K_r$ for a constructible complex $K_r \in D_{cons}((\text{Spec}\,k)_{\text{ét}}, \mathbb{Z}/\ell^r)$, see [BS15, Definition 6.5.1]. Moreover, $K \simeq \mathrm{R}\lim \nu^* K_r$ by completeness. We then have the following, cf. [Jan88, Proposition 1.6].

Lemma A.9. *Let $\pi_X : X \to \text{Spec}\,k$ be an algebraic k-scheme with a closed subset $Z \subset X$. Then there is a canonical short exact sequence*

(A.12)
$$0 \longrightarrow \mathrm{R}^1 \lim H^{i-1}_Z(X_{\text{ét}}, \pi^*_X K_r \otimes_{\mathbb{Z}/\ell^r} \mu^{\otimes n}_{\ell^r}) \longrightarrow H^i_Z(X, n) \longrightarrow \lim H^i_Z(X_{\text{ét}}, \pi^*_X K_r \otimes_{\mathbb{Z}/\ell^r} \mu^{\otimes n}_{\ell^r}) \longrightarrow 0,$$

where $\lim$ denotes the inverse limit functor along r. Moreover, this sequence is compatible with respect to pullbacks (A.9) and pushforwards (A.10) on the individual pieces.

Proof. Note that $R \lim$ and $R \Gamma_Z$ commute with each other. Since $K \simeq R \lim \nu^* K_r$, $H_Z^i(X, n)$ is the i-th cohomology of the complex

$$R \lim R \Gamma_Z(X_{\mathrm{proét}}, \pi_X^* \nu^* K_r \otimes \mu_{\ell^r}^{\otimes n}) \simeq R \lim R \Gamma_Z(X_{\mathrm{proét}}, \nu^* \pi_X^* K_r \otimes \mu_{\ell^r}^{\otimes n}).$$

where we use that ν^* and π_X^* commute. Since $\nu_* : \mathrm{Ab}(X_{\mathrm{proét}}) \to \mathrm{Ab}(X_{\mathrm{ét}})$ maps injectives to injectives and $\mathrm{id} \to R \nu_* \nu^*$ is an equivalence on bounded below complexes (see [BS15, Proposition 5.2.6(2)]), the above complex identifies to

$$R \lim R \Gamma_Z(X_{\mathrm{ét}}, \pi_X^* K_r \otimes \mu_{\ell^r}^{\otimes n})$$

The lemma follows then from the composed functor spectral sequence, where we note that $R \lim$ has cohomological dimension ≤ 1 on Ab. The compatibility with respect to pullbacks is obvious; the compatibility with respect to pushforwards follows from [Sch21b, Lemma 5.2]. $\square$

Lemma A.10 (Excision). *Let $f : X' \to X$ be an étale morphism between algebraic schemes. Let $Z \subset X$ and $Z' \subset X'$ be closed subschemes with $f(Z') = Z$ such that $f|_{Z'} : Z' \to Z$ is an isomorphism and $f(X' \setminus Z') \subset X \setminus Z$. Then*

$$f^* : H_Z^i(X, n) \xrightarrow{\simeq} H_{Z'}^i(X', n).$$

Proof. Since K is bounded (see [BS15, Lemma 6.5.3]), induction on its length together with the local to global spectral sequence reduces us to the case where K is a single sheaf. In this case the statement is proven for étale cohomology in [Mil80, p. 92, III.1.27]; the case of pro-étale cohomology follows from this by Lemma A.9. $\square$

Lemma A.11. *Let X and Y be smooth algebraic k-schemes. Let $f : X \to Y$ be a finite étale morphism of constant degree d. Then the composition*

$$H^i(Y, n) \xrightarrow{f^*} H^i(X, n) \xrightarrow{f_*} H^{i-2c}(Y, n - c)$$

is given by multiplication with d.

Proof. By Lemma A.9, it suffices to prove the statement for étale cohomology with coefficients in constructible complexes of $\mathbb{Z}/\ell^r$-modules. Induction on the length of the complex K reduces us by the canonical truncation functor and the exact triangle $\tau_{\leq j} K \to K \to \tau_{\geq j+1} K$ to the case where K is a sheaf concentrated in a single degree. (Here we may loose the condition that K is constructible; the statement still makes sense because on the étale site $f^!$ and hence Tr_f exist on the derived category of bounded $\mathbb{Z}/\ell^r$-modules, see [SGA4.3, p. 570, XVIII, Théorème 3.1.4].) In this case the result follows from [SGA4.3, p. 554, XVIII, Théorème 2.9, (Var 4)]. $\square$

A.7. **Compatibility of pushforwards and pullbacks.**

Lemma A.12. *Consider the following diagrams*

$$
\begin{array}{ccc}
X' \xrightarrow{g'} X & \qquad & H_{Z_{X'}}^{i-2c}(X', n-c) \xleftarrow{(g')^*} H_{Z_X}^{i-2c}(X, n-c) \\
\downarrow{f'} \quad \downarrow{f} & \qquad & \downarrow{f'_*} \qquad\qquad\qquad \downarrow{f_*} \\
Y' \xrightarrow{g} Y, & \qquad & H_{Z_{Y'}}^i(Y', n) \xleftarrow{g^*} H_{Z_Y}^i(Y, n),
\end{array}
$$

where the diagram on the left is a Cartesian diagram of smooth equi-dimensional algebraic k-schemes, f is proper of pure relative codimension $c = \dim Y - \dim X$, $Z_X \subset X$ and $Z_Y \subset Y$ are closed with $f(Z_X) \subset Z_Y$, $Z_{Y'} = g^{-1}(Z_Y)$, and $Z_{X'} = Z_{Y'} \times_Y Z_X \subset X'$.

(1) If g is an open immersion, then the diagram on the right commutes.

(2) If f is smooth of pure relative dimension and if $Z_X = X$ and $Z_Y = Y$, then the diagram on the right commutes.

Proof. By Lemma A.9, it suffices to prove the compatibilities in question or étale cohomology with values in constructible complexes of $\mathbb{Z}/\ell^r$-modules.

The case where g is an open immersion holds, because the isomorphism in (A.11) which is used to define the pushforward map is compatible with open immersions (cf. Lemma A.4) and the trace map is compatible with base change and hence in particular with respect to Zariski localization. This proves item (1). (In fact, the result holds by the arguments in [BO74, (1.2.2), (1.3.5), and p. 186] true more generally in the case where g is étale, but we won't need this.)

Next we deal with case (2), so that f is smooth of pure relative dimension $d = \dim X - \dim Y$. By Lemma A.3, there are natural isomorphisms of functors:

$$(A.13) \qquad f^! \simeq f^*(d)[2d], \quad (f')^! \simeq (f')^*(d)[2d].$$

By [Ver67, Proposition 4.5(2)], the natural map of functors $(g')^* f^! \overset{\simeq}{\to} (f')^! g^*$ is an isomorphism. Combining this with $\mathrm{R}\, f_* \mathrm{R}\, g'_* = \mathrm{R}\, g_* \mathrm{R}\, f'_*$, we get an isomorphism $\alpha : \mathrm{R}\, f_* \mathrm{R}\, g'_* (g')^* f^! \overset{\simeq}{\to} \mathrm{R}\, g_* \mathrm{R}\, f'_* (f')^! g^*$, which fits into a diagram

$$(A.14)$$

$$
\begin{array}{ccc}
\mathrm{R}\, f_* f^! & \overset{\theta_{g'}}{\longrightarrow} \mathrm{R}\, f_* \mathrm{R}\, g'_* (g')^* f^! \overset{\alpha}{\underset{\simeq}{\longrightarrow}} \mathrm{R}\, g_* \mathrm{R}\, f'_* (f')^! g^* \\
\Big\downarrow{\scriptstyle \mathrm{Tr}_f} & \Big\downarrow{\scriptstyle \mathrm{Tr}_{f'}} \\
\mathrm{id} & \overset{\theta_g}{\longrightarrow} \mathrm{R}\, g_* g^*,
\end{array}
$$

of functors on $D_{cons}(Y_{\text{ét}}, \mathbb{Z}/\ell^r)$. We claim that the above diagram is commutative. Recall that any complex $K \in D_{cons}(Y_{\text{ét}}, \mathbb{Z}/\ell^r)$ is bounded. If K has length $n > 1$, then the canonical truncation functors yield an exact triangle $\tau_{\le j} K \to K \to \tau_{\ge j+1} K$ such that for suitable j, $\tau_{\le j} K$ and $\tau_{\ge j+1} K$ have length less than n. As in the proof of Lemma A.11, it thus suffices by induction on the length to prove that (A.14) is commutative when applied to sheaves concentrated in degree zero: $K = F[0]$ for some $F \in \mathrm{Mod}(Y_{\text{ét}}, \mathbb{Z}/\ell^r)$. To simplify notation, we will identify $K = F[0]$ with F. We further use the identifications in (A.13). It follows that $\mathrm{R}\, f_* f^* F(d)[2d]$ is a complex concentrated in non-positive degrees, while $\mathrm{R}\, g_* g^* F$ is concentrated in non-negative degrees. This implies that any map $\mathrm{R}\, f_* f^* F(d)[2d] \to \mathrm{R}\, g_* g^* F$ in the derived category is uniquely determined by the induced map

$$\mathrm{R}^{2d} f_* f^* F(d) \longrightarrow g_* g^* F$$

on the 0-th cohomology sheaf, where $d = \dim X - \dim Y = -c$. (To see this, use the canonical truncation triangle $\tau_{\le j} \to \mathrm{id} \to \tau_{\ge j+1}$ for $j = 0$ and for $j = -1$.) Hence, in order to check that the diagram in question is commutative, it suffices to note that the trace map $\mathrm{Tr}_f : \mathrm{R}^{2d} f_* F(d) \to F$ is compatible with base change, see [SGA4.3, p. 553, XVIII, Théorème 2.9, (Var 2)]. This concludes the proof of the claim that (A.14) is commutative.

The commutative diagram of cohomology groups, claimed in the lemma now follows by precomposing the functors in (A.14) with $\mathrm{R}\,\Gamma(Y_{\text{ét}}, -)$, using (A.13), and applying this to the complex $\pi_Y^! K_r \otimes_{\mathbb{Z}/\ell^r} \mu_{\ell^r}^{\otimes n}$. This concludes the proof of the lemma. $\qquad\square$

A.8. Long exact sequence of triples and semi-purity.

