Monodromies and Poincaré series of quasihomogeneous complete intersections

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Abstract

We give a formula connecting the Saito duals of the reduced zeta functions of the monodromies of defining equations of a quasihomogeneous complete intersection, the Poincaré series of its coordinate ring, and orbit invariants with respect to the natural $\mathbb{C}^*$-action.

There were obtained a series of results which describe connections between Poincaré series of filtrations in coordinate rings and zeta functions of monodromy transformations of singularities (e.g., [CDG1, CDG2, E, EG]). In the last two papers which are devoted to quasihomogeneous singularities orbit invariants of the natural $\mathbb{C}^*$-action turn out to participate. The result of [EG] connects the Poincaré series of the weighted homogeneous filtration on the ring of germs of functions on a quasihomogeneous hypersurface singularity which is non-degenerate with respect to its Newton diagram, the Saito dual to the (reduced) zeta function of the classical monodromy transformation, and the orbit invariants. Up to now, the proofs of all these results essentially consist of computing both sides of the equations (in terms of a resolution or in terms of the weights of the variables) which turn out to coincide.

All previous results were obtained either for hypersurface singularities, or (in [E]) for very special (quasihomogeneous) complete intersections: surfaces in $\mathbb{C}^4$ with the first equation of the type A_1. Therefore using them it

*Partially supported by the DFG-programme "Global methods in complex geometry" (Eb 102/4–2), grants RFBR–01–01–00739, INTAS–00-259, NWO–RFBR–047.008.005. Keywords: quasihomogeneous complete intersection, Poincaré series, zeta function of monodromy. AMS Math. Subject Classification: 14B05, 14M10, 32S40.

was difficult to guess a possible form of a similar connection in more general cases. Here we give a generalization of the result of [EG] for quasihomogeneous complete intersection singularities (neither necessarily isolated nor non-degenerate with respect to the Newton diagram). This somewhat clarifies the result of [EG] and the formulae from [E] for certain complete intersections.

To each irreducible plane curve singularity C with g Puiseux pairs there corresponds a quasihomogeneous curve in $\mathbb{C}^{g+1}$ which is a complete intersection and has the same semigroup as C. Recently J. Stevens ([St]) showed that using this construction the result of [CDG1] can be deduced from Theorem 1 below.

Let $f_1, \ldots, f_k$ be quasihomogeneous functions on $\mathbb{C}^n$ of degrees $d_1, \ldots, d_k$ with respect to weights $q_1, \ldots, q_n$. Here $q_1, \ldots, q_n$ are positive integers with $\gcd(q_1, \ldots, q_n) = 1$, $f_j(\lambda^{q_1}x_1, \ldots, \lambda^{q_n}x_n) = \lambda^{d_j} f_j(x_1, \ldots, x_n)$, $\lambda \in \mathbb{C}$. We suppose that the equations $f_1 = f_2 = \ldots = f_k = 0$ define a complete intersection X in $\mathbb{C}^n$. There is a natural $\mathbb{C}^*$-action on the space $\mathbb{C}^n$ defined by $\lambda * (x_1, \ldots, x_n) = (\lambda^{q_1}x_1, \ldots, \lambda^{q_n}x_n)$, $\lambda \in \mathbb{C}^*$.

Let $A = \mathbb{C}[x]/(f_1, \ldots, f_k)$ be the coordinate ring of X. There is a natural grading on the ring A: A_s is the set of functions $g \in A$ such that $g(\lambda * x) = \lambda^s g(x)$. Let $P_X(t) = \sum_{s=0}^{\infty} \dim A_s \cdot t^s$ be the Poincaré series of the graded algebra $A = \oplus_{s=0}^{\infty} A_s$. One has

