ALGEBRAIC CYCLES AND REFINED UNRAMIFIED COHOMOLOGY

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Abstract. We introduce refined unramified cohomology groups. This notion allows us to give in arbitrary degree a cohomological interpretation of the failure of integral Hodge- or Tate-type conjectures, of ℓ-adic Griffiths groups, and of the subgroup of the Griffiths group that consists of torsion classes with trivial transcendental Abel–Jacobi invariant. Our approach simplifies and generalizes to cycles of arbitrary codimension previous results of Bloch–Ogus, Colliot-Thélène–Voisin, Voisin, and Ma that concerned cycles of low (co-)dimension. As an application, we give for any $i > 2$ the first example of a uniruled smooth complex projective variety for which the integral Hodge conjecture fails for codimension i-cycles in a way that cannot be explained by the failure on any lower-dimensional variety.

1. Introduction

Let X be a smooth complex projective variety. Then there is a cycle class map

$$\mathrm{cl}^i : \mathrm{CH}^i(X) \longrightarrow H^{i,i}(X, \mathbb{Z}),$$

where $H^{i,i}(X, \mathbb{Z}) \subset H^{2i}(X, \mathbb{Z})$ denotes the subspace of all classes that map to a class of Hodge type (i, i) in $H^{2i}(X, \mathbb{C})$. Describing the image of cl^i is equivalent to describing

$$Z^{2i}(X) := H^{i,i}(X, \mathbb{Z})/ \operatorname{im}(\mathrm{cl}^i).$$

The Lefschetz (1,1)-theorem says that $Z^2(X) = 0$ and the Hodge conjecture predicts that $Z^{2i}(X)$ is torsion for all i. For $i \geq 2$, it is known since the work of Atiyah–Hirzebruch [AH62] that the above cycle class map may fail to be surjective in general. Since then it has been a challenging problem to detect and explain this failure. Roughly speaking, the following techniques are known up till now:

- topological obstructions going back to [AH62] and extended by Totaro [To97];
- degeneration arguments going back to Kollár [BCC92], cf. [SV05, BeOt20a, Sh19].

In addition, using deep facts from K-theory, namely Bloch–Ogus theory [BO74] and the Bloch–Kato conjectures [Voe11], Colliot-Thélène and Voisin [CTV12] found a relation to

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unramified cohomology when $i = 2$:

$$Z^4(X)_{\mathrm{tors}} \simeq H^3_{nr}(X, \mathbb{Q}/\mathbb{Z})/H^3_{nr}(X, \mathbb{Q}). \tag{1.1}$$

Here $H^i_{nr}(X, A) := H^0(X, \mathcal{H}^i_X(A))$, where $\mathcal{H}^i_X(A)$ denotes the Zariski sheaf associated to $U \longmapsto H^i(U, A)$. Moreover, for abelian groups H and G we use the notation $G/H := \mathrm{coker}(H \to G)$ whenever there is no reason to confuse the map $H \to G$.

Another relation between unramified cohomology and algebraic cycles had earlier been given by Bloch–Ogus [BO74], who showed

$$\mathrm{Griff}^2(X) \simeq H^3_{nr}(X, \mathbb{Z})/H^3(X, \mathbb{Z}), \tag{1.2}$$

where $\mathrm{Griff}^i(X)$ denotes the Griffiths group of homologically trivial codimension i-cycles modulo algebraic equivalence.

In [Voi12] and [Ma17], Voisin and Ma found a geometric interpretation of unramified cohomology in degree four, by proving that there is an exact sequence

$$0 \longrightarrow \left(\frac{H^5(X, \mathbb{Z})}{N^2 H^5(X, \mathbb{Z})} \right)_{\mathrm{tors}} \longrightarrow \frac{H^4_{nr}(X, \mathbb{Q}/\mathbb{Z})}{H^4_{nr}(X, \mathbb{Q})} \longrightarrow \mathcal{T}^3(X) \longrightarrow 0, \tag{1.3}$$

where $\mathcal{T}^i(X) \subset \mathrm{Griff}^i(X)_{\mathrm{tors}}$ denotes the subgroup of torsion classes in the Griffiths group with trivial transcendental Abel–Jacobi invariant.

The cohomological interpretations in (1.1), (1.2) and (1.3) are very useful and important results, with many applications, see e.g. [MS83, CTV12, Pir12, Sch19, CT19, Dia20]. Nonetheless, it may be frustrating that they do not generalize to cycles of larger codimension, simply because the unramified cohomology groups of X are birational invariants, while $Z^{2i}(X)$, $\mathrm{Griff}^i(X)$, and $\mathcal{T}^{i+1}(X)$ are not birational invariants for $i > 2$.

1.1. **Refined unramified cohomology.** This paper introduces *refined unramified cohomology groups*, which generalize the classical urnamified cohomology groups. These groups are in general not birational invariants anylonger, but we will see that they are the natural playground when working with algebraic cycles.

To explain our definition, note that any variety X over $\mathbb{C}$ admits an increasing filtration

$$F_0X \subset F_1X \subset \cdots \subset F_{\dim X}X = X, \quad \text{where} \quad F_jX := \{x \in X \mid \mathrm{codim}_X(x) \leq j\}.$$

For instance, F_0X is the generic point of X, F_1X is the union of the generic point with all codimension one points of X and so on. Each F_jX is an inverse limit of schemes, which is however in general not a scheme itself. Nonetheless, for any abelian group A, we may define its cohomology

$$H^i(F_jX, A) := \varinjlim_{F_jX \subset U \subset X} H^i(U, A),$$

where $U \subset X$ runs through all Zariski open subsets containing F_jX and where $H^i(U, A) := H^i_{sing}(U(\mathbb{C}), A)$. We then define the j-th refined unramified cohomology groups of X by

$$H^i_{j,nr}(X, A) := \operatorname{im}(H^i(F_{j+1}X, A) \to H^i(F_jX, A)).$$

That is, the j-th refined unramified cohomology of X consists of all classes on F_jX that extend to all points of codimension $j + 1$.

Using [BO74], our first result is as follows; we formulate it here for smooth projective varieties over $\mathbb{C}$, but in the body of the paper, versions of all our results will be proven for any smooth (possibly non-proper) variety over an arbitrary algebraically closed field.

Theorem 1.1. *Let X be a smooth complex projective variety. Then for any i and any abelian group A, there is a canonical long exact sequence*

$$\dots H^{i+2j-1}_{j-1,nr}(X, A) \to H^{i+2j-1}_{j-2,nr}(X, A) \to H^j(X, \mathcal{H}^{i+j}_X(A)) \to H^{i+2j}_{j,nr}(X, A) \to H^{i+2j}_{j-1,nr}(X, A) \dots .$$

The above theorem shows that up to some extension data, refined unramified cohomology determines all Zariski cohomology groups $H^p(X, \mathcal{H}^q_X(A))$. The latter are sometimes called '*higher unramified cohomology groups*'. As a special case, we get:

Corollary 1.2. *Let X be a smooth complex projective variety of dimension d. For any $i \geq 0$, there are canonical isomorphisms*

$$H^i_{0,nr}(X, A) \simeq H^i_{nr}(X, A) \quad \textit{and} \quad H^{d+i}_{i,nr}(X, A) \simeq H^i(X, \mathcal{H}^d_X(A)).$$

Both groups that appear in the above corollary are birational invariants, cf. [CTV12].

For $m \geq j$, there are natural restriction maps $H^i_{m,nr}(X, A) \to H^i_{j,nr}(X, A)$. This allows us to define a decreasing filtration F^* on $H^i_{j,nr}(X, A)$ by

$$F^m H^i_{j,nr}(X, A) := \begin{cases} H^i_{j,nr}(X, A), & \text{if } m \leq j; \\ \operatorname{im}(H^i(F_mX, A) \to H^i_{j,nr}(X, A)), & \text{if } m > j. \end{cases}$$

For $A = \mathbb{Q}/\mathbb{Z}$, we further define the decreasing filtration G^* on $H^i_{j,nr}(X, \mathbb{Q}/\mathbb{Z})$ by

$$\alpha \in G^m H^i_{j,nr}(X, \mathbb{Q}/\mathbb{Z}) \iff \delta(\alpha) \in F^m H^{i+1}_{j,nr}(X, \mathbb{Z}),$$

where δ denotes the Bockstein map induced by $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$.

The main result of this paper is as follows, cf. Theorem 12.2 for a more general version.

Theorem 1.3. *Let X be a smooth complex projective variety. Then for any i there are canonical isomorphisms*

$$Z^{2i}(X)_{\text{tors}} \simeq \frac{H^{2i-1}_{i-2,nr}(X, \mathbb{Q}/\mathbb{Z}(i))}{H^{2i-1}_{i-2,nr}(X, \mathbb{Q}(i))}, \quad \operatorname{Griff}^i(X) \simeq \frac{H^{2i-1}_{i-2,nr}(X, \mathbb{Z}(i))}{H^{2i-1}(X, \mathbb{Z}(i))}, \quad \textit{and}$$

$$\mathcal{T}^i(X) \simeq \frac{H^{2i-2}_{i-3,nr}(X, \mathbb{Q}/\mathbb{Z}(i))}{G^i H^{2i-2}_{i-3,nr}(X, \mathbb{Q}/\mathbb{Z}(i))}.$$

The Tate twists in the above theorem may be ignored, but they are needed to get canonical isomorphisms.

Since $H^i_{0,nr}(X,A) \simeq H^i_{nr}(X,A)$, the above theorem generalizes (1.1), (1.2) and (1.3) to cycles of arbitrary codimension. Surprisingly, our proofs are simpler than the versions for cycles of low codimensions in [BO74, CTV12, Voi12, Ma17]. In particular, Theorem 1.3 is not using Bloch–Ogus theory [BO74], nor the Bloch–Kato conjectures beyond Merkurjev–Suslin's theorem [MS83] (which is only used in the description of $\mathcal{T}^i(X)$).

For any smooth complex projective variety X of dimension d, a description of $Z^{2d-2}(X)_{\mathrm{tors}}$ and $\mathcal{T}^{d-1}(X)$ in terms of $H^{d-3}(X,\mathcal{H}^d_X)$ and $H^{d-4}(X,\mathcal{H}^d_X)$ is given in[CTV12] and [Ma17], respectively. By Corollary 1.2, Theorem 1.3 contains these results as a special case.

Since $H^i_{0,nr}(X,A) \simeq H^i_{nr}(X,A)$, the above decreasing filtrations F^* and G^* on refined unramified cohomology yield in particular filtrations on classical unramified cohomology groups. In terms of these filtrations, Theorem 1.3 implies the following.

Corollary 1.4. *Let X be a smooth complex projective variety. Then there are canonical surjections*

$$Z^{2i}(X)_{\mathrm{tors}} \twoheadrightarrow \frac{F^{i-1}H^{2i-1}_{nr}(X,\mathbb{Q}/\mathbb{Z}(i))}{F^{i-1}H^{2i-1}_{nr}(X,\mathbb{Q}(i))}, \tag{1.4}$$

$$\mathrm{Griff}^i(X) \twoheadrightarrow gr_F^{i-1}H^{2i-1}_{nr}(X,\mathbb{Z}(i)), \quad \textit{and} \tag{1.5}$$

$$\mathcal{T}^i(X) \twoheadrightarrow \frac{F^{i-2}H^{2i-2}_{nr}(X,\mathbb{Q}/\mathbb{Z}(i))}{G^iF^{i-2}H^{2i-2}_{nr}(X,\mathbb{Q}/\mathbb{Z}(i))}. \tag{1.6}$$

Moreover, for any resolution $\widetilde{D}$ of a subvariety $D \subset X$ of codimension $c \geq 1$, the images of $Z^{2i-2c}(\widetilde{D})_{\mathrm{tors}}$, $\mathrm{Griff}^{i-c}(\widetilde{D})$, resp. $\mathcal{T}^{i-c}(\widetilde{D})$, lie in the kernel of the above surjections.

The above corollary shows that in arbitrary codimension i, unramified cohomology may be used to detect nontriviality of classes in $Z^{2i}(X)_{\mathrm{tors}}$, $\mathrm{Griff}^i(X)$, and $\mathcal{T}^i(X)$. In fact, it allows to detect nontriviality in the strong sense that the class does not come from any resolution of a (union of) proper subvarieties of X.

Item (1.6) in Corollary 1.4 generalizes a very recent result of Ma [Ma20],[1] who introduced a filtration on $H^i_{nr}(X,A)$ in the case where the Chow group of X is supported in dimension $< i$ and showed that two graded quotients of his filtration are subquotients of $\mathrm{Tors}\left(\mathrm{Griff}^{i/2+1}(X)\right)$ (if i is even) and $\mathrm{Tors}\left(\frac{H^i(X,\mathbb{Z})}{N^2H^i(X,\mathbb{Z})}\right)$, respectively. Note however that our result is more precise even in the special case where the Chow group of X is supported in dimension $< i$ where Ma's results apply.

[1]The preprint [Ma20] appeared on the arXiv when the results of this paper had already been obtained; our preprint appeared shortly afterwards.

1.2. **Applications to the integral Hodge conjecture for uniruled varieties.** Voisin [Voi06] proved that the integral Hodge conjecture holds for uniruled threefolds X (i.e. $Z^2(X)=0$) and conjectured that it should fail for codimension two cycles on rationally connected varieties of dimension at least four. This has later been proven in [CTV12] ($\dim X \geq 6$) and in full generality in [Sch19].

By taking products with $\mathbb{P}^n$, the examples in [CTV12, Sch19] yield also counterexamples to the integral Hodge conjecture on unirational varieties for cycles of codimension greater than two. However, from a philosophical point of view, these non-algebraic Hodge classes should still be seen as degree four classes, because they are Gysin pushforwards of non-algebraic degree four Hodge classes on a subvariety of $X\times\mathbb{P}^n$ (namely $X\times\{pt.\}$).

The tools of this paper allow us to go further by studying the integral Hodge conjecture for Hodge classes (of arbitrary degree) in the following strong sense.

Theorem 1.5. *For any $n\geq 1$, there is a smooth complex projective unirational variety Y of dimension $3n$ and an elliptic curve E such that $X:=E\times Y$ satisfies*

$$\operatorname{coker}\left(\bigoplus_{c\geq 1}\bigoplus_{\tilde{D}} Z^{2i-2c}(\tilde{D})_{\mathrm{tors}}\longrightarrow Z^{2i}(X)_{\mathrm{tors}}\right)\neq 0 \quad \textit{for all } 2\leq i\leq n+1, \tag{1.7}$$

where $\tilde{D}$ runs through all resolutions of all subvarieties $D\subset X$ of codimension $c\geq 1$.

The first examples where (a strengthening of) (1.5) holds for some given integer $i\geq 2$ were recently obtained by Benoist and Ottem [BeOt20b, Theorem 1.1], who used the topological obstruction from [AH62]. Their examples are however of non-negative Kodaira dimension and the method does not seem to apply to uniruled varieties.

1.3. **Applications to the Artin–Mumford invariant.** In [AM72], Artin and Mumford showed that for any smooth complex projective variety X, the torsion subgroup of $H^3(X,\mathbb{Z})$ is a birational invariant and used this to construct unirational threefolds that are not rational. For $i>3$, the torsion subgroup of $H^i(X,\mathbb{Z})$ is not a birational invariant. However, Voisin observed (see [Voi12, Remark 2.4]) that the Bloch–Kato conjecture proven by Voevodsky implies that the torsion subgroup of $H^{2i+1}(X,\mathbb{Z})/N^2H^{2i+1}(X,\mathbb{Z})$ is a birational invariant for all i. Our approach allows to identify this invariant with an unramified cohomology group as follows.

Theorem 1.6. *For any smooth complex projective variety X, there is a canonical isomorphism*

$$\operatorname{Tors}\left(\frac{H^i(X,\mathbb{Z})}{N^2H^i(X,\mathbb{Z})}\right)\simeq\frac{H^{i-1}_{nr}(X,\mathbb{Q}/\mathbb{Z})}{G^{\lceil i/2\rceil}H^{i-1}_{nr}(X,\mathbb{Q})}.$$

As an application, we prove that Voisin's generalization of the Artin–Mumford invariant is nontrivial for all $i\geq 1$.

Corollary 1.7. *For any positive integer i, there is a unirational smooth complex projective variety X with a torsion class in $H^{2i+1}(X,\mathbb{Z})$ that is non-zero in the quotient*

$$H^{2i+1}(X,\mathbb{Z})/N^2H^{2i+1}(X,\mathbb{Z}).$$

1.4. **Bloch's Abel–Jacobi map revisited.** Let X be a smooth projective variety over an algebraically closed field k and let ℓ be a prime invertible in k. In [Blo79], Bloch constructed a map

$$\lambda : \mathrm{CH}^i(X)[\ell^\infty] \longrightarrow H^{2i-1}_{\text{ét}}(X,\mathbb{Q}_\ell/\mathbb{Z}_\ell(i))$$

which in the case $k=\mathbb{C}$ coincides with the Abel–Jacobi map when restricted to homologically trivial torsion cycles. Bloch remarks in the introduction of his paper that his construction is '*surprisingly deep*' as he is forced to use the full strength of [BO74] together with the Weil conjectures proven by Deligne [Del74]. The approach of this paper allows to construct Bloch's map without Bloch–Ogus theory [BO74], but still relying on Deligne's proof of the Weil conjectures, see Theorem 11.2 below.

The map λ above induces a map

$$\lambda_{tr} : \mathrm{Griff}^i(X)[\ell^\infty] \longrightarrow H^{2i-1}_{\text{ét}}(X,\mathbb{Q}_\ell/\mathbb{Z}_\ell(i))/N^{i-1}H^{2i-1}_{\text{ét}}(X,\mathbb{Q}_\ell(i)).$$

If $k=\mathbb{C}$ then this map coincides with the transcendental Abel–Jacobi map. In contrast to the construction of λ, where the use of the Weil conjectures (as replacement of Hodge theory in the classical case) seems indispensable, we show in Section 9.1 and Proposition 11.5 that λ_{tr} can be defined using only some basic facts from étale cohomology: the Gysin sequence, the Bockstein sequence and Hilbert's theorem 90 for function fields. This observation seems to be new even in the case $k=\mathbb{C}$. In fact, given that our construction uses only such basic ingredients, it works in the case where X is any smooth (not necessarily proper) variety over k, which explains the remark alluded to above that our results generalize to the non-proper setting.

1.5. **Further applications.** The theory of refined unramified cohomology introduced in this paper yields a new tool to detect nontrivial classes in the Griffiths group, and in particular torsion classes that lie in the kernel of Griffiths' Abel–Jacobi map. This application appears in [Sch20b], where it is used to show for the first time that the torsion subgroup of Griffiths groups may be infinite.

1.6. **A spectral sequence that computes unramified cohomology.** In the appendix to this paper, we construct a spectral sequence that computes classical unramified cohomology of any smooth variety over an algebraically closed field k, see Theorem A.1. We will show that this spectral sequence essentially computes the filtration F^* from Corollary 1.4. Using Bloch–Ogus theory [BO74], we compute the E_2-page of the spectral sequence. Even though we will not use this spectral sequence in the body of the paper, it does give another view on the results in Corollary 1.4 and it has been important for us to find the correct generalization of unramified cohomology used in Theorems 1.1 and 1.3 above. We include the result in the appendix, as we believe that it might be of independent interest.