Lemma A.13 (Long exact sequence of triples)**.** *Let X be an algebraic scheme and let $Z \subset W \subset X$ be closed subsets. Then there is a canonical long exact sequence*

$$\ldots \longrightarrow H_Z^i(X, n) \overset{\iota_*}{\longrightarrow} H_W^i(X, n) \overset{rest_Z}{\longrightarrow} H_{W \setminus Z}^i(X \setminus Z, n) \overset{\partial}{\longrightarrow} H_Z^{i+1}(X, n) \longrightarrow \ldots,$$

such that for any morphism $f : X' \to X$ of algebraic schemes, with closed subsets $Z' \subset W' \subset X'$, the following holds:

(1) If $f^{-1}(Z) \subset Z'$ and $f^{-1}(W) \subset W'$, then pullback along f (as in (A.9)) induces a commutative ladder between the long exact sequence of the triple (X', W', Z') and that of (X, W, Z).

(2) *If X' and X are smooth, f is proper of pure relative codimension c, and $f(Z') = Z$ and $f(W') = W$, then pushforward along f (as in (A.10)) induces a commutative ladder between the long exact sequence of the triple (X', W', Z') and that of (X, W, Z), suitably shifted.*

Proof. If j is an open immersion, then $j^*_{comp} = j^*$ (see [BS15, Remark 6.5.10]) and so we will suppress the subscript *comp* for pullbacks along open immersions in what follows.

Let $i^W_Z : Z \to W$, $i_W : W \to X$, $j_{W \setminus Z} : W \setminus Z \to W$, and $j_{X \setminus Z} : X \setminus Z \to X$ denote the natural inclusions. By [BS15, Lemma 6.1.16], there is an exact triangle

$$\mathrm{R}(i_W)_* \mathrm{R}(i^W_Z)_*(i^W_Z)^!(i_W)^! \longrightarrow \mathrm{R}(i_W)_*(i_W)^! \longrightarrow \mathrm{R}(i_W)_* \mathrm{R}(j_{W \setminus Z})_*(j_{W \setminus Z})^*(i_W)^!.$$

We apply $\mathrm{R}\,\Gamma(X_{\mathrm{proét}}, -)$, and use $(i^W_Z)^!(i_W)^! \simeq i^!_Z$ and (A.6) to arrive at the triangle

$$\mathrm{R}\,\Gamma_Z(X_{\mathrm{proét}}, -) \longrightarrow \mathrm{R}\,\Gamma_W(X_{\mathrm{proét}}, -) \longrightarrow \mathrm{R}\,\Gamma((W \setminus Z)_{\mathrm{proét}}, (j_{W \setminus Z})^*(i_W)^!(-)).$$

If j is an open immersion, then $j^* = j^!$ (see e.g. [Sch21b, Lemma 5.1(2)]) and so

$$(j_{W \setminus Z})^*(i_W)^! \simeq (j_{W \setminus Z})^!(i_W)^! \simeq (j_{X \setminus Z} \circ i_{W \setminus Z})^! \simeq (i_{W \setminus Z})^!(j_{X \setminus Z})^! \simeq (i_{W \setminus Z})^!(j_{X \setminus Z})^*,$$

where $i_{W \setminus Z} : W \setminus Z \to X \setminus Z$ denotes the restriction of i_W. This implies that the last term in the above triangle identifies to $\mathrm{R}\,\Gamma_{W \setminus Z}((X \setminus Z)_{\mathrm{proét}}, j^*_{X \setminus Z}(-))$. The long exact sequence for the triple (X, W, Z) follows immediately from this. The compatibility properties with respect to pullbacks and pushforwards reduce by Lemma A.9 to the analogous results for étale cohomology, which are well-known, see e.g. [CTHK97, Definition 5.1.1 and §7.3(1)]. $\qquad\square$

Lemma A.14 (Semi-purity). *Let X be a smooth equi-dimensional algebraic k-scheme and let $i : Z \hookrightarrow X$ be a closed subset. Assume that the complex $K \in D_{cons}((\mathrm{Spec}\, k)_{\mathrm{proét}}, \widehat{\mathbb{Z}}_\ell)$ is concentrated in non-negative degrees. Then*

$$H^i_Z(X, n) = 0 \quad \text{for all } i < 2\,\mathrm{codim}_X Z.$$

Proof. By (A.6), we have

$$H^i_Z(X, n) \simeq H^i(Z_{\mathrm{proét}}, i^!(\pi^*_{comp} K \otimes_{\widehat{\mathbb{Z}}_\ell} \widehat{\mathbb{Z}}_\ell(n))),$$

where $\pi : X \to \mathrm{Spec}\, k$ is the structure morphism.

Assume first that Z is smooth. By étale excision (Lemma A.10) we may assume that Z is equi-dimensional of pure codimension c in X. By Lemma A.5, $i^*_{comp}(-c)[-2c] \xrightarrow{\simeq} i^!$, and so

$$H^i_Z(X, n) \simeq H^{i-2c}(Z_{\mathrm{proét}}, (\pi \circ i)^*_{comp} K \otimes_{\widehat{\mathbb{Z}}_\ell} \widehat{\mathbb{Z}}_\ell(n - c)).$$

Since K is concentrated in non-negative degrees, the above hyper-cohomology group vanishes for $i - 2c < 0$. This concludes the case where Z is smooth.

For the general case, note that we may by topological invariance (see Lemma A.8) assume that k is perfect. Hence, Z contains a open subset that is smooth over k. Using this and the fact that we know the result for Z smooth, we can proof the claim with the help of Lemma A.13 by induction on $\dim Z$. This concludes the lemma. $\qquad\square$

Lemma A.15. *Let X be a smooth algebraic scheme and let $i : Z \hookrightarrow X$ be a closed subset. Let further $W \subset X$ be a closed subset with complement $U = X \setminus W$ such that $Z \cap W$ has codimension at least j in X. If the complex K is concentrated in non-negative degrees, then the restriction map $H^i_Z(X, n) \to H^i_{Z \cap U}(U, n)$ is an isomorphism for all $i < 2j - 1$.*

Proof. Excision (see Lemma A.10) reduces us to the case where $W \subset Z$. By semi-purity (Lemma A.14), $H^i_W(X, n) = 0$ for all $i < 2j$ and so the result follows from long exact sequence in Lemma A.13. $\qquad\square$

A.9. **Cup products.** The pro-étale topos $\mathrm{Shv}(X_{\mathrm{proét}})$ of a scheme X is locally weakly contractible (see [BS15, Definition 3.2.1, Proposition 4.2.8]) which means in particular that it is locally coherent. From now on we assume that X is an algebraic scheme and so local coherence implies that X is coherent. By [SGA4.2, p. 336, Exposé VI, Proposition 9.0], it follows that $\mathrm{Shv}(X_{\mathrm{proét}})$ has enough points. Hence, any $F \in \mathrm{Mod}(X_{\mathrm{proét}}, \widehat{\mathbb{Z}}_\ell)$ admits a Godement resolution. More generally, for any bounded below complex $K \in D^+(X_{\mathrm{proét}}, \widehat{\mathbb{Z}}_\ell)$ there is a natural quasi-isomorphism $K \to \mathcal{G}^\bullet(K)$, given by the simple complex of the Godement double complex associated to K. The sheaves in the complex $\mathcal{G}^\bullet(K)$ are Γ_Z-acyclic and so $\mathrm{R}\,\Gamma_Z(X_{\mathrm{proét}}, K) = \Gamma_Z(X_{\mathrm{proét}}, \mathcal{G}^\bullet(K))$.

For any $F, G \in \mathrm{Mod}(X_{\mathrm{proét}}, \widehat{\mathbb{Z}}_\ell)$ and closed subschemes $Z, W \subset X$, we have a canonical map

$$(A.15) \qquad \Gamma_Z(X, F) \otimes_{\mathbb{Z}_\ell} \Gamma_W(X, G) \longrightarrow \Gamma_{Z \cap W}(X, F \otimes_{\widehat{\mathbb{Z}}_\ell} G),$$

given by the tensor product of sections. This induces a map

$$(A.16) \qquad \mathrm{R}\,\Gamma_Z(X_{\mathrm{proét}}, M) \otimes^{\mathbb{L}}_{\widehat{\mathbb{Z}}_\ell} \mathrm{R}\,\Gamma_W(X_{\mathrm{proét}}, L) \longrightarrow \mathrm{R}\,\Gamma_{Z \cap W}(X_{\mathrm{proét}}, M \otimes^{\mathbb{L}}_{\widehat{\mathbb{Z}}_\ell} L)$$

in the derived category for any constructible complexes $M, L \in D_{cons}(X_{\mathrm{proét}}, \widehat{\mathbb{Z}}_\ell)$. Using the above mentioned Godement resolutions, (A.16) may be described as follows. Recall that constructible complexes are bounded (see [BS15, Lemma 6.5.3]) and that there are enough $\widehat{\mathbb{Z}}_\ell$-flat objects in $\mathrm{Mod}(X_{\mathrm{proét}}, \widehat{\mathbb{Z}}_\ell)$, cf. [SGA4$\frac{1}{2}$, C.D., Chapter II, §3, no 2]. We may therefore choose a quasi-isomorphism $\mathcal{F}^\bullet(M) \to M$, where $\mathcal{F}^\bullet(M)$ is a complex of $\widehat{\mathbb{Z}}_\ell$-flat sheaves. Then the Godement resolution $\mathcal{F}^\bullet(M) \to \mathcal{G}^\bullet(\mathcal{F}^\bullet(M))$ is still a complex of flat $\widehat{\mathbb{Z}}_\ell$-modules. Hence, $\mathcal{G}^\bullet(\mathcal{F}^\bullet(M)) \otimes_{\widehat{\mathbb{Z}}_\ell} \mathcal{G}^\bullet(L)$ represents the derived tensor product $M \otimes^{\mathbb{L}}_{\widehat{\mathbb{Z}}_\ell} L$. Applying the Godement resolution once again, we get a quasi-isomorphism

$$\mathcal{G}^\bullet(\mathcal{F}^\bullet(M)) \otimes_{\widehat{\mathbb{Z}}_\ell} \mathcal{G}^\bullet(L) \longrightarrow \mathcal{G}^\bullet(\mathcal{G}^\bullet(\mathcal{F}^\bullet(M)) \otimes_{\widehat{\mathbb{Z}}_\ell} \mathcal{G}^\bullet(L)).$$

The cup product (A.16) may then explicitly be described by the natural map

$$\Gamma_Z(X_{\mathrm{proét}}, \mathcal{G}^\bullet(\mathcal{F}^\bullet(M))) \otimes_{\mathbb{Z}_\ell} \Gamma_W(X_{\mathrm{proét}}, \mathcal{G}^\bullet(L)) \longrightarrow \Gamma_{Z \cap W}(X_{\mathrm{proét}}, \mathcal{G}^\bullet(\mathcal{G}^\bullet(\mathcal{F}^\bullet(M)) \otimes_{\widehat{\mathbb{Z}}_\ell} \mathcal{G}^\bullet(L))).$$

In the following arguments, the above explicit description of (A.16) will be used throughout.

