

# **Braid Monodromy of Hypersurface Singularities**

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# Contents

<b>introduction</b>	<b>5</b>
<b>1 introduction to braid monodromy</b>	<b>9</b>
1.1 Polynomial covers and $\text{Br}_n$ -bundles . . . . .	10
1.1.1 Polynomial covers . . . . .	10
1.1.2 $\text{Br}_n$ -Bundles . . . . .	12
1.2 The braid monodromy of a plane algebraic curve . . . . .	14
1.2.1 The construction . . . . .	14
1.2.2 Braid equivalence . . . . .	16
1.3 The fundamental group of a plane algebraic curve . . . . .	16
1.3.1 Braid monodromy presentation . . . . .	17
1.3.2 braid monodromy generators . . . . .	20
1.4 braid monodromy of horizontal divisors . . . . .	22
1.4.1 braid monodromy presentation . . . . .	23
1.4.2 braid monodromy of local analytic divisors . . . . .	25
<b>2 braid monodromy of singular functions</b>	<b>27</b>
2.1 preliminaries on unfoldings . . . . .	27
2.1.1 versal unfolding . . . . .	28
2.1.2 discriminant set . . . . .	29
2.1.3 truncated versal unfolding . . . . .	30
2.1.4 bifurcation set . . . . .	30
2.2 discriminant braid monodromy . . . . .	31
2.2.1 basic properties . . . . .	32
2.2.2 invariance properties . . . . .	32
2.2.3 invariants . . . . .	33
<b>3 Hefez Lazzeri unfoldings</b>	<b>35</b>
3.1 discriminant and bifurcation hypersurface . . . . .	35
3.2 Hefez Lazzeri path system . . . . .	39
<b>4 singularities of type <math>A_n</math></b>	<b>41</b>
<b>5 results of Zariski type</b>	<b>49</b>
5.1 generalization of Morsification . . . . .	49
5.2 versal braid monodromy group . . . . .	51
5.3 comparison of braid monodromies . . . . .	53

5.4	Hefezi-Lazzeri base . . . . .	56
<b>6</b>	<b>braid monodromy of plane curve families</b>	<b>63</b>
6.1	parallel transport in the model family . . . . .	64
6.2	from tangled v-arcs to isosceles arcs . . . . .	70
6.3	from bisceles arcs to coiled isosceles arcs . . . . .	75
6.4	from coiled isosceles arcs to coiled twists . . . . .	79
6.5	from local w-arcs to coiled twists . . . . .	87
6.6	the length of bisceles arcs . . . . .	89
6.7	the discriminant family . . . . .	92
6.8	conclusion . . . . .	95
6.9	appendix on plane elementary geometry . . . . .	97
<b>7</b>	<b>braid monodromy induction to higher dimension</b>	<b>99</b>
7.1	preliminaries . . . . .	100
7.2	families of type $g_\alpha$ . . . . .	103
7.3	families of type $f_\alpha$ . . . . .	107
7.4	$l$ -companion models . . . . .	113
7.5	$l$ -companion monodromy . . . . .	117
<b>8</b>	<b>bifurcation braid monodromy of elliptic fibrations</b>	<b>123</b>
8.1	introduction . . . . .	123
8.2	bifurcation braid monodromy . . . . .	125
8.3	families of divisors in Hirzebruch surfaces . . . . .	126
8.4	families of elliptic surfaces . . . . .	133
8.5	Hurwitz stabilizer groups . . . . .	135
8.6	mapping class groups of elliptic fibrations . . . . .	137
<b>9</b>	<b>braid monodromy and fundamental groups</b>	<b>141</b>
9.1	fundamental groups . . . . .	141
9.2	Dynkin diagrams . . . . .	143
9.3	other functions . . . . .	144
9.4	conjectures and speculations . . . . .	144
<b>A</b>	<b>braid computations</b>	<b>147</b>
	<b>Bibliography</b>	<b>157</b>

## Introduction

Complex geometry can certainly be seen as a major source for the development and refinement of topological concepts and topological methods.

To exemplify this claim, we like to give two instances, which also will have impact on the proper topic of this work.

First there is the paper of Lefschetz on the topology of complex projective manifolds, which only later were adequately expressed in the language of algebraic topology. For example the Picard Lefschetz formula of ordinary double points is due to this paper.

Second we want to mention the theorem of van Kampen. It yields, in quite general situations, a presentation of the fundamental group of a union of spaces in terms of presentations of their fundamental groups. Originally conceived while investigating the fundamental group of plane curve complements, it is in its abstract form a standard topic of basic algebraic topology and a backbone for geometric and combinatorial group theory.

On the other hand new topological concepts are often tested in the realms of complex geometry. One may observe that many classifying spaces, Eilenberg-MacLane space in particular, have a natural complex structure and can thus be considered to belong to complex geometry.

A prominent example for the fruitful interplay of geometric, topological and combinatorial methods is singularity theory, into which the present work has to be subsumed.

Given a holomorphic function  $f$  or a holomorphic function germ it is standard procedure to consider a versal unfolding which is given by a function

$$F(x, z, u) = f(x) - z + \sum b_i u_i.$$

In case of a semi universal unfolding the unfolding dimension is given by the Milnor number  $\mu = \mu(f)$  and we get a diagram

$$\begin{array}{ccccccc} z, u_1, \dots, u_{\mu-1} & \xrightarrow{\mathbf{C}^\mu} & \mathcal{D} & = & \{(z, u) | F(0, z, u) = 0 = \nabla F(0, z, u)\} \\ \downarrow & & \downarrow & & & & \\ u_1, \dots, u_{\mu-1} & \xrightarrow{\mathbf{C}^{\mu-1}} & \mathcal{B} & = & \{u | F(\_, 0, u) \text{ is not Morse}\} & & \end{array}$$

The restriction  $p|_{\mathcal{D}}$  of the projection to the discriminant is a finite map, such that the branch set coincides with the bifurcation set  $\mathcal{B}$ .

One contribution of the present work is to show, that a suitable restriction of  $p$  to a subset of  $p^{-1}(\mathbf{C}^{\mu-1} \setminus \mathcal{B}) \setminus \mathcal{D}$  is a fibre bundle in a natural way. Its fibres are diffeomorphic to the  $\mu$ -punctured disc and its isomorphism type depends only on the right equivalence class of  $f$ .

When the focus was on the case of simple hypersurface singularities, this aspect was not needed, since there is a lot of additional structure one may resort to.

In this case the fundamental groups of discriminant complements of functions of type ADE are given by the Artin-Brieskorn groups of the same type. Moreover these groups have a natural presentation encoded by the Dynkin diagram of that type.

The complements of discriminants and of bifurcation sets were shown to be Eilenberg-MacLane spaces and homogeneous spaces. Moreover they were related to natural combinatorial structures via their Weyl groups.

More of this abundance of structure and relations will be used in chapter four. But sadly enough it only covers the simple singularities. We can observe that partial aspects can be generalized – especially to parabolic and hyperbolic singularities – but progress to arbitrary singularities has been sparse and slow.

On the other hand, parts of the theory prospered when they became the starting point of their own theory. Artin Brieskorn groups have lead to generalized Artin groups and the theory of Garside groups now subsumes them into a very active field of research.

Having succeeded in describing the discriminant complement in the case of simple singularities, Brieskorn, in [7], casts a light on some problems, which he intended for guidelines to the case of more general singularities. Among other problems he asked for the fundamental group and suggests to obtain these groups from a generic plane section using the theorem of Zariski and of van Kampen. But up to now, only in the case of simply elliptic singularities presentations of the fundamental group have been given.

Independently – initiated by Moishezon two decades ago – the study of complements of plane curves by the methods of Zariski and van Kampen has been revived and has found a lot of applications. Conceptionally recast as braid monodromy theory it has been successfully used for projective surfaces and symplectic four-manifolds alike by investigating branch curves of finite branched maps to  $\mathbf{P}^2$ .

The theory of braid monodromy has been generalized to the complements of hyperplane arrangements and it has found an interesting new interpretation in the theory polynomial coverings by Hansen.

The braid monodromy we develop in this work is based on this interpretation. In its context the fibre bundle obtained from  $p|_{\mathcal{D}}$  naturally gives rise to a braid monodromy homomorphism, which then can be made a braid monodromy invariant of the unfolded function  $f$ .

As in the case of plane curves the method of van Kampen leads to an explicit presentation of the fundamental group of the discriminant complement  $\mathbf{C}^\mu \setminus \mathcal{D}$  in terms of generators and relations.

Having accomplished this aim of more theoretical nature, we address next the problem to find the invariants and the group presentations for  $\pi_1(\mathbf{C}^\mu \setminus \mathcal{D})$  in case of polynomial functions of the kind given by  $f(x) = \sum x_i^{l_i+1}$ .

Pham investigated this class of function in the spirit of Lefschetz. He computed the homology of the regular fibre and then gave the global monodromy transformation thus generalizing the Picard Lefschetz situation  $l_i = 1$ .

Brieskorn exploited the same class of functions. He showed some of their links to be examples of exotic spheres. In his list [7] of problems he asks for the intersection lattice of  $f$ .

This problem has soon found a solution by a paper of Hefez and Lazzeri [19]. Their article has quite an impact on the present work, we owe them the description of a Milnor fibre and the choice of a natural geometrically distinguished path system.

We follow common convention by calling functions  $f$  of this class Brieskorn Pham polynomials.

We succeed to solve the Brieskorn problem of three decades ago in one go for the large and infinite class of Brieskorn Pham polynomials. Though generally speaking we follow the approach suggested by Brieskorn, our method to determine the presentation of the fundamental groups deviates in some essential aspects. To have explicit formula for the bifurcation divisor, we are forced to consider plane sections of  $\mathbf{C}^\mu$ , which fall short of the genericity conditions in even several ways. Nevertheless by a substantial amount of additional arguments and concepts, we finally get the desired results on the braid monodromy.

The presentations of fundamental groups thus obtained depend on the Brieskorn Pham polynomial chosen. They are natural generalizations of the presentations of Artin Brieskorn groups associated to the simple singularities. As in the case of simple singularities we can show, that they are determined by a intersection graph of  $f$ , given in [19]. Thus a further result has found an adequate generalization.

Its interesting to note, that also triangles, i.e. 2-simplices of the Dynkin diagram, make their contribution to the relations of the presentation. Surely one may expect, that the methods of combinatorial group theory will eventually provide a lot of additional properties of these groups.

With a chapter on elliptic fibrations we want to point to the fact, that also in the realm of compact manifolds the concept of braid monodromy may result in new and fruitful observations. Elliptic surfaces are good candidates, since in families almost always the fibration map deforms well, so we can make the singular value divisor of such a family the object of our braid monodromy considerations.

Concerning future developments we may only speculate. Nevertheless in the presence of such a lot of open problems we venture to finish our last chapter by some conjectures, the choice being led by personal interest and the newly gained insight.

We like to give a short outline of particular chapters.

The first two chapters are mainly of an introductory character. The first reviews braid monodromy. We start with braid monodromy of plane curves in the spirit of Moishezon and proceed like Hansen to get braid monodromy of horizontal divisors and of affine hypersurface germs. The result of van Kampen on fundamental groups is developed in each set up. Interspersed we mention results of Libgober on the complement of plane curve and applications by Moishezon and Teicher to the theory of branched covers of the projective plane.

In the second we review basic notions of singularity theory. We introduce discriminant divisors which we consider as a horizontal divisor over truncated versal unfoldings. We close the chapter with the definition of our new braid monodromy invariants for right equivalence classes of singular functions and the implications for the fundamental group of the discriminant complement.

With the third chapter we enter our computations of the braid monodromy of Brieskorn Pham polynomials. The equations of the discriminant and the bifurcation set of their unfoldings by linear polynomials are the main topic of this chapter. We then define a distinguished system of paths in regular fibres of a certain kind.

In the forth chapter the special case of singularities of type  $A_n$  is solved and the results prepared for later use in an inductive argument.

The fifth chapter the versal braid monodromy and provides the means to compute the braid monodromy of Brieskorn Pham polynomials from the versal braid monodromy of two one-parameter families of functions.

This is computed in the sixth chapter for one of the families in case of Brieskorn Pham polynomials defined on the plane. We have to develop a big machinery to distill from our geometric insight the concrete results we want to prove.

In the seventh chapter we conclude the computation of the braid monodromy by an inductive argument. Again we have to present more geometric notations and results.

The eighth chapter is devoted to the study of elliptic surfaces we mentioned before. We relate each family of elliptic surfaces with a family of divisors in Hirzebruch surfaces and can thus make use of a detailed study of plane polynomial functions.

In the final chapter we compute the fundamental group of discriminant complements in case of Brieskorn Pham polynomials. We consider and prove a close relationship to the Dynkin diagram found by Pham. Some immediate corollaries to general function are presented and all these results are used as motivation for the concluding conjectures.

It is my pleasure to express my thanks to Prof. Ebeling, who introduced me to the beautiful topic of singularity theory, and to my colleges in Hannover for their interest and many fruitful discussions.

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# Chapter 1

## introduction to braid monodromy

Given a singular curve  $\mathcal{C}$  in the affine plane  $\mathbf{C}^2$  it is natural to ask for the topology of the complement  $\mathbf{C}^2 \setminus \mathcal{C}$ . The study of its fundamental group  $\pi_1(\mathbf{C}^2 \setminus \mathcal{C})$  for various types of algebraic curves is a classical subject going back to the work of Zariski. An algorithm for its computation was given by van Kampen in [20]. It was obtained again by Moishezon as an application of his notion of braid monodromy, which he introduced in [31] and elaborated with Teicher in subsequent papers, eg. [33, 34]. Libgober [23] finally proved that the 2-complex associated to the braid monodromy even captures all homotopy properties of the curve complement  $\mathbf{C}^2 \setminus \mathcal{C}$ .