2. Notation

All schemes are separated. An algebraic scheme is a scheme of finite type over a field. A variety is an integral scheme of finite type over a field. An open subset $U \subset X$ of an algebraic scheme is called big if $\operatorname{codim}_X(X \setminus U) \geq 2$. For an algebraic scheme X, we denote by $X^{(i)}$ the set of all codimension i points of X.

An alteration of a k-variety X is a regular k-variety X' together with a generically finite proper morphism $\tau : X' \to X$. Alterations exist by the work of de Jong [deJ96]. By a result of Gabber, the degree of τ may be assumed to be coprime to any given prime that is invertible in k, see [IT14, Theorem 2.1].

Whenever G and H are abelian groups so that there is a canonical map $H \to G$ (and there is no reason to confuse this map with a different map), we write G/H as a short hand for $\operatorname{coker}(H \to G)$.

For an abelian group G, we denote by $G[\ell^r]$ the subgroup of ℓ^r-torsion elements, and by $G[\ell^\infty] := \bigcup_r G[\ell^r]$ the subgroup of elements that are ℓ^r-torsion for some $r \geq 1$. We further write $\operatorname{Tors}(G)$ or G_{tors} for the torsion subgroup of G.

3. Cohomology functor

Fix a field k and a prime ℓ.[2] For any n and $r \geq 1$, let $\mu_{\ell^r}^{\otimes n}$ be a group isomorphic to $\mathbb{Z}/\ell^r$ and equal to $\mathbb{Z}/\ell^r$ for $n = 0$. We further fix once and for all surjections $\mu_{\ell^{r'}}^{\otimes n} \twoheadrightarrow \mu_{\ell^r}^{\otimes n}$ and injections $\mu_{\ell^r}^{\otimes n} \hookrightarrow \mu_{\ell^{r'}}^{\otimes n}$ whenever $r' > r$ and define $\mathbb{Z}_\ell(n) := \lim_{\leftarrow} \mu_{\ell^r}^{\otimes n}$, $\mathbb{Q}_\ell(n) := \mathbb{Z}_\ell(n) \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\mathbb{Q}_\ell/\mathbb{Z}_\ell(n) := \lim_{\to} \mu_{\ell^r}^{\otimes n}$.

For each integer i we fix functors

$$\{\text{smooth algebraic } k\text{-schemes}\} \longrightarrow \{\text{abelian groups}\}, \quad X \longmapsto H^i(X, \mu_{\ell^r}^{\otimes n})$$

that are contravariant with respect to open immersions of schemes. We assume that $H^0(X, \mu_{\ell^r}^{\otimes n}) = \mu_{\ell^r}^{\otimes n}$ and $H^i(-\mu_{\ell^r}^{\otimes n}) = 0$ for all $i < 0$. Moreover, for all $r' \geq r$, the given maps $\mu_{\ell^r}^{\otimes n} \to \mu_{\ell^{r'}}^{\otimes n}$ and $\mu_{\ell^{r'}}^{\otimes n} \to \mu_{\ell^r}^{\otimes n}$ on H^0 need to extend to a map of functors:

$$H^i(-, \mu_{\ell^r}^{\otimes n}) \longrightarrow H^i(-, \mu_{\ell^{r'}}^{\otimes n}) \quad \text{and} \quad H^i(-, \mu_{\ell^{r'}}^{\otimes n}) \longrightarrow H^i(-, \mu_{\ell^r}^{\otimes n}).$$

[2]In our applications, ℓ will be invertible in k and k will be algebraically closed, but this is not strictly necessary for our formalism.

We use these maps to define the contravariant functors

$$H^i(-,\mathbb{Z}_\ell(n)) := \varprojlim_r H^i(-,\mu_{\ell^r}^{\otimes n}), \quad H^i(-,\mathbb{Q}_\ell(n)) := H^i(-,\mathbb{Z}_\ell(n)) \otimes_{\mathbb{Z}} \mathbb{Q}$$

and

$$H^i(-,\mathbb{Q}_\ell/\mathbb{Z}_\ell(n)) := \varinjlim_r H^i(-,\mu_{\ell^r}^{\otimes n}).$$

For a finitely generated field extension K of k, and whenever

$$A(n) \in \{\mu_{\ell^r}^{\otimes n}, \mathbb{Z}_\ell(n), \mathbb{Q}_\ell(n), \mathbb{Q}_\ell/\mathbb{Z}_\ell(n)\}, \tag{3.1}$$

we define

$$H^i(K,A(n)) := \varinjlim_{\emptyset\neq U\subset X} H^i(U,A(n)),$$

where X is a smooth k-variety with $k(X)=K$ and U runs through all non-empty open subsets of X. We assume that the following basic properties are satisfied:

P1 For any smooth variety U over k and any smooth closed subscheme $Z\subset U$ of pure codimension c and with complement V, there is a Gysin exact sequence

$$\dots \longrightarrow H^i(U,A(n)) \longrightarrow H^i(V,A(n)) \xrightarrow{\partial} H^{i+1-2c}(Z,A(n-c)) \xrightarrow{\iota_*} H^{i+1}(U,A(n)) \longrightarrow \dots$$

where ι_* is called pushforward map and ∂ is called residue map. For any open subset $U'\subset U$ with $V' := V\cap U'$ and $Z' := U'\cap Z$, the Gysin sequences for (U,Z) and (U',Z') are compatible via the natural restriction maps.

P2 For any smooth k-variety X, there is a long exact contravariantly functorial Bockstein sequence

$$\cdots \longrightarrow H^i(X,\mathbb{Z}_\ell(n)) \xrightarrow{\times \ell^r} H^i(X,\mathbb{Z}_\ell(n)) \longrightarrow H^i(X,\mu_{\ell^r}^{\otimes n}) \xrightarrow{\delta} H^{i+1}(X,\mathbb{Z}_\ell(n)) \xrightarrow{\times \ell^r} \cdots$$

where $H^i(X,\mathbb{Z}_\ell(n)) \longrightarrow H^i(X,\mu_{\ell^r}^{\otimes n})$ is given by reduction modulo ℓ^r, δ is called the Bockstein map and where $\delta\circ\iota_* = \iota_*\circ\delta$ for the pushforward map ι_* from Property (**P1**).

P3 Hilbert 90 for function fields holds in the sense that for any finitely generated field extension K of k there is a canonical surjection

$$\epsilon : K^* \twoheadrightarrow H^1(K,\mu_{\ell^r}^{\otimes 1})$$

with kernel $(K^*)^{\ell^r}$ and compatible with the given maps $H^1(K,\mu_{\ell^{r'}}^{\otimes 1}) \to H^1(K,\mu_{\ell^r}^{\otimes 1})$ for all $r'\geq r$. For any geometric discrete valuation ν on K that is trivial on k and with residue field κ, the natural composition

$$K^* \overset{\epsilon}{\twoheadrightarrow} H^1(K,\mu_{\ell^r}^{\otimes 1}) \xrightarrow{\partial} H^0(\kappa,\mu_{\ell^r}^{\otimes 0}) = \mathbb{Z}/\ell^r,$$

where ∂ is induced by (**P1**), coincides with the reduction modulo ℓ^r of $-\nu$.

Funtoriality of the Gysin sequence implies in particular

$$H^i(X \sqcup Y, A(n)) \simeq H^i(X, A(n)) \oplus H^i(Y, A(n))$$

where the isomorphism is induced by the pullback maps. Note also that property (**P2**) yields via direct limits an exact sequence

$$\cdots \longrightarrow H^i(X, \mathbb{Z}_\ell(n)) \longrightarrow H^i(X, \mathbb{Q}_\ell(n)) \longrightarrow H^i(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n)) \xrightarrow{\delta} H^{i+1}(X, \mathbb{Z}_\ell(n)) \longrightarrow \cdots$$

In addition to the above basic properties, the following slightly technical conditions need to be satisfied, where $A(n)$ denotes always one of the coefficients in (3.1):

P4 For any proper morphism $f : X \to Y$ between smooth k-varieties of relative dimension $c = \dim Y - \dim X$, there are pushforward maps

$$f_* : H^i(X, A(n)) \longrightarrow H^{i+2c}(Y, A(n+c)),$$

compatible with pullbacks whenever we replace Y by an open subset U and X by $f^{-1}(U)$. If f is a closed immersion, then f_* coincides with the map in (**P1**).

P5 Let $\tau : U' \to U$ be a finite morphism between smooth k-varieties of relative dimension $p = \dim U - \dim U'$. Let $Z \subset U$ be a smooth closed subscheme of pure codimension c such that $Z' = \tau^{-1}(Z)$ is a smooth closed subscheme of pure codimension $c' = c - p$ in U'. Let $V' := U' \setminus Z'$ and $V := U \setminus Z$. Then the following diagram commutes:

$$\begin{tikzcd}
H^{i-2p}(V', A(n-p)) \arrow[r, "\partial"] \arrow[d, "(\tau|_{V'})_*"] & H^{i+1-2c}(Z', A(n-c)) \arrow[d, "(\tau|_{Z'})_*"] \\
H^i(V, A(n)) \arrow[r, "\partial"] & H^{i+1-2c}(Z, A(n-c))
\end{tikzcd}$$

where the vertical arrows are the pushforward maps from (**P4**) and the horizontal maps are the residue maps from (**P1**).

P6 Homological and algebraic equivalence coincide for divisors on any smooth variety X over k in the sense that the map[3]

$$\iota_* : \bigoplus_{x \in X^{(1)}} [x]\mathbb{Z}_\ell \longrightarrow H^2(X, \mathbb{Z}_\ell(1))$$

induced by (**P1**) has as kernel the group of algebraically trivial $\mathbb{Z}_\ell$-divisors on X.

Remark 3.1. *Large parts of this paper work without (**P4**)–(**P6**). These properties are however necessary to ensure that the definition of Chow groups modulo rational resp. algebraic equivalence in Section 6.1 below coincide with the usual definitions. (For the*

[3]The existence of this map uses that (**P1**) implies $H^2(U, \mathbb{Z}_\ell(1)) \simeq H^2(X, \mathbb{Z}_\ell(1))$ for any big open subset $U \subset X$.

Chow group modulo algebraic equivalence, we will need the additional assumption that the ground field k is perfect, see Lemma 6.2 below.)

Proposition 3.2. *Assume that k is algebraically closed, ℓ is invertible in k and the above cohomology functor is given by étale cohomology, where $\mu_{\ell^r}^{\otimes n}$ denotes the (n-th twist) of the usual étale sheaf μ_{ℓ^r} of ℓ^r-th roots of unity. Then (**P1**)–(**P6**) are satisfied.*

Proof. This is well-known, we include the details for convenience of the reader.

Since ℓ is invertible in k and k is algebraically closed, $H^i(X, \mu_{\ell^r}^{\otimes n})$ is a finite group for any smooth k-variety X and any i, r and n, see [Mil80, p. 244, VI.5.5]. On the other hand, if $0 \to (F_r) \to (G_r) \to (H_r) \to 0$ is a short exact sequence of inductive systems with F_r finite for all r, then $0 \to \lim_{\leftarrow} F_r \to \lim_{\leftarrow} G_r \to \lim_{\leftarrow} H_r \to 0$ is exact, see e.g. [Fu11, Lemma 10.1.3]. Since the functors $\otimes_{\mathbb{Z}}\mathbb{Q}$ and $\lim_{\to}$ are exact anyways, this shows that it suffices to prove (**P1**), (**P4**) and (**P5**) in the case where $A(n) = \mu_{\ell^r}^{\otimes n}$.

To prove (**P1**), let (V, Z) be a smooth pair of pure codimension c. For any étale sheaf F on V, there is the long exact sequence

$$\ldots \longrightarrow H^i(U, F|_U) \longrightarrow H^i(V, F) \longrightarrow H^{i+1}_Z(X, F) \longrightarrow H^{i+1}(U, F|_U) \longrightarrow \ldots,$$

where $H^{i+1}_Z(X, F)$ denotes étale cohomology with support on Z, see [Mil80, p. 92, III.1.25]. Since Z is smooth of pure codimension c in V, there are canonical isomorphisms

$$H^{i+1}_Z(X, \mu_{\ell^r}^{\otimes n}) \simeq H^{i+1-2c}(Z, \mu_{\ell^r}^{\otimes n-c}),$$

see [Mil80, p. 244, VI.5.4(b)] or [Mil13, Theorem 16.1]. This establishes the Gysin sequence in (**P1**) and it is clear from the construction that this sequence is contravariantly functorial for open immersions of schemes, cf. proof of [Mil13, Theorem 16.1].

To prove (**P2**), note that for any r and s there are short exact sequences

$$0 \longrightarrow \mu_{\ell^s}^{\otimes n} \xrightarrow{\times \ell^r} \mu_{\ell^{s+r}}^{\otimes n} \longrightarrow \mu_{\ell^r}^{\otimes n} \longrightarrow 0.$$

Taking the associated long exact sequence of étale cohomology groups, we get a long exact sequence

$$\ldots \longrightarrow H^i(X, \mu_{\ell^s}^{\otimes n}) \xrightarrow{\times \ell^r} H^i(X, \mu_{\ell^{s+r}}^{\otimes n}) \longrightarrow H^i(X, \mu_{\ell^r}^{\otimes n}) \xrightarrow{\delta'} H^{i+1}(X, \mu_{\ell^s}^{\otimes n}) \longrightarrow \ldots.$$

We can take the inverse limit along s of the above long exact sequences and the aforementioned exactness property of inverse limits (together with the fact that $H^i(X, \mu_{\ell^s}^{\otimes n})$ is finite for all s, see [Mil80, p. 244, VI.5.5]), shows that we obtain the Bockstein sequence as claimed in (**P2**). It is clear that this is contravariantly functorial for open immersions of schemes. It remains to show that $\iota_* \circ \delta = \delta \circ \iota_*$ whenever $Z \hookrightarrow X$ is a closed immersion of a smooth subscheme of pure codimension c and where ι_* denotes the map from (**P1**).

It suffices to prove this on any finite level of $H^i(-,\mathbb{Z}_\ell(n))$ and so it suffices to prove that the following diagram commutes:

$$\begin{tikzcd} H^i(Z,\mu_{\ell^r}^{\otimes n}) \arrow[r, "\delta'"] \arrow[d, "\iota_*"] & H^{i+1}(Z,\mu_{\ell^s}^{\otimes n}) \arrow[d, "\iota_*"] \\ H^{i+2c}(X,\mu_{\ell^r}^{\otimes n+c}) \arrow[r, "\delta'"] & H^{i+2c+1}(X,\mu_{\ell^s}^{\otimes n+c}) \end{tikzcd}$$

where δ' is the boundary map above. This follows from the fact that the diagram

$$\begin{tikzcd} H^{i+2c}_Z(X,\mu_{\ell^r}^{\otimes n}) \arrow[r, "\delta'"] \arrow[d] & H^{i+2c+1}_Z(X,\mu_{\ell^s}^{\otimes n}) \arrow[d] \\ H^{i+2c}(X,\mu_{\ell^r}^{\otimes n+c}) \arrow[r, "\delta'"] & H^{i+2c+1}(X,\mu_{\ell^s}^{\otimes n+c}) \end{tikzcd}$$

commutes, where the boundary map δ' on cohomology with support is constructed as above, together with the fact that the isomorphism $H^{i+2c}_Z(X,\mu_{\ell^r}^{\otimes n})\simeq H^i(Z,\mu_{\ell^r}^{\otimes n})$ is canonical (cf. [Mil80, p. 244, VI.5.4(b)]) so that the Bockstein maps induced on $H^{i+2c}_Z(X,\mu_{\ell^r}^{\otimes n})$ and $H^i(Z,\mu_{\ell^r}^{\otimes n})$ are compatible with each other. This proves (**P2**).

Property (**P3**) is a direct consequence of the Kummer sequence

$$0\longrightarrow\mu_{\ell^r}\longrightarrow\mathbb{G}_m\xrightarrow{\times\ell^r}\mathbb{G}_m\longrightarrow 0$$

and Hilbert's theorem 90, which asserts that $H^1_{\text{ét}}(K,\mathbb{G}_m)=0$. The claim about the residue map on $H^1(K,\mu_{\ell^r}^{\otimes 1})$ follows for instance from [GS06, Proposition 7.5.1] together with the fact that the residue maps in Galois cohomology is the negative of the map defined via the Gysin sequence in étale cohomology, see e.g. [CT95, §3.3].

To prove (**P4**), let X be a smooth k-variety of dimension d. By Poincaré duality, there is a canonical perfect pairing

$$H^i(X,\mu_{\ell^r}^{\otimes n})\times H^{2d-i}_c(X,\mu_{\ell^r}^{\otimes d-n})\longrightarrow H^{2d}_c(X,\mu_{\ell^r}^{\otimes d})\simeq\mathbb{Z}/\ell^r,$$

where H^*_c denotes compactly supported cohomology, see [Mil13, §24]. For $f:X\to Y$ proper, there are pullback maps

$$f^*:H^i_c(Y,\mu_{\ell^r}^{\otimes n})\longrightarrow H^i_c(X,\mu_{\ell^r}^{\otimes n})$$

on compactly supported cohomology, which by the above duality theorem induce pushforward maps

$$f_*:H^i(X,\mu_{\ell^r}^{\otimes n})\longrightarrow H^{i+2c}(Y,\mu_{\ell^r}^{\otimes n+c}).$$

Since the Poincaré duality theorem is functorial for restrictions via open immersions, it follows that the pushforward maps f_* defined above are compatible when replacing Y by an open subset U and X by $X\times_Y U$. Finally, the pushforward constructed above coincides in the case where f is a closed immersion with the one from (**P1**), see [Mil13, 24.2]. This proves (**P4**).

To prove (**P5**), let $d := \dim V$. Then $\dim V' = d - p$ and $\dim Z = \dim Z' = d - c$. The diagram in question is by Poincaré duality dual to the diagram

$$\begin{tikzcd} H_c^{2d-i}(V', A(-n+d)) & H_c^{2d-i-1}(Z', A(-n+d)) \arrow[l] \\ H_c^{2d-i}(V, A(-n+d)) \arrow[u, "(\tau|_{V'})^*"] & H_c^{2d-i-1}(Z, A(-n+d)) \arrow[l] \arrow[u, "(\tau|_{Z'})^*"'] \end{tikzcd} .$$

Here the vertical arrows are boundary maps in the long exact sequence associated to the short exact sequence

$$0 \longrightarrow j_! \mu_{\ell^r}^{\otimes n} \longrightarrow \mu_{\ell^r}^{\otimes n} \longrightarrow i_* \mu_{\ell^r}^{\otimes n} \longrightarrow 0$$

of constructible étale sheaves on U (resp. U'), where $i : Z \to U$ and $j : V \to U$ (resp. $i : Z' \to U'$ and $j : V' \to U'$) denote the inclusions, cf. [Mil80, p. 94, III.1.30]. The commutativity of the above diagram follows thus from the fact that cohomology with compact support is a delta functor, see [Mil80, p. 93, III.1.29(b)].