$$P_X(t) = \frac{\prod_{j=1}^{k}(1 - t^{d_j})}{\prod_{i=1}^{n}(1 - t^{q_i})}.$$

For $0 \leq j \leq k$, let $X^{(j)}$ be the complete intersection given by the equations $f_1 = \ldots = f_j = 0$ ($X^{(0)} = \mathbb{C}^n$, $X^{(k)} = X$). The restriction of the function f_j ($j = 1, \ldots, k$) to the variety $X^{(j-1)}$ defines a locally trivial fibration $X^{(j-1)} \setminus X^{(j)} \to \mathbb{C}^*$. Let $F^{(j)} = f_j^{-1}(1) \cap X^{(j-1)}$ be the (Milnor) fibre of this fibration (the fibre $F^{(j)}$ is not necessarily smooth) and $h^{(j)} : F^{(j)} \to F^{(j)}$ be the classical monodromy transformation of it. For a map $h : Z \to Z$ of a topological space Z, let $\zeta_h(t)$ be its *zeta function*

$$\zeta_h(t) = \prod_{p \geq 0} \left\{ \det \left(\mathrm{id} - t \cdot h_*|_{H_p(Z)} \right) \right\}^{(-1)^p}.$$

If, in the definition, we use the actions of the operators h_* on the homology groups $\widetilde{H}_p(Z)$ reduced modulo a point, we get the *reduced* zeta function

$$\widetilde{\zeta}_h(t) = \frac{\zeta_h(t)}{(1 - t)}.$$

Let

$$\widetilde{\zeta}_j(t) := \widetilde{\zeta}_{h^{(j)}}(t).$$

If both $X^{(j)}$ and $X^{(j-1)}$ have isolated singularities at the origin then $\widetilde{H}_p(F^{(j)})$ is non-trivial only for $p = n - j$ and therefore, if $n - j \geq 1$,

$$\left(\widetilde{\zeta}_j(t)\right)^{(-1)^{n-j}} = \det\left(\mathrm{id} - t \cdot h_*^{(j)}|_{H_{n-j}(F^{(j)})}\right)$$

is the characteristic polynomial of the classical monodromy operator $h_*^{(j)}$.

One can show (see below) that $(h_*^{(j)})^{d_j} = \mathrm{id}$ and therefore $\widetilde{\zeta}_j(t)$ can be written in the form

$$\prod_{\ell | d_j} (1 - t^\ell)^{\alpha_\ell}, \;\; \alpha_\ell \in \mathbb{Z}.$$

Following K. Saito [S1, S2], we define the Saito dual to $\widetilde{\zeta}_j(t)$ to be the rational function

$$\widetilde{\zeta}_j^*(t) = \prod_{m | d_j} (1 - t^m)^{-\alpha_{(d_j/m)}}$$

(note that different degrees d_j are used for different j).

Let $Y^{(j)} = (X^{(j)} \setminus \{0\})/\mathbb{C}^*$ be the space of orbits of the $\mathbb{C}^*$-action on $X^{(j)} \setminus \{0\}$. There is a natural stratification of the variety $Y^{(j)}$ defined by the types of the $\mathbb{C}^*$-orbits: $Y_m^{(j)}$ is the set of orbits for which the isotropy group is the cyclic group of order m. (The stratum $Y_m^{(j)}$ can be nonempty only if m is equal to the greatest common divisor of a subset of the weights $q_1, \ldots, q_n$.) Let

$$\mathrm{Or}_X(t) := \prod_{m \geq 1} (1 - t^m)^{\chi(Y_m^{(k)})}$$

be the product of cyclotomic polynomials with exponents corresponding to the partition of the complete intersection $X = X^{(k)}$ into parts of different orbit types; here $\chi(Z)$ denotes the Euler characteristic of a topological space Z.

Theorem 1

$$\prod_{j=1}^{k} \widetilde{\zeta}_j^*(t) = P_X(t) \cdot \mathrm{Or}_X(t).$$

Proof. For $I \subset I_0 = \{1, \dots, n\}$, $I \neq \emptyset$, let $|I|$ be the number of elements of I, let $T_I = \{x \in \mathbb{C}^n \,|\, x_i = 0 \text{ for } i \notin I, x_i \neq 0 \text{ for } i \in I\}$ be the ("coordinate") complex torus of dimension $|I|$, and let $m_I = \gcd(q_i,\ i \in I)$. The integer m_I is the order of the isotropy group of the $\mathbb{C}^*$-action on the torus T_I. Let $X_I^{(j)} = X^{(j)} \cap T_I$, $Y_I^{(j)} = X_I^{(j)}/\mathbb{C}^*$. Note that if m_I does not divide d_j then $Y_I^{(j)}$ is empty. One has $Y^{(j)} = \bigcup_{I:|I|\geq 1} Y_I^{(j)}$. Therefore