Before generalizing the considerations to complements of divisors in affine space, we present the interpretation given in [10] of the process by which the braid monodromy of a curve  $\mathcal{C}$  is defined. It is close to the approach in [23], but uses a self-contained argument based on Hansen's theory of polynomial covering maps, [17], [18].

Given a simple Weierstrass polynomial  $f : X \times \mathbf{C} \rightarrow \mathbf{C}$  of degree  $n$ , we consider the complement of its zero locus  $Y = X \times \mathbf{C} \setminus \{f(x, z) = 0\}$ . In Theorem 1.3, we show that the projection  $p = \text{pr}_1|_Y : Y \rightarrow X$  is a fiber bundle map, with structure group the braid group  $\text{Br}_n$ , and monodromy the homomorphism from  $\pi_1(X)$  to  $\text{Br}_n$  induced by the coefficient map of  $f$ .

This result can be applied when a plane curve  $\mathcal{C}$  is defined by a polynomial  $f$ , and  $X = \mathbf{C} \setminus \{y_1, \dots, y_s\}$  is the set of regular values of a generic linear projection, so that by restriction to  $X \times \mathbf{C}$  the polynomial  $f$  becomes simple Weierstrass of degree  $n$ . The braid monodromy of  $\mathcal{C}$  is simply the coefficient homomorphism,  $a_* : F_s \rightarrow \text{Br}_n$ .

Obviously  $a_*$  depends on the choice of a generic projection, of loops representing a basis of  $F_s$ , of an identification of mapping classes with braid group generators, and of basepoints. However, the braid-equivalence class of the monodromy – the double coset  $[a_*] \in \text{Br}_s \setminus \text{Hom}(F_s, \text{Br}_n) / \text{Br}_n$ , where  $\text{Br}_s$  acts on the left by the Artin representation, and  $\text{Br}_n$  acts on the right by conjugation – is uniquely determined by  $\mathcal{C}$ .

**Remark 1.1:** Recall that the braid monodromy depends not only on the number and types of the singularities of a curve but is also sensitive to their relative positions as is shown by the famous example of Zariski [42], [43] consisting of two sextics, both with six cusps, one with all cusps on a conic, the other not.

It even captures more than the fundamental group of the curve complement as is shown in [10], and one may hope that it detects to some extend the homeomorphism type of the complement or the ambient homeomorphism type of the curve.

When passing to higher dimensions we assume to be given a Weierstrass polynomial  $f : \mathbf{C}^r \times \mathbf{C} \rightarrow \mathbf{C}$  defining a horizontal divisor  $\mathcal{D}$  over  $\mathbf{C}^r$ . If  $X := \mathbf{C}^r \setminus \mathcal{B}$  is the set of regular values, the complement of the bifurcation divisor  $\mathcal{B}$  of the branched covering  $\mathcal{D} \rightarrow \mathbf{C}^r$ , then the restriction of  $f$  to  $X \times \mathbf{C}$  is a simple Weierstrass polynomial of degree  $n$  equal to the degree of the covering. The braid monodromy is again the coefficient homomorphism  $a_* : \pi_1(X) \rightarrow \text{Br}_n$ . Also the method to compute the fundamental group of the plane curve complement extends to the given situation and provides the tool to get the fundamental group  $\pi_1(\mathbf{C}^r \setminus \mathcal{D})$ .

We push the generalization even further to include the case of analytic germs. With a generic choice of local coordinates the Weierstrass preparation theorem can be applied and provides us with a Weierstrass polynomial which is simple in the complement of the germ of a divisor. Again the subsequent definitions generalize.

## 1.1 Polynomial covers and $\text{Br}_n$ -bundles

We begin by reviewing polynomial covering maps. These were introduced by Hansen in [17], and studied to some detail in his book [18]. Together with the by now classical book of Birman [5] it should serve as the basic reference for this section. We then consider the relation between bundles of punctured discs, whose structure group is Artin's braid group  $\text{Br}_n$ , and polynomial  $n$ -fold covers.

### 1.1.1 Polynomial covers

Let  $X$  be a path-connected topological manifold. A *Weierstrass polynomial* of degree  $n$  is a map  $f : X \times \mathbf{C} \rightarrow \mathbf{C}$  given by

$$f(x, z) = z^n + \sum_{i=1}^n a_i(x)z^{n-i},$$

with continuous coefficient maps  $a_i : X \rightarrow \mathbf{C}$ . If  $f$  has no multiple roots for any  $x \in X$ , then  $f$  is called a *simple Weierstrass polynomial*.

Given such  $f$ , the restriction of the first-coordinate projection map  $X \times \mathbf{C} \rightarrow X$  to the subspace

$$E = E(f) = \{(x, z) \in X \times \mathbf{C} \mid f(x, z) = 0\}$$

defines an  $n$ -fold topological cover  $\pi = \pi_f : E \rightarrow X$ , the *polynomial covering map* associated to  $f$ .

Since  $f$  has no multiple roots, the *coefficient map*

$$a = (a_1, \dots, a_n) : X \rightarrow \mathbf{C}^n$$

takes values in the complement  $B^n = \mathbf{C}^n \setminus \Delta_n$  of the discriminant set  $\Delta_n$ , which is a tautology by the definition of  $\Delta_n$  as the set of coefficient  $n$ -tuples such that the corresponding polynomial of degree  $n$  has at least one multiple root.

Over  $B^n$ , there is a tautological  $n$ -fold polynomial covering

$$\pi_n := \pi_{f_n} : E(f_n) \rightarrow B^n, \quad (1.1)$$

determined by the tautological Weierstrass polynomial

$$f_n : \mathbf{C}^n \times \mathbf{C} \longrightarrow \mathbf{C}, \quad (x_1, \dots, x_n, z) \mapsto z^n + \sum_{i=1}^n x_i z^{n-i}.$$

The polynomial cover  $\pi_f : E(f) \rightarrow X$  can then be identified with the pull-back of  $\pi_n : E(f_n) \rightarrow B^n$  along the coefficient map  $a : X \rightarrow B^n$ .

This can be interpreted on the level of fundamental groups as follows. The fundamental group of the configuration space,  $B^n$ , of  $n$  unordered points in  $\mathbf{C}$  is the group  $\text{Br}_n$  of braids on  $n$  strands. The map  $a$  determines the *coefficient homomorphism*  $a_* : \pi_1(X) \rightarrow \text{Br}_n$ , unique up to conjugacy.

Recall that the structure group of a topological  $n$ -sheeted cover is the permutation group  $\Sigma_n$  and the associated *cover monodromy* is a homomorphism from the fundamental group of the base to  $\Sigma_n$ . So it is immediate that the monodromy of the polynomial cover  $\pi : E \rightarrow X$  factors through the coefficient homomorphism  $a_*$  and the canonical surjection  $\text{Br}_n \rightarrow \Sigma_n$ . In fact this condition is sufficient, i.e. one may characterize polynomial covers by this factorization property of their permutation monodromy map.

Assume now that the simple Weierstrass polynomial  $f$  is *completely solvable*, i.e.  $f$  factors as

$$f(x, z) = \prod_{i=1}^n (z - b_i(x)),$$

with continuous roots  $b_i : X \rightarrow \mathbf{C}$ . Since the Weierstrass polynomial  $f$  is simple, the *root map*  $b = (b_1, \dots, b_n) : X \rightarrow \mathbf{C}^n$  takes values in the complement  $P^n$  of the braid arrangement  $\mathcal{A}_n = \{(w_1, \dots, w_n) | w_i \neq w_j \forall i < j\}$  in  $\mathbf{C}^n$ . Over  $P^n$ , there is a canonical  $n$ -fold covering map,  $\tilde{\pi}_n = \pi_{\tilde{f}_n} : E(\tilde{f}_n) \rightarrow P^n$ , determined by the Weierstrass polynomial  $\tilde{f}_n(w, z) = (z - w_1) \cdots (z - w_n)$ . Evidently, the cover  $\pi_f : E \rightarrow X$  is the pull-back of  $\tilde{\pi}_n : E(\tilde{f}_n) \rightarrow P^n$  along the root map  $b : X \rightarrow P^n$ .

The fundamental group of the configuration space,  $P^n$ , of  $n$  ordered points in  $\mathbf{C}$  is the group,  $\text{PBr}_n = \ker(\text{Br}_n \rightarrow \Sigma_n)$ , of pure braids on  $n$  strands. The map  $b$  determines the *root homomorphism*  $b_* : \pi_1(X) \rightarrow \text{PBr}_n$ , unique up to conjugacy. The polynomial covers which are trivial covers (in the usual sense) are precisely those for which the coefficient homomorphism  $a_*$  has image in the subgroup  $\text{PBr}_n$  of  $\text{Br}_n$ .

### 1.1.2 $\text{Br}_n$ -Bundles

The group  $\text{Br}_n$  is isomorphic to the mapping class group  $\text{Map}^n(D^2)$  of orientation-preserving diffeomorphisms of the disc  $D^2$ , permuting a collection of  $n$  marked points. A natural way to fix an isomorphism is the choice of a 'frame' as in [34]. But it pays off immediately if we choose instead a different geometric object (which incidentally is called 'bush' in [34]).

**Definition 1.2:** A *geometrically distinguished* system of paths in  $D^2$  with respect to marked points  $y_i, 1 \leq i \leq n$ , is a finite sequence of paths  $p_i : [0, 1] \rightarrow D^2$ , such that:

- i)  $y_0 := p_i(0)$  is the unique point in the intersection of two paths,  $p_i(1) = y_i$ ,
- ii) on some loop around  $y_0$  in  $\mathbf{R}^2$  the intersection points with the  $p_i$  are in bijection with the indices  $i = 1, \dots, n$ , preserving the order.

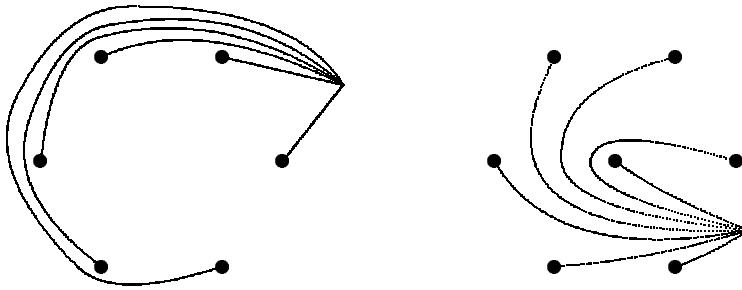


Figure 1.1: Examples of geometrically distinguished path systems

They give rise to a basis  $t_1, \dots, t_n$  of free generators of  $\pi_1(D^2 \setminus \{y_1, \dots, y_n\}, y_0)$ , which accordingly is called *geometrically distinguished*, too.

The free basis of the fundamental group is unambiguously represented by loops which are obtained by replacing each path  $p_i$  by a sufficiently close noose embedded into  $D^2 \setminus \{y_1, \dots, y_n\}$ , based at  $y_0$ , and linking the marked point  $y_i$  once.



Figure 1.2: Noose associated to a path

Upon identifying  $\pi_1(D^2 \setminus \{n \text{ points}\})$  in such a way with the free group  $F_n$ , the action of  $\text{Br}_n$  on  $\pi_1$  yields the *Artin representation*,  $\alpha_n : \text{Br}_n \rightarrow \text{Aut}(F_n)$ . As shown by Artin, this representation is faithful. Hence, we may identify a braid  $\beta \in \text{Br}_n$  with the corresponding braid automorphism,  $\alpha_n(\beta) \in \text{Aut}(F_n)$ .

Now let  $f : X \times \mathbf{C} \rightarrow \mathbf{C}$  be a simple Weierstrass polynomial. Let  $\pi_f : E(f) \rightarrow X$  be the corresponding polynomial  $n$ -fold cover, and  $a : X \rightarrow B^n$  the coefficient map.

Consider the complement

$$Y = Y(f) = X \times \mathbf{C} \setminus E(f),$$

and let  $p = p_f : Y(f) \rightarrow X$  be the restriction of  $\text{pr}_1 : X \times \mathbf{C} \rightarrow X$  to  $Y$ .

The fibre over a single point we denote by  $\mathbf{C}_n := \mathbf{C} \setminus \{n \text{ points}\}$ .

**Theorem 1.3** *The map  $p : Y \rightarrow X$  is a locally trivial bundle, with structure group  $\text{Br}_n$  and fiber  $\mathbf{C}_n$ . Upon identifying  $\pi_1(\mathbf{C}_n)$  with  $F_n$ , the monodromy of this bundle may be written as  $\alpha_n \circ a_*$ , where  $a_* : \pi_1(X) \rightarrow \text{Br}_n$  is the coefficient homomorphism.*

Moreover, if  $f$  is completely solvable, the structure group reduces to  $\text{PBr}_n$ , and the monodromy is  $\alpha_n \circ b_*$ , where  $b_* : \pi_1(X) \rightarrow \text{PBr}_n$  is the root homomorphism.

*Proof:* We first prove the theorem for the configuration spaces, and their canonical Weierstrass polynomials. Start with  $X = P^n$ ,  $f = \tilde{f}_n$ , and the canonical cover  $\tilde{\pi}_n : E(\tilde{f}_n) \rightarrow P^n$ . Clearly,  $Y(\tilde{f}_n) = \mathbf{C}^{n+1} \setminus E(\tilde{f}_n)$  is equal to the configuration space  $P^{n+1}$ . Let  $p_{\tilde{f}_n} : P^{n+1} \rightarrow P^n$  be the restriction of  $\text{pr}_1 : \mathbf{C}^n \times \mathbf{C} \rightarrow \mathbf{C}^n$ . As shown by Faddell and Neuwirth [16], this is a bundle map, with fiber  $\mathbf{C}_n$ , and monodromy the restriction of the Artin representation to  $\text{PBr}_n$ .