It remains to prove (**P6**). For this we denote by $\mathrm{NS}(X)$ the group of divisors modulo algebraic equivalence on X. The cycle class map then yields a map

$$c_1 : \mathrm{NS}(X) \otimes \mathbb{Z}_\ell \to H^2(X, \mathbb{Z}_\ell(1))$$

and we need to show that this map is injective. If X is proper, then this is well-known, see e.g. [Mil80, p. 216, V.3.28]. Otherwise, let $\overline{X}$ be a proper normal compactification of X. Let further $\tau : \overline{X}' \to \overline{X}$ be an alteration of degree prime to ℓ, which exists by [IT14, Theorem 2.1], and let $X' := \tau^{-1}(X)$. We get a commutative diagram

$$\begin{tikzcd} \mathrm{NS}(\overline{X}') \otimes \mathbb{Z}_\ell \arrow[r, "c_1"] \arrow[d, "\text{restr.}"] & H^2(\overline{X}', \mathbb{Z}_\ell(1)) \arrow[d, "\text{restr.}"] \\ \mathrm{NS}(X') \otimes \mathbb{Z}_\ell \arrow[r, "c_1"] \arrow[d, "(\tau|_{X'})_*"] & H^2(X', \mathbb{Z}_\ell(1)) \arrow[d, "(\tau|_{X'})_*"] \\ \mathrm{NS}(X) \otimes \mathbb{Z}_\ell \arrow[r, "c_1"] & H^2(X, \mathbb{Z}_\ell(1)) \end{tikzcd}$$

where the first horizontal map is injective, by what we have seen above. We claim that it suffices to show that the horizontal arrow in the middle is injective. Indeed, let $\alpha \in \mathrm{NS}(X) \otimes \mathbb{Z}_\ell$. Then $c_1(\tau^*\alpha) = \tau^* c_1(\alpha)$ and so injectivity of the horizontal arrow in the middle implies $c_1(\alpha) \neq 0$ unless $\tau^*\alpha = 0$ which in turn implies $\tau_*\tau^*\alpha = \deg \tau \cdot \alpha = 0$ and so $\alpha = 0$ since $\deg \tau$ is coprime to ℓ.

By the localization sequence for class groups, the kernel of $\mathrm{NS}(\overline{X}') \otimes \mathbb{Z}_\ell \to \mathrm{NS}(X') \otimes \mathbb{Z}_\ell$ is generated by classes of divisors supported on $\overline{X}' \setminus X$. Similarly, the Gysin sequence (see (**P1**)) shows that the kernel of the restriction map $H^2(\overline{X}', \mathbb{Z}_\ell(1)) \to H^2(X', \mathbb{Z}_\ell(1))$ is generated by the cycle classes of these divisors. Altogether, this shows that the horizontal arrow in the middle of the above diagram is injective, as we want. This proves (**P6**).

This concludes the proof of the proposition. $\square$

Proposition 3.3. *Assume that $k = \mathbb{C}$ and the above cohomology functor is given by the singular cohomology of the underlying analytic space: $X \longmapsto H^i_{sing}(X(\mathbb{C}), A(n))$. This functor is defined for any abelian group A, where $A(n) := A \otimes_{\mathbb{Z}} \mathbb{Z}(n)$ and $\mathbb{Z}(n)$ denotes the n-th Tate twist. Then (**P1**)–(**P6**) are satisfied for any abelian group A. Moreover, we may in (**P2**) and (**P6**) replace $\mathbb{Z}_\ell$ by $\mathbb{Z}$ and $\mu_{\ell^r}^{\otimes n}$ by $\mathbb{Z}/\ell^r(n)$ throughout.*

Proof. The Gysin sequence (**P1**) is a consequence of the long exact sequence of pairs and the Thom isomorphism. Property (**P3**) follows from the analogous property for étale cohomology by the comparison theorem between étale cohomology and singular cohomology, see e.g. [Mil13, Theorem 21.1]. The remaining properties are also well-known and follow by a similar argument as in Proposition 3.2. $\square$

4. Basic definitions and simple consequences

Usually, $A(n)$ will always denote one of the coefficients in (3.1). In the special case where $k = \mathbb{C}$ and H^* denotes singular cohomology of the underlying analytic space, $A(n)$ may as well denote $A \otimes \mathbb{Z}(n)$, where A is an arbitrary abelian group and $\mathbb{Z}(n)$ is the n-th Tate twist, cf. Proposition 3.3.

4.1. **Filtration on schemes and their cohomology.** For any variety X over k, we consider the filtration

$$\emptyset \subset F_0X \subset F_1X \subset \cdots \subset F_{\dim X}X = X, \quad \text{where} \quad F_jX := \{x \in X \mid \operatorname{codim}_X(x) \leq j\}.$$

That is, F_0X is the generic point of X and for $j \geq 1$, F_jX is the union of $F_{j-1}X$ with all points of codimension j. In general F_jX is an inverse limit of schemes. Since the restriction maps between two nested open subsets $V \subset U \subset X$ are in general not affine, this inverse limit is in general not a scheme. Nonetheless, we may define its cohomology naturally as follows.

Definition 4.1. *For any smooth variety X over k and for any $j \in \mathbb{Z}$, we define*

$$H^i(F_jX, A(n)) := \varinjlim_{F_jX \subset U \subset X} H^i(U, A(n)).$$

In the above definition, U runs through all open subsets of X that consist all points of codimension $\leq j$.

Property (**P1**) together with the fact that taking direct limits of abelian groups is an exact functor immediately yields the following important lemma.

Lemma 4.2. *Let X be a smooth variety over k. Then for any $j, n \in \mathbb{Z}$, there is a long exact sequence*

$$\dots \longrightarrow \bigoplus_{x \in X^{(j)}} H^{i-2j}(\kappa(x), A(n-j)) \xrightarrow{\iota_*} H^i(F_jX, A(n)) \xrightarrow{f} H^i(F_{j-1}X, A(n))$$
$$\xrightarrow{\partial} \bigoplus_{x \in X^{(j)}} H^{i+1-2j}(\kappa(x), A(n-j)) \xrightarrow{\iota_*} H^{i+1}(F_jX, A(n)) \longrightarrow \dots,$$

*where ι_* (resp. ∂) is induced by the pushforward (resp. residue) map from the Gysin exact sequence (**P1**), while f is the natural restriction map given by functoriality.*

Lemma 4.3. *For any smooth variety X over k and any integer n,*

1. *$H^i(F_jX, A(n)) \simeq H^i(X, A(n))$, for all $j \geq \lceil i/2 \rceil$;*
2. *if $H^i(K, A(n)) = 0$ for any finitely generated field extension K of k with $i >$ trdeg(K/k), then $H^i(F_jX, A(n)) = 0$ for all $j < i - \dim X$.*

Proof. Since H^i vanishes for $i < 0$, Lemma 4.2 implies

$$H^i(F_jX, A(n)) \simeq H^i(F_{j-1}X, A(n))$$

for all j with $j > \lceil i/2 \rceil$. This proves the first item in the lemma by induction on j, because $H^i(F_jX, A(n)) = H^i(X, A(n))$ for $j \geq \dim(X)$.

To prove the second item, note that our assumption implies by Lemma 4.2 that $H^i(F_jX, A(n)) \simeq H^i(F_{j-1}X, A(n))$ for all j with $j < i - \dim X$. Hence, $H^i(F_jX, A(n)) \simeq H^i(F_0X, A(n))$ for all j with $j < i - \dim X$. But $H^i(F_0X, A(n)) = 0$ for all $i > \dim X$ by assumption because F_0X is the generic point of X. This proves the lemma. $\square$

4.2. Filtrations on cohomology.

Definition 4.4. *For any smooth variety X over k, we define a decreasing filtration F^* on $H^i(F_jX, A(n))$ by*

$$F^m H^i(F_jX, A(n)) := \begin{cases} H^i(F_jX, A(n)), & \text{if } m \leq j; \\ \operatorname{im}\left(H^i(F_mX, A(n)) \longrightarrow H^i(F_jX, A(n))\right), & \text{if } m > j. \end{cases}$$

In addition to the filtration F^*, we also need the following refinement.

Definition 4.5. *For any smooth variety X over k, we define a decreasing filtration G^* on $H^i(F_jX, \mu_{\ell^r}^{\otimes n})$ by*

$$\alpha \in G^m H^i(F_jX, \mu_{\ell^r}^{\otimes n}) \iff \delta(\alpha) \in F^m H^{i+1}(F_jX, \mathbb{Z}_\ell(n))$$

*where δ denotes the Bockstein map from property (**P2**). We further define the filtration G^* on $H^i(F_jX, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))$ by*

$$G^m H^i(F_jX, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n)) := \varinjlim_r G^m H^i(F_jX, \mu_{\ell^r}^{\otimes n}).$$

As an immediate consequence of the definition and Lemma 4.3 above, we get:

Lemma 4.6. *Let X be a smooth variety over k. Then:*

- $F^m H^i(F_j X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n)) \subset G^m H^i(F_j X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))$ *for all* $m \geq j$;
- $G^m H^i(F_j X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n)) = G^{m+1} H^i(F_j X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))$ *for all* $m \geq \lceil (i+1)/2 \rceil$.

We will also need the filtration on $F^m H^i(F_j X, \mu_{\ell^r}^{\otimes n})$ that is induced by G^* as follows.

Definition 4.7. *For any smooth variety X over k, we define a decreasing filtration G^* on $F^m H^i(F_j X, \mu_{\ell^r}^{\otimes n})$ for $m \geq j$, by*

$$G^h F^m H^i(F_j X, \mu_{\ell^r}^{\otimes n}) = \mathrm{im}(G^h H^i(F_m X, \mu_{\ell^r}^{\otimes n}) \to H^i(F_j X, \mu_{\ell^r}^{\otimes n})).$$

As before, we define the filtration G^* on $F^m H^i(F_j X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))$ by

$$G^h F^m H^i(F_j X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n)) := \varinjlim_r G^h F^m H^i(F_j X, \mu_{\ell^r}^{\otimes n}).$$

4.3. **Refined unramified cohomology groups.** The following is a direct consequence of Definition 4.1 and Lemma 4.3.

Lemma 4.8. *We have*

$$F^{j+1} H^i(F_j X, A(n)) = \ker\left(\partial : H^i(F_j X, A(n)) \longrightarrow \bigoplus_{x \in X^{(j+1)}} H^{i-1-2j}(\kappa(x), A(n-j-1))\right).$$

This lemma is the motivation for the following.

Definition 4.9. *For any smooth variety X over k, we define the j-th refined unramified cohomology of X with values in $A(n)$ by*

$$H^i_{j,nr}(X, A(n)) := F^{j+1} H^i(F_j X, A(n)).$$

Remark 4.10. *Lemma 4.8 shows that $H^i_{0,nr}(X, A(n))$ coincides with the traditional unramified cohomology groups, cf. Section 4.4 below.*

For $m \geq j$, there are canonical restriction maps

$$H^i_{m,nr}(X, A(n)) \longrightarrow H^i_{j,nr}(X, A(n))$$

and Definition 4.4 yields the following decreasing filtration F^* on $H^i_{j,nr}(X, A(n))$:

$$F^m H^i_{j,nr}(X, A(n)) := \begin{cases} H^i_{j,nr}(X, A(n)), & \text{if } m \leq j; \\ \mathrm{im}\left(H^i_{m,nr}(X, A(n)) \longrightarrow H^i_{j,nr}(X, A(n))\right), & \text{if } m > j. \end{cases}$$

Similarly, Definition 4.5 yields a decreasing filtration G^* on $H^i_{j,nr}(X, A(n))$, where $A(n) = \mu_{\ell^r}^{\otimes n}$ or $A(n) = \mathbb{Q}_\ell/\mathbb{Z}_\ell(n)$, which for $m > j$ is given by

$$G^m H^i_{j,nr}(X, A(n)) := G^m F^{j+1} H^i(F_j X, A(n))$$

4.4. **(Classical) unramified cohomology groups.** Let X be a smooth variety over a field k. The (classical) unramified cohomology of X with values in $A(n)$ is defined as

$$H^i_{nr}(X, A(n)) := H^i_{0-nr}(F_0X, A(n)) := F^1H^i(F_0X, A(n)).$$

By definition, we have $H^i(F_0X, A(n)) = H^i(k(X), A(n))$ and so Lemma 4.8 says that

$$H^i_{nr}(X, A(n)) = \ker\left(\partial : H^i(k(X), A(n)) \longrightarrow \bigoplus_{x \in X^{(1)}} H^{i-1}(\kappa(x), A(n-1))\right).$$

It follows from [BO74] that our definition of (classical) unramified cohomology coincides with the definition $H^0(X, \mathcal{H}^i_X(A(n)))$ of Bloch–Ogus, used in the introduction.

The unramified cohomology groups defined above inherit natural decreasing filtrations F^* and G^*, given by ($j \geq 1$):

$$F^jH^i_{nr}(X, A(n)) := F^jH^i(F_0X, A(n)) \quad \text{and} \quad G^jH^i_{nr}(X, A(n)) := G^jF^1H^i(F_0X, A(n)),$$

where G^* is only defined if $A(n) = \mu_{\ell^r}^{\otimes n}$ or $A(n) = \mathbb{Q}_\ell/\mathbb{Z}_\ell(n)$.

By its very definition, $H^i_{nr}(X, A(n))$ is the subgroup of $H^i(k(X), A(n))$ that consists of all classes that lift to a big open subset of X. More generally, $F^jH^i_{nr}(X, A(n))$ for $j \geq 1$ consists of all classes that lift to an open subset whose complement has dimension at least $j + 1$ in X.

Lemma 4.11. *The natural map $H^i(X, A(n)) \to H^i_{nr}(X, A(n))$ is surjective for $i \leq 2$.*

Proof. The result is trivial for $i \leq 0$ and it follows for $i > 0$ from the isomorphism $H^i(X, A(n)) \simeq H^i(F_1X, A(n))$ for $i = 1, 2$, see Lemma 4.3. □

4.5. **Coniveau filtration.** Following Grothendieck, one defines the coniveau filtration $N^*H^i(X, A(n))$ by

$$N^jH^i(X, A(n)) := \ker\left(H^i(X, A(n)) \longrightarrow H^i(F_{j-1}X, A(n))\right).$$

4.6. **Torsion-freeness in low degree.** Property (**P3**) has the following well-known consequence cf. [Blo10, Section 5].

Lemma 4.12. *Let K be a finitely generated field extension of k. Then*

$$H^1(K, \mathbb{Z}_\ell(0)) \quad \textit{and} \quad H^2(K, \mathbb{Z}_\ell(1))$$

have no ℓ-torsion (hence are torsion-free).

Proof. Taking direct limits of abelian groups is exact, so that property (**P2**) implies that

$$H^i(K, \mathbb{Z}_\ell(n))[\ell^r] \simeq \operatorname{coker}(H^{i-1}(K, \mathbb{Z}_\ell(n)) \longrightarrow H^{i-1}(K, \mu_{\ell^r}^{\otimes n})).$$

The above group vanishes for $i = 2$ and $n = 1$ by property (**P3**). It also vanishes for $i = 1$ and any n, as in this case we have by property (**P2**) an exact sequence

$$H^0(K, \mathbb{Z}_\ell(n)) = \mathbb{Z}_\ell(n) \xrightarrow{\times \ell^r} H^0(K, \mathbb{Z}_\ell(n)) = \mathbb{Z}_\ell(n) \longrightarrow H^0(K, \mu_{\ell^r}^{\otimes n}) = \mu_{\ell^r}^{\otimes n}$$

where the last arrow must be surjective because $\mathbb{Z}_\ell(n) \simeq \mathbb{Z}_\ell$ and $\mu_{\ell^r}^{\otimes n} \simeq \mathbb{Z}/\ell^r$. $\square$

4.7. A consequence of (P4) and (P5).

Lemma 4.13. *Let X be a smooth variety over k. Let $w \in X^{(p-1)}$ with closure $W \subset X$ and let $\tau : \tilde{W} \to W$ be the normalization. Then the following diagram commutes for each integer i and n*

$$\begin{tikzcd} H^i(\kappa(w), A(n)) = H^i(k(\widetilde{W}), A(n)) \arrow[r, "\partial"] \arrow[d, hook] & \bigoplus_{\tilde{w} \in \widetilde{W}^{(1)}} H^{i-1}(\kappa(\tilde{w}), A(n-1)) \arrow[d, "\tau_*"] \\ \bigoplus_{x \in X^{(p-1)}} H^i(\kappa(x), A(n)) \arrow[r, "\partial \circ \iota_*"] & \bigoplus_{x \in X^{(p)}} H^{i-1}(\kappa(x), A(n-1)), \end{tikzcd}$$

*where the vertical arrow on the left is the natural inclusion, the vertical arrow on the right is induced by the proper pushforward maps from (**P4**), the upper horizontal arrow is induced by the residue map in (**P1**) and the lower horizontal arrow is given by the composition:*

$$\bigoplus_{x \in X^{(p-1)}} H^i(\kappa(x), A(n)) \xrightarrow{\iota_*} H^{i+2p}(F_{p-1}X, A(n+p)) \xrightarrow{\partial} \bigoplus_{x \in X^{(p)}} H^{i-1}(\kappa(x), A(n-1)),$$

*where ι_* resp. ∂ is the pushforward resp. residue map induced by (**P1**).*

Proof. Fix a point $x \in X^{(p)}$ that is contained in W and let $U \subset X$ be an open neighbourhood of x whose complement in X has codimension at least p. Up to shrinking U, we may assume that $Z := U \cap \overline{\{x\}}$ is smooth. In general, $W \cap U$ will not be smooth but we may assume that it is smooth away from Z and that the preimage

$$U' := \tau^{-1}(W \cap U) \subset \widetilde{W}$$

in the normalization $\widetilde{W}$ is smooth. Up to shrinking U further (without changing $x \in U$ and $\operatorname{codim}_X(X \setminus U) \geq p$), we may also assume that the preimage

$$Z' = \tau^{-1}(Z) \subset U'$$

is smooth and of pure codimension one in U'. Then τ induces a finite morphism of smooth pairs $\tau|_{U'} : (U', Z') \to (U, Z)$ such that $\tau(U' \setminus Z')$ is smooth. By properties (**P4**) and (**P5**), we thus get a commutative diagram, where $V := U \setminus Z$ and $V' := U' \setminus Z'$:

$$\begin{tikzcd} H^i(V', A(n)) \arrow[r, "\partial"] \arrow[d, "(\tau|_{V'})_*"] & H^{i-1}(Z', A(n-1)) \arrow[d, "(\tau|_{Z'})_*"] \\ H^{i+2p-2}(V, A(n)) \arrow[r, "\partial"] & H^{i-1}(Z, A(n-1)) \end{tikzcd}.$$

The lemma is an immediate consequence of this. ☐

5. Comparison to higher unramified cohomology of Bloch–Ogus

In this section we aim to compare the refined unramified cohomology groups defined in Section 4.3 to higher unramified cohomology groups, which by Lemma 4.13 and [BO74] may be defined as follows:

$$H^j(X, \mathcal{H}_X^{i+j}(A(n))) := \frac{\ker(\partial \circ \iota_* : \oplus_{x\in X^{(j)}} H^i(\kappa(x)) \to \oplus_{x \in X^{(j+1)}} H^{i-1}(\kappa(x)))}{\operatorname{im}(\partial \circ \iota_* : \oplus_{x\in X^{(j-1)}} H^{i+1}(\kappa(x)) \to \oplus_{x \in X^{(j)}} H^{i}(\kappa(x)))},$$

where $H^*(\kappa(x))$ is a short hand for $H^*(\kappa(x), A(n-c))$, where $c = \operatorname{codim}_X(x)$.