We denote from now on by

$$(A.17) \qquad H^i_Z(X, \mathbb{Z}_\ell(n)) := \mathrm{R}^i\,\Gamma_Z(X_{\mathrm{proét}}, \pi_X^* \widehat{\mathbb{Z}}_\ell(n)) \simeq H^i_{Z,cont}(X_{\mathrm{ét}}, (\mu_{\ell^r}^{\otimes n})_r)$$

twisted ℓ-adic pro-étale cohomology with support, which coincides with continuous ℓ-adic étale cohomology with support by Lemma A.1. As before we write $H^i_Z(X, n) := H^i_Z(X_{\mathrm{proét}}, \pi_{comp}^* K \otimes_{\widehat{\mathbb{Z}}_\ell} \widehat{\mathbb{Z}}_\ell(n))$.

Lemma A.16 (Cup products). *Let X be an algebraic k-scheme and let $Z, W \subset X$ be closed subsets. Then there are cup product maps*

$$H^i_Z(X, \mathbb{Z}_\ell(n)) \otimes_{\mathbb{Z}_\ell} H^j_W(X, m) \longrightarrow H^{i+j}_{Z \cap W}(X, n+m), \quad \alpha \otimes \beta \longmapsto \alpha \cup \beta$$

which are functorial with respect to pullbacks.

Proof. We apply (A.16) to $M = \widehat{\mathbb{Z}}_\ell(m)$ and $L = K \otimes_{\widehat{\mathbb{Z}}_\ell} \widehat{\mathbb{Z}}_\ell(n)$. The lemma thus follows by taking cohomology, because

$$\widehat{\mathbb{Z}}_\ell(n) \otimes^{\mathbb{L}}_{\widehat{\mathbb{Z}}_\ell} (\pi_{comp}^* K \otimes_{\widehat{\mathbb{Z}}_\ell} \widehat{\mathbb{Z}}_\ell(m)) \simeq \pi_{comp}^* K \otimes_{\widehat{\mathbb{Z}}_\ell} \widehat{\mathbb{Z}}_\ell(n+m)$$

since $\widehat{\mathbb{Z}}_\ell(n)$ is a locally free $\mathbb{Z}_\ell$-module (of rank 1). $\square$

Lemma A.17. *Let X be an algebraic k-scheme and let $Z, Z', W, W' \subset X$ be closed with $Z \subset Z'$ and $W \subset W'$. Then the cup product maps from Lemma A.16 fit into a commutative diagram*

$$
\begin{array}{ccc}
H_Z^i(X, \mathbb{Z}_\ell(n)) \otimes_{\mathbb{Z}_\ell} H_W^j(X, m) & \xrightarrow{\cup} & H_{Z\cap W}^{i+j}(X, n+m) \\
\downarrow & & \downarrow \\
H_{Z'}^i(X, \mathbb{Z}_\ell(n)) \otimes_{\mathbb{Z}_\ell} H_{W'}^j(X, m) & \xrightarrow{\cup} & H_{Z'\cap W'}^{i+j}(X, n+m),
\end{array}
$$

where the horizontal maps are the natural maps given by the pushforward along the identity on X.

Proof. Recall that the pushforward with respect to the identity corresponds to the map induced by the natural inclusion $\Gamma_Z(X, -) \hookrightarrow \Gamma_{Z'}(X, -)$, see Remark A.7. Clearly, the product map (A.15) fits into a similar commutative diagram. Passing to the derived category, we get for bounded below complexes of sheaves $M, L \in D(X_{\text{proét}}, \widehat{\mathbb{Z}}_\ell)$ a commutative diagram

$$
\begin{array}{ccc}
\mathrm{R}\,\Gamma_Z(X_{\text{proét}}, M) \otimes_{\widehat{\mathbb{Z}}_\ell}^{\mathbb{L}} \mathrm{R}\,\Gamma_W(X_{\text{proét}}, L) & \longrightarrow & \mathrm{R}\,\Gamma_{Z\cap W}(X_{\text{proét}}, M \otimes_{\widehat{\mathbb{Z}}_\ell}^{\mathbb{L}} L) \\
\downarrow & & \downarrow \\
\mathrm{R}\,\Gamma_{Z'}(X_{\text{proét}}, M) \otimes_{\widehat{\mathbb{Z}}_\ell}^{\mathbb{L}} \mathrm{R}\,\Gamma_{W'}(X_{\text{proét}}, L) & \longrightarrow & \mathrm{R}\,\Gamma_{Z'\cap W'}(X_{\text{proét}}, M \otimes_{\widehat{\mathbb{Z}}_\ell}^{\mathbb{L}} L).
\end{array}
$$

Applying this to $M = \widehat{\mathbb{Z}}_\ell(m)$ and $L = K \otimes_{\widehat{\mathbb{Z}}_\ell} \widehat{\mathbb{Z}}_\ell(n)$, the lemma follows by taking cohomology. $\qquad\square$

Lemma A.18. *Let $f : X \to Y$ be a proper morphism between smooth equi-dimensional algebraic k-schemes. Assume that f is either a closed immersion or a smooth map. Let $M, L \in D_{cons}(Y_{\text{proét}}, \widehat{\mathbb{Z}}_\ell)$. Then the following diagram is commutative*

(A.18)
$$
\begin{array}{ccc}
(\mathrm{R}\,f_* f^! M) \otimes_{\widehat{\mathbb{Z}}_\ell}^{\mathbb{L}} L & \xrightarrow[\simeq]{\psi} & \mathrm{R}\,f_* f^!(M \otimes_{\widehat{\mathbb{Z}}_\ell}^{\mathbb{L}} L) \\
{\scriptstyle \mathrm{Tr}_f \otimes \mathrm{id}} \downarrow & & \downarrow {\scriptstyle \mathrm{Tr}_f} \\
M \otimes_{\widehat{\mathbb{Z}}_\ell}^{\mathbb{L}} L & \xrightarrow{=} & M \otimes_{\widehat{\mathbb{Z}}_\ell}^{\mathbb{L}} L,
\end{array}
$$

where ψ is a natural isomorphism (described in the proof below).

Proof. Recall that the derived tensor product of complete complexes is again complete, see [BS15, Lemma 6.5.5]. Since f is proper, the projection formula in [BS15, Lemma 6.7.14] implies that the canonical map

(A.19)
$$
\mathrm{R}\,f_* f^! M \otimes_{\widehat{\mathbb{Z}}_\ell}^{\mathbb{L}} L \xrightarrow{\simeq} \mathrm{R}\,f_*(f^! M \otimes_{\widehat{\mathbb{Z}}_\ell}^{\mathbb{L}} f_{comp}^* L)
$$

is an isomorphism, see also [SGA4.3, p. 358, XVII, 5.2.9]. By Lemmas A.3 and A.5, our assumptions imply that there is a natural isomorphism $f^! \simeq f^*(d)[2d]$, where $d := \dim X - \dim Y$. Using this we find that the above isomorphism induces a natural isomorphism

$$
\psi : \mathrm{R}\,f_* f^! M \otimes_{\widehat{\mathbb{Z}}_\ell}^{\mathbb{L}} L \xrightarrow{\simeq} \mathrm{R}\,f_* f^!(M \otimes_{\widehat{\mathbb{Z}}_\ell}^{\mathbb{L}} L).
$$

It remains to see that (A.18) is commutative.

Assume first that f is a closed immersion. In this case, $\mathrm{R}\,f_* f^!$ is the derived functor associated to the functor $\mathcal{F} \longmapsto \mathcal{H}_X^0(Y, \mathcal{F})$ that maps a sheaf $\mathcal{F} \in \mathrm{Mod}(X_{\text{proét}}, \widehat{\mathbb{Z}}_\ell)$ to the subsheaf of local sections with

support contained in X. Moreover, the trace map corresponds on the non-derived level to the natural inclusion $\mathcal{H}^0_X(Y, \mathcal{F}) \to \mathcal{F}$. Using this, we find that (A.18) arises by deriving the following diagram

$$
\begin{array}{ccc}
\mathcal{H}^0_X(Y, \mathcal{F}) \otimes_{\widehat{\mathbb{Z}}_\ell} \mathcal{G} & \longrightarrow & \mathcal{H}^0_X(Y, \mathcal{F} \otimes_{\widehat{\mathbb{Z}}_\ell} \mathcal{G}) \\
\downarrow & & \downarrow \\
\mathcal{F} \otimes_{\widehat{\mathbb{Z}}_\ell} \mathcal{G} & \xrightarrow{=} & \mathcal{F} \otimes_{\widehat{\mathbb{Z}}_\ell} \mathcal{G},
\end{array}
$$

where the upper horizontal arrow is given by noting that if s is a local section of $\mathcal{F}$ whose support is contained in X and t is a local section of $\mathcal{G}$, then the local section $s \otimes t$ of $\mathcal{F} \otimes_{\widehat{\mathbb{Z}}_\ell} \mathcal{G}$ has support contained in X. The above diagram is clearly commutative and so is the derived version in (A.18).