Next consider  $X = B^n$ ,  $f = f_n$ , and the canonical cover  $\pi_n : E(f_n) \rightarrow B^n$ . Forgetting the order of the points defines a covering projection from the ordered to the unordered configuration space,  $\kappa_n : P^n \rightarrow B^n$ .

In coordinates,  $\kappa_n(w_1, \dots, w_n) = (x_1, \dots, x_n)$ , where  $x_i = (-1)^i s_i(w_1, \dots, w_n)$ , and  $s_i$  are the elementary symmetric functions. By Vieta's formulas, we have

$$\tilde{f}_n(w, z) = f_n(\kappa_n(w), z).$$

Let  $Y^{n+1} = Y(f_n)$  and  $p_{f_n} : Y^{n+1} \rightarrow B^n$ . By the above formula, we see that  $\kappa_n \times \text{id} : P^n \times \mathbf{C} \rightarrow B^n \times \mathbf{C}$  restricts to a map  $\bar{\kappa}_{n+1} : Y(\tilde{f}_n) \rightarrow Y(f_n)$ , which fits into the fiber product diagram

$$\begin{array}{ccc} P^{n+1} & \xrightarrow{p_{\tilde{f}_n}} & P^n \\ \downarrow \bar{\kappa}_{n+1} & & \downarrow \kappa_n \\ Y^{n+1} & \xrightarrow{p_{f_n}} & B^n \end{array}$$

where the vertical maps are principal  $\Sigma_n$ -bundles.

Since the bundle map  $p_{\tilde{f}_n} : P^{n+1} \rightarrow P^n$  is naturally equivariant with respect to the  $\Sigma_n$ -actions, the map on quotients,  $p_{f_n} : Y^{n+1} \rightarrow B^n$ , is also a bundle map, with fiber  $\mathbf{C}_n$ , and monodromy action the Artin representation of  $\text{Br}_n$ . This finishes the proof in the case of the canonical Weierstrass polynomials over configuration spaces.

Now let  $f : X \times \mathbf{C} \rightarrow \mathbf{C}$  be an arbitrary simple Weierstrass polynomial. We then have the following Cartesian square:

$$\begin{array}{ccc} Y & \longrightarrow & Y^{n+1} \\ \downarrow p & & \downarrow p_{f_n} \\ X & \longrightarrow & B^n \end{array}$$

In other words,  $p : Y \rightarrow X$  is the pull back of the bundle  $p_{f_n} : Y^{n+1} \rightarrow B^n$  along the coefficient map  $a$ . Thus,  $p$  is a bundle map, with fiber  $\mathbf{C}_n$ , and monodromy representation  $\alpha_n \circ a_*$ . When  $f$  is completely solvable, the bundle  $p : Y \rightarrow X$  is the pull back of  $p_{\tilde{f}_n} : P^{n+1} \rightarrow P^n$  along the root map  $b$ . Since  $a_* = b_*$  then, the monodromy is as claimed.  $\square$

Let us summarize the above discussion of braid bundles over configuration spaces. From the Faddell-Neuwirth theorem [16], it follows that  $P^n$  is a  $K(\pi_1, 1)$  space. Since  $B^n$  is covered by  $P^n$ , it is also an  $K(\pi_1, 1)$  space.

So in both cases the groups are discrete and any of their representations as groups of diffeomorphisms determines a locally trivial bundle over the corresponding configuration space  $P^n$ , resp.  $B^n$ .

**Example 1.4:** We considered two bundles over  $P^n$  obtained in this way:

- (i)  $\tilde{\pi}_n : E(\tilde{f}_n) \rightarrow P^n$ , by the trivial representation of  $\text{PBr}_n$  on  $\{1, \dots, n\}$ ;
- (ii)  $p_{\tilde{f}_n} : P^{n+1} \rightarrow P^n$ , by the geometric Artin representation of  $\text{PBr}_n$  on  $\mathbf{C}_n$ .

Over  $B^n$  we have even seen three instances of bundles of this kind:

- (iii)  $\kappa_n : P^n \rightarrow B^n$ , by the canonical surjection  $\text{Br}_n \rightarrow \Sigma_n$ , acting by left translations on the discrete set  $\Sigma_n$ ,
- (iv)  $\pi_n : E(f_n) \rightarrow B^n$ , by the above, followed by the permutation representation of  $\Sigma_n$  on  $\{1, \dots, n\}$ ;
- (v)  $p_n : Y^{n+1} \rightarrow B^n$ , by the (geometric) Artin representation of  $\text{Br}_n$  on  $\mathbf{C}_n$ .

Note finally, that  $\pi_1(Y^{n+1})$  is isomorphic to  $\text{Br}_{n,1} = F_n \times_{\alpha_n} \text{Br}_n$ , the group of braids on  $n+1$  strands that fix the endpoint of the last strand, and that  $Y^{n+1}$  is a  $K(\text{Br}_{n,1}, 1)$  space.

## 1.2 The braid monodromy of a plane algebraic curve

We are now ready to define the braid monodromy of an algebraic curve in the complex plane. The construction, based on classical work of Zariski and van Kampen, is due to Moishezon [31]. Though we want to include the results of Libgober [23], [24], [25], we prefer to interpret the construction in the context established in the previous section.

### 1.2.1 The construction

Let  $\mathcal{C}$  be a reduced curve in the affine plane  $\mathbf{C}^2$  defined by a polynomial  $f$ . Let  $\pi : \mathbf{C}^2 \rightarrow \mathbf{C}$  be a linear projection, such that no fibre of  $\pi$  is a component of  $\mathcal{C}$ , and let  $\mathcal{S} = \{y_1, \dots, y_s\}$  be the set of points in  $\mathbf{C}$  for which the corresponding fiber of  $\pi$  is tangent to  $\mathcal{C}$  or contains a singular point of  $\mathcal{C}$ .

In case we assume  $\pi$  to be *generic* with respect to  $\mathcal{C}$ , we mean that, for each  $k$ , the line  $\mathcal{L}_k = \pi^{-1}(y_k)$  passes transversally through regular points of  $\mathcal{C}$  except for either a single regular point  $v_k$  at which it is a simple tangent or for one singular point  $v_k$  at which it is not contained in the tangent cone.

Let  $\mathcal{L}$  denote the union of the lines  $\mathcal{L}_k$ , and let  $y_0$  be a basepoint in  $\mathbf{C} \setminus \mathcal{S}$ . The definition of the braid monodromy of  $\mathcal{C}$  depends on two observations:

- (i) The restriction of the projection map,  $p : \mathbf{C}^2 \setminus \mathcal{C} \cup \mathcal{L} \rightarrow \mathbf{C} \setminus \mathcal{S}$ , is a locally trivial bundle.

Identify the fiber  $p^{-1}(y_0)$  with  $\mathbf{C}_n$  for the appropriate  $n$  and fix a basepoint  $v_0 \in \mathbf{C}_n$ . The monodromy of  $\mathcal{C}$  is, by definition, the holonomy of the bundle,  $\rho : \pi_1(\mathbf{C} \setminus \mathcal{S}, y_0) \rightarrow \text{Aut}(\pi_1(\mathbf{C}_n, v_0))$ . Upon identifying  $\pi_1(\mathbf{C} \setminus \mathcal{S}, y_0)$  with  $F_s$ , and  $\pi_1(\mathbf{C}_n, v_0)$  with  $F_n$  using geometrically distinguished systems of paths with respect to the exceptional points, this can be written as  $\rho : F_s \rightarrow \text{Aut}(F_n)$ .

- (ii) The image of  $\rho$  is contained in the braid group  $\text{Br}_n$  (viewed as a subgroup of  $\text{Aut}(F_n)$  via the Artin embedding  $\alpha_n$ ).

**Definition 1.5:** The homomorphism  $\alpha : F_s \rightarrow \text{Br}_n$  determined by  $\alpha_n \circ \alpha = \rho$  is called *braid monodromy homomorphism* of  $\mathcal{C}$  with respect to  $\pi$ .

In case  $\pi$  is generic,  $\alpha : F_s \rightarrow \text{Br}_n$  is simply called *braid monodromy homomorphism* of  $\mathcal{C}$ .

We shall present a self-contained proof of the two assertions, and, in the process, identify the map  $\alpha$ . The first assertion is well-known, and can also be proved by standard techniques (using blow-up and Ehresmann's criterion—see [11], page 123), but our approach sheds some light on the underlying topology of the situation.

We may assume – after a linear change of variables in  $\mathbf{C}^2$  if necessary – that  $\pi$  is the projection map  $\text{pr}_1$  onto the first coordinate. In the chosen coordinates, the defining polynomial  $f$  of  $\mathcal{C}$  may be written as  $f(x, z) = z^n + \sum_{i=1}^n a_i(x)z^{n-i}$ . Since  $\mathcal{C}$  is reduced, for each  $x \notin \mathcal{S}$ , the equation  $f(x, z) = 0$  has  $n$  distinct roots. Thus  $f$  is a simple Weierstrass polynomial over  $\mathbf{C} \setminus \mathcal{S}$ , and

$$\pi = \pi_f : \mathcal{C} \setminus \mathcal{L} \cap \mathcal{C} \rightarrow \mathbf{C} \setminus \mathcal{S} \quad (1.2)$$

is the associated polynomial  $n$ -fold cover.

Note that  $Y(f) = ((\mathbf{C} \setminus \mathcal{S}) \times \mathbf{C}) \setminus (\mathcal{C} \setminus \mathcal{L} \cap \mathcal{C})$  is equal to  $\mathbf{C}^2 \setminus \mathcal{C} \cup \mathcal{L}$ . By theorem 1.3, the restriction of  $\text{pr}_1$  to  $Y(f)$ ,

$$p : \mathbf{C}^2 \setminus \mathcal{C} \cup \mathcal{L} \rightarrow \mathbf{C} \setminus \mathcal{S}, \quad (1.3)$$

is a bundle map, with structure group  $\text{Br}_n$ , fiber  $\mathbf{C}_n$ , and monodromy homomorphism

$$\alpha = a_* : \pi_1(\mathbf{C} \setminus \mathcal{S}) \rightarrow \text{Br}_n. \quad (1.4)$$

This proves assertions (i) and (ii), and implies immediately:

**Proposition 1.6** *The braid monodromy of a plane algebraic curve coincides with the coefficient homomorphism of the associated polynomial cover.*

### 1.2.2 Braid equivalence

The braid monodromy of a plane algebraic curve is not unique, but rather, depends on the choices made in defining it. This indeterminacy was studied to its full extend by Libgober in [24], [25]. To make the analysis more precise, we first need a definition.

**Definition 1.7:** Two homomorphisms  $\alpha : F_s \rightarrow \text{Br}_n$  and  $\alpha' : F_s \rightarrow \text{Br}_n$  are *braid equivalent* if there exist an automorphisms  $\psi \in \text{Aut}(F_s)$  in the image of the Artin representation and  $\phi \in \text{Br}_n$  such that  $\alpha'(\psi(g)) = \phi^{-1} \cdot \alpha(g) \cdot \phi$ , for all  $g \in F_s$ . In other words, the following diagram commutes

$$\begin{array}{ccc} F_s & \xrightarrow{\alpha} & \text{Br}_n \\ \downarrow \psi & & \downarrow \text{conj}_\phi \\ F_s & \xrightarrow{\alpha'} & \text{Br}_n. \end{array}$$

**Theorem 1.8** *The braid monodromy of a plane algebraic curve  $\mathcal{C}$  is well-defined up to braid-equivalence, and so are the braid monodromies with respect to a fixed linear projection.*

*Proof:* First fix the projection. The identification  $\pi_1(\mathbf{C} \setminus \mathcal{S}) = F_s$  depends on the choice of a distinguished system of paths, and any two such choices yield monodromies which differ by a braid automorphism of  $F_s$ , see [24]. Furthermore, there is the choice of basepoints, and any two such choices yield monodromies differing by a conjugation in  $\text{Br}_n$ .

Finally, one must analyze the effect of a change in the choice of generic projection. Let  $\pi$  and  $\pi'$  be two such projections, with critical sets  $\mathcal{S}$  and  $\mathcal{S}'$ , and braid monodromies  $a_* : \pi_1(\mathbf{C} \setminus \mathcal{S}) \rightarrow \text{Br}_n$  and  $a'_* : \pi_1(\mathbf{C} \setminus \mathcal{S}') \rightarrow \text{Br}_n$ . Libgober [25] shows that there is a homeomorphism  $h : \mathbf{C} \rightarrow \mathbf{C}$ , isotopic to the identity, and taking  $\mathcal{S}$  to  $\mathcal{S}'$ , for which the isomorphism  $h_* : \pi_1(\mathbf{C} \setminus \mathcal{S}) \rightarrow \pi_1(\mathbf{C} \setminus \mathcal{S}')$  induced by the restriction of  $h$  satisfies  $a'_* \circ h_* = a_*$ . From the construction, we see that  $h$  can be taken to be the identity outside a ball of large radius (containing  $\mathcal{S} \cup \mathcal{S}'$ ). Thus, once the identifications of source and target with  $F_s$  using distinguished systems of paths are made,  $h_*$  can be written as a braid automorphism of  $F_s$ :  $h_* = \psi$ , since  $\text{Br}_n$  acts transitively on the isotopy classes of distinguished path systems. We obtain  $a'_* \circ \psi = a_*$ , completing the proof.  $\square$

Thus, we may regard the braid monodromy of  $\mathcal{C}$  as a braid-equivalence class, i.e., as a double coset  $[a_*] \in \text{Br}_s \backslash \text{Hom}(F_s, \text{Br}_n) / \text{Br}_n$ , uniquely determined by  $\mathcal{C}$ . In fact, it follows from [25] that  $[a_*]$  depends only on the equisingular isotopy class of the curve.