Theorem 5.1. *Let X be a smooth variety over k. Then for any i, there is a canonical long exact sequence*

$$\dots \to H^{i+2j-1}_{j-1,nr}(X, A(n)) \to H^{i+2j-1}_{j-2,nr}(X, A(n)) \to H^j(X, \mathcal{H}_X^{i+j}(A(n))) \to H^{i+2j}_{j,nr}(X, A(n)) \to \dots$$

Proof. Let $[\xi] \in H^j(X, \mathcal{H}_X^{i+j}(A(n)))$ with $\xi \in \oplus_{x\in X^{(j)}} H^i(\kappa(x))$ and $\partial \circ \iota_*(\xi) = 0$. By the Gysin sequence (Lemma 4.2), the condition $\partial \circ \iota_*(\xi) = 0$ is equivalent to

$$\iota_*\xi \in F^{j+1}H^{2j+i}(F_jX).$$

If $\xi = \partial \circ \iota_*(\zeta)$ for some $\zeta \in \oplus_{x\in X^{(j-1)}} H^{i+1}(\kappa(x))$, then

$$\iota_*\xi = \iota_* \circ \partial \circ \iota_*(\zeta) = 0$$

by the exactness of the Gysin sequence. It follows that there is a well-defined map

$$H^j(X, \mathcal{H}_X^{i+j}(A(n))) \longrightarrow H^{i+2j}_{j,nr}(X, A(n)), \quad [\xi] \longmapsto \iota_*\xi \tag{5.1}$$

Any class in the image of this map lies in the kernel of

$$H^{i+2j}_{j,nr}(X, A(n)) \longrightarrow H^{i+2j}_{j-1,nr}(X, A(n)) \tag{5.2}$$

because $\iota_*\xi$ vanishes on $F_{j-1}X$ by Lemma 4.2. Conversely, any class $\alpha \in H^{i+2j}_{j,nr}(X, A(n))$ in the kernel of the above restriction map is by Lemma 4.2 of the form $\alpha = \iota_*\xi$ for some $\xi \in \oplus_{x\in X^{(j)}} H^i(\kappa(x))$. The fact that $\alpha \in H^{i+2j}_{j,nr}(X, A(n)) \subset H^{i+2j}(F_jX, A(n))$ is unramified implies $\partial \circ \iota_*(\xi) = 0$, and so α lies in the image of (5.1). Hence, the composition of (5.1) and (5.2) is exact.

Let now $[\xi] \in H^j(X, \mathcal{H}_X^{i+j}(A(n)))$ with $\xi \in \oplus_{x\in X^{(j)}} H^i(\kappa(x))$ and $\partial \circ \iota_* \xi = 0$ be a class in the kernel of (5.1). By the exactness of the Gysin sequence, this means that $\xi = \partial\alpha$ for some $\alpha \in H^{i+2j-1}(F_{j-1}X, A(n))$. Hence, the natural sequence

$$H^{i+2j-1}(F_{j-1}X, A(n)) \xrightarrow{\partial} H^j(X, \mathcal{H}_X^{i+j}(A(n))) \xrightarrow{\iota_*} H^{i+2j}_{j,nr}(X, A(n)) \tag{5.3}$$

is exact. The image of

$$\iota_* : \oplus_{x\in X^{(j-1)}} H^{i+1}(\kappa(x), A(n)) \longrightarrow H^{i+2j-1}(F_{j-1}X, A(n))$$

lies in the kernel of the first map in (5.3). By the Gysin sequence, it follows that (5.3) descends to an exact sequence

$$H^{i+2j-1}_{j-2,nr}(X, A(n)) \xrightarrow{\partial} H^j(X, \mathcal{H}_X^{i+j}(A(n))) \xrightarrow{\iota_*} H^{i+2j}_{j,nr}(X, A(n)). \tag{5.4}$$

Let $[\alpha] \in H^{i+2j-1}_{j-2,nr}(X, A(n))$ with $\alpha \in H^{i+2j-1}(F_{j-1}X, A(n))$ and assume that

$$\partial\alpha = 0 \in H^j(X, \mathcal{H}_X^{i+j}(A(n))).$$

This means that there is a class $\zeta \in \oplus_{x\in X^{(j-1)}} H^{i+1}(\kappa(x))$ with $\partial(\alpha - \iota_*\zeta) = 0$. Hence, up to replacing α by $\alpha - \iota_*\zeta$, we may assume $\partial\alpha = 0$ and so

$$[\alpha] \in F^j H^{i+2j-1}(F_{j-2}X, A(n)).$$

Conversely, any class in $F^j H^{i+2j-1}(F_{j-2}X, A(n))$ clearly maps to zero in $H^j(X, \mathcal{H}_X^{i+j}(A(n)))$. Hence, the kernel of the first map in (5.4) agrees with the image of the canonical restriction map

$$H^{i+2j-1}_{j-1,nr}(X, A(n)) \longrightarrow H^{i+2j-1}_{j-2,nr}(X, A(n)).$$

This concludes the proof of the theorem. □

6. ℓ-adic Chow groups

6.1. **Chow groups with ℓ-adic coefficients.** Properties (**P1**) and (**P3**) allow us to define Chow groups with $\mathbb{Z}_\ell$-coefficients of codimension i-cycles of a smooth variety X over a field k in terms of the cohomology functor H^* by

$$\mathrm{CH}^i(X)_{\mathbb{Z}_\ell} := \frac{\bigoplus_{x\in X^{(i)}} [x]\mathbb{Z}_\ell}{\mathrm{im}\left(\bigoplus_{x\in X^{(i-1)}} \kappa(x)^* \xrightarrow{\epsilon} \bigoplus_{x\in X^{(i-1)}} H^1(\kappa(x), \mathbb{Z}_\ell(1)) \xrightarrow{\partial\circ\iota_*} \bigoplus_{x\in X^{(i)}} [x]\mathbb{Z}_\ell\right) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell},$$

where we use the canonical identification $H^0(-, \mathbb{Z}_\ell(0)) = \mathbb{Z}_\ell$ and where $\partial \circ \iota_*$ denotes the composition

$$\bigoplus_{x\in X^{(i-1)}} H^1(\kappa(x), \mathbb{Z}_\ell(1)) \xrightarrow{\iota_*} H^{2i-1}(F_{i-1}X, \mathbb{Z}_\ell(1)) \xrightarrow{\partial} \bigoplus_{x\in X^{(i)}} [x]\mathbb{Z}_\ell.$$

It follows from Lemma 4.13 (which itself is a consequence of properties (**P4**) and (**P5**)) that this coincides with $\mathrm{CH}^i(X) \otimes \mathbb{Z}_\ell$ where $\mathrm{CH}^i(X)$ denotes the usual Chow group of algebraic cycles of codimension i on X modulo rational equivalence.

Similarly, property (**P1**) allows us to define

$$A^i(X)_{\mathbb{Z}_\ell} := \frac{\bigoplus_{x\in X^{(i)}}[x]\mathbb{Z}_\ell}{\operatorname{im}\left(\bigoplus_{x\in X^{(i-1)}} H^1(\kappa(x),\mathbb{Z}_\ell(1)) \xrightarrow{\partial\circ\iota_*} \bigoplus_{x\in X^{(i)}}[x]\mathbb{Z}_\ell\right)}.$$

We will see in Lemma 6.2 below that property (**P6**) ensures that $A^i(X)_{\mathbb{Z}_\ell}$ coincides with the Chow group of algebraic cycles with $\mathbb{Z}_\ell$-coefficients modulo algebraic equivalence.

Remark 6.1. *The above presentation suggests that with coefficients in $\mathbb{Z}_\ell$ the difference between rational and algebraic equivalence is encoded by the failure of surjectivity of*

$$K^*\otimes \varprojlim_r \mathbb{Z}/\ell^r \longrightarrow \varprojlim_r \left(K^*/(K^*)^{\ell^r}\right).$$

Lemma 6.2. *Assume that k is perfect. Then for any smooth (not necessarily proper) variety X over k, $A^i(X)_{\mathbb{Z}_\ell} = A^i(X)\otimes_{\mathbb{Z}}\mathbb{Z}_\ell$, where $A^i(X)$ denotes the Chow group of algebraic cycles of codimension i on X modulo algebraic equivalence.*

Proof. We aim to describe the group $\operatorname{Alg}^i(X)\otimes\mathbb{Z}_\ell$ of algebraically trivial codimension i-cycles with coefficients in $\mathbb{Z}_\ell$. By definition, $\operatorname{Alg}^i(X)\otimes\mathbb{Z}_\ell$ is generated by

$$\tau_*\operatorname{Alg}^1(W')\otimes\mathbb{Z}_\ell \subset \bigoplus_{x\in X^{(i)}}[x]\mathbb{Z}_\ell,$$

where $W\subset X$ runs through all closed subvarieties of codimension $i-1$ with normalization $\tau: W'\to W$. Since the ground field k is perfect by assumptions, any regular algebraic scheme over k is smooth. Since W' is normal, it is thus smooth in codimension one and so $\operatorname{Alg}^1(W')\otimes\mathbb{Z}_\ell$ does not change if we replace W' by its smooth locus. Property (**P6**) thus implies that $\operatorname{Alg}^i(X)\otimes\mathbb{Z}_\ell$ is generated by cycles of the form $\tau_*(z)$ where $z\in\bigoplus_{w\in(W')^{(1)}}[w]\mathbb{Z}_\ell$ such that

$$\iota_*z = 0 \in H^2(F_1W',\mathbb{Z}_\ell(1)),$$

where we note that the latter group makes sense because W' is regular in codimension one. By Lemma 4.2, $\iota_*z=0$ is equivalent to $z=\partial\xi$ for some $\xi\in H^1(k(W'),\mathbb{Z}_\ell(1))$. Hence, $\operatorname{Alg}^i(X)\otimes\mathbb{Z}_\ell$ is generated by cycles of the form $\tau_*\partial\xi$ with $\xi\in H^1(k(W'),\mathbb{Z}_\ell(1))$. It thus follows from Lemma 4.13 that $\operatorname{Alg}^i(X)\otimes\mathbb{Z}_\ell$ is given by the image of the composition

$$\bigoplus_{x\in X^{(i-1)}} H^1(\kappa(x),\mathbb{Z}_\ell(1)) \xrightarrow{\iota_*} H^{2i-1}(F_{i-1}X,\mathbb{Z}_\ell(1)) \xrightarrow{\partial} \bigoplus_{x\in X^i}[x]\mathbb{Z}_\ell.$$

This proves the lemma. □

6.2. **ℓ^r-torsion in the Chow groups.** The properties formulated in Section 3 imply the existence of the following commutative diagram with exact rows:

$$\begin{tikzcd}
\bigoplus_{x\in X^{(i-1)}} \kappa(x)^*\otimes_{\mathbb{Z}}\mathbb{Z}_\ell \arrow[r, "\times\ell^r"] \arrow[d, "\epsilon"] & \bigoplus_{x\in X^{(i-1)}} \kappa(x)^*\otimes_{\mathbb{Z}}\mathbb{Z}_\ell \arrow[r] \arrow[d, "\epsilon"] & \bigoplus_{x\in X^{(i-1)}} \kappa(x)^*/(\kappa(x)^*)^{\ell^r} \arrow[d, "\simeq"] \\
\bigoplus_{x\in X^{(i-1)}} H^1(\kappa(x),\mathbb{Z}_\ell(1)) \arrow[r, "\times\ell^r"] \arrow[d, "\partial\circ\iota_*"] & \bigoplus_{x\in X^{(i-1)}} H^1(\kappa(x),\mathbb{Z}_\ell(1)) \arrow[r] \arrow[d, "\partial\circ\iota_*"] & \bigoplus_{x\in X^{(i-1)}} H^1(\kappa(x),\mu_{\ell^r}^{\otimes 1}) \arrow[d, "\partial\circ\iota_*"] \\
\bigoplus_{x\in X^{(i)}} \mathbb{Z}_\ell[x] \arrow[r, "\times\ell^r"] \arrow[d] & \bigoplus_{x\in X^{(i)}} \mathbb{Z}_\ell[x] \arrow[r] \arrow[d] & \bigoplus_{x\in X^{(i)}} \mathbb{Z}/\ell^r[x] \\
\mathrm{CH}^i(X)_{\mathbb{Z}_\ell} \arrow[r, "\times\ell^r"] \arrow[d] & \mathrm{CH}^i(X)_{\mathbb{Z}_\ell} \arrow[d] & \\
A^i(X)_{\mathbb{Z}_\ell} \arrow[r, "\times\ell^r"] & A^i(X)_{\mathbb{Z}_\ell} &
\end{tikzcd}$$

The following is motivated by [Blo79, §2].

Lemma 6.3. *There are canonical isomorphisms*

$$\phi_r : \mathrm{CH}^i(X)[\ell^r] \xrightarrow{\simeq} \frac{\ker\left(\partial\circ\iota_* : \bigoplus_{x\in X^{(i-1)}} H^1(\kappa(x),\mu_{\ell^r}^{\otimes 1}) \longrightarrow \bigoplus_{x\in X^{(i)}} \mathbb{Z}/\ell^r[x]\right)}{\ker\left(\partial\circ\iota_*\circ\epsilon : \bigoplus_{x\in X^{(i-1)}} \kappa(x)^*\otimes_{\mathbb{Z}}\mathbb{Z}_\ell \longrightarrow \bigoplus_{x\in X^{(i)}} \mathbb{Z}_\ell[x]\right)}$$

and

$$\psi_r : A^i(X)[\ell^r] \xrightarrow{\simeq} \frac{\ker\left(\partial\circ\iota_* : \bigoplus_{x\in X^{(i-1)}} H^1(\kappa(x),\mu_{\ell^r}^{\otimes 1}) \longrightarrow \bigoplus_{x\in X^{(i)}} \mathbb{Z}/\ell^r[x]\right)}{\ker\left(\partial\circ\iota_* : \bigoplus_{x\in X^{(i-1)}} H^1(\kappa(x),\mathbb{Z}_\ell(1)) \longrightarrow \bigoplus_{x\in X^{(i)}} \mathbb{Z}_\ell[x]\right)}.$$

Proof. Note that the first arrow in the third row of the above diagram is injective, while the last arrows in the first two rows are surjective by property (**P3**). The result is therefore an immediate consequence of the snake lemma and the presentation of $\mathrm{CH}^i(X)_{\mathbb{Z}_\ell}$ and $A^i(X)_{\mathbb{Z}_\ell}$ given in Section 6.1. □

6.3. **Cycle class maps.** Lemma 4.2 yields a map

$$\iota_* : \bigoplus_{x\in X^{(i)}} \mathbb{Z}_\ell[x] \longrightarrow H^{2i}(X,\mathbb{Z}_\ell(i))$$

which is zero on the image of $\partial : \bigoplus_{x\in X^{(i-1)}} H^1(\kappa(x),\mathbb{Z}_\ell(1)) \to \bigoplus_{x\in X^{(i)}} \mathbb{Z}_\ell[x]$. It thus follows from the presentation of Chow groups in Section 6.1 that there is a well-defined cycle class map

$$\mathrm{cl}^i : \mathrm{CH}^i(X)_{\mathbb{Z}_\ell} \longrightarrow H^{2i}(X,\mathbb{Z}_\ell(i))$$

which factors through the canonical surjection $\mathrm{CH}^i(X)_{\mathbb{Z}_\ell} \twoheadrightarrow A^i(X)_{\mathbb{Z}_\ell}$.

7. Defect of integral Hodge- or Tate-type conjectures

Definition 7.1. *We define the ℓ^r-torsion defect of integral Hodge- or Tate-type conjectures by*

$$Z^i(X)[\ell^r] := \operatorname{coker}\left(\mathrm{cl}^i : \mathrm{CH}^i(X)_{\mathbb{Z}_\ell} \longrightarrow H^{2i}(X, \mathbb{Z}_\ell(i))\right)[\ell^r].$$

Theorem 7.2. *Let X be a smooth variety over k. Then there is a natural isomorphism*

$$Z^i(X)[\ell^r] \simeq \frac{H^{2i-1}_{i-2,nr}(X, \mu_\ell^{\otimes i})}{H^{2i-1}_{i-2,nr}(X, \mathbb{Z}_\ell(i))}.$$

The image of the ℓ^r-torsion classes $H^{2i}(X, \mathbb{Z}_\ell(i))[\ell^r]$ inside $Z^i(X)[\ell^r]$ corresponds via this isomorphism to the subspace

$$\frac{F^i H^{2i-1}_{i-2,nr}(X, \mu_\ell^{\otimes i})}{H^{2i-1}_{i-2,nr}(X, \mathbb{Z}_\ell(i))} \subset \frac{H^{2i-1}_{i-2,nr}(X, \mu_\ell^{\otimes i})}{H^{2i-1}_{i-2,nr}(X, \mathbb{Z}_\ell(i))}$$

Remark 7.3. *In the case $i = 2$, the first assertion of the theorem is due to* [CTV12], *but note that even in this case, the second assertion which allows to tell apart torsion Hodge classes from non-torsion Hodge classes seems new, cf.* [CTV12, §5.6].