Assume now that f is smooth. Here it is convenient to apply further reduction steps first. In fact, one can formally reduce the problem easily to the case where $M = \widehat{\mathbb{Z}}_\ell$ (by applying the result to various choices for L and comparing the results). Applying the canonical truncation triangle $\tau_{\leq j} L \to L \to \tau_{\geq j+1} L$, we further reduce by induction on the length of L to the case where $L = \mathcal{G}[0]$ is a sheaf $\mathcal{G} \in \mathrm{Mod}(X_{\mathrm{pro\acute{e}t}}, \widehat{\mathbb{Z}}_\ell)$ placed in degree zero. The complexes

$$
(\mathrm{R}\, f_* f^! M) \otimes^{\mathbb{L}}_{\widehat{\mathbb{Z}}_\ell} L = (\mathrm{R}\, f_* f^! \widehat{\mathbb{Z}}_\ell) \otimes^{\mathbb{L}}_{\widehat{\mathbb{Z}}_\ell} \mathcal{G}
$$

and

$$
\mathrm{R}\, f_* f^! (M \otimes^{\mathbb{L}}_{\widehat{\mathbb{Z}}_\ell} L) = \mathrm{R}\, f_* f^! \mathcal{G}
$$

are then concentrated in non-positive degrees, while $M \otimes^{\mathbb{L}}_{\widehat{\mathbb{Z}}_\ell} L = \mathcal{G}$ is concentrated in degree zero. It follows that the vertical arrows in (A.18) factor through the respective 0-th cohomology sheaves and so it suffices to prove commutativity of the following diagram:

(A.20)
$$
\begin{array}{ccc}
\mathrm{R}^{2d} f_* \widehat{\mathbb{Z}}_\ell(d) \otimes_{\widehat{\mathbb{Z}}_\ell} \mathcal{G} & \longrightarrow & \mathrm{R}^{2d} f_* \mathcal{G}(d) \\
{\scriptstyle \mathrm{Tr}_f \otimes \mathrm{id}} \downarrow & & \downarrow {\scriptstyle \mathrm{Tr}_f} \\
\widehat{\mathbb{Z}}_\ell \otimes_{\widehat{\mathbb{Z}}_\ell} \mathcal{G} & \xrightarrow{=} & \mathcal{G}.
\end{array}
$$

This follows directly from the construction of the trace map, see e.g. [Sch21b, (6.10)] and [Ver67, p. 190, (2)]. This concludes the proof of the lemma. $\qquad\square$

Lemma A.19 (Projection formula). *Let $f : X \to Y$ be a proper morphism between smooth equi-dimensional k-schemes of relative codimension $c = \dim Y - \dim X$. Let $Z_Y, W_Y \subset Y$ be closed and let $Z_X = f^{-1}(Z_Y)$ and $W_X = f^{-1}(W_Y)$. Assume that f is either a closed immersion or a smooth map. Then the following holds:*

(1) *For any $\alpha \in H^j_{W_X}(X, \mathbb{Z}_\ell(m))$ and $\beta \in H^i_{Z_Y}(Y, n)$, we have*

$$
f_*(\alpha \cup f^* \beta) = f_* \alpha \cup \beta \in H^{i+j+2c}_{Z_Y \cap W_Y}(Y, n + m + c),
$$

where $f_* \alpha \in H^{j+2c}_{W_Y}(Y, \mathbb{Z}_\ell(m+c))$ and $f^* \beta \in H^i_{Z_X}(X, n)$.

(2) *For any $\alpha \in H^i_{Z_Y}(Y, \mathbb{Z}_\ell(n))$ and $\beta \in H^j_{W_X}(X, m)$, we have*

$$
f_*(f^* \alpha \cup \beta) = \alpha \cup f_* \beta \in H^{i+j+2c}_{Z_Y \cap W_Y}(Y, n + m + c),
$$

where $f^* \alpha \in H^i_{Z_X}(X, \mathbb{Z}_\ell(n))$ and $f_* \beta \in H^{j+2c}_{W_Y}(Y, m + c)$.

Proof. Let $M, L \in D_{cons}((\operatorname{Spec} k)_{\text{proét}}, \widehat{\mathbb{Z}}_\ell)$ be constructible complexes of $\widehat{\mathbb{Z}}_\ell$-modules on the pro-étale site $(\operatorname{Spec} k)_{\text{proét}}$, and let $\pi_X : X \to \operatorname{Spec} k$ and $\pi_Y : Y \to \operatorname{Spec} k$ be the structure morphisms. Since $Z_X = f^{-1}(Z_Y)$,

$$\mathrm{R}\,\Gamma_{Z_X}(X, (\pi_X)^*_{comp}M(-c)[-2c]) = \mathrm{R}\,\Gamma_{Z_Y}(Y, \mathrm{R}\,f_*(\pi_X)^*_{comp}M(-c)[-2c]).$$

Since X and Y are smooth, Lemmas A.2 and A.3 imply $(\pi_X)^*_{comp}M(-c)[-2c] \simeq f^!(\pi_Y)^*_{comp}M$ and so

(A.21) $$\mathrm{R}\,\Gamma_{Z_X}(X, (\pi_X)^*_{comp}M(-c)[-2c]) = \mathrm{R}\,\Gamma_{Z_Y}(Y, \mathrm{R}\,f_*f^!(\pi_Y)^*_{comp}M).$$

We then consider the following diagram

(A.22)
$$\mathrm{R}\,\Gamma_{Z_Y}(Y, \mathrm{R}\,f_*f^!(\pi_Y)^*_{comp}M) \otimes^{\mathbb{L}}_{\mathbb{Z}_\ell} \mathrm{R}\,\Gamma_{W_Y}(Y, (\pi_Y)^*_{comp}L) \xrightarrow{\cup} \mathrm{R}\,\Gamma_{Z_Y \cap W_Y}(Y, \mathrm{R}\,f_*f^!(\pi_Y)^*_{comp}M \otimes^{\mathbb{L}}_{\mathbb{Z}_\ell} (\pi_Y)^*_{comp}L)$$

$$\downarrow {\scriptstyle \mathrm{Tr}_f \otimes \mathrm{id}} \qquad\qquad\qquad\qquad\qquad\qquad \downarrow {\scriptstyle \mathrm{Tr}_f \otimes \mathrm{id}}$$

$$\mathrm{R}\,\Gamma_{Z_Y}(Y, (\pi_Y)^*_{comp}M) \otimes^{\mathbb{L}}_{\mathbb{Z}_\ell} \mathrm{R}\,\Gamma_{W_Y}(Y, (\pi_Y)^*_{comp}L) \xrightarrow{\cup} \mathrm{R}\,\Gamma_{Z_Y \cap W_Y}(Y, (\pi_Y)^*_{comp}M \otimes^{\mathbb{L}}_{\widehat{\mathbb{Z}}_\ell} (\pi_Y)^*_{comp}L),$$

which is clearly commutative.

By Lemma A.18, there is a commutative diagram

$$\mathrm{R}\,f_*f^!(\pi_Y)^*_{comp}M \otimes^{\mathbb{L}}_{\widehat{\mathbb{Z}}_\ell} (\pi_Y)^*_{comp}L \xrightarrow[\psi]{\simeq} \mathrm{R}\,f_*f^!(\pi_Y)^*_{comp}(M \otimes^{\mathbb{L}}_{\widehat{\mathbb{Z}}_\ell} L)$$

$$\downarrow {\scriptstyle \mathrm{Tr}_f \otimes \mathrm{id}} \qquad\qquad\qquad\qquad\qquad \downarrow {\scriptstyle \mathrm{Tr}_f}$$

$$(\pi_Y)^*_{comp}M \otimes^{\mathbb{L}}_{\widehat{\mathbb{Z}}_\ell} (\pi_Y)^*_{comp}L \xrightarrow{\simeq} (\pi_Y)^*_{comp}(M \otimes^{\mathbb{L}}_{\widehat{\mathbb{Z}}_\ell} L).$$

Applying $\mathrm{R}\,\Gamma_{Z_Y \cap W_Y}(Y, -)$ to this and combining the resulting commutative square with (A.22), we get a commutative diagram of the form

$$\mathrm{R}\,\Gamma_{Z_Y}(Y, \mathrm{R}\,f_*f^!(\pi_Y)^*_{comp}M) \otimes^{\mathbb{L}}_{\mathbb{Z}_\ell} \mathrm{R}\,\Gamma_{W_Y}(Y, (\pi_Y)^*_{comp}L) \xrightarrow{\cup} \mathrm{R}\,\Gamma_{Z_Y \cap W_Y}(Y, \mathrm{R}\,f_*f^!(\pi_Y)^*_{comp}(M \otimes^{\mathbb{L}}_{\widehat{\mathbb{Z}}_\ell} L))$$

$$\downarrow {\scriptstyle \mathrm{Tr}_f \otimes \mathrm{id}} \qquad\qquad\qquad\qquad\qquad\qquad \downarrow {\scriptstyle \mathrm{Tr}_f}$$

$$\mathrm{R}\,\Gamma_{Z_Y}(Y, (\pi_Y)^*_{comp}M) \otimes^{\mathbb{L}}_{\mathbb{Z}_\ell} \mathrm{R}\,\Gamma_{W_Y}(Y, (\pi_Y)^*_{comp}L) \xrightarrow{\cup} \mathrm{R}\,\Gamma_{Z_Y \cap W_Y}(Y, (\pi_Y)^*_{comp}(M \otimes^{\mathbb{L}}_{\widehat{\mathbb{Z}}_\ell} L)).$$

The term in the right upper corner of the above diagram is by (A.21) (applied to the complex $M \otimes^{\mathbb{L}}_{\widehat{\mathbb{Z}}_\ell} L$ and the closed subset $Z_X \cap W_X \subset X$) given by

$$\mathrm{R}\,\Gamma_{Z_Y \cap W_Y}(Y, \mathrm{R}\,f_*f^!(\pi_Y)^*_{comp}(M \otimes^{\mathbb{L}}_{\widehat{\mathbb{Z}}_\ell} L)) = \mathrm{R}\,\Gamma_{Z_X \cap W_X}(X, (\pi_X)^*_{comp}(M \otimes^{\mathbb{L}}_{\widehat{\mathbb{Z}}_\ell} L)(-c)[-2c])$$

Item (1) in the lemma therefore follows by applying the above argument to $M = \widehat{\mathbb{Z}}_\ell(m)$ and $L = K \otimes_{\widehat{\mathbb{Z}}_\ell} \widehat{\mathbb{Z}}_\ell(n)$, and taking cohomology of the above commutative square. Item (2) follows similarly by applying the above argument to $L = \widehat{\mathbb{Z}}_\ell(m)$ and $M = K \otimes_{\widehat{\mathbb{Z}}_\ell} \widehat{\mathbb{Z}}_\ell(n)$ and using the anti-commutativity of cup products. This concludes the proof of the lemma. $\square$

A.10. **Cycle class.**

Lemma A.20 (Cycle class). *Let X be an equi-dimensional smooth algebraic k-scheme and let $\Gamma \in Z^c(X)$ be a cycle of codimension c. Let $Z \subset X$ be any closed subscheme that contains the support of Γ. Then there is a cycle class*

$$\mathrm{cl}^X_Z(\Gamma) \in H^{2c}_Z(X, \mathbb{Z}_\ell(c))$$

with the following properties:

(1) If $W \subset X$ is closed with $Z \subset W$, then $\mathrm{cl}^X_W(\Gamma)$ is the image of $\mathrm{cl}^X_Z(\Gamma)$ via the natural map $H^{2c}_Z(X, \mathbb{Z}_\ell(c)) \to H^{2c}_W(X, \mathbb{Z}_\ell(c))$.