## 1.3 The fundamental group of a plane algebraic curve

We now give the braid monodromy presentation of the fundamental group of the complement of a plane algebraic curve  $\mathcal{C}$ . This presentation first appeared in the classical work of van Kampen and Zariski [20], [43], and has been much studied since, e.g. by Moishezon, Teicher [34, 40], Libgober [23, 24], Rudolph [38] and many more.

### 1.3.1 Braid monodromy presentation

We want to find a presentation of  $\pi_1(\mathbf{C}^2 \setminus \mathcal{C})$ . As a first approximation we give a presentation of  $\pi_1(\mathbf{C}^2 \setminus \mathcal{C} \cup \mathcal{L})$  which can be derived with the help of the previous discussions. The essential step is then to extract enough information out of the embedding  $\mathbf{C}^2 \setminus \mathcal{C} \cup \mathcal{L} \rightarrow \mathbf{C}^2 \setminus \mathcal{C}$  to determine the presentation of  $\pi_1(\mathbf{C}^2 \setminus \mathcal{C})$  sought for.

The first necessary observation is the following:

**Lemma 1.9** *Given a Weierstrass polynomial  $f : X \times \mathbf{C} \rightarrow \mathbf{C}$  there is a topological section  $s : X \rightarrow X \times \mathbf{C}$  to the projection  $X \times \mathbf{C} \xrightarrow{\text{pr}} X$  with image in the complement of the zero set of  $f$ .*

*Proof:* It is a well known fact, that the zeroes of a monic polynomial are bounded by the sum of the absolute values of all coefficients. Since the coefficients are continuous functions on  $X$ , this bound – considered as a complex valued function on  $X$  – defines a continuous section with image disjoint from  $f^{-1}(0)$ .  $\square$

Since the  $s$ -punctured complex line  $\mathbf{C} \setminus \mathcal{S} \cong \mathbf{C}_s$  is a  $K(F_s, 1)$ , the long exact homotopy sequence of the bundle  $p : \mathbf{C}^2 \setminus \mathcal{C} \cup \mathcal{L} \rightarrow \mathbf{C} \setminus \mathcal{S}$  yields a short exact sequence which is split due to the preceding lemma:

$$1 \rightarrow \pi_1(\mathbf{C}_n) \rightarrow \pi_1(\mathbf{C}^2 \setminus \mathcal{C} \cup \mathcal{L}) \xrightarrow[\overleftarrow{p_*}]{} \pi_1(\mathbf{C} \setminus \mathcal{S}) \rightarrow 1.$$

Moreover the action is given by the braid monodromy homomorphism  $a_*$  of (1.4), so  $\pi_1(\mathbf{C}^2 \setminus \mathcal{C} \cup \mathcal{L})$  is the semi-direct product via  $a_*$ , a presentation of which can be derived from presentations of  $\pi_1(\mathbf{C}_n)$  and  $\pi_1(\mathbf{C} \setminus \mathcal{S})$ .

As remarked before, these groups are free, but in the given geometric set up the identifications with an abstractly presented free groups  $F_n = \langle t_1, \dots, t_n | \rangle$  and  $F_s = \langle \gamma_1, \dots, \gamma_s | \rangle$  determined by geometrically distinguished bases are privileged.

Having chosen such geometrically distinguished bases for  $\mathbf{C}_n$  and  $\mathbf{C} \setminus \mathcal{S}$ , which amounts to a distinguished choice of isomorphisms  $\pi_1(\mathbf{C}_n) \cong \langle t_1, \dots, t_n | \rangle$  and  $\pi_1(\mathbf{C} \setminus \mathcal{S}) \cong \langle \gamma_1, \dots, \gamma_s | \rangle$ , the split sequence above naturally induces an isomorphism

$$\pi_1(\mathbf{C}^2 \setminus \mathcal{C} \cup \mathcal{L}) = \langle t_1, \dots, t_n, \gamma_1, \dots, \gamma_s \mid \gamma_k^{-1} \cdot t_i \cdot \gamma_k = a_*(\gamma_k)(t_i) \rangle.$$

To proceed we are in need of a result relating the fundamental groups given the injective map  $\mathbf{C}^2 \setminus \mathcal{C} \cup \mathcal{L} \rightarrow \mathbf{C}^2 \setminus \mathcal{C}$ , where  $\mathcal{L}$  is a divisor with no component in common with  $\mathcal{C}$ . This is very much in the spirit of the Zariski van Kampen results although we owe it mostly to [4], to which we have added only a distinctive topological flavour.

**Definition 1.10:** Let  $D$  be a reduced divisor in affine space  $\mathbf{C}^n$ . An element  $g$  of  $\pi_1(\mathbf{C}^n \setminus D)$  is called a *simple geometric element* if there is an embedded oriented disc in  $\mathbf{C}^n$ , which intersects  $D$  transversally in a unique point, such that the orientations of  $D$  and the disc give the orientation of  $\mathbf{C}^n$  and such that the oriented boundary is freely homotopic to  $g$ .

**Definition 1.11:** Let  $D$  in  $\mathbf{C}^n$  be a reduced divisor and  $p$  a point of  $\mathbf{C}^n$  in the complement of  $D$ . Let  $D_0$  be an irreducible component of  $D$ . Then an element  $g$  of  $\pi_1(\mathbf{C}^n \setminus D, p)$  is called the *simple geometric element associated to  $D_0$*  if  $g$  is freely homotopic to the oriented boundary of a disc intersecting  $D$  transversally in a unique point of  $D_0$ .

There are of course simple geometric elements in abundance, examples of which are provided by the nooses associated to a system of paths.

**Lemma 1.12** *The simple geometric elements associated to the same component are conjugate and any element conjugate to a simple geometric one is itself a simple geometric element associated to the same component.*

*Proof:* The open part  $D_{reg}$  of the given component consisting of points which are regular and not contained in any other component is path connected. So if transversal discs to points in  $D_{reg}$  are given, these points are connected by a path embedded in  $D_{reg}$ . Along this path  $D_{reg}$  is a submanifold of real codimension two, so a normal disc bundle exists, which shows that the boundaries of both discs are freely homotopic. Hence the geometric elements are freely homotopic as well, which implies the first claim. The second is obvious.  $\square$

**Lemma 1.13** *Let  $U$  be a smooth connected complex variety. Let  $D_1$  and  $D_2$  be divisors with no irreducible component in common. Then the naturally induced map  $\pi_1(U - D_1 \cup D_2) \rightarrow \pi_1(U - D_1)$  is surjective and*

- i) *For any simple geometric element in  $\pi_1(U - D_1)$  associated to an irreducible component  $J$  of  $D_1$ , there is a lift in  $\pi_1(U - D_1 \cup D_2)$  which is a simple geometric element associated to  $J$ .*
- ii) *The simple geometric elements of  $\pi_1(U - D_1 \cup D_2)$  associated to  $D_2$  generate the kernel.*

*Proof:* Since a path is of real dimension 1 and the divisor  $D_2$  is of real codimension two, any path in the complement of  $D_1$  is isotopic in  $U - D_1$  to a path disjoint from  $D_2$ , so surjectivity holds as claimed.

To address the claim i), let any simple geometric element in  $\pi_1(U - D_1)$  be given which is associated to an irreducible component  $J$  of  $D_1$ . Choose any simple geometric element in  $\pi_1(U - D)$  associated to  $J$ . Its image in  $\pi_1(U - D_1)$  is still a simple geometric element associated to  $J$ , hence freely homotopic to the given one. By 1.12 they are even conjugate.

Since surjectivity is already established we may find a preimage of the conjugating element by which we can conjugate the chosen geometric element to get a preimage of the given element. So by 1.12 we have found a preimage which itself is a geometric element.

We prove the last claim by induction on the number of irreducible components of  $D_2$ : If we have just a single component  $J$  we know the claim to be true in case  $\dim U = 1$  and  $J$  consists only of an isolated point. If  $\dim U \geq 2$  any element in the kernel is represented by a path isotopic in  $U - D_1$  to the constant path. By a general position check we may assume that the isotopy is transversal to  $J$ .

Hence a suitable modification can be found which is supported in the complement of  $J$  and exhibits the path to be isotopic to a concatenation of boundaries of discs normal to  $J$  and segments connecting these. So its class in  $\pi_1(U - D_1 \cup J)$  is represented by a product of simple geometric elements and their inverses.

Suppose finally  $D_2 = D'_2 \cup J$ , so by induction hypothesis the simple geometric elements associated to  $D'_2$  generate the kernel of  $\pi_1(U - D_1 \cup D'_2) \rightarrow \pi_1(U - D_1)$  and the simple geometric elements associated to  $J$  generate the kernel of

$$\pi_1(U - D_1 \cup D_2) \rightarrow \pi_1(U - D_1 \cup D'_2).$$

By *i*) the simple geometric elements associated to  $D'_2$  lift to simple geometric elements associated to  $D'_2$ , hence we can conclude that the kernel of the composed map

$$\pi_1(U - D_1 \cup D_2) \rightarrow \pi_1(U - D_1 \cup D'_2) \rightarrow \pi_1(U - D_1)$$

is generated by simple geometric elements associated to  $D_2$  as claimed.  $\square$

We can now apply this technical lemma to the curves  $\mathcal{C}$  and  $\mathcal{L}$  in  $\mathbf{C}^2$ .

**Lemma 1.14** *The fundamental group  $\pi_1(\mathbf{C}^2 \setminus \mathcal{C})$  of the complement of the curve is the quotient of  $\pi_1(\mathbf{C}^2 \setminus \mathcal{C} \cup \mathcal{L})$  by the normal closure of  $F_s$  considered as a subgroup by the presentation above, thus it is presented as*

$$\pi_1(\mathbf{C}^2 \setminus \mathcal{C}) = \langle t_1, \dots, t_n \mid t_i = a_*(\gamma_k)(t_i) \rangle.$$

**Remark 1.15:** Some of the given relations may be trivial, e.g. if  $a_*(\gamma_k) = \sigma_1$  then  $t_i = a_*(\gamma_k)(t_i) = t_i$  for  $i > 2$ .

*Proof of lemma 1.14:* First by 1.13 the map  $i_* : \pi_1(\mathbf{C}^2 \setminus \mathcal{C} \cup \mathcal{L}) \rightarrow \pi_1(\mathbf{C}^2 \setminus \mathcal{C})$  is surjective. By the construction  $F_s$  is considered a subgroup via

$$\begin{aligned} F_s &\cong \pi_1(\mathbf{C} \setminus \mathcal{S}, y_0) &\xrightarrow{s_*} \pi_1(\mathbf{C}^2 \setminus \mathcal{C} \cup \mathcal{L}, s(y_0)) \\ &\cong \pi_1(\mathbf{C}^2 \setminus \mathcal{C} \cup \mathcal{L}). \end{aligned}$$

Since  $s(\mathbf{C})$  is disjoint from  $\mathcal{C}$  and a section to the projection  $\mathbf{C}^2 \rightarrow \mathbf{C}$ , it is contractible and all elements of  $\pi_1(\mathbf{C} \setminus \mathcal{S})$  are therefore mapped to the trivial class. So as claimed the normal closure of  $F_s$  is contained in the kernel of  $i_*$ .

We are left to prove that it actually coincides with this kernel. As we know by 1.13 the kernel is generated by simple geometric elements associated to the irreducible components of  $\mathcal{L}$ . Pick  $\mathcal{L}_k$  among the irreducible components, then a simple geometric element in  $\pi_1(\mathbf{C} \setminus \mathcal{S})$  associated to  $y_k$  is mapped by  $s_*$  to a simple geometric element in  $\pi_1(\mathbf{C}^2 \setminus \mathcal{C} \cup \mathcal{L})$  associated to  $\mathcal{L}_k$ . So by 1.12 each simple geometric element associated to an irreducible component of  $\mathcal{L}$  is conjugated to an element in the image of  $s_*$  and thus in the normal closure of  $F_s$  as claimed.

Finally we have to derive the given presentation. We start with a presentation of  $\pi_1(\mathbf{C}^2 \setminus \mathcal{C} \cup \mathcal{L})$ . Since  $\gamma_1, \dots, \gamma_s$  generate the subgroup  $F_s$  we get a presentation for  $\pi_1(\mathbf{C}^2 \setminus \mathcal{C})$  by adding  $\gamma_1, \dots, \gamma_s$  to the set of relations. In the third step we get rid of generators  $\gamma_i$  and relations  $\gamma_i$  simultaneously and must replace  $\gamma_i$  by the identity in the remaining relations. In fact we get the presentation as claimed.  $\square$

**Remark 1.16:** The group  $G(a_*)$  defined by presentation (1.14) is the quotient of  $F_n$  by the normal subgroup generated by  $\{\gamma(t) \cdot t^{-1} \mid \gamma \in \text{im}(a_*), t \in F_n\}$ . In other words,  $G(a_*)$  is the maximal quotient of  $F_n$  on which the representation  $a_* : F_s \rightarrow \text{Br}_n$  acts trivially.

For use in a later chapter we prove a slight variation of this claim and compute the fundamental group of the complement of  $\mathcal{C}$  and a single component of  $\mathcal{L}$ .

**Lemma 1.17** Suppose  $\mathcal{L}_1$  is a single component of  $\mathcal{L}$ , then the fundamental group  $\pi_1(\mathbf{C}^2 \setminus \mathcal{C} \cup \mathcal{L}_1)$  is generated by  $i_*(\pi_1(\mathbf{C}_n))$  and any simple geometric element associated to  $\mathcal{L}_1$ .