Proof of Theorem 7.2. By Lemma 4.2, we have an exact sequence

$$\bigoplus_{x \in X^{(i)}} [x]\mathbb{Z}_\ell \xrightarrow{\iota_*} H^{2i}(X, \mathbb{Z}_\ell(i)) \longrightarrow H^{2i}(F_{i-1}X, \mathbb{Z}_\ell(i)) \longrightarrow \bigoplus_{x \in X^{(i)}} H^1(\kappa(x), \mathbb{Z}_\ell(1)).$$

By Lemma 4.12, the last term is torsion-free and so we get an isomorphism

$$Z^i(X)[\ell^r] \simeq H^{2i}(F_{i-1}X, \mathbb{Z}_\ell(i))[\ell^r].$$

By property (**P2**), the Bockstein map thus induces an isomorphism

$$Z^i(X)[\ell^r] \simeq \operatorname{coker}\left(H^{2i-1}(F_{i-1}X, \mathbb{Z}_\ell(i)) \longrightarrow H^{2i-1}(F_{i-1}X, \mu_{\ell^r}^{\otimes i})\right).$$

Let now

$$\alpha \in \operatorname{coker}\left(H^{2i-1}(F_{i-1}X, \mathbb{Z}_\ell(i)) \longrightarrow H^{2i-1}(F_{i-1}X, \mu_{\ell^r}^{\otimes i})\right).$$

Since $H^{2i-1}(F_iX, \mu_{\ell^r}^{\otimes i}) \simeq H^{2i-1}(X, \mu_{\ell^r}^{\otimes i})$, this class lifts to $F^i H^{2i-1}(F_{i-1}X, \mu_{\ell^r}^{\otimes i})$ if and only if $\delta(\alpha) \in H^{2i}(F_{i-1}X, \mathbb{Z}_\ell(i))[\ell^r]$ lifts to an ℓ^r-torsion class in $H^{2i}(X, \mathbb{Z}_\ell(i))$. Hence, the image of the ℓ^r-torsion classes $H^{2i}(X, \mathbb{Z}_\ell(i))[\ell^r]$ inside $Z^i(X)[\ell^r]$ corresponds via the above isomorphism to the subspace

$$\frac{F^i H^{2i-1}(F_{i-1}X, \mu_{\ell^r}^{\otimes i})}{H^{2i-1}(F_{i-1}X, \mathbb{Z}_\ell(i))} \subset \frac{H^{2i-1}(F_{i-1}X, \mu_{\ell^r}^{\otimes i})}{H^{2i-1}(F_{i-1}X, \mathbb{Z}_\ell(i))}$$

The theorem thus follows if we can prove that the natural map

$$\frac{H^{2i-1}(F_{i-1}X, \mu_{\ell^r}^{\otimes i})}{H^{2i-1}(F_{i-1}X, \mathbb{Z}_\ell(i))} \longrightarrow \frac{F^{i-1} H^{2i-1}(F_{i-2}X, \mu_{\ell^r}^{\otimes i})}{F^{i-1} H^{2i-1}(F_{i-2}X, \mathbb{Z}_\ell(i))} = \frac{H^{2i-1}_{i-2,nr}(X, \mu_\ell^{\otimes i})}{H^{2i-1}_{i-2,nr}(X, \mathbb{Z}_\ell(i))}$$

is an isomorphism. Since the above map is clearly surjective, it suffices to show that it is injective. It thus suffices to show that any element

$$\alpha \in \ker\left(H^{2i-1}(F_{i-1}X,\mu_{\ell^r}^{\otimes i})\longrightarrow H^{2i-1}(F_{i-2}X,\mu_{\ell^r}^{\otimes i})\right)$$

satisfies

$$\delta(\alpha)=0\in H^{2i}(F_{i-1}X,\mathbb{Z}_\ell(i))[\ell^r],$$

because this implies by (**P2**) that α lifts to an integral class. By Lemma 4.2, $\alpha=\iota_*\xi$ for some

$$\xi\in\bigoplus_{x\in X^{(i-1)}}H^1(\kappa(x),\mu_{\ell^r}^{\otimes 1}).$$

Since δ commutes with ι_* by property (**P2**), we find $\delta(\alpha)=\iota_*(\delta(\xi))$. On the other hand

$$\delta(\xi)\in\bigoplus_{x\in X^{(i-1)}}H^2(\kappa(x),\mathbb{Z}_\ell(1))$$

is ℓ^r-torsion by property (**P2**), while the above direct sum is torsion-free by Lemma 4.12. Hence $\delta(\alpha)=0$, which concludes the proof of the theorem. □

Corollary 7.4. *Let X be a smooth variety over k. For all $i\geq 2$, there is a surjection*

$$\operatorname{coker}\left(\bigoplus_{c\geq 1}\bigoplus_{\tilde D}Z^{2i-2c}(\tilde D)[\ell^r]\longrightarrow Z^{2i}(X)[\ell^r]\right)\twoheadrightarrow\frac{F^{i-1}H^{2i-1}_{nr}(X,\mu_{\ell^r}^{\otimes i})}{F^{i-1}H^{2i-1}_{nr}(X,\mathbb{Z}_\ell(i))},$$

which is an isomorphism for $i=2$. Here $\tilde D$ runs through all alterations of all closed subvarieties $D\subset X$ of codimension c. The image of $H^{2i}(X,\mathbb{Z}_\ell(i))[\ell^r]$ inside $Z^{2i}(X)[\ell^r]$ surjects via the above map onto $F^iH^{2i-1}_{nr}(X,\mu_{\ell^r}^{\otimes i})/F^{i-1}H^{2i-1}_{nr}(X,\mathbb{Z}_\ell(i))$.

Proof. By Theorem 7.2, there is a natural surjection

$$Z^{2i}(X)[\ell^r]\simeq\frac{H^{2i-1}_{i-2,nr}(X,\mu_\ell^{\otimes i})}{H^{2i-1}_{i-2,nr}(X,\mathbb{Z}_\ell(i))}\twoheadrightarrow\frac{F^{i-1}H^{2i-1}_{nr}(X,\mu_{\ell^r}^{\otimes i})}{F^{i-1}H^{2i-1}_{nr}(X,\mathbb{Z}_\ell(i))}$$

which is an isomorphism for $i=2$ and such that the image of $H^{2i}(X,\mathbb{Z}_\ell(i))[\ell^r]$ inside $Z^{2i}(X)[\ell^r]$ maps via the above surjection onto $F^iH^{2i-1}_{nr}(X,\mu_{\ell^r}^{\otimes i})/F^{i-1}H^{2i-1}_{nr}(X,\mathbb{Z}_\ell(i))$.

Let now $D\subset X$ be a closed subvariety of codimension c and let $\tau_*:\widetilde D\to D$ be an alteration. The image of

$$Z^{2i-2c}(\tilde D)[\ell^r]\simeq\frac{F^{i-c-1}H^{2i-2c-1}(F_{i-c-2}\tilde D,\mu_\ell^{\otimes i})}{F^{i-c-1}H^{2i-2c-1}(F_{i-c-2}\tilde D,\mathbb{Z}_\ell(i))}\longrightarrow\frac{F^{i-1}H^{2i-1}(F_0X,\mu_\ell^{\otimes i})}{F^{i-1}H^{2i-1}(F_0X,\mathbb{Z}_\ell(i))}$$

is contained in the image of

$$\iota_*\circ\tau_*:\frac{H^{2i-2c-1}(F_0\tilde D,\mu_\ell^{\otimes i})}{H^{2i-2c-1}(F_0\tilde D,\mathbb{Z}_\ell(i))}\longrightarrow\frac{F^{i-1}H^{2i-1}(F_0X,\mu_\ell^{\otimes i})}{F^{i-1}H^{2i-1}(F_0X,\mathbb{Z}_\ell(i))},$$

which is zero by (**P1**) and (**P4**) as soon as $c>0$. This proves the corollary. □

The proof of Theorem 7.2 has the following consequence, which we want to record here.

Corollary 7.5. *Let X be a smooth variety over k. Then*

$$G^iH^{2i-1}(F_{i-1}X,\mu_{\ell^r}^{\otimes i})=H^{2i-1}(F_{i-1}X,\mu_{\ell^r}^{\otimes i}).$$

Proof. By Lemma 4.2, we have an exact sequence

$$\bigoplus_{x\in X^{(i)}}[x]\mathbb{Z}_\ell\xrightarrow{\iota_*}H^{2i}(X,\mathbb{Z}_\ell(i))\longrightarrow H^{2i}(F_{i-1}X,\mathbb{Z}_\ell(i))\longrightarrow\bigoplus_{x\in X^{(i)}}H^1(\kappa(x),\mathbb{Z}_\ell(1)).$$

For any $\alpha\in H^{2i-1}(F_{i-1}X,\mu_{\ell^r}^{\otimes i})$, the class $\delta(\alpha)\in H^{2i}(F_{i-1}X,\mathbb{Z}_\ell(i))$ is torsion and so Lemma 4.12 implies that it maps to zero in $\bigoplus_{x\in X^{(i)}}H^1(\kappa(x),\mathbb{Z}_\ell(1))$. Hence, $\alpha\in G^iH^{2i-1}(F_{i-1}X,\mu_{\ell^r}^{\otimes i})$, as we want. □

8. The ℓ-adic Griffiths group

For any smooth variety X over k, we define the Griffiths group $\mathrm{Griff}^i(X)_{\mathbb{Z}_\ell}$ with coefficients in $\mathbb{Z}_\ell$ by

$$\mathrm{Griff}^i(X)_{\mathbb{Z}_\ell}:=\ker\left(\mathrm{cl}^i:A^i(X)_{\mathbb{Z}_\ell}\longrightarrow H^{2i}(X,\mathbb{Z}_\ell(i))\right).$$

Using the definition of $A^i(X)_{\mathbb{Z}_\ell}$ from Section 6.1, we get

$$\mathrm{Griff}^i(X)_{\mathbb{Z}_\ell}=\frac{\ker\left(\iota_*:\bigoplus_{x\in X^{(i)}}\mathbb{Z}_\ell[x]\longrightarrow H^{2i}(X,\mathbb{Z}_\ell(i))\right)}{\mathrm{im}\left(\partial\circ\iota_*:\bigoplus_{x\in X^{(i-1)}}H^1(\kappa(x),\mathbb{Z}_\ell(1))\longrightarrow\bigoplus_{x\in X^{(i)}}[x]\mathbb{Z}_\ell\right)}. \tag{8.1}$$

At least if k is perfect, the above group coincides by Lemma 6.2 with the usual Griffiths group of homologically trivial cycles modulo algebraic equivalence.

Theorem 8.1. *Let X be a smooth variety over the field k. Then there is a natural isomorphism*

$$\mathrm{Griff}^i(X)_{\mathbb{Z}_\ell}\simeq\frac{H^{2i-1}_{i-2,nr}(X,\mathbb{Z}_\ell(i))}{H^{2i-1}(X,\mathbb{Z}_\ell(i))}.$$

Proof. By Lemmas 4.2 and 4.3, we have exact sequences

$$H^{2i-1}(X,\mathbb{Z}_\ell(i))\longrightarrow H^{2i-1}(F_{i-1}X,\mathbb{Z}_\ell(i))\xrightarrow{\partial}\bigoplus_{x\in X^{(i)}}\mathbb{Z}_\ell[x]\xrightarrow{\iota_*}H^{2i}(X,\mathbb{Z}_\ell(i))$$

and

$$\bigoplus_{x\in X^{(i-1)}}H^1(\kappa(x),\mathbb{Z}_\ell(1))\xrightarrow{\iota_*}H^{2i-1}(F_{i-1}X,\mathbb{Z}_\ell(i))\xrightarrow{f}H^{2i-1}(F_{i-2}X,\mathbb{Z}_\ell(i)).$$

This shows by (8.1) that $\mathrm{Griff}^i(X)_{\mathbb{Z}_\ell}$ is isomorphic to

$$\mathrm{im}\left(\frac{H^{2i-1}(F_{i-1}X,\mathbb{Z}_\ell(i))}{H^{2i-1}(X,\mathbb{Z}_\ell(i))}\longrightarrow\frac{H^{2i-1}(F_{i-2}X,\mathbb{Z}_\ell(i))}{H^{2i-1}(X,\mathbb{Z}_\ell(i))}\right),$$

which proves the theorem by definition of $H^{2i-1}_{i-2,nr}(X,\mathbb{Z}_\ell(i))$. □

The above theorem has the following immediate consequence, which generalizes [BO74, 7.5].

Corollary 8.2. *Let X be a smooth variety over the field k. Then there is a natural surjective map*

$$\operatorname{coker}\left(\bigoplus_{c\geq 1}\bigoplus_{\tilde{D}} \operatorname{Griff}^{i-c}(\tilde{D})_{\mathbb{Z}_\ell} \longrightarrow \operatorname{Griff}^i(X)_{\mathbb{Z}_\ell}\right) \twoheadrightarrow gr_F^{i-1}H^{2i-1}_{nr}(X,\mathbb{Z}_\ell(i)),$$

which is an isomorphism if $i=2$. Here $\tilde{D}$ runs through all alterations of all subvarieties $D\subset X$ of codimension $c\geq 1$.

Proof. By Theorem 8.1, there is a natural surjection,

$$\operatorname{Griff}^i(X)_{\mathbb{Z}_\ell} \simeq \frac{H^{2i-1}_{i-2,nr}(X,\mathbb{Z}_\ell(i))}{H^{2i-1}(X,\mathbb{Z}_\ell(i))} \longrightarrow \frac{F^{i-1}H^{2i-1}(F_0X,\mathbb{Z}_\ell(i))}{H^{2i-1}(X,\mathbb{Z}_\ell(i))} = gr_F^{i-1}H^{2i-1}_{nr}(X,\mathbb{Z}_\ell(i))$$

which is an isomorphism for $i=2$. As in the proof of Corollary 7.4, it follows that the image of $\operatorname{Griff}^{i-c}(\tilde{D})_{\mathbb{Z}_\ell}$ in $\operatorname{Griff}^i(X)_{\mathbb{Z}_\ell}$ lies in the kernel of this surjection. This concludes the proof. ☐

8.1. **Canonical extensions.** Combining Theorems 7.2 and 8.1, we obtain the following result of independent interest.

Corollary 8.3. *Let X be a smooth variety over k. Then there is a canonical short exact sequence*

$$0\longrightarrow \operatorname{Griff}^i(X)_{\mathbb{Z}_\ell}\otimes \mathbb{Z}/\ell^r \longrightarrow \frac{H^{2i-1}_{i-2,nr}(X,\mu_{\ell^r}^{\otimes i})}{H^{2i-1}(X,\mu_{\ell^r}^{\otimes i})} \longrightarrow \frac{Z^{2i}(X)[\ell^r]}{H^{2i}(X,\mathbb{Z}_\ell(i))[\ell^r]} \longrightarrow 0.$$

Proof. The image of $H^{2i-1}(X,\mu_{\ell^r}^{\otimes i})$ in $Z^{2i}(X)[\ell^r]$ is exactly the image of $H^{2i}(X,\mathbb{Z}_\ell(i))[\ell^r]$ (because the map is induced by the Bockstein map). This shows by Theorems 7.2 and 8.1 that there is an exact sequence

$$\operatorname{Griff}^i(X)_{\mathbb{Z}_\ell}\otimes \mathbb{Z}/\ell^r \longrightarrow \frac{H^{2i-1}_{i-2,nr}(X,\mu_{\ell^r}^{\otimes i})}{H^{2i-1}(X,\mu_{\ell^r}^{\otimes i})} \longrightarrow \frac{Z^{2i}(X)[\ell^r]}{H^{2i}(X,\mathbb{Z}_\ell(i))[\ell^r]} \longrightarrow 0 \tag{8.2}$$

is exact and it remains to show that the first arrow is injective. For this, let $z\in \bigoplus_{x\in X^{(i)}}\mathbb{Z}_\ell[x]$ with $\iota_*z=0$ be a homologically trivial cycle on X and let $[z]\in \operatorname{Griff}^i(X)$. By Lemma 4.2, there is a class $\alpha\in H^{2i-1}(F_{i-1}X,\mathbb{Z}_\ell(i))$ with $\partial\alpha = z$. The above map in question then maps $[z]$ to the image of α in

$$\frac{F^{i-1}H^{2i-1}(F_{i-2}X,\mu_{\ell^r}^{\otimes i})}{H^{2i-1}(X,\mu_{\ell^r}^{\otimes i})} = \frac{H^{2i-1}_{i-2,nr}(X,\mu_{\ell^r}^{\otimes i})}{H^{2i-1}(X,\mu_{\ell^r}^{\otimes i})}$$

If this vanishes, then there is a class $\xi \in \bigoplus_{x\in X^{(i-1)}} H^1(\kappa(x),\mu_{\ell^r}^{\otimes 1})$ such that

$$\partial(\overline{\alpha}+\iota_*\xi)=0\in \bigoplus_{x\in X^{(i)}} \mathbb{Z}/\ell^r[x],$$

where $\overline{\alpha}$ denotes the image of α in $H^{2i-1}(F_{i-1}X,\mu_{\ell^r}^{\otimes i})$. By property (**P3**), we can pick a lift $\xi'\in \bigoplus_{x\in X^{(i-1)}} H^1(\kappa(x),\mathbb{Z}_\ell(1))$ of ξ and find that

$$\partial(\alpha+\iota_*\xi)\in \bigoplus_{x\in X^{(i)}} \mathbb{Z}_\ell[x]$$

is zero modulo ℓ^r. The above cycle is algebraically equivalent to $z=\partial\alpha$ and so z has trivial image in $\operatorname{Griff}^i(X)\otimes \mathbb{Z}/\ell^r$. This shows that the first map in (8.2) is injective and so the second short exact sequence from the corollary follows. This concludes the proof. $\square$

9. Torsion classes in the Griffiths group with trivial Abel–Jacobi invariant

9.1. **An elementary construction of the transcendental Abel–Jacobi map on torsion cycles.** In this section we show that our formal set-up suffices to define a map

$$\lambda_{tr}:\operatorname{Griff}^i(X)[\ell^\infty]\longrightarrow \frac{H^{2i-1}(X,\mathbb{Q}_\ell/\mathbb{Z}_\ell(i))}{N^{i-1}H^{2i-1}(X,\mathbb{Q}_\ell(i))}.$$

We will see in Proposition 11.5 below that if k is algebraically closed and ℓ is invertible in k, then this coincides with the map that is induced by Bloch's map

$$\lambda:\operatorname{CH}^i(X)[\ell^r]\longrightarrow H^{2i-1}(X,\mathbb{Q}_\ell/\mathbb{Z}_\ell(i))$$

from [Blo79], who used the Gersten conjectures as well as the Weil conjectures in his construction. In particular, for $k=\mathbb{C}$, Proposition 11.5 below will show that λ_{tr} coincides with the transcendental Abel–Jacobi map.

To begin with, note that by Lemmas 4.2 and 4.3 we have exact sequences

$$H^{2i-1}(X,\mathbb{Z}_\ell(i))\longrightarrow H^{2i-1}(F_{i-1}X,\mathbb{Z}_\ell(i))\xrightarrow{\partial}\bigoplus_{x\in X^{(i)}}\mathbb{Z}_\ell[x]\xrightarrow{\iota_*}H^{2i}(X,\mathbb{Z}_\ell(i)) \tag{9.1}$$

and

$$\bigoplus_{x\in X^{(i-1)}} H^1(\kappa(x),\mathbb{Z}_\ell(1))\xrightarrow{\iota_*}H^{2i-1}(F_{i-1}X,\mathbb{Z}_\ell(i))\xrightarrow{f}H^{2i-1}(F_{i-2}X,\mathbb{Z}_\ell(i))$$
$$\xrightarrow{\partial}\bigoplus_{x\in X^{(i-1)}} H^2(\kappa(x),\mathbb{Z}_\ell(1)).$$

Let now $[z]\in\operatorname{Griff}^i(X)[\ell^r]$ for some r and some $z\in\bigoplus_{x\in X^{(i)}}\mathbb{Z}_\ell[x]$. Then z is homologically trivial and so we may choose a lift $\alpha\in H^{2i-1}(F_{i-1}X,\mathbb{Z}_\ell(i))$ via (9.1). This is well-defined up to classes that come from $H^{2i-1}(X,\mathbb{Z}_\ell(i))$. Since $[z]$ is ℓ^r-torsion, (8.1) implies that $\partial(\ell^r\alpha-\iota_*\xi)=0$ for some $\xi\in\bigoplus_{x\in X^{(i-1)}}H^1(\kappa(x),\mathbb{Z}_\ell(1))$. Hence there is a class $\beta\in H^{2i-1}(X,\mathbb{Z}_\ell(i))$ with

$$\beta=\ell^r\alpha-\iota_*\xi\in F^iH^{2i-1}(F_{i-1}X,\mathbb{Z}_\ell(i)).$$

We then define

$$\lambda_{tr}(z):=[\beta/\ell^r]\in\frac{H^{2i-1}(X,\mathbb{Q}_\ell/\mathbb{Z}_\ell(i))}{N^{i-1}H^{2i-1}(X,\mathbb{Q}_\ell(i))}.$$

Lemma 9.1. *The map λ_{tr} above is well-defined.*

Proof. It is clear that $\lambda_{tr}(z)$ does not depend on the choice of α, as this would change β by a class in $\ell^r\cdot H^{2i-1}(X,\mathbb{Z}_\ell(i))$. The class ξ is well-defined up to classes $\zeta\in\bigoplus_{x\in X^{(1)}}H^1(\kappa(x),\mathbb{Z}_\ell(1))$ with $\partial(\iota_*\zeta)=0$. This changes β by

$$\iota_*\zeta\in F^iH^{2i-1}(F_{i-1}X,\mathbb{Z}_\ell(i))\simeq H^{2i-1}(X,\mathbb{Z}_\ell(i))$$

and hence by a class in $N^{i-1}H^{2i-1}(X,\mathbb{Z}_\ell(i))$. In particular,

$$[\beta/\ell^r]\in\frac{H^{2i-1}(X,\mathbb{Q}_\ell/\mathbb{Z}_\ell(i))}{N^{i-1}H^{2i-1}(X,\mathbb{Q}_\ell(i))}$$

remains unchanged. Finally, if we replace z by a cycle z' that is algebraically equivalent to z, then, by (8.1), $z-z'=\partial\iota_*\zeta$ for some $\zeta\in\bigoplus_{x\in X^{(1)}}H^1(\kappa(x),\mathbb{Z}_\ell(1))$. Hence, α needs to be replaced by $\alpha-\iota_*\zeta$ and we may by what we have seen above replace ξ by $\xi+\zeta$, so that the class β does not change at all via this process. This proves the lemma. $\square$

As a consequence of our construction, we note the following for future reference.