(2) If $\Gamma = \Gamma_1 + \Gamma_2$ for cycles $\Gamma_i \in Z^c(X)$ with $\mathrm{supp}(\Gamma_i) \subset Z$, then $\mathrm{cl}^X_Z(\Gamma) = \mathrm{cl}^X_Z(\Gamma_1) + \mathrm{cl}^X_Z(\Gamma_2)$.

Proof. Recall from Lemma A.1 that $H^{2c}_Z(X, \mathbb{Z}_\ell(c)) = H^{2c}_{Z,cont}(X_{\text{ét}}, \mathbb{Z}_\ell(c))$ agrees with continuous étale cohomology with values in the inverse system $(\mu^{\otimes c}_{\ell^r})_r$. By additivity of continuous étale cohomology, it suffices to deal with the case where X is integral, hence a variety. One then defines $\mathrm{cl}^X_Z(\Gamma)$ linearly in Γ and hence reduces to the case where Γ is a prime cycle, i.e. a subvariety of codimension one. Let us first assume that $Z = \mathrm{supp}\,\Gamma$. By (A.17) and [Jan88, Theorem 3.23], there is a canonical isomorphism

$$(A.23) \qquad H^{2c}_Z(X, \mathbb{Z}_\ell(c)) \simeq \varprojlim_r H^{2c}_Z(X, \mu^{\otimes c}_{\ell^r})$$

and the class $\mathrm{cl}^X_Z(\Gamma)$ is defined as limit of the cycle classes of Γ in $H^{2c}_Z(X, \mu^{\otimes c}_{\ell^r})$ from [SGA4$\frac{1}{2}$, p. 143, 2.2.10], where one uses compatibility for different values of r, cf. [Jan88, p. 221]. In general, if Z is any closed subset with $|\Gamma| := \mathrm{supp}\,\Gamma \subset Z$, then we define $\mathrm{cl}^X_Z(\Gamma)$ as image of $\mathrm{cl}^X_{|\Gamma|}(\Gamma)$ via the natural pushforward map $H^{2c}_{|\Gamma|}(X, \mathbb{Z}_\ell(c)) \to H^{2c}_Z(X, \mathbb{Z}_\ell(c))$. Condition (1) follows then from the functoriality of pushforwards, while item (2) follows directly from the construction together with the linearity of pushforwards. This proves the lemma. $\qquad\square$

Lemma A.21. *Let $f : X \to Y$ be a morphism between smooth equi-dimensional algebraic k-schemes.*

(1) *Let $\Gamma \in Z^i(Y)$ be a cycle on Y with support $Z_Y := \mathrm{supp}(\Gamma)$. Assume that $Z_X := f^{-1}(Z_Y) \subset X$ has pure codimension i. Then $f^* \mathrm{cl}^Y_{Z_Y}(\Gamma) = \mathrm{cl}^X_{Z_X}(f^*\Gamma)$.*

(2) *Let $\Gamma \in Z^i(X)$ with support $Z_X := \mathrm{supp}(\Gamma)$. Assume that f is proper of pure relative codimension c. Set $Z_Y := f(Z_X)$ with the reduced scheme structure. Then $f_* \mathrm{cl}^X_{Z_X}(\Gamma) = \mathrm{cl}^Y_{Z_Y}(f_*\Gamma)$.*

Proof. To prove item (1), we note that by the construction of the cycle class via the isomorphism (A.23), it suffices to prove the compatibility result for étale cohomology with coefficients in $\mu^{\otimes i}_{\ell^r}$ with $r \geq 1$. Hence, item (1) follows from [SGA4$\frac{1}{2}$, p. 148, Théorème 2.3.8.(ii)].

We prove item (2) next. Lemma A.8 reduces us to the case where the ground field k is perfect. By Lemma A.15, we may then assume that Z_Y and Z_X are smooth and pure-dimensional of pure codimensions i and $i + c$, respectively (this uses that any reduced scheme is generically regular, hence smooth because k is perfect). Lemma A.5 yields canonical isomorphisms

$$H^{2i}_{Z_X}(X, \mathbb{Z}_\ell(i)) \simeq H^0(Z_X, \mathbb{Z}_\ell(0)) \quad \text{and} \quad H^{2i+2c}_{Z_Y}(Y, \mathbb{Z}_\ell(i+c)) \simeq H^0(Z_Y, \mathbb{Z}_\ell(0)).$$

It follows from the construction of the pushforward f_* that it is compatible via the above isomorphism with the pushforward

$$(f|_{Z_X})_* : H^0(Z_X, \mathbb{Z}_\ell(0)) \longrightarrow H^0(Z_Y, \mathbb{Z}_\ell(0)).$$

Since Z_X and Z_Y are smooth, the above groups decompose according to the irreducible components of Z_X and Z_Y, respectively. Using this, we reduce to the case where Z_X and Z_Y are integral, and we need to show that $(f|_{Z_X})_*$ is given by multiplication with $\deg(f|_{Z_X})$. Up to shrinking Z_Y further, we may assume that $f|_{Z_X} : Z_X \to Z_Y$ is étale (this uses that k is perfect) and so the claim follows from Lemma A.11. This proves item (2) and concludes the proof of the lemma. $\qquad\square$

Lemma A.22. *Let X be a smooth equi-dimensional algebraic scheme and let for $i = 1, 2$, $\Gamma_i \in \mathrm{CH}^{c_i}(X)$ be a cycle and let $Z_i \subset X$ be closed of codimension c_i. Assume that Z_1 and Z_2 meet dimensionally transversely, so that $Z = Z_1 \cap Z_2$ is either empty or has codimension $c = c_1 + c_2$ in X. Then*

$$\mathrm{cl}^X_Z(\Gamma_1 \cdot \Gamma_2) = \mathrm{cl}^X_{Z_1}(\Gamma_1) \cup \mathrm{cl}^X_{Z_2}(\Gamma_2) \in H^{2c}_Z(X, \mathbb{Z}_\ell(c)).$$

Proof. Since the cycle class in Jannsen's continuous étale cohomology from Lemma A.20 is constructed via the isomorphism (A.23), and because the cup product constructed in Lemma A.16 is compatible with reduction modulo ℓ^r, it suffices to prove the lemma modulo ℓ^r, i.e. for étale cohomology with coefficients in $\mu_{\ell^r}^{\otimes c}$ In this case the result in question follows from [SGA4$\frac{1}{2}$, Theorem 2.3.8(iii), Remarque 2.3.9] together with the fact that the multiplicities of the intersection product $\Gamma_1 \cdot \Gamma_2$ may be computed by Serre's Tor formula, see [Ful98, §20.4 and Example 7.1.2]. This concludes the proof of the lemma. $\square$

Lemma A.23. *Let X be a smooth equi-dimensional algebraic scheme over a perfect field k. Let $\Gamma \in Z^c(X)$ be a cycle with support $Z := \mathrm{supp}(\Gamma)$. Assume that there is a closed subset $W \subset X$ of codimension $c-1$ and with $Z \subset W$, such that Γ, viewed as a cycle on W, is rationally equivalent to zero on W. Then there is a class*

$$\varphi \in H_{W \setminus Z}^{2c-1}(X \setminus Z, \mathbb{Z}_\ell(c)) \quad with \quad \partial\varphi = \mathrm{cl}_Z^X(\Gamma) \in H_Z^c(X, \mathbb{Z}_\ell(c)).$$

Proof. We endow W with the canonical reduced scheme structure and consider the normalization $\tau : W' \to W$. The assumptions imply that there is a rational function φ' on W' such that $\Gamma = \tau_* \mathrm{div}(\varphi')$.

By Lemma A.15, the groups in question do not change when we remove from X a closed subset of codimension at least $c+1$. Since W' is regular in codimension one, we may thus without loss of generality assume that W' is regular, hence smooth. For the same reason, we may assume that Z as well as the preimage $Z' = \tau^{-1}(Z)$ are smooth. We thus have a pushforward map $\tau_* : H^1(W', \mathbb{Z}_\ell(1)) \to H_W^{2c-1}(X \setminus Z, \mathbb{Z}_\ell(c))$, given by (A.10). The rational function φ' yields via Jannsen's Kummer sequence [Jan88, 3.27] a class $(\varphi') \in H^1(W' \setminus Z', \mathbb{Z}_\ell(1))$ and we let

$$\varphi := \tau_*(1/\varphi') \in H_{W \setminus Z}^{2c-1}(X \setminus Z, \mathbb{Z}_\ell(c)).$$

The image of (φ') via the residue map $H^1(W' \setminus Z', \mathbb{Z}_\ell(1)) \to H_{Z'}^2(W', \mathbb{Z}_\ell(1)) \simeq H^0(Z', \mathbb{Z}_\ell(0))$ coincides with the cycle given by $-\mathrm{div}(\varphi')$ (indeed, it suffices to prove this modulo ℓ^r, hence for étale cohomology with coefficients in μ_{ℓ^r}, where it follows from [SGA4$\frac{1}{2}$, p. 147, (cycle), Lemme 2.3.6] and the anticommutativity of the diagram in [SGA4$\frac{1}{2}$, p. 138, (cycle), 2.1.3]). Since the pushforwards from (A.10) induce by Lemma A.13 commutative ladders between the respective long exact sequences of triples, we find that

$$\partial(\varphi) = \partial(\tau_*(1/\varphi')) = \tau_*(\partial(1/\varphi')) = -\tau_*(\partial(\varphi')) = \mathrm{cl}_Z^X(\tau_* \mathrm{div}(\varphi')) = \mathrm{cl}_Z^X(\Gamma).$$

This concludes the lemma. $\square$

Lemma A.24. *In the notation of Lemma A.23, we have $\mathrm{cl}_W^X(\Gamma) = 0 \in H_W^{2c}(X, \mathbb{Z}_\ell(c))$.*

Proof. By Lemma A.21, $\mathrm{cl}_W^X(\Gamma)$ is the image of $\mathrm{cl}_Z^X(\Gamma)$ via the natural map $H_Z^{2c}(X, \mathbb{Z}_\ell(c)) \to H_W^{2c}(X, \mathbb{Z}_\ell(c))$. The assertion in the lemma thus follows from the long exact sequence of triples (see Lemma A.13) together with the fact that $\partial\varphi = \mathrm{cl}_Z^X(\Gamma) \in H_Z^{2c}(X, \mathbb{Z}_\ell(c))$. $\square$

A.11. **Proof of Proposition 3.2.**

Proof of Proposition 3.2. By Lemma A.1, item (2) follows from item (3) applied to the complex K given by the pro-étale sheaf $\lim \nu^* F_r$ placed in degree zero. We further claim that item (1) follows from item (2). To see this, note that étale cohomology commutes with direct limits in the coefficients, see [Stacks21, Tag 09YQ]. It thus suffices to prove item (1) in the case where there is some integer r so that $\mathcal{F} = \pi_X^* F$ is a ℓ^r-torsion étale sheaf on $X_{\text{ét}}$. The result then follows by [Jan88, (3.1)] from item (2) applied to the constant inverse system $(\mathcal{F})_r$.