*Proof:* By the same argument as before, we get a presentation

$$\pi_1(\mathbf{C}^2 \setminus \mathcal{C}) = \langle t_1, \dots, t_n \mid \gamma_1^{-1} \cdot t_i \gamma_1 = a_*(\gamma_1)(t_i), t_i = a_*(\gamma_k)(t_i), k > 1 \rangle.$$

A simple geometric element associated to  $\mathcal{L}_1$  is then in the conjugation class of  $\gamma_1$ , thus represented by a word  $w\gamma_1w^{-1}$ . Due to the relations  $\gamma_1^{-1} \cdot t_i \cdot \gamma_1 = a_*(\gamma_1)(t_i)$  we may assume that  $w$  does not contain the letter  $\gamma_1$ , since they can be moved to the end of  $w$  and then cancel through  $\gamma_1$ . But then it is obvious that the fundamental group is generated by the  $t_i$  and  $w\gamma_1w^{-1}$ .  $\square$

**Remark 1.18:** Suppose  $a'_* : F_s \rightarrow \text{Br}_n$  is related to  $a_*$  by a commutative diagram

$$\begin{array}{ccccc} F_s & \xrightarrow{a_*} & \text{Br}_n & \xrightarrow{\alpha_n} & \text{Aut}(F_n) \\ \psi \downarrow & & & & \downarrow \text{conj}_\phi \\ F_s & \xrightarrow{a'_*} & \text{Br}_n & \xrightarrow{\alpha_n} & \text{Aut}(F_n) \end{array}$$

with  $\psi \in \text{Aut}(F_s)$ ,  $\phi \in \text{Aut}(F_n)$  such that the restriction of  $\phi$  to  $\alpha_n(\text{Br}_n)$  is an isomorphism of  $\alpha_n(\text{Br}_n)$ . Then  $G(a_*)$  is isomorphic to  $G(a'_*)$ . Indeed, this condition can be written as  $\phi(a_*(g)(t) \cdot t^{-1}) = a'_*(\psi(g))(\phi(t)) \cdot \phi(t)^{-1}$ ,  $\forall g \in F_s$ ,  $\forall t \in F_n$ . Thus  $\phi \in \text{Aut}(F_n)$  induces an isomorphism  $\bar{\phi} : G(a_*) \rightarrow G(a'_*)$ .

Since  $a_*$ ,  $a'_*$  need not to be braid equivalent we see that the fundamental groups of complements can be isomorphic, in fact the curve complements can be homotopy equivalent, for curves which have different braid monodromies. An example of this kind was given in [10].

### 1.3.2 braid monodromy generators

We now make the presentation (1.14) more precise. To this end recall that the braid group  $\text{Br}_n$  can be presented by generators  $\sigma_1, \dots, \sigma_{n-1}$  subjected to relations  $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$  ( $1 \leq i < n-1$ ),  $\sigma_i \sigma_j = \sigma_j \sigma_i$  ( $|i-j| > 1$ ), see [5], [18]. The Artin representation  $\alpha_n : \text{Br}_n \rightarrow \text{Aut}(F_n)$  is then given in terms of  $F_n = \langle t_1, \dots, t_n \mid \rangle$  by:

$$\sigma_i(t_j) = \begin{cases} t_i t_{i+1} t_i^{-1} & \text{if } j = i, \\ t_i & \text{if } j = i+1, \\ t_j & \text{otherwise.} \end{cases}$$

So far we chose systems of paths in  $\mathbf{C} \setminus \mathcal{S}$  and the fibre  $\mathbf{C}_n$  over  $y_0$  only. But of course we may do so in any fibre  $p^{-1}(y), y \in \mathbf{C} \setminus \mathcal{S}$  – though not in a coherent way, as that would amount to a global trivialization of the bundle.

As we have represented the elements  $\gamma_i$  by nooses we may divide them into small loops  $\omega_i$  based at the neck, and long ropes  $\eta_i$ .

Like  $\gamma_i$  induce braid automorphism upon the choice of a system of paths for  $\mathbf{C}_n$ , so do the  $\omega_i$  upon a choice for the fibre over the neck of the noose.

The identification of  $\eta_i$  with an element in  $\text{Br}_n$  depends on both choices. Note that restricted to  $\eta_i$  the bundle is trivializable, hence one choice is sufficient as a global choice. So we can compare both choices.

In fact the different choices up to isotopy form a simply transitive orbit under the Artin representation, hence an ordered pair determines uniquely the corresponding transition braid.

These constructions fit naturally together to yield

$$a_*(\gamma_k) = \beta_k^{-1} \alpha_k \beta_k,$$

where  $\alpha_k \in \text{Br}_n$  is the monodromy along  $\omega_k$  and  $\beta_k \in \text{Br}_n$  is the transition along  $\eta_k$ .

So as one would like to express these braids in terms of the standard generators  $\sigma_i$  of  $\text{Br}_n$ , one may try to accomplish this in two steps.

**Step 1** The structure of the isolated singularities in the fibre at  $y_k$  determines the local braid  $\alpha_k \in \text{Br}_n$ , upon a choice of a geometrically distinguished system of paths for the fibre at  $y_k^0$ , the neck of the noose and base point of  $\omega_k$ .

This braid may be obtained from the Puiseux series expansion of the defining polynomial of  $\mathcal{C}$  at each intersection point with  $\pi^{-1}(y_k^0)$ . The actual algorithm is implicit in the work of Brieskorn, Knörrer [8] and Eisenbud, Neumann [14].

**Step 2** A transition braid  $\beta_k$  depends on the relation between the choices of systems of paths in the fibres to be compared along the path.

The pull back of the bundle along the path  $\eta_k$  is a trivial  $\text{Br}_n$ -bundle, hence the mapping class groups of the two fibres at the endpoints are identified by parallel transport. Since both are identified with the abstract braid group we get an isomorphism of braid groups.

This is actually obtained by inner conjugation since the isomorphisms are all defined in terms of distinguished systems of paths.

The important point to note is that in case the local braid  $\alpha_k$  involves few strings we don't need to understand the parallel transport resp.  $\beta_k$  to full extend, but only  $\beta_k$  up to the stabilizer of the local braid under the conjugation action, in order to determine  $\beta_k^{-1} \alpha_k \beta_k$ .

Let us try the first step on some specific singularities and exemplify the final remark of the second step.

**Example 1.19:** Consider the plane curve  $\mathcal{C} : z^p - x^q = 0$ . The fundamental group of its complement was determined by Oka [35]. A look at Oka's computation

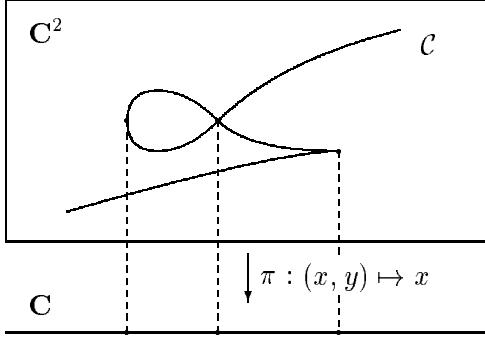


Figure 1.3: vertical tangent, node and cusp

reveals a natural choice of system of paths such that the braid monodromy generator is  $(\sigma_1 \cdots \sigma_{p-1})^q \in \text{Br}_p$ . For instance, to a simple tangency yields  $\sigma_1$ , a node yields  $\sigma_1^2$ , and a cusp yields  $\sigma_1^3$ . Hence the individual monodromies around the special points  $y_k$  are conjugated to powers of half-twists, the exponent being 1 in the case of tangency points, 2 in the case of ordinary nodes, and 3 in the case of ordinary cusps.

**Example 1.20:** If the monodromy  $\alpha$  of  $\omega$  is conjugated to a power of a half twist as above, then this twist is determined by an isotopy class of arcs between the two punctures involved. Of course  $\beta\alpha\beta^{-1}$  is a half twist again, so in fact we need to determine the image of the arc under the parallel transport along  $\eta$  only.

We close this section with two remarks touching only on the surface of recent developments.

**Remark 1.21:** The program presented so far is well adapted to the study of branch curves of generic projections of projective surfaces to complex projective plane. Such curves have only singular points which are ordinary cusps or nodes.

The methods have also been generalized to the symplectic set up.

**Remark 1.22:** In the case of branch curves Moishezon and Teicher, [32, 40], developed a degeneration technique. They start with a projective embedding of the complex surface  $X$ , and deforms the image of this embedding to a singular configuration  $X_0$  consisting of a union of planes intersecting along lines. The branch curve of a projection of  $X_0$  to  $\mathbf{C}^2$  is therefore a union of lines; the manner in which the smoothing of  $X_0$  affects this curve can be studied explicitly, by considering a certain number of standard local models near the various points of  $X_0$  where three or more planes intersect.

## 1.4 braid monodromy of horizontal divisors

We want now to generalize the considerations for plane algebraic curves in two aspects. We go to higher dimensions, i.e. affine algebraic divisors in complex affine

space, and include the local case of analytic space germs in  $\mathbf{C}^{r+1}, 0$ . The latter is in fact a generalization of the local considerations at singularities  $v_k$  of a plane curve by means of Puiseux series.

So we work with a polynomial  $f$  or a holomorphic function germ of vanishing order  $n$  in  $r+1$  variables. For simplicity we assume that the projection is along the distinguished coordinate  $z$  and that  $f$  is monic of degree  $n$  in the polynomial ring over  $\mathbf{C}[x_1, \dots, x_r]$ , resp.  $z$ -general in  $\mathbf{C}[x, z]$ . Then in the local analytic case we may use the Weierstrass preparation theorem to end up in either case with the divisor given by the zero set of a Weierstrass polynomial function  $f : \mathbf{C}^r \times \mathbf{C} \rightarrow \mathbf{C}$ ,

$$f(x, z) = z^n + \sum_{i=0}^{n-1} a_i(x) z^i,$$

with  $a_i \in \mathbf{C}[x]$  resp.  $a_i \in \mathbf{m}_r := \{a \in \mathcal{O}_r \mid a(0) = 0\}$ .

Let us resume the local analytic case later merely addressing the necessary improvements and stick to the affine case now.

The set  $\mathcal{S} \subset \mathbf{C}^r$  of points for which the corresponding fibre is tangent to the divisor  $\mathcal{H}$  defined by  $f$  or contains a singular point of  $\mathcal{H}$  is the zero locus of the resultant of  $f$  and  $\partial_z f$  which eliminates  $z$  and coincides with the pull back of the discriminant  $\Delta_n$  in  $\mathbf{C}^n$  along the coefficient map  $a = (a_0, \dots, a_{n-1}) : \mathbf{C}^r \rightarrow \mathbf{C}^n$ .

Let  $\mathcal{L}$  denote the linear extension of  $\mathcal{S}$  to  $\mathbf{C}^r \times \mathbf{C}$  by pull back. In generalization of the plane curve case we get a locally trivial bundle

$$\mathbf{C}^{r+1} \setminus \mathcal{H} \cup \mathcal{L} \longrightarrow \mathbf{C}^r \setminus \mathcal{S}$$

with structure group  $\text{Br}_n$ , fibre  $\mathbf{C}_n$  and holonomy

$$\rho : \pi_1(\mathbf{C}^r \setminus \mathcal{S}, y_0) \rightarrow \text{Aut}(\pi_1(\mathbf{C}_n, v_0)) \quad (1.5)$$

which is the coefficient homomorphism  $a_* : \pi_1(\mathbf{C}^r \setminus \mathcal{S}, y_0) \rightarrow \text{Br}_n$ , if  $\text{Br}_n$  is identified with its image under the Artin representation using a suitable geometrically distinguished system of paths. So we define:

**Definition 1.23:** The homomorphism  $\alpha : \pi_1(\mathbf{C}^r \setminus \mathcal{S}) \rightarrow \text{Br}_n$  determined by the composition  $\alpha_n \circ \alpha = \rho$  is called *braid monodromy homomorphism* of the hypersurface  $\mathcal{H}$  projected along  $z$ .

It is only well defined up to a certain isomorphisms of the source and conjugation in the target, but to proceed we won't have to specify them.

#### 1.4.1 braid monodromy presentation

To get a presentation for  $\pi_1(\mathbf{C}^{r+1} \setminus \mathcal{H})$  we argue as in the curve case. So we start with the long exact homotopy sequence of the bundle  $\mathbf{C}^{r+1} \setminus \mathcal{H} \cup \mathcal{L} \rightarrow \mathbf{C}^r \setminus \mathcal{S}$ .

This time our argument must be a bit more substantial to get a short exact sequence out of it:

**Lemma 1.24** *The boundary map  $\pi_2(\mathbf{C}^r \setminus \mathcal{S}) \rightarrow \pi_1(\mathbf{C}_n)$  is trivial.*

*Proof:* Of course the image of the boundary map is normal and abelian. We get the proof now in cases  $n = 1$  and  $n > 1$  separately. If  $n = 1$  then  $f$  is a linear polynomial and  $\mathcal{S}$  is empty so  $\pi_2(\mathbf{C}^r \setminus \mathcal{S}) = 0$ . If  $n > 1$  the group  $\pi_1(\mathbf{C}_n)$  is free of rank at least two. But then only the trivial subgroup is normal and abelian.  $\square$

Since the splitting argument 1.9 still applies, we can base the proof of the following lemma on the short split exact sequence:

$$1 \rightarrow \pi_1(\mathbf{C}_n) \rightarrow \pi_1(\mathbf{C}^{r+1} \setminus \mathcal{H} \cup \mathcal{L}) \xrightarrow{s_*} \pi_1(\mathbf{C}^r \setminus \mathcal{S}) \rightarrow 1,$$

with action given by the braid monodromy homomorphism  $a_*$  (1.5).

**Lemma 1.25** *The fundamental group  $\pi_1(\mathbf{C}^{r+1} \setminus \mathcal{H})$  of the complement of an affine divisor  $\mathcal{H}$  is the quotient of  $\pi_1(\mathbf{C}^{r+1} \setminus \mathcal{H} \cup \mathcal{L})$  by the normal closure of  $\pi_1(\mathbf{C}^r \setminus \mathcal{S})$  considered as a subgroup by the section  $s_*$ , thus it is presented as*

$$\pi_1(\mathbf{C}^{r+1} \setminus \mathcal{H}) = \langle t_1, \dots, t_n \mid t_i = a_*(\gamma_k)(t_i) \rangle,$$

where  $\gamma_1, \dots, \gamma_m$  is any system of generators for  $\pi_1(\mathbf{C}^r \setminus \mathcal{S})$ .