Corollary 9.2. *A class $[z]\in\operatorname{Griff}^i(X)[\ell^r]$ lies in the kernel of λ_{tr} if and only if there is a class $\alpha\in H^{2i-1}(F_{i-1}X,\mathbb{Z}_\ell(i))$ with $\partial\alpha=z$ and such that the image of α in $H^{2i-1}(F_{i-2}X,\mathbb{Z}_\ell(i))$ is ℓ^r-torsion. Moreover, if this holds for one $\alpha\in H^{2i-1}(F_{i-1}X,\mathbb{Z}_\ell(i))$ with $\partial\alpha=z$, then for any $\alpha'\in H^{2i-1}(F_{i-1}X,\mathbb{Z}_\ell(i))$ such that $\partial\alpha'$ and $\partial\alpha$ give the same class in $A^i(X)_{\mathbb{Z}_\ell}$, the image of α' in $H^{2i-1}(F_{i-2}X,\mathbb{Z}_\ell(i))$ is ℓ^r-torsion.*

9.2. **The kernel of the transcendental Abel–Jacobi map on torsion cycles.**

Definition 9.3. *We define*

$$\mathcal{T}^i(X)[\ell^r]:=\ker\left(\lambda_{tr}:\operatorname{Griff}^i(X)[\ell^r]\longrightarrow\frac{H^{2i-1}(X,\mathbb{Q}_\ell/\mathbb{Z}_\ell(i))}{N^{i-1}H^{2i-1}(X,\mathbb{Q}_\ell(i))}\right)$$

where λ_{tr} is the transcendental Abel–Jacobi map defined in Section 9.1.

Theorem 9.4. *Let X be a smooth variety over k. Then there are natural isomorphisms*

$$\mathcal{T}^i(X)[\ell^r]\simeq\frac{H^{2i-2}(F_{i-2}X,\mu_{\ell^r}^{\otimes i})}{G^iH^{2i-2}(F_{i-2}X,\mu_{\ell^r}^{\otimes i})}.$$

The above theorem will be deduced from the following result.

Proposition 9.5. *Let X be a smooth variety over k. Then there is a canonical isomorphism*

$$\mathcal{T}^i(X)[\ell^r] \simeq \frac{H^{2i-1}(F_{i-2}X, \mathbb{Z}_\ell(i))[\ell^r]}{F^i H^{2i-1}(F_{i-2}X, \mathbb{Z}_\ell(i))[\ell^r]}.$$

Proof. By Corollary 9.2, there is a well-defined map

$$\varphi : \mathcal{T}^i(X)[\ell^r] \longrightarrow \frac{H^{2i-1}(F_{i-2}X, \mathbb{Z}_\ell(i))[\ell^r]}{F^i H^{2i-1}(F_{i-2}X, \mathbb{Z}_\ell(i))[\ell^r]},$$

given as follows: if $[z] \in \operatorname{Griff}^i(X)[\ell^r]$ lies in the kernel of λ_{tr}, then $\varphi([z]) := [f(\alpha)]$ for any class $\alpha \in H^{2i-1}(F_{i-1}X, \mathbb{Z}_\ell(i))[\ell^r]$ with $[\partial \alpha] = [z] \in \operatorname{Griff}^i(X)[\ell^r]$, where

$$f : H^{2i-1}(F_{i-1}X, \mathbb{Z}_\ell(i)) \longrightarrow H^{2i-1}(F_{i-2}X, \mathbb{Z}_\ell(i))$$

denotes the restriction map. Note that the fact that we quotient out the subgroup $F^i H^{2i-1}(F_{i-2}X, \mathbb{Z}_\ell(i))[\ell^r]$ from $H^{2i-1}(F_{i-2}X, \mathbb{Z}_\ell(i))[\ell^r]$ is necessary to make the above map independent of the choice of α. It remains to see that φ is an isomorphism.

To see that φ is injective, assume that in the above construction $f(\alpha)$ lifts to a class in $H^{2i-1}(X, \mathbb{Z}_\ell(i))$. Then there is a class $\epsilon \in \bigoplus_{x \in X^{(i-1)}} H^1(\kappa(x), \mathbb{Z}_\ell(1))$ such that

$$\alpha - \iota_* \epsilon \in H^{2i-1}(F_{i-1}X, \mathbb{Z}_\ell(i))$$

lifts to $H^{2i-1}(X, \mathbb{Z}_\ell(i))$. Since $\partial \alpha$ and $\partial(\alpha - \iota_*\epsilon) = 0$ have the same image in $\operatorname{Griff}^i(X)_{\mathbb{Z}_\ell}$, it follows that

$$[z] = [\partial \alpha] = [\partial(\alpha - \iota_* \epsilon)] = 0 \in \operatorname{Griff}^i(X)_{\mathbb{Z}_\ell},$$

as we want.

Next, we claim that φ is surjective. For this, let $\beta \in H^{2i-1}(F_{i-2}X, \mathbb{Z}_\ell(i))[\ell^r]$. Then $\partial \beta \in \bigoplus_{x \in X^{(i-1)}} H^2(\kappa(x), \mathbb{Z}_\ell(1))$ is torsion and so it must vanish by Lemma 4.12. Hence, $\beta = f(\alpha)$ for some $\alpha \in H^{2i-1}(F_{i-1}X, \mathbb{Z}_\ell(i))$. The cycle $z = \partial \alpha$ is then homologically trivial by exactness of (9.1) and the class $[z] \in \operatorname{Griff}^i(X)_{\mathbb{Z}_\ell}$ of z satisfies $\varphi([z]) = f(\alpha) = \beta$. This concludes the proof of the proposition □

Proof of Theorem 9.4. The theorem will follow from Proposition 9.5 if we can show that the Bockstein map

$$\delta : H^{2i-2}(F_{i-2}X, \mu_{\ell^r}^{\otimes i}) \longrightarrow H^{2i-1}(F_{i-2}X, \mathbb{Z}_\ell(i))$$

from property (**P2**) induces an isomorphism

$$\frac{H^{2i-2}(F_{i-2}X, \mu_{\ell^r}^{\otimes i})}{G^i H^{2i-2}(F_{i-2}X, \mu_{\ell^r}^{\otimes i})} \simeq \frac{H^{2i-1}(F_{i-2}X, \mathbb{Z}_\ell(i))[\ell^r]}{F^i H^{2i-1}(F_{i-2}X, \mathbb{Z}_\ell(i))[\ell^r]}.$$

By (**P2**), the image of δ is $H^{2i-1}(F_{i-2}X, \mathbb{Z}_\ell(i))[\ell^r]$ and so it suffices to show that

$$\delta^{-1}\left(F^i H^{2i-1}(F_{i-2}X, \mathbb{Z}_\ell(i))[\ell^r]\right) = G^i H^{2i-2}(F_{i-2}X, \mu_{\ell^r}^{\otimes i}),$$

which is exactly the definition of G^i. This proves the theorem. □

Corollary 9.6. *Let X be a smooth variety over k and let $i \geq 3$. Then there is a natural surjective map*

$$\operatorname{coker}\left(\bigoplus_{c\geq 1}\bigoplus_{\tilde{D}} \mathcal{T}^{i-c}(\tilde{D})[\ell^r] \longrightarrow \mathcal{T}^i(X)[\ell^r]\right) \twoheadrightarrow \frac{F^{i-2}H^{2i-2}_{nr}(X,\mu_{\ell^r}^{\otimes i})[\ell^r]}{G^iF^{i-2}H^{2i-2}_{nr}(X,\mu_{\ell^r}^{\otimes i})[\ell^r]},$$

where $\tilde{D}$ runs through all alterations of all subvarieties $D \subset X$ of codimension $c \geq 1$.

Proof. By Theorem 8.1, there is a natural surjection

$$\mathcal{T}^i(X)[\ell^r] \simeq \frac{H^{2i-2}(F_{i-2}X,\mu_{\ell^r}^{\otimes i})}{G^iH^{2i-2}(F_{i-2}X,\mu_{\ell^r}^{\otimes i})} \twoheadrightarrow \frac{F^{i-2}H^{2i-2}_{nr}(X,\mu_{\ell^r}^{\otimes i})}{G^iF^{i-2}H^{2i-2}_{nr}(X,\mu_{\ell^r}^{\otimes i})}.$$

As in the proof of Corollary 7.4, it follows that the image of $\mathcal{T}^{i-c}(\tilde{D})[\ell^r]$ in $\mathcal{T}^i(X)[\ell^r]$ lies in the kernel of this surjection. This concludes the proof. $\square$

9.3. **Consequences of the Merkurjev–Suslin theorem.** To deduce some further consequences from Theorem 9.4, we will need to assume that the following property holds:

P7 for any finitely generated field extension K of k, $H^3(K,\mathbb{Z}_\ell(2))$ is torsion-free.

Proposition 9.7. *Assume that (**P7**) holds. Then the natural map*

$$\frac{H^{2i-2}(F_{i-2}X,\mu_{\ell^r}^{\otimes i})}{G^iH^{2i-2}(F_{i-2}X,\mu_{\ell^r}^{\otimes i})} \longrightarrow \frac{F^{i-2}H^{2i-2}(F_{i-3}X,\mu_{\ell^r}^{\otimes i})}{G^iF^{i-2}H^{2i-2}(F_{i-3}X,\mu_{\ell^r}^{\otimes i})} = \frac{H^{2i-2}_{i-3,nr}(X,\mu_{\ell^r}^{\otimes i})}{G^iH^{2i-2}_{i-3,nr}(X,\mu_{\ell^r}^{\otimes i})}$$

is an isomorphism.

Proof. The map in question is surjective by definition.

Let now $\alpha \in H^{2i-2}(F_{i-2}X,\mu_{\ell^r}^{\otimes i})$ so that the image $\alpha' \in H^{2i-2}(F_{i-3}X,\mu_{\ell^r}^{\otimes i})$ of α is contained in $G^iF^{i-2}H^{2i-2}(F_{i-3}X,\mu_{\ell^r}^{\otimes i})$. By Definition 4.7, this means that there is a lift $\alpha'' \in H^{2i-2}(F_{i-2}X,\mu_{\ell^r}^{\otimes i})$ of α' such that

$$\delta(\alpha'') \in H^{2i-1}(F_{i-2}X,\mathbb{Z}_\ell(i))$$

lifts to a class $\beta \in H^{2i-1}(X,\mathbb{Z}_\ell(i))$. Then $\alpha - \alpha''$ lies in the kernel of

$$H^{2i-2}(F_{i-2}X,\mu_{\ell^r}^{\otimes i}) \longrightarrow H^{2i-2}(F_{i-3}X,\mu_{\ell^r}^{\otimes i}).$$

Lemma 4.2 thus implies that

$$\alpha - \alpha'' = \iota_*\xi \in H^{2i-2}(F_{i-2}X,\mu_{\ell^r}^{\otimes i})$$

for some $\xi \in \bigoplus_{x\in X^{i-2}} H^2(\kappa(x),\mu_{\ell^r}^{\otimes 2})$. The class

$$\delta(\iota_*\xi) = \iota_*(\delta(\xi)) \in \bigoplus_{x\in X^{i-2}} H^3(\kappa(x),\mathbb{Z}_\ell(2))$$

is torsion by property (**P2**) and so it vanishes, because $H^3(\kappa(x),\mathbb{Z}_\ell(2))$ is torsion-free by our assumptions. This shows that $\delta(\alpha)=\delta(\alpha'')$. Since $\delta(\alpha'')$ extends to the class $\beta\in H^{2i-1}(X,\mathbb{Z}_\ell(i))$, the same holds for $\delta(\alpha)$ and so

$$\alpha\in G^iH^{2i-2}(F_{i-2}X,\mu_{\ell^r}^{\otimes i}).$$

This proves that the map in question is injective, as we want. □

Corollary 9.8. *Assume that (**P7**) holds. Then for any smooth variety X, there is a natural isomorphism*

$$\mathcal{T}^i(X)[\ell^r]\simeq\frac{H^{2i-2}_{i-3,nr}\left(X,\mu_{\ell^r}^{\otimes i}\right)}{G^iH^{2i-2}_{i-3,nr}\left(X,\mu_{\ell^r}^{\otimes i}\right)}.$$

Proof. This is an immediate consequence of Theorem 9.4 and Proposition 9.7. □

10. The second piece of the coniveau filtration

Theorem 10.1. *Let X be a smooth variety over k and let $i\geq 0$. Then there is a natural surjection*

$$\varphi:\left(\frac{H^i(X,\mathbb{Z}_\ell(n))}{N^2H^i(X,\mathbb{Z}_\ell(n))}\right)_{\mathrm{tors}}\twoheadrightarrow\frac{G^{\lceil i/2\rceil}H^{i-1}_{nr}(X,\mathbb{Q}_\ell/\mathbb{Z}_\ell(n))}{H^{i-1}_{nr}(X,\mathbb{Q}_\ell(n))}$$

which maps the image of $H^i(X,\mathbb{Z}_\ell(n))_{\mathrm{tors}}$ on the left onto the image of $H^{i-1}(X,\mathbb{Q}_\ell/\mathbb{Z}_\ell(n))$ on the right. Moreover, if $H^{i-2}(K,\mathbb{Z}_\ell(n))$ is torsion-free for any finitely generated field extension K of k, then φ is an isomorphism.

Proof. Let $\alpha\in H^i(X,\mathbb{Z}_\ell(n))$ which becomes torsion modulo $N^2H^i(X,\mathbb{Z}_\ell(n))$. Then there is a big open subset $U\subset X$ such that $\alpha|_U\in H^i(U,\mathbb{Z}_\ell(n))$ is torsion. By the Bockstein sequence (**P2**), $\alpha|_U$ corresponds to a class

$$[\beta]\in\frac{H^{i-1}(U,\mathbb{Q}_\ell/\mathbb{Z}_\ell(n))}{H^{i-1}(U,\mathbb{Q}_\ell(n))}\simeq H^i(U,\mathbb{Z}_\ell(n))_{\mathrm{tors}}$$

with $\beta\in H^{i-1}(U,\mathbb{Q}_\ell/\mathbb{Z}_\ell(n))$. Since $U\subset X$ is big, β gives rise to a class

$$\varphi([\alpha]):=[\beta]\in\frac{H^{i-1}_{nr}(X,\mathbb{Q}_\ell/\mathbb{Z}_\ell(n))}{H^{i-1}_{nr}(X,\mathbb{Q}_\ell(n))}.$$

Since α is defined on X, it follows from the definition of G^* that

$$\varphi([\alpha])\in\frac{G^{\lceil i/2\rceil}H^{i-1}_{nr}(X,\mathbb{Q}_\ell/\mathbb{Z}_\ell(n))}{H^{i-1}_{nr}(X,\mathbb{Q}_\ell(n))},$$

where we use $H^{i-1}_{nr}(X,\mathbb{Q}_\ell)\subset G^{\lceil i/2\rceil}H^{i-1}_{nr}(X,\mathbb{Q}_\ell/\mathbb{Z}_\ell(n))$ from Lemma 4.6. Lemma 4.6 also implies that any class in the above quotient is hit by some $\alpha\in H^i(X,\mathbb{Z}_\ell(n))$. Hence, we have defined the map φ in the proposition and we have seen that it is surjective.

It remains to see that φ is injective under our additional assumption formulated in the theorem. To this end, assume that $\varphi([\alpha])=0$. This implies that there is a non-empty open subset $V \subset U$ such that $[\beta]|_V = 0$. Up to shrinking U (without changing the fact that it is big), this means that there is a class $\beta' \in H^{i-1}(U, \mathbb{Q}_\ell(n))$ with $\beta'|_V = \beta|_V$. Replacing β by $\beta - \beta'$ does not change the class $[\beta]$ in the quotient $\frac{H^{i-1}(U,\mathbb{Q}/\mathbb{Z})}{H^{i-1}(U,\mathbb{Q}_\ell(n))}$ and so we may without loss of generality assume that $\beta' = 0$. Then we have

$$\beta \in \ker\left(H^{i-1}(U, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n)) \longrightarrow H^{i-1}(V, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))\right).$$

Up to shrinking U (without changing the fact that it is big), we may assume that $D := U \setminus V$ is smooth and we denote the inclusion by $\iota : D \to U$. Then $\beta|_V = 0$ implies by (**P1**) that $\beta = \iota_*\gamma$ for some $\gamma \in H^{i-3}(D, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))$. Since $H^{i-2}(k(D), \mathbb{Z}_\ell(n))$ is torsion-free by assumption, we may up to shrinking D (by shrinking U) assume that

$$[\gamma] = 0 \in \frac{H^{i-3}(D, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))}{H^{i-3}(D, \mathbb{Q}_\ell(n))} \simeq H^{i-2}(D, \mathbb{Z}_\ell(n))_{\mathrm{tors}}.$$

That is, γ lifts to a class with rational coefficients and so

$$[\beta] = [\iota_*\gamma] = 0 \in \frac{H^{i-1}(U, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))}{H^{i-1}(U, \mathbb{Q}_\ell(n))}.$$

This shows that φ is injective, hence an isomorphism as we have noted above that it is also surjective. Finally, it is clear from the definition that ϕ identifies the image of $H^i(X, \mathbb{Z}_\ell(n))_{\mathrm{tors}}$ in

$$\left(\frac{H^i(X, \mathbb{Z}_\ell(n))}{N^2 H^i(X, \mathbb{Z}_\ell(n))}\right)_{\mathrm{tors}}$$

with the image of $H^{i-1}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))$ in

$$G^{\lceil i/2 \rceil} H^{i-1}_{nr}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))/H^{i-1}_{nr}(X, \mathbb{Q}_\ell(n)).$$

This proves the theorem. □

11. Bloch's map revisited

From now on, we assume that k is algebraically closed, ℓ is invertible in k and the cohomology functor H^* from Section 3 is given by étale cohomology. Then all properties listed in Section 3 (including (**P1**)–(**P6**)) hold true, see Proposition 3.2.