Altogether we have thus seen that it suffices to prove item (3). To this end we use the notation $H_Z^i(X, n) := H_Z^*(X_{\text{proét}}, (\pi_X)_{comp}^* K \otimes_{\widehat{\mathbb{Z}}_\ell} \widehat{\mathbb{Z}}_\ell(n))$ from above. The pullback maps from Lemma A.6 make $(X, Z) \longmapsto H_Z^i(X, n)$ into a functor as we want. If the complex K is concentrated in non-negative degrees, then semi-purity (i.e. C5) follows from Lemma A.14.

Item $C1$ follows from Lemma A.10, item $C2$ follows from Lemma A.13, and item $C3$ follows from Lemmas A.6 and A.12. It thus remains to prove item $C4$. The action of cycles is given by cup product and the cycle class map, see Lemmas A.16 and A.20. Item $C4$a follows then from Lemma A.24.

To prove $C4$b, let $\iota : W \hookrightarrow X$ be a closed embedding with W smooth and irreducible of codimension c and let $\Gamma = [W] \in Z^c(X)$. Let $\alpha \in H^i_Z(X, n)$ and note that $\mathrm{cl}^X_X(X) \in H^0(X, \mathbb{Z}_\ell(0))$ satisfies $\mathrm{cl}^X_X(X) \cup \alpha = \alpha$. The projection formula in Lemma A.19 thus yields

$$\iota_* \iota^* \alpha = \iota_*(\iota^* \mathrm{cl}^X_X(X) \cup \iota^* \alpha) = (\iota_* \iota^* \mathrm{cl}^X_X(X)) \cup \alpha.$$

By Lemma A.21, $\iota_* \iota^* \mathrm{cl}^X_X(X) = \iota_* \mathrm{cl}^W_W(W) = \mathrm{cl}^X_W(\Gamma)$ and so the above equation reads $\iota_* \iota^* \alpha = \mathrm{cl}^X_W(\Gamma) \cup \alpha$, as we want.

Item $C4$c follows from Lemma A.17.

Item $C4$d follows from the functoriality of the cup product in Lemma A.16 with respect to pullbacks (hence with respect to restrictions), together with the compatibility of the cycle class map with flat pullback from Lemma A.21.

The formula $\mathrm{cl}^X_W(\Gamma) \cup (\mathrm{cl}^X_{W'}(\Gamma') \cup \alpha) = \mathrm{cl}^X_{W \cap W'}(\Gamma \cdot \Gamma') \cup \alpha$ in item $C4$e follows from the associativity of cup products in (A.16) together with the formula $\mathrm{cl}^X_W(\Gamma) \cup \mathrm{cl}^X_{W'}(\Gamma') = \mathrm{cl}^X_{W \cap W'}(\Gamma \cdot \Gamma')$ from Lemma A.22, where we use that $W = \mathrm{supp}\,\Gamma$ meets $W' = \mathrm{supp}\,\Gamma'$ properly by assumption.

It remains to prove $C4$f. If $f : X' \to X$ is flat, then the formula $f^*(\mathrm{cl}^X_W(\Gamma) \cup \alpha) = \mathrm{cl}^X_{f^{-1}(W)}(f^*\Gamma) \cup f^*\alpha$ follows from the compatibility of cup products with pullbacks (see Lemma A.16) together with the fact that $f^* \mathrm{cl}^X_W(\Gamma) = \mathrm{cl}^X_{f^{-1}(W)}(f^*\Gamma)$ by Lemma A.21. Finally, if $f : X' \to X$ is proper, the formula $f_*(\mathrm{cl}^{X'}_{W'}(\Gamma) \cup f^*\alpha) = \mathrm{cl}^X_{f(W')}(f_*\Gamma) \cup \alpha$ follows from item (1) in Lemma A.19 together with Lemma A.21, while $f_*(f^* \mathrm{cl}^X_W(\Gamma) \cup \alpha) = \mathrm{cl}^X_W(\Gamma) \cup f_*\alpha$ follows from item (2) in Lemma A.19.

This concludes the proof of the proposition. $\qquad\square$

Appendix B. Comparison with action on algebraic cycles

B.1. **Comparison to the action on Chow groups.** Let X be a smooth projective equi-dimensional scheme over a perfect field k and let ℓ be a prime that is invertible in k. For $Z \subset X$ closed, we let as before

$$H^i_Z(X, \mathbb{Z}_\ell(n)) := \mathrm{R}^i\, \Gamma_Z(X_{\mathrm{pro\acute{e}t}}, \widehat{\mathbb{Z}}_\ell(n)) \quad \text{and} \quad H^i(X, \mathbb{Z}_\ell(n)) := H^i_X(X, \mathbb{Z}_\ell(n)).$$

The theory of refined unramified cohomology developed in [Sch21b] relies on an ℓ-adic Borel–Moore cohomology theory $H^i_{BM}(Z, \mathbb{Z}_\ell(n))$ for each algebraic k-scheme Z, see [Sch21b, Section 4 and Proposition 6.6]). If X is a equi-dimensional smooth algebraic k-scheme and $Z \subset X$ is closed of codimension c (not necessarily of pure dimension), then the Borel–Moore cohomology of Z is by (A.6) and Lemmas A.2 and A.3 given by

$$H^i_{BM}(Z, \mathbb{Z}_\ell(n)) = H^{i+2c}_Z(X, \mathbb{Z}_\ell(n + c)).$$

In particular, $H^i_{BM}(X, \mathbb{Z}_\ell(n) = H^i(X, \mathbb{Z}_\ell(n))$ for X smooth and equi-dimensional.

We let $\mathrm{CH}^i(X)_{\mathbb{Z}_\ell} := \mathrm{CH}^i(X) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$. The coniveau filtration N^* on $\mathrm{CH}^i(X)_{\mathbb{Z}_\ell}$ is the decreasing filtration given by the condition that a class $[z] \in \mathrm{CH}^i(X)_{\mathbb{Z}_\ell}$ lies in $N^j \mathrm{CH}^i(X)_{\mathbb{Z}_\ell}$ if and only if $[z]$ can be represented by a cycle z that is homologically trivial on a closed subset $W \subset X$ of codimension j, i.e. $\mathrm{supp}\, z \subset W$ with $\mathrm{cl}^X_W(z) = 0 \in H^{2i}_W(X, \mathbb{Z}_\ell(i))$, see [Blo85, Jan00] or [Sch21b, Definition 7.3]. For instance, $N^0 \mathrm{CH}^i(X)_{\mathbb{Z}_\ell}$ is the space of ℓ-adic cycles with trivial cycle class in $H^{2i}(X, \mathbb{Z}_\ell(i))$. Moreover, $N^{i-1} \mathrm{CH}^i(X)_{\mathbb{Z}_\ell}$ is zero if k is finitely generated (or an inseparable extension thereof) and it is the space of algebraically trivial ℓ-adic cycles of codimension i if k is algebraically closed, see [Sch21b, Proposition 6.6 and Lemma 7.5] or [Jan00, Lemmas 5.7 and 5.8] for the corresponding rational statements.

We define

$$A^i(X)_{\mathbb{Z}_\ell} := \mathrm{CH}^i(X)_{\mathbb{Z}_\ell} / N^{i-1} \mathrm{CH}^i(X)_{\mathbb{Z}_\ell}.$$

By what we have said above, this is the ℓ-adic Chow group of algebraic cycles modulo rational equivalence if k is finitely generated (or the perfect closure of a such a field), while it is the group of ℓ-adic cycles modulo algebraic equivalence if k is algebraically closed. We further let $A_0^i(X)_{\mathbb{Z}_\ell} \subset A^i(X)_{\mathbb{Z}_\ell}$ be the subspace of cycles with trivial cycle class on X, i.e.

$$(\text{B.1}) \qquad\qquad A_0^i(X)_{\mathbb{Z}_\ell} = N^0\,\mathrm{CH}^i(X)_{\mathbb{Z}_\ell}/N^{i-1}\,\mathrm{CH}^i(X)_{\mathbb{Z}_\ell}.$$

By [Sch21b, Lemma 4.7 and Proposition 6.9], there is a canonical isomorphism

$$(\text{B.2}) \qquad\qquad \frac{H^{2i-1}_{i-2,nr}(X,\mathbb{Z}_\ell(i))}{H^{2i-1}(X,\mathbb{Z}_\ell(i))} \xrightarrow{\ \simeq\ } A_0^i(X)_{\mathbb{Z}_\ell}.$$

By item (2) in Corollary 6.8, the action of correspondences on refined unramified cohomology descends to an action on the left hand side of (B.2), where we use $H^{2i-1}_{m,nr}(X,\mathbb{Z}_\ell(i)) = H^{2i-1}(X,\mathbb{Z}_\ell(i))$ for $m \geq \dim X$. There is also a natural action on the right hand side of (B.2). The main result of this appendix shows that both actions agree with each other.