$$\mathrm{Or}_X(t) = \prod_{I:|I|\geq 1} (1 - t^{m_I})^{\chi(Y_I^{(k)})}.$$

The monodromy transformation $h^{(j)} : F^{(j)} \to F^{(j)}$ can be defined as the transformation

$$x \mapsto e^{2\pi i/d_j} * x \quad (x \in F^{(j)} \subset \mathbb{C}^n,\ e^{2\pi i/d_j} \in \mathbb{C}^*).$$

Let $F_I^{(j)} = F^{(j)} \cap T_I$, $h_I^{(j)} = h^{(j)}|_{F_I^{(j)}}$ (the monodromy transformation $h^{(j)}$ maps $F_I^{(j)}$ to itself). One has

$$\zeta_j(t) = \prod_{I:|I|\geq 1} \zeta_{h_I^{(j)}}(t).$$

Suppose $m_I | d_j$. The natural projection $X^{(j-1)} \setminus X^{(j)} \to Y^{(j-1)} \setminus Y^{(j)}$ restricted to $F_I^{(j)} \subset X^{(j-1)} \setminus X^{(j)}$ is a (d_j/m_I)-fold covering over $Y_I^{(j-1)} \setminus Y_I^{(j)}$. The monodromy transformation $h_I^{(j)}$ is a covering transformation of it and acts as a cyclic permutation of the d_j/m_I points of a fibre. Therefore

$$\zeta_{h_I^{(j)}}(t) = (1 - t^{d_j/m_I})^{\chi(Y_I^{(j-1)} \setminus Y_I^{(j)})}.$$

This formula also makes sense if m_I does not divide d_j since in this case $Y_I^{(j-1)} \setminus Y_I^{(j)} = \emptyset$ and therefore the exponent is equal to 0. Thus

$$\begin{aligned} \widetilde{\zeta}_j(t) &= (1-t)^{-1} \cdot \prod_{I:|I|\geq 1} (1 - t^{d_j/m_I})^{\chi(Y_I^{(j-1)}) - \chi(Y_I^{(j)})}, \\ \widetilde{\zeta}_j^*(t) &= (1 - t^{d_j}) \cdot \prod_{I:|I|\geq 1} (1 - t^{m_I})^{\chi(Y_I^{(j)}) - \chi(Y_I^{(j-1)})}. \end{aligned}$$

Therefore

$$\begin{aligned}\prod_{j=1}^{k} \widetilde{\zeta}_j^*(t) &= \prod_{j=1}^{k}(1-t^{d_j}) \cdot \prod_{I:|I|\geq 1} (1-t^{m_I})^{\chi(Y_I^{(k)})-\chi(Y_I^{(0)})} \\ &= \mathrm{Or}_X(t) \cdot \prod_{j=1}^{k}(1-t^{d_j}) \cdot \prod_{I:|I|\geq 1} (1-t^{m_I})^{-\chi(T_I/\mathbb{C}^*)}.\end{aligned}$$

Now $\chi(T_I/\mathbb{C}^*) = 0$ if and only if $|I| \geq 2$ and $\chi(T_{\{i\}}/\mathbb{C}^*) = 1$ since $T_{\{i\}}/\mathbb{C}^* = \mathrm{pt}$, $i = 1, \ldots, n$; $m_{\{i\}} = q_i$. Therefore

$$\prod_{j=1}^{k} \widetilde{\zeta}_j^*(t) = \mathrm{Or}_X(t) \cdot \prod_{j=1}^{k}(1-t^{d_j}) \cdot \prod_{i=1}^{n}(1-t^{q_i})^{-1}.$$

□

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