Bloch [Blo79] used Bloch–Ogus theory [BO74] and the Weil conjectures, proven by Deligne [Del74], to construct a map

$$\lambda : \mathrm{CH}^i(X)[\ell^\infty] \longrightarrow H^{2i-1}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(i)) \tag{11.1}$$

which agrees with the Abel–Jacobi map on homologically trivial cycles in the case where $k = \mathbb{C}$, cf. [Blo79, Proposition 3.7].

In this section we use the ideas of the present paper to construct Bloch's map without Bloch–Ogus theory [BO74], but still relying on Deligne's proof of the Weil conjectures. In fact, we will only use that étale cohomology has the basic properties outlined in

Section 3, together with the following consequence of the Weil conjectures and the smooth specialization theorem in étale cohomology, which goes back to Bloch [Blo79].

Lemma 11.1. *Let k be a an algebraically closed field and let ℓ be a prime that is invertible in k. Let X be a smooth projective variety over k. Then the image of*

$$\ker\left(\partial\circ\iota_*\circ\epsilon:\bigoplus_{x\in X^{(i-1)}}\kappa(x)^*\otimes_{\mathbb{Z}}\mathbb{Z}_\ell\longrightarrow\bigoplus_{x\in X^{(i)}}\mathbb{Z}_\ell[x]\right)$$

via the composition

$$\bigoplus_{x\in X^{(i-1)}}\kappa(x)^*\otimes_{\mathbb{Z}}\mathbb{Z}_\ell\xrightarrow{\epsilon}\bigoplus_{x\in X^{(i-1)}}H^1(\kappa(x),\mathbb{Z}_\ell(1))\xrightarrow{\iota}F^iH^{2i-1}(F_{i-1}X,\mathbb{Z}_\ell(i))$$

is torsion.

Proof. Our proof is similar to [Blo79, Lemma 2.4] with some simplifications as we avoid Bloch–Ogus theory.

Let $\xi\in\bigoplus_{x\in X^{(i-1)}}\kappa(x)^*\otimes_{\mathbb{Z}}\mathbb{Z}_\ell$ with $\partial(\iota_*(\epsilon(\xi)))=0$. By Lemma 4.2, we get

$$\iota_*(\epsilon(\xi))\in F^iH^{2i-1}(F_{i-1}X,\mathbb{Z}_\ell(i))\simeq H^{2i-1}(X,\mathbb{Z}_\ell(i)).$$

If k is the algebraic closure of a finite field, then X and ξ are both defined over $\mathbb{F}_q$ for some finite field $\mathbb{F}_q\subset k$. In particular, $X=X_0\times_{\mathbb{F}_q}k$ and the Frobenius F (given by $x\longmapsto x^q$ on X_0 and by id on k) satisfies

$$F(\iota_*(\epsilon(\xi)))=\iota_*(\epsilon(\xi^q))=q\cdot\iota_*(\epsilon(\xi)).$$

By the Weil conjectures [Del74], q cannot appear as an eigenvalue of the action of F on $H^{2i-1}(X,\mathbb{Q}_\ell(i))$ and so $\iota_*(\epsilon(\xi))$ must be torsion, as claimed.

If k is not the algebraic closure of a finite field, then the result in question follows from spreading out the problem over a finitely generated field, which allows us to specialize to a finite field and so the smooth proper base change theorem yields the result. This proves the lemma. □

Taking the direct limit of the isomorphism from Lemma 6.3, we obtain an isomorphism

$$\phi:\mathrm{CH}^i(X)[\ell^\infty]\xrightarrow{\simeq}\frac{\ker\left(\partial\circ\iota_*:\bigoplus_{x\in X^{(i-1)}}H^1(\kappa(x),\mathbb{Q}_\ell/\mathbb{Z}_\ell(1))\longrightarrow\bigoplus_{x\in X^{(i)}}\mathbb{Q}_\ell/\mathbb{Z}_\ell[x]\right)}{\ker\left(\partial\circ\iota_*\circ\epsilon:\bigoplus_{x\in X^{(i-1)}}\kappa(x)^*\otimes_{\mathbb{Z}}\mathbb{Q}_\ell\longrightarrow\bigoplus_{x\in X^{(i)}}\mathbb{Q}_\ell[x]\right)}.$$

Theorem 11.2. *There is a well-defined map*

$$\lambda':\mathrm{CH}^i(X)[\ell^\infty]\longrightarrow H^{2i-1}(X,\mathbb{Q}_\ell/\mathbb{Z}_\ell(i)),$$

given by

$$\lambda'(\phi^{-1}([\xi])):=-\iota_*\xi\in F^iH^{2i-1}(F_{i-1}X,\mathbb{Q}_\ell/\mathbb{Z}_\ell(i))\simeq H^{2i-1}(X,\mathbb{Q}_\ell/\mathbb{Z}_\ell(i)).$$

Proof. The composition

$$H^{2i-1}(X,\mathbb{Z}_\ell(i))_{\mathrm{tors}} \longrightarrow H^{2i-1}(X,\mu_{\ell^r}^{\otimes i}) \longrightarrow H^{2i-1}(X,\mathbb{Q}_\ell/\mathbb{Z}_\ell(i))$$

is trivial, so that Lemma 11.1 implies that for any $\xi \in \bigoplus_{x\in X^{(i-1)}} \kappa(x)^* \otimes_{\mathbb{Z}} \mathbb{Q}_\ell$ with $\partial(\iota_*(\epsilon(\xi))) = 0$, the class

$$\iota_*(\epsilon(\xi)) \in F^iH^{2i-1}(F_{i-1}X,\mathbb{Q}_\ell/\mathbb{Z}_\ell(i)) \simeq H^{2i-1}(X,\mathbb{Q}_\ell/\mathbb{Z}_\ell(i)).$$

vanishes. This concludes the proof. □

The minus sign in Theorem 11.2 is necessary to make our definition compatible with λ_{tr} defined in Section 9.1; a similar sign issue appeared to Bloch, see [Blo79, p. 112].

Lemma 11.3. *The map λ' constructed above coincides with the map (11.1) constructed by Bloch: $\lambda = \lambda'$.*

Proof. This follows directly from Lemma 4.13 by comparing our construction with Bloch's construction via diagram (2.2) in [Blo79], where we recall that Bloch included the minus sign in [Blo79, p. 112]. □

Remark 11.4. *The above construction of Bloch's map does not use Bloch–Ogus theory. This has for instance the advantage that functoriality of our construction is clear, while it requires some work in Bloch's original approach, see* [Blo79, Section 3].

Proposition 11.5. *The map*

$$\lambda_{tr} : \mathrm{Griff}^i(X)[\ell^r] \longrightarrow \frac{H^{2i-1}(X,\mathbb{Q}_\ell/\mathbb{Z}_\ell(i))}{N^{i-1}H^{2i-1}(X,\mathbb{Q}_\ell(i))}$$

constructed in Section 9.1 is induced by Bloch's map in (11.1).

Proof. Let $z \in \bigoplus_{x\in X^{(i)}}[x]\mathbb{Z}_\ell$ be a homologically trivial cycle. Then $\partial\alpha = z$ for some $\alpha \in H^{2i-1}(F_{j-1}X,\mathbb{Z}_\ell(i))$. Assume that z is ℓ^r-torsion modulo algebraic equivalence. As in Section 9.1, we find classes $\beta \in H^{2i-1}(X,\mathbb{Z}_\ell(i))$ and $\xi \in \bigoplus_{x\in X^{(i-1)}} H^1(\kappa(x),\mathbb{Z}_\ell(i))$ with

$$\beta = \ell^r\cdot\alpha - \iota_*\xi \in F^iH^{2i-1}(F_{j-1}X,\mathbb{Z}_\ell(i)).$$

In particular, $\partial\xi/\ell^r = z$ and so $\psi_r([z]) = [\xi/\ell^r]$, where ψ_r is the isomorphism from Lemma 6.3. By our construction of Bloch's map, we thus find

$$\lambda([z]) = \lambda'([z]) = -\iota_*\xi/\ell^r \in F^iH^{2i-1}(F_{j-1}X,\mathbb{Q}_\ell/\mathbb{Z}_\ell(i)) \simeq H^{2i-1}(X,\mathbb{Q}_\ell/\mathbb{Z}_\ell(i)).$$

On the other hand,

$$\lambda_{tr}([z]) = [\beta/\ell^r] \in H^{2i-1}(X,\mathbb{Q}_\ell/\mathbb{Z}_\ell(i))/N^{i-1}H^{2i-1}(X,\mathbb{Q}_\ell(i))$$

by our construction of λ_{tr} in Section 9.1. The result thus follows from the fact that

$$\beta/\ell^r + \iota_*\xi/\ell^r = \alpha = 0 \in H^{2i-1}(F_{i-1}X,\mathbb{Q}_\ell/\mathbb{Z}_\ell(i))$$

because α is an integral class. This concludes the proposition. $\square$

11.1. **Comparison to the Deligne cycle class map.** Let now $k=\mathbb{C}$ and let X be a smooth complex projective variety. We further denote by H^* singular cohomology, which is compatible with étale cohonology for finite or ℓ-adic coefficients, see [Mil13, §21]. There is a Deligne cycle class map

$$\mathrm{cl}^i_{\mathcal{D}}:\mathrm{CH}^i(X)\longrightarrow H^{2i}_{\mathcal{D}}(X,\mathbb{Z}(i))$$

where $H^{2i}_{\mathcal{D}}(X,\mathbb{Z}(i))$ denotes the $2i$-th Deligne cohomology, see [Voi02, §12.3], which fits into a canonical extension

$$0\longrightarrow J^{2i-1}(X)\longrightarrow H^{2i}_{\mathcal{D}}(X,\mathbb{Z}(i))\xrightarrow{\pi} H^{i,i}(X,\mathbb{Z})\longrightarrow 0 \tag{11.2}$$

see [Voi02, Corollary 12.27]. The Deligne cycle class map induces via the above extension the usual cycle class map (i.e. $\mathrm{cl}^i=\pi\circ\mathrm{cl}^i_{\mathcal{D}}$) and the Abel–Jacobi map on homologically trivial cycles. The construction of Deligne cohomology implies (see e.g. proof of [Voi12, Lemma 3.3]) that there is a canonical isomorphism

$$\mathrm{Tors}(H^{2i}_{\mathcal{D}}(X,\mathbb{Z}(i)))\simeq H^{2i-1}(X,\mathbb{Q}/\mathbb{Z}). \tag{11.3}$$

Proposition 11.6. *Let $k=\mathbb{C}$. Fix an isomorphism $\mathbb{Q}_\ell/\mathbb{Z}_\ell(n)\simeq\mathbb{Q}_\ell/\mathbb{Z}_\ell$ by taking $e^{2\pi i/l^r}$ as a generator of the group $\mu_{\ell^r}(\mathbb{C})$ of ℓ^r-th roots of unity in $\mathbb{C}$. Then Bloch's map*

$$\lambda:\mathrm{CH}^i(X)[\ell^\infty]\longrightarrow H^{2i-1}(X,\mathbb{Q}_\ell/\mathbb{Z}_\ell(i))\simeq H^{2i-1}(X,\mathbb{Q}_\ell/\mathbb{Z}_\ell)$$

is the one induced by the Deligne cycle class map above.

Proof. Recall that λ coincides with the Abel–Jacobi map on homologically trivial cycles, see [Blo79, Proposition 3.7]. One also checks that the restriction of π in (11.2) to the torsion subgroup (11.3) is given by the Bockstein map

$$H^{2i-1}(X,\mathbb{Q}/\mathbb{Z})\longrightarrow H^{2i}(X,\mathbb{Z})$$

induced by $0\to\mathbb{Z}\to\mathbb{Q}\to\mathbb{Q}/\mathbb{Z}\to 0$. In view of the extension (11.2) it thus suffices to prove that the composition

$$\mathrm{CH}^i(X)[\ell^\infty]\xrightarrow{\lambda} H^{2i-1}(X,\mathbb{Q}_\ell/\mathbb{Z}_\ell(i))\xrightarrow{\delta} H^{2i}(X,\mathbb{Z}_\ell(i))$$

coincides with the usual cycle class map. To this end, let $z\in\bigoplus_{x\in X^{(i)}}[x]\mathbb{Z}_\ell$ be ℓ^r-torsion modulo rational equivalence. Then $\ell^r z=\partial(\iota_*(\epsilon(\xi)))$ for some $\xi\in\bigoplus_{x\in X^{(i-1)}}\kappa(x)^*\otimes_{\mathbb{Z}}\mathbb{Z}_\ell$ and we have $\lambda([z])=\iota_*(\epsilon(\xi)/\ell^r)$ by Lemma 11.3. The result in question thus follows from

$$\delta(\iota_*(\epsilon(\xi)/\ell^r))=\iota_*z\in H^{2i}(X,\mathbb{Z}_\ell(i)),$$

which is a direct consequence of the explicit construction of the Bockstein map δ for étale (or singular) cohomology. This concludes the proof. $\square$

Remark 11.7. *Voisin states in* [Voi12, Proposition 3.4] *a similar result for the transcendental Deligne cycle class map, but it seems that there is a sign missing in loc. cit., as her result is not compatible with* [Blo79, Proposition 3.7].

12. Proof of main results

12.1. **Proof of Theorem 1.1 and Corollary 1.2.** The following is a corollary of Theorem 5.1.

Corollary 12.1. *Let k be an algebraically closed field and let ℓ be a prime invertible in k. Let X be a smooth (not necessarily proper) variety over k and take for the cohomology functor from Section 3 étale cohomology. Then for any i and whenever $A(n)$ is one of the coefficients in (3.1), there are canonical isomorphisms*

$$H^i_{0,nr}(X, A(n)) \simeq H^0(X, \mathcal{H}^i_X(A(n))) \quad \textit{and} \quad H^{d+i}_{i,nr}(X, A(n)) \simeq H^i(X, \mathcal{H}^d_X(A(n))),$$

where $\mathcal{H}^i_X(A(n))$ denotes the Zariski sheaf on X associated to the presheaf $U \longmapsto H^i(U, A(n))$.

Proof. The first isomorphism is a direct consequence of Lemma 4.8 and [BO74]; alternatively, it could also be deduced from [BO74] and Theorem 5.1.

For the second isomorphism, note that the fact that k is algebraically closed implies $H^i(K, A(n)) = 0$ whenever K is a finitely generated field extension of k of transcendence degree less than i. Lemma 4.3 thus implies

$$H^i_{j,nr}(X, A(n)) = 0 \quad \text{for all } j < i - \dim X.$$

The result in question is then an immediate consequence of Theorem 5.1 together with the fact that the group $H^i(X, \mathcal{H}^j(A(n)))$ defined in Section 5 is indeed the cohomology of the Zariski sheaf $\mathcal{H}^j_X(A(n))$ by [BO74]. This concludes the proof. □

Proof of Theorem 1.1. Theorem 1.1 follows from Proposition 3.3 and Theorem 5.1. □

Proof of Corollary 1.2. Using Proposition 3.3, the corollary follows from Theorem 1.1 by the same argument as in Corollary 12.1. This concludes the proof. □

12.2. **Proof of Theorems 1.3 and 1.6.**

Theorem 12.2. *Let k be an algebraically closed field and let ℓ be a prime invertible in k. Let X be a smooth (not necessarily proper) variety over k and take for the cohomology functor from Section 3 étale cohomology. Then for any i there are canonical* *isomorphisms*

$$Z^{2i}(X)[\ell^\infty] \simeq \frac{H^{2i-1}_{i-2,nr}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(i))}{H^{2i-1}_{i-2,nr}(X, \mathbb{Q}_\ell(i))};$$

$$\mathrm{Griff}^i(X) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \simeq \frac{H^{2i-1}_{i-2,nr}(X, \mathbb{Z}_\ell(i))}{H^{2i-1}(X, \mathbb{Z}_\ell(i))};$$

$$\mathcal{T}^i(X)[\ell^\infty] \simeq \frac{H^{2i-2}_{i-3,nr}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(i))}{G^i H^{2i-2}_{i-3,nr}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(i))};$$

and

$$\left(\frac{H^i(X, \mathbb{Z}_\ell(n))}{N^2 H^i(X, \mathbb{Z}_\ell(n))}\right)_{\text{tors}} \simeq \frac{G^{\lceil i/2 \rceil} H^{i-1}_{nr}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))}{H^{i-1}_{nr}(X, \mathbb{Q}_\ell(n))}.$$

Proof. By Proposition 3.2, the properties outlined in Section 3 are satisfied for étale cohomology (because k is algebraically closed and ℓ is invertible in k). By Lemma 6.2, $A^i(X)_{\mathbb{Z}_\ell}$ defined in Section 6.1 coincides with the Chow group of codimension i cycles on X modulo algebraic equivalence and with coefficients in $\mathbb{Z}_\ell$. By taking direct limits, the first two isomorphisms follow therefore from Theorems 7.2 and 8.1.

By Merkrujev–Suslin's proof of the Bloch–Kato conjecture in degree two [MS83], $H^3(K, \mathbb{Z}_\ell(2))$ is torsion-free for any field extension K of k (cf. proof of Lemma 4.12) and so Proposition 9.7 and Theorem 9.4 imply the third isomorphism above after taking direct limits. Finally, the last isomorphism in question follows from Theorem 10.1 and Voevodsky's proof of the Bloch–Kato conjecture [Voe11], which implies that $H^i(K, \mathbb{Z}_\ell(i-1))$ for all i and any field extension K of k. □

Proof of Theorem 1.3. Using Proposition 3.3, Theorem 1.3 follows by the same argument as in Theorem 12.2, where $\mathbb{Z}_\ell(n)$, $\mathbb{Q}_\ell(n)$ and $\mathbb{Q}_\ell/\mathbb{Z}_\ell(n)$ are replaced by $\mathbb{Z}(n)$, $\mathbb{Q}(n)$ and $\mathbb{Q}/\mathbb{Z}(n)$, respectively. This proves the theorem. □

Proof of Corollary 1.4. This follows easily from Theorem 1.3; for more details, see Corollaries 7.4, 8.2 and 9.6. □

Proof of Theorem 1.6. Using Proposition 3.3, the result follows from Theorem 10.1 where the the coefficients $\mathbb{Z}_\ell(n)$, $\mathbb{Q}_\ell(n)$ and $\mathbb{Q}_\ell/\mathbb{Z}_\ell(n)$ are replaced by $\mathbb{Z}(n)$, $\mathbb{Q}(n)$ and $\mathbb{Q}/\mathbb{Z}(n)$, respectively. □

13. Applications

We will need the following result that is proven with methods from [Sch19].

Theorem 13.1. *For any positive integer n, there is a smooth projective unirational variety Y of dimension $3n$ such that the composition*

$$H^{2i}(Y, \mathbb{Z}/2) \longrightarrow H^{2i}(\mathbb{C}(Y), \mathbb{Z}/2) \longrightarrow H^{2i}(\mathbb{C}(Y), \mathbb{Q}/\mathbb{Z})$$

is nonzero for all $i = 1, \dots, n$.