Proposition B.1. *Let X and Y be smooth projective equi-dimensional schemes over a perfect field k, let $d_X := \dim(X)$ and let ℓ be a prime that is invertible in k. Then the bilinear pairing*

$$\mathrm{CH}^c(X \times Y) \times \frac{H^{2i-1}_{i-2,nr}(X,\mathbb{Z}_\ell(i))}{H^{2i-1}(X,\mathbb{Z}_\ell(i))} \longrightarrow \frac{H^{2(i+c-d_X)-1}_{i+c-d_X-2,nr}(Y,\mathbb{Z}_\ell(i+c-d_X))}{H^{2(i+c-d_X)-1}(Y,\mathbb{Z}_\ell(i+c-d_X))}$$

induced by the pairing on refined unramified cohomology in Corollary 6.8 agrees via the isomorphism in (B.2) with the natural action on $A_0^i(X)_{\mathbb{Z}_\ell} = N^0\,\mathrm{CH}^i(X)_{\mathbb{Z}_\ell}/N^{i-1}\,\mathrm{CH}^i(X)_{\mathbb{Z}_\ell}$.

The above proposition relies on the results recalled in Appendix A, together with the following result (we will only need the special case where $W = U = X$, but state and prove the more general version below).

Lemma B.2 (Compatibility of cup products with residue maps)**.** *Let X be an algebraic k-scheme. Let $Z \subset W \subset X$ and $Y \subset X$ be closed subsets. Let $U \subset X$ be an open subset with $W \cap Y \subset U$ and $Z \cap Y \subset U$.*

Then for any $\beta \in H^j_Y(U,\mathbb{Z}_\ell(m))$, the following diagram is commutative:

$$
\begin{array}{ccc}
H^i_W(U \setminus Z, n) & \xrightarrow{\quad\partial\quad} & H^{i+1}_Z(U, n) \\[4pt]
\downarrow{\scriptstyle \cup\beta|_{U\setminus Z}} & & \downarrow{\scriptstyle \cup\beta} \\[8pt]
H^{i+j}_{W\cap Y}(U \setminus Z, n+m) & & H^{i+j+1}_{Z\cap Y}(U, n+m) \\[4pt]
\simeq\uparrow & & \simeq\uparrow \\[8pt]
H^{i+j}_{W\cap Y}(X \setminus (Z\cap Y), n+m) & \xrightarrow{\quad\partial\quad} & H^{i+j+1}_{Z\cap Y}(X, n+m),
\end{array}
$$

where the cohomology groups in question are those from item (3) in Proposition 3.2 (and the notation $H^i_Z(U,n) := H^i_{Z\cap U}(U,n)$ etc. is used), the horizontal maps are parts of the long exact sequence in Lemma A.13, the upper vertical maps are the cup product maps from Lemma A.16 and the lower vertical maps are the isomorphisms given by pullback and excision (where we use $W \cap Y \subset U$ and $Z \cap Y \subset U$).

Proof. We write $Z_Y := Z \cap Y$, and $W_Y := W \cap Y$. For $U \subset X$ open, we will further use the notation $\mathrm{R}\Gamma_Z(U,-) := \mathrm{R}\Gamma_{Z\cap U}(U,-)$ and so on. Let then $M, L \in D_{cons}(X_{\mathrm{pro\acute{e}t}}, \widehat{\mathbb{Z}}_\ell)$ and consider the exact triangle

$$\mathrm{R}\Gamma_Z(U, M) \longrightarrow \mathrm{R}\Gamma_W(U, M) \longrightarrow \mathrm{R}\Gamma_{W\setminus Z}(U \setminus Z, M),$$

in $D(\mathrm{Mod}_{\mathbb{Z}_\ell})$ from the proof of Lemma A.13, where by slight abuse of notation we do not distinguish between M and the pullback to the respective open subsets above.

We may take the derived tensor product of the above triangle with the complex $\mathrm{R}\,\Gamma_Y(U, L)$. This gives rise to an exact triangle

$$\mathrm{R}\,\Gamma_Z(U, M) \otimes^{\mathbb{L}}_{\mathbb{Z}_\ell} \mathrm{R}\,\Gamma_Y(U, L) \longrightarrow \mathrm{R}\,\Gamma_W(U, M) \otimes^{\mathbb{L}}_{\mathbb{Z}_\ell} \mathrm{R}\,\Gamma_Y(U, L)$$
$$\longrightarrow \mathrm{R}\,\Gamma_{W\setminus Z}(U \setminus Z, M) \otimes^{\mathbb{L}}_{\mathbb{Z}_\ell} \mathrm{R}\,\Gamma_Y(U, L).$$

The maps in (A.16) yield a map from this triangle to the sequence

$$(\mathrm{B.3}) \qquad \mathrm{R}\,\Gamma_{Z_Y}(U, M \otimes^{\mathbb{L}}_{\mathbb{Z}_\ell} L) \longrightarrow \mathrm{R}\,\Gamma_{W_Y}(U, M \otimes^{\mathbb{L}}_{\mathbb{Z}_\ell} L) \longrightarrow \mathrm{R}\,\Gamma_{W_Y\setminus Z_Y}(U \setminus Z, M \otimes^{\mathbb{L}}_{\mathbb{Z}_\ell} L),$$

such that the corresponding diagram commutes (as it commutes before taking derived functors). Note that the sequence in (B.3) is a priori not an exact triangle. However, the natural restriction maps yield a map from the exact triangle

$$\mathrm{R}\,\Gamma_{Z_Y}(X, M \otimes^{\mathbb{L}}_{\mathbb{Z}_\ell} L) \longrightarrow \mathrm{R}\,\Gamma_{W_Y}(X, M \otimes^{\mathbb{L}}_{\mathbb{Z}_\ell} L) \longrightarrow \mathrm{R}\,\Gamma_{W_Y\setminus Z_Y}(X \setminus Z_Y, M \otimes^{\mathbb{L}}_{\mathbb{Z}_\ell} L)$$

to the sequence in (B.3) such that the corresponding diagram commutes. The lemma follows from this by setting $M := \widehat{\mathbb{Z}}_\ell(m)$ and $L := (\pi_X)^*_{comp} K \otimes_{\widehat{\mathbb{Z}}_\ell} \widehat{\mathbb{Z}}_\ell(n)$ and taking cohomology, where we note that the restriction map from the above triangle to (B.3) yields isomorphisms in cohomology by Lemma A.10, because $W_Y \subset W_U$ and $Z_Y \subset Z_U$. (This shows in fact that (B.3) is isomorphic to an exact triangle, hence is an exact triangle itself.) $\qquad\square$

Proof of Proposition B.1. Let $[\alpha] \in H^{2i-1}_{i-2,nr}(X, \mathbb{Z}_\ell(i))$ and let $[\Gamma] \in \mathrm{CH}^c(X \times Y)$. Applying either Theorem 2.1 to Γ or item (2) from Corollary 6.5 to $[\alpha]$ (cf. proof of Corollary 6.8), we can assume that there is a representative $\alpha \in H^{2i-1}(U, \mathbb{Z}_\ell(i))$ for some open subset $U \subset X$ whose complement $R = X \setminus U$ is pure-dimensional of codimension i and such that $R \times Y$ meets $W := \mathrm{supp}\,\Gamma$ properly. Let $S := q((R \times Y) \cap W)$ and $V = Y \setminus S$. Then the class

$$\Gamma(W)_*(\alpha) \in H^{2(i+c-d_X)-1}(V, \mathbb{Z}_\ell(i+c-d_X))$$

from Lemma 4.1 represents $[\Gamma]_*[\alpha] \in H^{2(i+c-d_X)-1}_{i+c-d_X-2,nr}(Y, \mathbb{Z}_\ell(i+c-d_X))$. This yields via (B.2) an (ℓ-adic) cycle on Y and we aim to show that this cycle is $[\Gamma]_*[z]$, where $[z] \in A^i_0(X)_{\mathbb{Z}_\ell}$ is the class represented by $[\alpha]$ via (B.2).

To begin with, we aim to describe the cycle z on X explicitly. To this end, let

$$\partial \alpha \in H^{2i}_R(X, \mathbb{Z}_\ell(i)),$$

where ∂ denotes the residue map from Lemma A.13. Consider further the natural map

$$H^{2i}_R(X, \mathbb{Z}_\ell(i)) \longrightarrow \bigoplus_{x \in R^{(0)}} H^0(x, \mathbb{Z}_\ell) = \bigoplus_{x \in R^{(0)}} [x]\mathbb{Z}_\ell,$$

given by pullback and purity (see Lemma A.5), where we note that R is generically smooth (as k is perfect). The image of $\partial \alpha$ via the above map is a cycle $z \in Z^i(X)_{\mathbb{Z}_\ell}$ with coefficients in $\mathbb{Z}_\ell$ whose support is given by some components of R. In fact, Lemmas A.5 and A.15 imply that the above map is an isomorphism and we get

$$(\mathrm{B.4}) \qquad \mathrm{cl}^X_R(z) = \partial \alpha \in H^{2i}_R(X, \mathbb{Z}_\ell(i)),$$

which defines z uniquely. It follows directly from the construction of the map in (B.2) (see [Sch21b, Proposition 6.9]) that the class $[z] \in A^i_0(X)_{\mathbb{Z}_\ell}$ is via (B.2) represented by

$$[\alpha] \in H^{2i-1}_{i-2,nr}(X, \mathbb{Z}_\ell(i))/H^{2i-1}X, \mathbb{Z}_\ell(i)).$$

Similarly, the unramified class $[\Gamma]_*[\alpha] \in H^{2(i+c-d_X)-1}_{i+c-d_X-2,nr}(Y, \mathbb{Z}_\ell(i + c - d_X))$ corresponds to the cycle z' with $\operatorname{supp} z' \subset S$ on Y, which is uniquely determined by

$$(B.5) \qquad \operatorname{cl}^Y_S(z') = \partial(\Gamma(W)_*(\alpha)) \in H^{2(i+c-d_X)}_S(Y, \mathbb{Z}_\ell(i + c - d_X)).$$

By the construction of $\Gamma(W)_*(\alpha)$, we have

$$\Gamma(W)_*(\alpha) = q_*(\operatorname{exc}(\operatorname{cl}^{X \times Y}_W(\Gamma) \cup p^*\alpha)),$$

where $\operatorname{exc} : H^{2i-1+2c}_W(U \times Y, \mathbb{Z}_\ell(i + c)) \xrightarrow{\simeq} H^{2i-1+2c}_W((X \times Y) \setminus W_R, \mathbb{Z}_\ell(i + c))$ with $W_R := W \cap R \times Y$ is the isomorphism given by excision. By item (2) in Lemma A.13, we find

$$\partial(\Gamma(W)_*(\alpha)) = q_* \left(\partial \left(\operatorname{exc} \left(\operatorname{cl}^{X \times Y}_W(\Gamma) \cup p^*\alpha \right) \right) \right).$$