*Proof:* By 1.13 the map  $i_* : \pi_1(\mathbf{C}^{r+1} \setminus \mathcal{H} \cup \mathcal{L}) \rightarrow \pi_1(\mathbf{C}^{r+1} \setminus \mathcal{H})$  is surjective. Since  $s(\mathbf{C}^r)$  is contractible the image by  $s_*$  of  $\pi_1(\mathbf{C}^r \setminus \mathcal{S})$  and its normal closure must be trivial in  $\pi_1(\mathbf{C}^{r+1} \setminus \mathcal{H})$ . As in the curve case 1.14 we can give an argument relying on 1.13 that this is the kernel of  $i_*$ .

The presentation is then obtained using the split sequence,  $s_*$ , and  $i_*$ . Let us suppose we are given an arbitrary presentation

$$\pi_1(\mathbf{C}^r \setminus \mathcal{S}) = \langle \gamma_1, \dots, \gamma_s \mid r_1, \dots, r_m \rangle,$$

where  $r_1, \dots, r_m$  are words in the listed generators. Since the fibre of the bundle has fundamental group presented as  $\pi_1(\mathbf{C}_n) = \langle t_1, \dots, t_n \mid \rangle$ , we get with the split exact sequence:

$$\pi_1(\mathbf{C}^{r+1} \setminus \mathcal{H} \cup \mathcal{L}) = \langle t_1, \dots, t_n, \gamma_1, \dots, \gamma_s \mid r_1, \dots, r_m, \gamma_k t_i \gamma_k^{-1} = a_*(\gamma_k)(t_i) \rangle$$

As in the curve case the results on the kernel of  $i_*$  imply, that we get a presentation for  $\pi_1(\mathbf{C}^{r+1} \setminus \mathcal{H})$  if we add the relations  $\gamma_1, \dots, \gamma_s$ . So we may drop generators  $\gamma_k$  and replace  $\gamma_k$  in all relations by the identity, to get a Tietze equivalent presentation. Because  $\gamma_1, \dots, \gamma_s, r_1, \dots, r_m$  become trivial relations, we may as well discard them and end up with

$$\pi_1(\mathbf{C}^{r+1} \setminus \mathcal{H}) = \langle t_1, \dots, t_n \mid t_i = a_*(\gamma_k)(t_i) \rangle,$$

as claimed, since in the beginning we may have chosen any set of generators for  $\pi_1(\mathbf{C}^r \setminus \mathcal{S})$ .  $\square$

In fact, to get presentations of fundamental groups as in lemma 1.25, we may go back to the case of curves:

Consider a generic fibre  $L$  of a generic fibration of  $\mathbf{C}^r \setminus \mathcal{S}$ . Then  $\pi_1(L \setminus \mathcal{S}_L)$  surjects onto  $\pi_1(\mathbf{C}^r \setminus \mathcal{S})$ . So given generators  $\gamma_1, \dots, \gamma_s$  for  $\pi_1(L \setminus \mathcal{S}_L)$  we get by the last result a presentation

$$\pi_1(\mathbf{C}^{r+1} \setminus \mathcal{H}) = \langle t_1, \dots, t_n \mid t_i = a_*(\gamma_k)(t_i) \rangle,$$

If we define in the plane  $p^{-1}(L)$  the curve  $\mathcal{C}_L := \mathcal{H} \cap p^{-1}(L)$  then by our arguments the above presentation is also valid for  $\pi_1(p^{-1}(L) \setminus \mathcal{C}_L)$ . This is of course one of the well known Zariski van Kampen results.

#### 1.4.2 braid monodromy of local analytic divisors

To get the analogous results in the case of local analytic divisors, we must go through the same procedure, but we are forced to make some more choices in the construction, which we later have to show to have no bearing on the definitions.

We already reached the stage at which we have to deal with a Weierstrass polynomial  $f(x, z) = z^n + \sum a_i(x)z^i$ ,  $a_i(0) = 0$ . It defines the given analytic germ  $\mathcal{H}$  and for sufficiently small  $\rho$  the coefficients  $a_i$  are defined and bounded for  $|x| < \rho$ .

Since the divisor was assumed to be reduced, the set of points  $|x| < \rho$  such that  $f(x, z)$  is a simple Weierstrass polynomial is the complement of a proper analytic subset  $\mathcal{S}_\rho$ , which is defined by the resultant of  $f$  and  $\partial_z f$  with respect to  $z$ . So we get an associated polynomial cover over  $X_\rho := B_\rho \setminus \mathcal{S}_\rho$ :

$$f : X_\rho \times \mathbf{C} \rightarrow \mathbf{C}, (x, z) \mapsto z^n + \sum a_i(x)z^i.$$

By the choice of  $\rho$ , the zeroes of  $f$  are uniformly bounded for  $x \in B_\rho$ , hence there is a section  $s$  to the projection  $\pi : X_\rho \times \mathbf{C} \rightarrow X_\rho$  which avoids the zero set of  $f$ .

As in the previous cases we have a locally trivial bundle

$$Y_\rho \rightarrow X_\rho, \quad Y_\rho := X_\rho \times \mathbf{C} \setminus f^{-1}(0),$$

for which a holonomy  $\pi_1(X_\rho, y_0) \rightarrow \text{Aut}(\pi_1(\mathbf{C}_n, v_0))$  is given after a choice of base points,  $y_0 \in X_\rho$  and  $v_0$  in the fibre  $\pi^{-1}(y_0)$ .

For  $\rho$  below some finite bound, the isomorphism type of  $\pi_1(X_\rho, y_0)$  is well defined, hence we may define:

**Definition 1.26:** For  $\rho$  sufficiently small, the map  $a_* : \pi_1(X_\rho, y_0) \rightarrow \text{Br}_n$  induced by the coefficient map of  $Y_\rho \rightarrow X_\rho$  is called *braid monodromy homomorphism* of the germ  $\mathcal{H}$  projected along  $z$ .

It is well defined at least up to isomorphisms of the source and conjugation in the target, which is all we need in the sequel.

Moreover the local analogue of lemma 1.25 holds true:

**Lemma 1.27** Suppose  $\mathcal{H}$  represents the germ of a divisor in  $\mathbf{C}^{r+1}, 0$ , then for  $\varepsilon$  and  $\rho = \rho(\varepsilon)$  sufficiently small, the isomorphism class  $\pi_1(B_\varepsilon \times B_\rho \setminus \mathcal{H})$  of fundamental groups is presented as

$$\pi_1(B_\varepsilon \times B_\rho \setminus \mathcal{H}) = \langle t_1, \dots, t_n \mid t_i = a_*(\gamma_k)(t_i) \rangle,$$

where  $\gamma_1, \dots, \gamma_m$  is any system of generators for  $\pi_1(X_\rho, y_0)$ .

*Proof:* Let  $\varepsilon$  and  $\rho$  be small enough to make sure that  $B_\varepsilon \times B_\rho \cap \mathcal{H}$  is a branched cover of degree  $n$  over  $B_\rho$  and branched along an analytic subset with complement  $X_\rho$ .

Then  $B_\varepsilon \times B_\rho \setminus \mathcal{H}$  is a strong deformation retract of  $Y_\rho$  and the claim follows along the line of argument of lemma 1.25.  $\square$

**Remark 1.28:** If  $\mathcal{H}$  is given by a quasi homogeneous polynomial  $f$  with a good  $\mathbf{C}^*$ -action, then so is  $\mathcal{S}_\rho$ . In that case the  $\mathbf{C}^*$ -action can be used to show that  $B_\varepsilon \times B_\rho \setminus \mathcal{H}$  is a strong deformation retract of  $\mathbf{C}^{r+1} \setminus f^{-1}(0)$  by a map which respects the projection to  $B_\varepsilon$  resp.  $\mathbf{C}^r$ .

Hence all braid monodromy considerations for the hypersurface germ  $\mathcal{H}$  are equal to those for the affine hypersurface defined by  $f$ .

## Chapter 2

# braid monodromy of singular functions

In this chapter we review the notion of semi universal unfoldings of singular function germs and show how a natural polynomial cover arises in this set-up. The corresponding braid monodromy homomorphisms and braid monodromy groups are then associated invariantly to equivalence classes of function germs via their versal unfoldings.

In particular a presentation for the fundamental group of the discriminant complement will be derived from the new invariants.

But to get that far, we have to make quick digression through the theory of unfoldings of singular functions touching on such diverse notions as discriminant sets, truncated unfoldings, and bifurcation sets.

### 2.1 preliminaries on unfoldings

The basic objects in singularity theory we start with are holomorphic function germs on affine coordinate space,  $f : \mathbf{C}^n, 0 \rightarrow \mathbf{C}$ , which form the  $\mathbf{C}$ -algebra  $\mathcal{O}_n$ . It is a local algebra with maximal ideal

$$\mathfrak{m}_n = \{f \in \mathcal{O}_n \mid f(0) = 0\}$$

Since a function on abstract affine space is identified with different elements of  $\mathcal{O}_n$  depending on a choice of coordinates, it is natural to consider such elements to be equivalent, more precisely:

**Definition 2.1:** Given elements  $f, g \in \mathcal{O}_n$  are called *right equivalent* or simply *equivalent*,  $f \sim g$ , if there is a holomorphic map  $\rho : \mathbf{C}^n, 0 \rightarrow \mathbf{C}^n, 0$  such that  $f(x) = g(\rho(x))$  and  $\rho$  is biholomorphic.

If  $X \subset \mathfrak{m}_n$  is an equivalence class and  $f \in X$ , then  $f$  is called a function of type  $X$ .

**Example 2.2:** The classes of simple singularities according to Arnold, cf. [1], are represented by

$$A_k: x_1^{k+1} + x_2^2 + \cdots + x_n^2,$$

$$D_k: x_1^2 x_2 + x_2^{k-1} + x_3^2 + \cdots + x_n^2,$$

$$E_6: x_1^3 + x_2^4 + x_3^2 + \cdots + x_n^2,$$

$$E_7: x_1^3 + x_1 x_2^3 + x_3^2 + \cdots + x_n^2,$$

$$E_8: x_1^3 + x_2^5 + x_3^2 + \cdots + x_n^2,$$

### 2.1.1 versal unfolding

Proceeding deeper into the theory we introduce secondary objects to a given function germ  $f \in \mathcal{O}_n$ :

**Definition 2.3:** A function germ  $F$  on affine coordinate germ  $\mathbf{C}^n \times \mathbf{C}^k, 0$  is called *unfolding* of  $f \in \mathcal{O}_n$ , if  $F_0 = f$  for  $F_0 := F|_{\mathbf{C}^n \times \{0\}, 0}$ . Then  $k$  is called the *unfolding dimension* and  $\mathbf{C}^k, 0$  the base or the parameter space of the unfolding.

This notion should be understood as a family of function germs in  $\mathcal{O}_n$  parameterized over a space germ  $\mathbf{C}^k, 0$ . This interpretation is also at the base of the equivalence notion induced on unfoldings.

**Definition 2.4:** Suppose  $F, G \in \mathcal{O}_{n+k}$  are unfoldings of  $f \in \mathcal{O}_n$ . Then  $F, G$  are called *equivalent*, if there is a holomorphic map germ  $\rho : \mathbf{C}^{n+k}, 0 \rightarrow \mathbf{C}^n, 0$  such that  $G(u, x) = F(u, \rho(u, x)), \rho(0, x) = x$ .

And as with families one can consider the pull back of an unfolding along a map to its base: Suppose  $F \in \mathcal{O}_{n+k}$  is an unfolding of  $f \in \mathcal{O}_n$  and  $\varphi : \mathbf{C}^l, 0 \rightarrow \mathbf{C}^k, 0$  is a map germ, we call  $G \in \mathcal{O}_{n+l}$  the unfolding of  $f$  *induced* from  $F$  by  $\varphi$ , if  $G(v, x) = F(\varphi(v), x)$ .

Thus prepared we can now introduce the concept best suited for classification in singularity theory.

**Definition 2.5:** If  $F \in \mathcal{O}_{n+k}$  is an unfolding of  $f \in \mathcal{O}_n$  then  $F$  is called *versal*, if each unfolding  $G \in \mathcal{O}_{n+l}$  of  $f$  is equivalent to an unfolding induced from  $F$ .

A versal unfolding is called *miniversal* if it is versal and of minimal dimension.

Naturally one would ask to induce a given unfolding in a unique way but experience taught to be content with uniqueness only of the differential of the pull back map.

Accordingly the miniversal unfoldings which have got this property are also called *semi universal*.

**Remark 2.6:** Versality is an open condition in the following sense. If a representative of a versal unfolding  $F$  is given and is defined for  $u, 0$ , then  $G$  given by  $G(v, x) := F(u + v, x)$  is an unfolding of the function  $x \mapsto F(u, x)$ .

The case of functions with an isolated singularity – which we are interested in exclusively – is characterized by the finite codimension in  $\mathcal{O}_n$  of the *Jacobian ideal* generated by the partial derivatives,  $J(f) = \langle \partial_1 f, \dots, \partial_n f \rangle$ .

In this case specific miniversal unfoldings and the mutual relations of versal unfoldings can be described according to [1], chapter 8:

**Proposition 2.7** *Given  $f \in \mathcal{m}_n$  and  $b_1, \dots, b_k \in \mathcal{O}_n$  such that the classes of  $b_1, \dots, b_k$  modulo  $J(f)$  form a generating set (a basis) of  $\mathcal{O}_n/J(f)$ . A versal (semi universal) unfolding of  $f$  is then given by  $F(u_1, \dots, u_k, x) := f(x) + u_1 b_1(x) + \dots + u_k b_k(x)$ .*

*Proof:* By hypothesis the function  $b_i$  and the ideal  $J(f)$  span the tangent space of  $\mathcal{O}_n$  at  $f$ . The ideal  $J(f)$  can be shown to be the tangent space to the orbit of action by biholomorphic maps of the source of  $f$ . The point of the proof is thus – as indicated in [1] – that the infinitesimal transversality to the orbit of  $f$  lifts to local transversality to all orbits sufficiently close.