Proof. By the proof of [Sch19, Theorem 1.5], there is a unirational smooth complex projective threefold T together with a morphism $f : T \to \mathbb{P}^2$ whose generic fibre is a conic, such that the following holds:

- the class $\alpha = (x_1/x_0, x_2/x_0) \in H^2(\mathbb{C}(\mathbb{P}^2), \mathbb{Z}/2)$ has the property that $f^*\alpha \in H^2_{nr}(T, \mathbb{Z}/2)$ is unramified and nontrivial;
- there is a specialization T_0 of T such that the specialization $f_0 : T_0 \to \mathbb{P}^2$ of f has the property that its generic fibre has a $\mathbb{C}(\mathbb{P}^2)$-rational point in its smooth locus.

Let us now consider $Y := T^n$, which is a smooth complex projective variety of dimension $3n$ that is unirational. Let $\mathrm{pr}_j : Y \to T$ denote the projection onto the j-th factor and consider the class

$$\gamma_i := \mathrm{pr}_1^* f^*\alpha \cup \mathrm{pr}_2^* f^*\alpha \cup \dots \cup \mathrm{pr}_i^* f^*\alpha \in H^{2i}(\mathbb{C}(Y), \mathbb{Z}/2).$$

Since α is of degree two, the unramified class $f^*\alpha$ must admit a lift to a class in $H^2(T, \mathbb{Z}/2)$, see Lemma 4.11. Hence, γ_i admits a lift to a class in $H^{2i}(Y, \mathbb{Z}/2)$ and so

$$\gamma_i \in F^i H^{2i}_{nr}(Y, \mathbb{Z}/2).$$

It remains to show that the image γ_i' of γ_i in $H^{2i}(\mathbb{C}(Y), \mathbb{Q}/\mathbb{Z})$ is nonzero for all $i = 1, \dots, n$. By construction of the class γ_i, it suffices to prove that γ_n' is nonzero and our argument is similar to the proofs of [Sch19, Proposition 6.1] and [Sch20a, Theorem 5.3(3)].

Assume for a contradiction that $\gamma' := \gamma_n'$ is zero in $H^{2i}(\mathbb{C}(Y), \mathbb{Q}/\mathbb{Z})$. Let us then specialize T to T_0. Then Y specializes to a projective variety Y_0 together with a morphism $Y_0 \to (\mathbb{P}^2)^n$ whose generic fibre admits a rational point in its smooth locus. The specialization γ_0' of γ' vanishes, because γ' vanishes by assumption. It follows that the restriction of γ_0' to the rational point in the smooth locus of the generic fibre of $Y_0 \to (\mathbb{P}^2)^n$ is zero. This restriction in turn computes explicitly as the image of

$$\mathrm{pr}_1^* \alpha \cup \mathrm{pr}_2^* \alpha \cup \dots \cup \mathrm{pr}_n^* \alpha \in H^{2n}\left(\mathbb{C}\left((\mathbb{P}^2)^n\right), \mathbb{Z}/2\right)$$

in $H^{2n}(\mathbb{C}((\mathbb{P}^2)^n), \mathbb{Q}/\mathbb{Z})$. But this class is nonzero, as one can check by computing successive residues. This is a contradiction, which concludes the proof. □

13.1. Integral Hodge conjecture for uniruled varieties.

Proof of Theorem 1.5. By Theorem 13.1, there is a unirational smooth complex projective variety Y of dimension $3n$ such that $H^{2i}(Y, \mathbb{Q}/\mathbb{Z}) \longrightarrow H^{2i}(\mathbb{C}(Y), \mathbb{Q}/\mathbb{Z})$ is nonzero for all $i = 1, \dots, n$. It thus follows from a theorem of Colliot-Thélène [CT19, Theorem 1.1] that there is an elliptic curve E such that the product $X = Y \times E$ has the property that

$$H^{2i+1}(X, \mathbb{Q}/\mathbb{Z}) \longrightarrow H^{2i+1}(\mathbb{C}(X), \mathbb{Q}/\mathbb{Z})$$

is nonzero for all $i = 1, \dots, n$. Since the Chow group of zero-cycles of X is supported on a curve (i.e. $\mathrm{CH}_0(\{pt.\} \times E) \to \mathrm{CH}_0(Y \times E)$ is surjective), the rational unramified cohomology groups of X above degree one vanish by a simple Bloch–Srinivas decomposition of the diagonal argument, see e.g. [CTV12, Proposition 3.3.(i)].[4] The result thus follows from Corollary 1.4. □

13.2. **Artin–Mumford invariant beyond degree three.**

Proof of Corollary 1.7. Rationally connected varieties have no rational unramified cohomology in positive degrees, see e.g. [CTV12, Proposition 3.3.(i)]. The claim in Corollary 1.7 follows therefore directly from Theorems 1.6 (resp. 10.1) and 13.1. □

Remark 13.2. *Corollary 1.7 does not rely on the Bloch–Kato conjecture, because the surjectivity of φ in Theorem 10.1 is a formal consequence – only injectivity relies on* [Voe11].

Appendix A. A spectral sequence that computes unramified cohomology

Let X be a smooth variety over an algebraically closed field k. If $k \neq \mathbb{C}$ we fix a prime ℓ in k and a positive integer r, and if $k = \mathbb{C}$, then we fix an abelian group A. We then use the unifying notation

$$A(n) := \begin{cases} A \otimes_{\mathbb{Z}} \mathbb{Z}(n) & \text{if } k = \mathbb{C}; \\ \mu_{\ell^r}^{\otimes n} & \text{if } k \neq \mathbb{C}; \end{cases}$$

where $\mathbb{Z}(n) \simeq \mathbb{Z}$ denotes the n-th Tate twist of $\mathbb{Z}$ and

$$H^i(X, A(n)) := \begin{cases} H^i_{\mathrm{sing}}(X(\mathbb{C}), A(n)), & \text{if } k = \mathbb{C}; \\ H^i_{\text{ét}}(X, \mu_{\ell^r}^{\otimes n}), & \text{if } k \neq \mathbb{C}. \end{cases} \tag{A.1}$$

Let further $\pi : X_{\text{ét}} \to X_Z$ (resp. $\pi : X(\mathbb{C}) \to X(\mathbb{C})_{Zar}$) be the continuous map from the étale site to the Zariski site of X if $k \neq \mathbb{C}$ (resp. the analytic site to the Zariski site of $X(\mathbb{C})$ if $k = \mathbb{C}$). Consider the Leray spectral sequence

$$'E_2^{p,q} \Longrightarrow H^{p+q}(X, A(n)) \quad \text{with} \quad 'E_2^{p,q} = H^p(X, \mathcal{H}^q_X(A(n))) \quad \text{and} \quad \mathcal{H}^i_X(A(n)) = R^i\pi_* A(n).$$

[4]This step does not use the Bloch–Kato conjectures, as we are only concerned about the vanishing of unramified cohomology with rational coefficients and so torsion-freeness of $H^i_{nr}(X, \mathbb{Z})$ is not needed.

Bloch and Ogus [BO74] proved that this is the second page of the coniveau spectral sequence.[5] This implies in particular

$$H^i_{nr}(X, A(n)) \simeq H^0(X, \mathcal{H}^i_X(A(n))). \tag{A.2}$$

We show in this appendix that one can reverse the roles of étale (resp. singular) and unramified cohomology in the above spectral sequence and construct a spectral sequence which converges to $H^{p+q}_{nr}(X, A(n))$ instead of $H^{p+q}(X, A(n))$. The result is not used explicitly anywhere else in this paper, but it was the starting point of our work and we believe that the result could be of independent interest. Our result is motivated by the idea that the natural filtration on unramified cohomology from Definition 4.4 should be the shadow of a spectral sequence.

Theorem A.1. *Let X be a smooth variety over a separably closed field k and let $H^i(-, A(n))$ be the functor from (A.1). There is a convergent spectral sequence*

$$E_2^{p,q} \Longrightarrow H^{p+q}_{nr}(X, A(n))$$

where

- *for $0 \leq p < q - 1$:*
$$E_2^{p,q} = H^{p+1}(X, \mathcal{H}^q_X(A(n)));$$
- *for $p = q - 1 > 0$:*
$$E_2^{q-1,q} = \frac{\ker(\mathrm{cl}^q \otimes A(n-q) : \mathrm{CH}^q(X) \otimes A(n-q) \longrightarrow H^{2q}(X, A(n)))}{\sim_{\mathrm{alg}}};$$
- *for $p = q > 0$:*
$$E_2^{p,p} = \frac{H^{2p}(X, A(n))}{\mathrm{im}(\mathrm{cl}^p \otimes A(n-p))};$$
- *for $p = q + 1 > 0$:*
$$E_2^{p,p-1} = H^{2p-1}(X, A(n));$$

$E_2^{0,0} = A(n)$, and where $E_2^{p,q} = 0$ for all other p and q. The filtration $(F')^*$ on $H^{p+q}_{nr}(X, A(n))$ that is induced by this spectral sequence coincides with F^* from Section 4.4 for all $q \geq p \geq 1$.

By Lemma 4.3, the above theorem shows in particular that

$$E_\infty^{p,q} = gr^p_F H^{p+q}_{nr}(X, A(n)) \quad \text{for all } q > p \geq 1,$$
$$E_\infty^{p,p} = \mathrm{im}(H^{2p}(X, A(n)) \longrightarrow H^{2p}_{nr}(X, A(n)));$$
$$E_\infty^{p+1,p} = \mathrm{im}(H^{2p+1}(X, A(n)) \longrightarrow H^{2p+1}_{nr}(X, A(n))).$$

[5] Bloch and Ogus assumed that A is a ring, see [BO74, Example 2.3], but the result holds more generally because any abelian group is the direct limit of (finite) sums of rings.

The following table illustrates (one part of) the E_2-page of the above spectral sequence.

$$
\begin{array}{ccccc}
E_2^{0,3} & E_2^{1,3} & \frac{\ker(\mathrm{cl}^3 \otimes A(n-3))}{\sim_{\mathrm{alg}}} & \frac{H^6(X,A(n))}{\mathrm{im}(\mathrm{cl}^3 \otimes A(n-3))} & H^7(X,A(n)) \\
E_2^{0,2} & \frac{\ker(\mathrm{cl}^2 \otimes A(n-2))}{\sim_{\mathrm{alg}}} & \frac{H^4(X,A(n))}{\mathrm{im}(\mathrm{cl}^2 \otimes A(n-2))} & H^5(X,A(n)) & \\
\frac{\ker(\mathrm{cl}^1 \otimes A(n-1))}{\sim_{\mathrm{alg}}} & \frac{H^2(X,A(n))}{\mathrm{im}(\mathrm{cl}^1 \otimes A(n-1))} & H^3(X,A(n)) & & \\
A(n) & H^1(X,A(n)) & & &
\end{array}
$$

A.1. **An exact couple.** Let X be a smooth variety of dimension d over an algebraically closed field k. In the above notation, we define

(A.3)

$$
E_1^{p,q} := \begin{cases} \bigoplus_{x\in X^{(p+1)}} H^{q-p-1}(\kappa(x), A(n-p-1)) & \text{for } 0 < p < q \leq d; \\ \operatorname{coker}\left(H^q(k(X), A(n)) \xrightarrow{\partial} \bigoplus_{x\in X^{(1)}} H^{q-1}(\kappa(x), A(n-1))\right) & \text{if } p = 0 \text{ and } q > 0; \\ H^{p+q}(X, A(n)) & \text{if } p = q \text{ or } p = q+1; \\ 0 & \text{otherwise.} \end{cases}
$$

We next define

$$
D_1^{p,q} := \begin{cases} H^{p+q}_{nr}(X, A(n)) & \text{for } p \leq 0; \\ H^{p+q}(F_pX, A(n)) & \text{for } 0 < p \leq q; \\ 0 & \text{otherwise.} \end{cases}
$$

For all p and q, we have natural maps

$$f : D_1^{p,q} \longrightarrow D_1^{p-1,q+1}.$$

There are also natural maps

$$\iota_* : E_1^{p,q} \longrightarrow D_1^{p+1,q},$$

which are induced by the Gysin pushforward maps for $0 \leq p < q \leq d$ and zero otherwise.

Finally, we define

$$\partial : D_1^{p,q} \longrightarrow E_1^{p,q}$$

to be the natural residue map from Lemma 4.2 if $0 < p < q$, to be the natural isomorphism

$$D_1^{p,p} = H^{2p}(F_pX, A(n)) \longrightarrow H^{2p}(X, A(n))$$

given by Lemma 4.3 if $p = q$, and to be zero otherwise.

Proposition A.2. *With the above definitions, the sequence*

$$\dots \longrightarrow D_1^{p,q} \xrightarrow{\partial} E_1^{p,q} \xrightarrow{\iota_*} D_1^{p+1,q} \xrightarrow{f} D_1^{p,q+1} \xrightarrow{\partial} E_1^{p,q+1} \longrightarrow \dots$$

is exact for all p and q.

Proof. This is a simple consequence of Lemma 4.2 and the way we defined the maps above. We leave it to the reader to check the details. ☐

Putting

$$D_1 := \bigoplus_{p,q} D_1^{p,q} \quad \text{and} \quad E_1 := \bigoplus_{p,q} E_1^{p,q}$$

we get by Proposition A.2 an exact couple

$$\begin{tikzcd} D_1 \arrow[rr, "f"] & & D_1 \arrow[dl, "\partial"] \\ & E_1 \arrow[ul, "\iota_*"] & \end{tikzcd},$$

where f is of bidegree $(-1, 1)$, ∂ is of bidegree $(0, 0)$ and ι_* is of bidegree $(1, 0)$.

This implies

Corollary A.3. *There is a convergent spectral sequence*

$$E_1^{p,q} \Longrightarrow H_{nr}^{p+q+1}(X, A(n)),$$

where the differential is given by

$$d_1 = \begin{cases} \partial \circ \iota_* & \text{for } 0 < p < q \leq d \text{ with } q \neq p+1; \\ \mathrm{cl}^q_{A(n-q)} & \text{if } q = p+1; \\ 0 & \text{otherwise,} \end{cases}$$

and $\mathrm{cl}^q_{A(n-q)} : \bigoplus_{x \in X^{(q)}} A(n-q)[x] \longrightarrow H^{2q}(X, A(n))$ denotes the cycle class map.

Proof. Since $D_1^{p,q} = 0$ for $p > q$, the above exact couple yields a converging spectral sequence which computes

$$\varinjlim_m D_1^{p-m,q+m} = H_{nr}^{p+q}(X, A(n)),$$

see [Wei94]. The differential of this spectral sequence is given by

$$d_1 = \partial \circ \iota_* : E_1^{p,q} \longrightarrow E_1^{p+1,q}.$$

This proves the corollary by our definition of ∂ and ι_* on the E_1-page. ☐

Remark A.4. *The main point in the above construction is the fact that we chopped off the exact sequences from Lemma 4.2 in such a way that $D_1^{p,q}=0$ for $p>q$ holds and so the spectral sequence converges to the direct limit of the groups $D_1^{p-m,q+m}$. In fact, Lemma 4.2 also gives on the nose an exact couple*

$$\begin{tikzcd} {'D_1} \arrow[rr, "f"] & & {'D_1} \arrow[dl, "\partial"] \\ & {'E_1} \arrow[ul, "\iota_*"] & \end{tikzcd},$$

with ${'D_1}^{p,q}=H^{p+q}(F_pX,A(n))$ and ${'E_1}\bigoplus_{x\in X^{(p+1)}}H^{q-p-1}(\kappa(x),A(n-p-1))$, but the associated spectral sequence converges to the inverse limit of the groups $D_1^{p-m,q+m}$, which is $H^{p+q}(X,A(n))$. In fact, the spectral sequence constructed this way coincides by Lemma 4.13 with the ${'E_1}$-page of the coniveau spectral sequence, see e.g. [BO74, Proposition 3.9] *where this spectral sequence had been constructed starting with slightly different long exact sequences on the level of homology.*

The following table illustrates (parts of) the E_1-page of the spectral sequence from Corollary A.3:

$$\frac{\bigoplus_{x\in X^{(1)}}H^2(\kappa(x),A(n-1))}{\partial(H^3(k(X),A(n-1)))}\xrightarrow{\partial\circ\iota_*}\bigoplus_{x\in X^{(2)}}H^1(\kappa(x),A(n-2))\xrightarrow{\partial\circ\iota_*}\bigoplus_{x\in X^{(3)}}[x]A(n-3)\xrightarrow{\mathrm{cl}_A}\cdots$$

$$\frac{\bigoplus_{x\in X^{(1)}}H^1(\kappa(x),A(n-1))}{\partial(H^2(k(X),A(n-1)))}\xrightarrow{\partial\circ\iota_*}\bigoplus_{x\in X^{(2)}}[x]A(n-2)\xrightarrow{\mathrm{cl}_A}H^4(X,A(n))\xrightarrow{0}\cdots$$

$$\bigoplus_{x\in X^{(1)}}[x]A(n-1)\xrightarrow{\mathrm{cl}_A}H^2(X,A(n))\xrightarrow{0}H^3(X,A(n))$$

$$H^0(X,A(n))\xrightarrow{0}H^1(X,A(n))$$

Proof of Theorem 7.2. Let $E_2^{p,q}\implies H_{nr}^{p+q}(X,A(n))$ be the spectral sequence, given by the E_2-page of the spectral sequence from Corollary A.3. By Lemma 4.13, $E_2^{p,q}={'E_2^{p+1,q}}$ for $0\leq p<q-1$, where ${'E_2^{p,q}}$ denotes the E_2-terms of the coniveau spectral sequence. It thus follows from [BO74, (0.2)] that

$$E_2^{p,q}=H^{p+1}(X,\mathcal{H}_X^q(A(n)))\ \text{ for } 0\leq p<q-1.$$

By Lemma 6.2,

$$\operatorname{coker}\left(\partial\circ\iota_*:\bigoplus_{x\in X^{(p+1)}}H^1(\kappa(x),A(n-p-1))\longrightarrow\bigoplus_{x\in X^{(p+1)}}[x]A(n)\right)$$

identifies naturally with $(\mathrm{CH}^{p+1}(X)\otimes A(n-p-1))/\sim_{\mathrm{alg}}$ (c.f. [BO74, 7.3]) and so the E_2-term is given as claimed in Theorem A.1. Moreover, by the theory of exact couples (see e.g. [Wei94]), the filtration F' on $H^{p+q}_{nr}(X,A(n))$ that is induced by the above spectral sequence is given by

$$(F')^pH^{p+q}_{nr}(X,A(n))=\mathrm{im}(D_1^{p,q}\longrightarrow\varinjlim_m D_1^{p-m,q+m})$$

and so

$$(F')^pH^{p+q}_{nr}(X,A(n))=\mathrm{im}\left(H^{p+q}(F_pX,A(n))\longrightarrow H^{p+q}_{nr}(X,A(n))\right)$$

for all $0<p\leq q$. Hence,

$$(F')^pH^{p+q}_{nr}(X,A(n))=F^pH^{p+q}_{nr}(X,A(n))$$

for all $q\geq p\geq 1$ by the definition of F^* on $H^i_{nr}(X,A)$. This concludes the proof. $\square$

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