Since $\operatorname{cl}^{X \times Y}_W(\Gamma) \in H^{2c}_W(X \times Y, \mathbb{Z}_\ell(c))$, Lemma B.2 implies that

$$\partial(\operatorname{exc}(\operatorname{cl}^{X \times Y}_W(\Gamma) \cup p^*\alpha)) = \operatorname{cl}^{X \times Y}_W(\Gamma) \cup \partial(p^*\alpha) \in H^{2(i+c)}_{W_R}(X \times Y, \mathbb{Z}_\ell(i + c)),$$

where $W_R = W \cap (R \times Y)$. By item (1) in Lemma A.13,

$$\partial(p^*\alpha) = p^*(\partial\alpha) = p^* \operatorname{cl}^X_R(z) \in H^{2i}_{R \times Y}(X \times Y, \mathbb{Z}_\ell(i)),$$

where $\operatorname{cl}^X_R(z) = \partial\alpha$ is from (B.4). By Lemma A.21, $p^* \operatorname{cl}^X_R(z) = \operatorname{cl}^{X \times Y}_{R \times Y}(p^*z)$ and so

$$\partial(\operatorname{cl}^{X \times Y}_W(\Gamma) \cup p^*\alpha) = \operatorname{cl}^{X \times Y}_W(\Gamma) \cup \operatorname{cl}^{X \times Y}_{R \times Y}(p^* z) \in H^{2(i+c)}_{W_R}(X \times Y, \mathbb{Z}_\ell(i + c)).$$

By Lemma A.22,

$$\operatorname{cl}^{X \times Y}_W(\Gamma) \cup \operatorname{cl}^X_{R \times Y}(p^*z) = \operatorname{cl}^{X \times Y}_{W_R}(\Gamma \cdot p^*z) \in H^{2(i+c)}_{W_R}(X \times Y, \mathbb{Z}_\ell(i + c)).$$

Since W and $R \times Y$ meet properly by assumption, W_R has codimension $c + i$ and the natural restriction map together with purity (Lemma A.5) yields a map

$$H^{2(i+c)}_{W_R}(X \times Y, \mathbb{Z}_\ell(i + c)) \longrightarrow \bigoplus_{x \in W^{(0)}_R} H^0(x, \mathbb{Z}_\ell) = \bigoplus_{x \in W^{(0)}_R} [x] \cdot \mathbb{Z}_\ell$$

which, by Lemma A.15, is an isomorphism as before. The image of $\operatorname{cl}^{X \times Y}_{W_R}(\Gamma \cdot p^* z)$ via this map is given by the cycle $\Gamma \cdot p^*z$, where multiplicities are computed via Serre's Tor formula, see [Ful98, §20.4]. The cycle

$$z' \in \bigoplus_{x \in S_{(d_Y + d_X - i - c)}} [x] \cdot \mathbb{Z}_\ell$$

from (B.5) is thus given by the pushforward of $\Gamma \cdot p^*z$ via q: $z' = q_*(\Gamma \cdot p^*z)$, which proves the proposition. $\qquad\square$

B.2. Transcendental Abel–Jacobi maps are motivic.

Let X be a smooth equi-dimensional scheme over a perfect field k and let ℓ be a prime invertible in k. We let $H^i(X, \mathbb{Q}_\ell(n)) := H^i(X_{\text{proét}}, \widehat{\mathbb{Z}}_\ell(n)) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ and $H^i(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n)) = \operatorname{colim}_r H^i(X_{\text{proét}}, \nu^*\mu^{\otimes n}_{\ell^r})$, where we note that $H^i(X_{\text{proét}}, \nu^*\mu^{\otimes n}_{\ell^r}) \simeq H^i(X_{\text{ét}}, \mu^{\otimes n}_{\ell^r})$, cf. Lemma A.1.

Recall the ℓ-adic cycle group $A^i_0(X)_{\mathbb{Z}_\ell}$ from (B.1). We denote by $A^i_0(X)[\ell^\infty]$ the torsion subgroup of $A^i_0(X)_{\mathbb{Z}_\ell}$. By [Sch21b, §7.5], there is a transcendental Abel–Jacobi map on torsion cycles

$$\lambda^i_{tr} : A^i_0(X)[\ell^\infty] \longrightarrow H^{2i-1}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(i))/N^{i-1}H^{2i-1}(X, \mathbb{Q}_\ell(i)),$$

where N^* denotes Grothendieck's coniveau filtration on cohomology, i.e. $\alpha \in H^{2i-1}(X, \mathbb{Q}_\ell(i))$ lies in $N^j H^{2i-1}(X, \mathbb{Q}_\ell(i))$ if α vanishes on the complement of a closed codimension j set of X. Correspondences between smooth projective equi-dimensional k-schemes act on both sides of the above Abel–Jacobi map and the main result in this section is that these actions are compatible with the map λ^i_{tr}.

Corollary B.3. *Let X and Y be smooth projective equi-dimensional schemes over a perfect field k, let $d_X := \dim(X)$ and let ℓ be a prime that is invertible in k. Let $[\Gamma] \in \mathrm{CH}^c(X \times Y)$ be a correspondence. Then the following diagram is commutative:*

$$
\begin{array}{ccc}
A_0^i(X)[\ell^\infty] & \xrightarrow{\ \lambda_{tr}^i\ } & \dfrac{H^{2i-1}(X,\mathbb{Q}_\ell/\mathbb{Z}_\ell(i))}{N^{i-1}H^{2i-1}(X,\mathbb{Q}_\ell(i))} \\[4mm]
\Big\downarrow{\scriptstyle [\Gamma]_*} & & \Big\downarrow{\scriptstyle [\Gamma]_*} \\[4mm]
A_0^{i+c-d_X}(Y)[\ell^\infty] & \xrightarrow{\ \lambda_{tr}^{i+c-d_X}\ } & \dfrac{H^{2i+2c-2d_X-1}(Y,\mathbb{Q}_\ell/\mathbb{Z}_\ell(i+c-d_X))}{N^{i+c-d_X-1}H^{2i+2c-2d_X-1}(Y,\mathbb{Q}_\ell(i+c-d_X))}.
\end{array}
$$

Proof. By [Sch21b, Lemma 7.15], the map λ_{tr}^i can be described as follows. Let $[z] \in A_0^i(X)[\ell^\infty]$ and let $[\alpha] \in H_{i-2,nr}^{2i-1}(X,\mathbb{Z}_\ell(i))$ be a class that represents $[z]$ via the isomorphism in (B.2). This means that whenever $[\alpha]$ can be represented by a class $\alpha \in H^i(U,\mathbb{Z}_\ell(i))$ on some open subset $U \subset X$ whose complement $R = X \setminus U$ has codimension $2i$, then the residue

$$
\partial \alpha \in H_R^{2i}(X,\mathbb{Z}_\ell(i)) \simeq \bigoplus_{x \in R_{(d_X-i)}} \mathbb{Z}_\ell[x]
$$

is a cycle z that represents the class $[z]$. Since $[z]$ is torsion, there is some $r \geq 1$ such that $\ell^r[z] = 0 \in A^i(X)_{\mathbb{Z}_\ell}$. This implies that $\ell^r[\alpha]$ lifts to a class $[\beta] \in H^{2i-1}(X,\mathbb{Z}_\ell(i))$. Then, $[\beta/\ell^r]$ gives rise to a class in $H^{2i-1}(X,\mathbb{Q}_\ell(i))$ and hence in $H^{2i-1}(X,\mathbb{Q}_\ell/\mathbb{Z}_\ell(i))$ and we have

$$
\lambda_{tr}^i([z]) = [\beta/\ell^r] \in H^{2i-1}(X,\mathbb{Q}_\ell/\mathbb{Z}_\ell(i))/N^{i-1}H^{2i-1}(X,\mathbb{Q}_\ell(i)).
$$

By Proposition B.1, the cycle $[\Gamma]_*[z]$ corresponds via (B.2) to the class

$$
[\Gamma]_*[\alpha] \in H_{i+c-d_X-2,nr}^{2i+2c+2d_X-1}(X,\mathbb{Z}_\ell(i+c+d_X)).
$$

Applying the moving lemma (Theorem 2.1) to Γ or item (2) from Corollary 6.5 to $[\alpha]$, we can assume that there is a representative $\alpha \in H^i(U,\mathbb{Z}_\ell(i))$ of $[\alpha]$ as above such that $R \times Y$ meets $W := \operatorname{supp}\Gamma$ in codimension at least $i+c$. We then let $S = q(W \cap (R \times Y))$ and $V = Y \setminus S$ and find by item (1) in Corollary 6.8 that $[\Gamma]_*[\alpha]$ is represented by

$$
\Gamma(W)_*(\alpha) \in H^{2i+2c-2d_X-1}(V,\mathbb{Z}_\ell(i+c-d_X))
$$

from Lemma 4.1. By item (2) in Corollary 6.8 we find that $\ell^r \cdot \Gamma(W)_*(\alpha) = \Gamma(W)_*(\ell^r \cdot \alpha)$ extends to the class $\Gamma(W)_*(\beta) \in H^{2i+2c-2d_X-1}(Y,\mathbb{Z}_\ell(i+c-d_X))$. Hence, by the description of λ_{tr}^i given above, we find that

$$
\lambda_{tr}^i([\Gamma]_*[z]) = [\Gamma(W)_*\beta/\ell^r] \in \frac{H^{2i+2c-2d_X-1}(Y,\mathbb{Q}_\ell/\mathbb{Z}_\ell(i+c-d_X))}{N^{i+c-d_X-1}H^{2i+2c-2d_X-1}(X,\mathbb{Q}_\ell(i+c-d_X))}.
$$

The class $\Gamma(W)_*(\beta/\ell^r) \in H^{2i+2c-2d_X-1}(Y,\mathbb{Q}_\ell/\mathbb{Z}_\ell(i+c-d_X))$ from Lemma 4.1 agrees by construction with $[\Gamma]_*(\beta/\ell^r)$, i.e. with the image of β/ℓ^r via the action of the correspondence Γ. This shows that the diagram in question is commutative, which concludes the proof of the corollary. $\qquad\square$

Acknowledgements

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