**Proposition 2.8** *Suppose  $F \in \mathcal{O}_{n+k}$  and  $G \in \mathcal{O}_{n+l}$  are versal unfoldings of  $f \in \mathcal{O}_n$  and  $k \geq l$ . Then  $F$  is equivalent to some unfolding induced from  $G$  by a map germ  $\varphi : \mathbf{C}^k, 0 \rightarrow \mathbf{C}^l, 0$  of full rank.*

*Proof:* This result can be derived from the previous by a careful analysis of all the definitions involved.

### 2.1.2 discriminant set

The discriminant set is the germ of subsets in the base of a versal unfolding  $F \in \mathcal{O}_{n+k}$  of a singular function  $f \in \mathcal{O}$  given by those parameters  $u$  for which 0 is a critical value of the function  $F_u : x \mapsto F(u, x)$ .

To give a precise meaning to this description, we examine the situation for a representative of the germ  $F$ . W.l.o.g. assume  $f : \mathbf{C}^n, 0 \rightarrow \mathbf{C}, 0$  to be a germ with an isolated critical point at 0. Choose sufficiently small neighbourhoods of zero  $M = \{x | \|x\| \leq \rho\} \subset \mathbf{C}^n$  and  $U = \{u | \|u\| \leq \delta\} \subset \mathbf{C}^k$ , for which a representative  $\tilde{F}$  is defined.

By the curve selection lemma we may assume  $\rho$  and  $\delta = \delta(\rho)$  sufficiently small such that the level set  $\{x | \tilde{F}(u, x) = 0\}$  is non-singular on the boundary  $\partial M$  of the ball  $M$  and is transverse to  $\partial M$  for every  $u \in U$ , cf. [28].

If we distinguish  $u \in U$  into singular and non-singular parameters according to  $V_u := \{x | F(u, x) = 0\} \cap M$  being singular or not, we define the discriminant set by means of any representative of  $F$ :

**Definition 2.9:** The *discriminant set* in the base of the versal unfolding  $F \in \mathcal{O}_{n+k}$  of a singularity  $f \in \mathcal{O}_n$  is represented by the singular parameters in  $U$  for a representative of  $F$ .

Of course any other representative of  $F$  coincides with the chosen one for sufficiently small neighbourhoods and so does the corresponding set of singular parameters.

**Example 2.10:** Let  $f(x) = x^3$ ,  $F(u, x) = x^3 + u_1 x + u_0$ . The discriminant set for  $f$  is precisely the set of pairs  $u_1, u_0$  such that the polynomial  $x^3 + u_1 x + u_0$  has multiple roots. Hence the discriminant is the hypersurface germ cut out by the equation  $27u_0^2 + 4u_1^3 = 0$ .

This examples exposes a common feature of all discriminant sets:

**Proposition 2.11** *The discriminant set in the base of a versal unfolding  $F$  of a singular function  $f$  is an irreducible hypersurface germ.*

*Proof:* The proof uses suitable representatives again and exploits the fact, that the discriminant is the image of the critical set, which is irreducible and analytic, under a proper finite map, cf. [13]  $\square$

### 2.1.3 truncated versal unfolding

Let  $f : \mathbf{C}^n, 0 \rightarrow \mathbf{C}, 0$  be a function germ. Instead of unfoldings  $F \in \mathcal{O}_{n+k}$ , which can be considered as families of elements in  $\mathcal{O}_n$ , we now confine ourselves to unfoldings  $F$  such that  $F(u, 0) \equiv 0$ , which can be considered as families of elements in  $\mathbf{m}_n$ .

The definitions of equivalence, induced unfoldings, and versality carry over without modifications to the present case.

**Definition 2.12:** If  $F \in \mathbf{m}_{n+k}$  is an unfolding of  $f \in \mathbf{m}_n$  then  $F$  is called a *truncated versal unfolding*, if  $F(u, 0) \equiv 0$  and each unfolding  $G \in \mathbf{m}_{n+l}$  of  $f$  with  $G(v, 0) \equiv 0$  is equivalent to an unfolding induced from  $F$ .

The relation to versal unfoldings in the ordinary sense is readily given:

**Proposition 2.13** *Suppose  $F \in \mathcal{O}_{n+k}$  and  $F^\# \in \mathbf{m}_{n+k-1}$  are unfoldings of  $f \in \mathbf{m}_k$  related by  $F(u, x) = F^\#(u^\#, x) + u_0$ . Then  $F^\#$  is a truncated versal unfolding of  $f$  if and only if  $F$  is a versal unfolding of  $f$ .*

This stems from the fact that a miniversal truncated unfolding is given by  $F^\#(u, x) = f(x) + u_1 b_1 + \dots + u_{k-1} b_{k-1}$  if for  $\mathbf{m}_n \rightarrow \mathbf{m}_n/J(f)$  the  $b_i$  map to a basis of the quotient, cf. prop. 2.7.

Given a truncated versal unfolding  $F^\#$  we call the versal unfolding  $F$  of the proposition the *completed versal unfolding* corresponding to  $F^\#$ .

### 2.1.4 bifurcation set

A function is said to be a *Morse function*, if it has only non-degenerate critical points and their values are distinct. For the definition of the bifurcation set as the set of parameters in the base of a truncated versal unfolding  $F^\#$ , for which the corresponding function is not Morse, we have once again to resort to a local representative:

**Definition 2.14:** The *bifurcation set* in the base of a truncated versal unfolding  $F^\#$  of a singular function  $f$  is represented by the set of parameters for which the corresponding function restricted to a sufficiently small ball  $M$  is not a Morse function.

Again one should convince oneself that this definition does not depend on the choice of representatives and sufficiently small neighbourhoods.

**Example 2.15:** By  $F^\#(u, x) = x^4 + u_2x^2 + u_1x$  a truncated versal unfolding of  $f(x) = x^4$  is given. This polynomial has degenerate critical points if and only if its derivative has multiple roots, i.e.  $27u_1^2 + 8u_2^3 = 0$ . Moreover one can verify that critical values coincide only if  $F_u$  is an even function, i.e.  $u_1 = 0$ . The bifurcation set is thus the union of a cusp corresponding to functions with degenerate critical points and a line corresponding generically to functions with common values of distinct critical points.

The example is but an instance of general facts which are more involved than in the case of discriminants and which are summarized below.

**Proposition 2.16** *In the base of a truncated versal unfolding the set of parameters such that the corresponding function has a degenerate critical points (not of type  $A_1$ ) defines an analytic hypersurface germ. It is empty only for  $f$  of type  $A_1$  and irreducible otherwise.*

*Proof:* Similar to the proof of prop. 2.11 one has to use the fact that the determinant of the Hessian is transversal to the critical set and thus determines a smooth analytic germ of dimension  $k - 1$  which can be shown to project properly and finitely to the set under scrutiny, cf. [41],[37].  $\square$

The complement of this *strict bifurcation variety* consists of parameters corresponding to functions with non-degenerate critical points only, at least two of which have a common value. Its closure goes by several names, e.g. *Maxwell stratum* [2], or *conflict variety* [41], and its decomposition into irreducible components was determined by Wirthmüller [41], cf. also [37]:

**Proposition 2.17** *The conflict variety in the base of a truncated versal unfolding is an analytic hypersurface germ, it is*

- i) empty, if  $f$  is of type  $A_1, A_2$ ,
- ii) the union of three irreducible components, if  $f$  is of type  $D_4$ ,
- iii) the union of two irreducible components, if  $f$  is of type  $D_\mu, \mu \geq 5$ ,
- iv) irreducible in all remaining cases.

## 2.2 discriminant braid monodromy

We finally draw closer to our proper objective. For any given  $f \in \mathbf{m}_n$  with isolated singularity consider a truncated versal unfolding  $F^\#$  and the corresponding completed versal unfolding  $F$ .

The discriminant is an analytic hypersurface germ, in the base  $\mathbf{C}^k, 0$  of  $F$ , hence given by a reduced holomorphic function  $\Delta$ . Since  $f$  is assumed to have an isolated singularity only, its singular value 0 is isolated, too. Therefore  $\Delta$  is  $u_0$ -regular and we may define:

**Definition 2.18:** The *braid monodromy homomorphism* associated to the truncated versal unfolding  $F^\#$  is the braid monodromy homomorphism of the discriminant in the base of the corresponding completed versal unfolding  $F$  projected along  $u_0$ .

It is this notion which is the central one of this paper and we will devote the rest of the section to some of its general properties, before we start the investigation for specific classes of singularities.

### 2.2.1 basic properties

First we identify the range of the braid monodromy homomorphism. We actually link it to the *Milnor number*  $\mu$  of the singular function  $f$ , which can be defined as the multiplicity of the critical value.

**Lemma 2.19** *The function  $\Delta$  defining the discriminant divisor is  $u_0$ -regular of order equal to the Milnor number of  $f$ .*

*Proof:* This follows from the fact that in the base of a versal unfolding of the given kind the line  $u_1 = \dots = u_{k-1} = 0$  is not in the tangent cone of the discriminant. We have just to add the information that the Milnor number gives the multiplicity of the discriminant.  $\square$

**Corollary 2.20** *The range of the braid monodromy homomorphism is  $\text{Br}_{\mu(f)}$ .*

Next we identify the source with the help of the bifurcation set  $\mathcal{S}_\rho$  in the base of  $F^\#$  restricted to a ball of sufficiently small radius  $\rho$  centered at the origin.

**Proposition 2.21** *The braid monodromy homomorphism of a truncated versal unfolding  $F^\#$  is defined on the isomorphism class  $\pi_1(B_\rho \setminus \mathcal{S}_\rho)$  for  $\rho$  sufficiently small.*

*Proof:* Note first that by the Weierstrass preparation theorem  $\Delta$  can be assumed to be a Weierstrass polynomial. By the construction of the braid monodromy homomorphism of projected hypersurface germs, it suffices to show, that the the discriminant set coincides with the set of parameters for which  $\Delta$  fails to be a simple Weierstrass polynomial, at least for  $\rho$  sufficiently small.

Now the multiplicity of a critical value is one, if and only if it is the value of a single non-degenerate critical point. Therefore the number of critical values drops for sufficiently small  $\rho$ , if and only if the corresponding function is not a Morse function.  $\square$

### 2.2.2 invariance properties

Most important are the invariance properties of the braid monodromy homomorphism. We have already noted, that the assignment of a braid monodromy homomorphism to an affine hypersurface germ projected along a suitable coordinate is not well-defined. But if we consider equivalence classes up to isomorphisms of the source and conjugation in the range, then the induced map is well defined.

In fact with this interpretation we get the invariance on the class of all truncated versal unfoldings of equivalent functions with isolated singularity.

**Proposition 2.22** *Given braid monodromy homomorphisms  $\alpha(F), \alpha(G)$  associated to truncated versal unfoldings  $F^\#$  and  $G^\#$  of equivalent functions  $f, g$ , then there exists a commutative diagram*

$$\begin{array}{ccc} \pi_1(B_\rho^F \setminus \mathcal{S}_\rho^F) & \xrightarrow{\alpha(F)} & \text{Br}_{\mu(f)} \\ \cong \downarrow & & \downarrow \text{conj}_\phi \\ \pi_1(B_\rho^G \setminus \mathcal{S}_\rho^G) & \xrightarrow{\alpha(G)} & \text{Br}_{\mu(g)}, \end{array}$$

where  $\mu(f) = \mu(g)$  and  $\text{conj}_\phi$  is conjugation by  $\phi \in \text{Br}_\mu$ .

*Proof:* Suppose w.l.o.g. that  $k \geq l$  for the unfolding dimensions of  $F$  and  $G$ . By hypothesis and prop. 2.8 there is a map germ of full rank  $\varphi^\# : \mathbf{C}^{k-1}, 0 \rightarrow \mathbf{C}^{l-1}, 0$  and a map germ  $\rho^\# : \mathbf{C}^{k-1} \times \mathbf{C}^n, 0 \rightarrow \mathbf{C}^n, 0$  with  $\rho^\#$  restricted to  $0 \times \mathbf{C}^n, 0$  biholomorphic and

$$F^\#(u, x) = G^\#(\varphi^\#(u), \rho^\#(u, x)).$$

The map germs  $\varphi : u \mapsto (u_0, \varphi^\#(u))$  and  $\rho : u, x \mapsto \rho^\#(u, x)$  then have analogous properties with

$$F(u, x) = G(\varphi(u), \rho(u, x)).$$

Hence we arrive at a commutative square of germs

$$\begin{array}{ccc} \mathbf{C}^k, 0 & \xrightarrow{\varphi} & \mathbf{C}^l, 0 \\ \downarrow \pi_{u_0} & & \downarrow \pi_{u_0} \\ \mathbf{C}^{k-1}, 0 & \xrightarrow{\varphi^\#} & \mathbf{C}^{l-1}, 0, \end{array}$$

which induces a commutative square for sufficiently small representatives.

Since the pull back of the discriminant for  $G$  along  $\varphi$  yields the discriminant for  $F$ , the bottom map  $\varphi^\#$  induces an isomorphism of fundamental groups of local bifurcation complements.

Moreover the associated polynomial cover for  $F$  is the pull back by  $\varphi, \varphi^\#$  of the polynomial cover associated to  $G$ . Hence the holonomy is the same.

The assertion of the proposition then follows, since conjugation on the right corresponds to different choices of geometrically distinguished systems of paths used for an identification of the holonomy group with the abstract group  $\text{Br}_\mu$ .  $\square$

### 2.2.3 invariants

Though keeping in mind the invariance along equivalence classes of singular functions, we will for linguistic reasons define an invariant of singular functions:

**Definition 2.23:** The class of braid monodromy homomorphisms associated to a truncated versal unfolding of a singular function  $f$  with isolated singularity is called the *braid monodromy homomorphism* of  $f$ .

An invariant which is easier to handle is obtained by considering the image of the braid monodromy homomorphism only, up to conjugation of course.

**Definition 2.24:** The braid monodromy group associated to  $f$  is the conjugacy class of subgroups of  $\text{Br}_{\mu(f)}$  given as the image of a braid monodromy homomorphism.

Supposing we have generators for the braid monodromy group associated to  $f$ , we get a presentation of the local fundamental group of the discriminant complement:

**Lemma 2.25** *Suppose the braid monodromy group associated to a singular function  $f$  is given as the conjugacy class determined by a subgroup of  $\text{Br}_\mu$  generated by braids  $\beta_1, \dots, \beta_n$ . Then the isomorphism class  $\pi_1(B_\varepsilon \times B_\rho \setminus \mathcal{H}_\Delta)$  for  $\varepsilon, \rho(\varepsilon)$  sufficiently small is represented by the finitely presented group*

$$\langle t_1, \dots, t_\mu \mid t_i^{-1} \beta_j(t_i) \rangle.$$

*Proof:* If braids  $\beta_j$  generate the image of the braid monodromy homomorphism, their preimages  $\gamma_j$  and generators of the kernel generate the source. Hence the claim follows from 1.27, since the trivial braid yields the trivial relation only.  $\square$

In fact we should also include an algebraic observation, which reduces the number of relations dramatically in case the generators are conjugated to braids non-trivial only on a few strands.

First we note that the choice of the generators  $\beta_j$  and of generators of the free group does not matter.

**Lemma 2.26** *Suppose  $B$  is a subgroup of  $\text{Br}_n$  which acts on generators  $t_1, \dots, t_n$  of a free group  $F_n$  by the Artin representation. Then the normal closure of the subgroup of  $F_n$  generated by*

$$w_i^{-1} \beta_j(w_i)$$

*is independent of the choice of a finite set of generators  $\{\beta_j\}$  of  $B$  and a finite set of generators  $\{w_i\}$  of  $F_n$ .*

Now we can use this to reduce the number of relations in case a braid generator  $\beta$  is conjugated to a twist.

**Lemma 2.27** *Suppose  $\beta = \beta_0 \sigma_1^l \beta_0^{-1} \in \text{Br}_n$ , then the normal subgroup generated by  $t_i^{-1} \sigma(t_i)$ ,  $i = 1, \dots, n$ , is equal to the normal subgroup generated by*

$$\beta_0(t_1^{-1}) \beta_0(\sigma_1^l(t_1)), \beta_0(t_2^{-1}) \beta_0(\sigma_1^l(t_2)).$$

*Proof:* By the previous lemma the first normal subgroup is equal to the normal subgroup generated by  $\beta_0(t_i^{-1}) \beta_0(\sigma_1^l(t_i))$ . Since these elements are trivial except for  $i \in \{1, 2\}$ , the claim follows.  $\square$

## Chapter 3

# Hefez Lazzeri unfoldings

The singular functions to which this study is devoted are the Brieskorn-Pham polynomials in arbitrary dimensions.

**Definition 3.1:** A polynomial  $f \in \mathbf{C}[x_1, \dots, x_n]$  is called a *Brieskorn Pham polynomial*, if there are positive integers  $l_1, \dots, l_n$  and

$$f(x_1, \dots, x_n) = x_1^{l_1+1} + \cdots + x_n^{l_n+1}.$$

They exhibit a lot of symmetry, most apparent the invariance of the polynomial under multiplication of the coordinate  $x_i$  with  $(l_i + 1)^{\text{st}}$  roots of 1. But more important to us is the invariance of its singular values under multiplication with  $l_i^{\text{th}}$  unit roots, for this invariance persists to the singular values of all functions obtained by linear perturbation terms. It is due to these symmetries that singular values and the bifurcation can be given by explicit polynomials, whereas in general such description seems quite unattainable.

In [19], Hefez and Lazzeri exploited the unfolding over the linear perturbation terms to some extend, computing the intersection lattice for Brieskorn polynomials. In order to extend their exploits, we review their results on the discriminant of  $f$  unfolded over the space of linear monomials and determine the bifurcation divisor, too. Moreover we show how this sheds light on various geometric aspects of the unfolding and of the corresponding perturbed functions.

### 3.1 discriminant and bifurcation hypersurface

Our current objective is to compute the corresponding discriminant and bifurcation divisor, the discriminant for the unfolding of  $f \in \mathbf{C}[x_1, \dots, x_n]$  given by

$$F(x, \alpha, z) := f(x) + z - \sum_{i=1}^n \alpha_i (l_i + 1) x_i,$$

and the bifurcation divisor for the unfolding

$$F^\#(x, \alpha) := f(x) - \sum_{i=1}^n \alpha_i (l_i + 1) x_i.$$

The partial derivatives are given by

$$\partial_i F(x, \alpha, z) = \partial_i F^\#(x, \alpha) = (l_i + 1)(x_i^{l_i} - \alpha_i), \quad i = 1, \dots, n.$$

Of course we shouldn't neglect the obvious quasi-homogeneity apparent in this situation:

**Lemma 3.2** *The functions  $F, \partial_i F$  are weighted homogeneous of degree one, resp.  $\frac{l_i}{l_i+1}$ , if  $wt(x_i) = \frac{1}{l_i+1}$ ,  $wt(z) = 1$  and  $wt(\alpha_i) = \frac{l_i}{l_i+1}$ .*

It implies in particular that the critical set, the discriminant divisor and the bifurcation divisor are quasi-homogeneous in their ambient affine spaces with a good  $\mathbf{C}^*$ -action.

We adopt now the following notation referring to roots of unity:

$$\xi_i : \text{the primitive } l_i^{\text{th}} \text{ root of 1 of least angle in } ]0, 2\pi[.$$

Then we can state the two central results of this section:

**Lemma 3.3** *The polynomial defining the critical value divisor is given by the expansion of the formal product*

$$\prod_{1 \leq \nu_1 \leq l_1} \cdots \prod_{1 \leq \nu_n \leq l_n} \left( -z + \sum_{i=1}^n l_i \xi_i^{\nu_i} \alpha_i^{\frac{l_i+1}{l_i}} \right).$$

*Proof:* The discriminant is the set of points  $(\alpha_1, \dots, \alpha_n, z)$  such that  $F$  and its partial derivatives  $\partial_{x_i} F$  vanish simultaneously at a point  $(x, \alpha, z)$ . Since the discriminant is known to be an algebraic hypersurface we are thus looking for a reduced monic polynomial  $p_\Delta \in \mathbf{C}[\alpha_1, \dots, \alpha_n, z]$  with zero set equal to the discriminant set.

Therefore we try to eliminate the variables  $x_i$  from the system of equations  $F = \partial_i F = 0$ . In a first step we replace  $x_i^{l_i}$  by  $\alpha_i$  in the equation  $F = 0$  and get instead

$$z = \sum_{i=1}^n \alpha_i l_i x_i.$$

Now it is helpful to consider the Galois extension of  $\mathbf{C}(\alpha_1, \dots, \alpha_n)$  defined by the polynomials  $x_i^{l_i} - \alpha_i$  which we denote by  $\mathbf{C}(a_1, \dots, a_n)$  with  $a_i^{l_i} = \alpha_i$ .

Then the system of equations is easily solvable in  $\mathbf{C}(a_1, \dots, a_n)[x_1, \dots, x_n, z]$  with  $x_i = \xi_i^{\nu_i} a_i$  to be eliminated to get

$$z = \sum_{i=1}^n \alpha_i l_i x_i = \sum_{i=1}^n l_i \xi_i^{\nu_i} a_i^{l_i+1}, \quad 0 \leq \nu_i < l_i.$$

So the corresponding discriminant is simply given by the polynomial

$$\prod_{1 \leq \nu_i \leq l_i} \left( -z + \sum_{i=1}^n l_i \xi_i^{\nu_i} a_i^{l_i+1} \right).$$

It is in fact a polynomial in  $\mathbf{C}[\alpha_1, \dots, \alpha_n, z]$  since it is invariant under the action of the Galois group of the extension which acts by multiplication of  $\xi_i$  on  $a_i$ .  $\square$

In particular the Milnor number  $\mu$  of  $f$ , which coincides with the  $z$ -degree of the discriminant, is thus shown to be

$$\mu = \prod_{i=1}^n l_i.$$

Similarly we get a description of the bifurcation divisor:

**Lemma 3.4** *The polynomial defining the bifurcation divisor in the base of  $F^\#$  is given by the expansion of the formal product*

$$\prod_{\substack{1 \leq \nu_i \leq l_i \\ 1 \leq \eta_i \leq l_i \\ \nu \neq \eta}} \left( \sum_{i=1}^n l_i \alpha_i^{\frac{l_i+1}{l_i}} (\xi_i^{\nu_i} - \xi_i^{\eta_i}) \right).$$

*Proof:* Of course the polynomial is uniquely defined up to a constant as the discriminant with respect to the variable  $z$  of the polynomial defining the discriminant divisor, given in lemma 3.3.

Passing to the Galois extension of  $\mathbf{C}(\alpha_1, \dots, \alpha_n)$  once again, we have to compute the discriminant of a polynomial which is a product of linear factors in  $\mathbf{C}(a_1, \dots, a_n)[z]$ . So up to a constant the discriminant is the product of the squares of the mutual differences between the zeroes of distinct factors.

$$discr_z(p_\Delta) = \prod_{\substack{1 \leq \nu_i, \eta_i \leq l_i \\ \nu \neq \eta}} \left( \sum_{i=1}^n l_i a_i^{l_i+1} (\xi_i^{\nu_i} - \xi_i^{\eta_i}) \right).$$

Since this polynomial is Galois invariant, the claim follows as above.  $\square$

Due to the importance of the bifurcation divisor this polynomial deserves a proper name, it will be denote by  $p_B$  and henceforth be called the *Hefez-Lazzeri polynomial* of  $f$ .

We end this section with some corollaries, highlighting some of the nice geometric properties of the Hefez-Lazzeri unfoldings of Brieskorn Pham polynomials.

**Lemma 3.5** *The polynomial  $p_B$  vanishes to order exactly  $(l_i^2 - 1)\mu/l_i$  along the hyperplane  $\alpha_i = 0$ .*

*Proof:* We have to show that  $\alpha_i$  exponentiated to the given order is a divisor of  $p_B$  while higher powers are not. From the factorization of  $p_B$  in  $\mathbf{C}[a_1, \dots, a_n]$ , we see that  $a_i$  divides a factor if and only if  $\nu_j = \eta_j$  for all  $j \neq i$ . An easy check shows that  $(l_i^2 - 1)\mu$  is the vanishing order of  $a_i$ . So the claim follows, for  $a_i^{l_i} = \alpha_i$ .  $\square$

**Lemma 3.6** *The leading coefficient of  $p_B \in \mathbf{C}[\alpha_1, \dots, \alpha_n]$  with respect to the variable  $\alpha_n$  is the  $l_n^{th}$  power of the Hefez Lazzeri polynomial of the Brieskorn Pham function  $g(x_1, \dots, x_{n-1}) = x_1^{l_1+1} + \dots + x_{n-1}^{l_{n-1}+1}$ .*

*Proof:* In the factorization of  $p_B$  in  $\mathbf{C}[a_1, \dots, a_n]$  we note that a factor is either a polynomial in  $a_n$  with constant leading coefficient or does not contain  $a_n$  at all. We conclude that up to a constant the leading coefficient with respect to the variable  $a_n$  is the product of those factors not containing  $a_n$ . So we get this coefficient by collecting the factors with  $\nu_i = \eta_i$  from the factorization of  $p_B$ :

$$\prod_{\nu_n = \eta_n \leq l_n} \prod_{\substack{\nu_i, \eta_i \leq l_i \\ i < n \\ \nu \neq \eta}} \left( \sum_{i=1}^n l_i a_i^{l_i+1} (\xi_i^{\nu_i} - \xi_i^{\eta_i}) \right) = \prod_{\substack{\nu_i, \eta_i \leq l_i \\ i < n \\ \nu \neq \eta}} \left( \sum_{i=1}^{n-1} l_i a_i^{l_i+1} (\xi_i^{\nu_i} - \xi_i^{\eta_i}) \right)^{l_n}$$

Compare this product with the proof of 3.4 and the claim is immediate.  $\square$

**Lemma 3.7** *Given a Brieskorn Pham polynomial  $f$ , then the generic degenerations in its Hefez-Lazzeri unfolding are:*

- i) *at a generic point of the coordinate hyperplane  $\alpha_i = 0$ , the corresponding function has  $l'_i = \mu/l_i$  critical points of type  $A_{l_i}$  with distinct critical values,*
- ii) *at any point on the coordinate hyperplanes, the corresponding function is of Brieskorn Pham type,*
- iii) *at a generic point of any other component of the Lazzeri discriminant, the corresponding projection has critical points of type  $A_1$  with at least two of common value.*

*Proof:* The Hessian of the function  $F^\#(x, \alpha)$  with respect to the variables  $x$  only is given by the diagonal matrix with entries  $l_i(l_i + 1)x^{l_i-1}\delta_{i,j}$ , which is of full rank outside the hyperplanes  $x_i = 0$ .

From the gradient of  $F^\#$  with respect to  $x$  we deduce, that there is a critical point on the  $i^{th}$  hyperplane  $x_i = 0$  if and only if the parameter  $\alpha$  is on the hyperplane  $\alpha_i = 0$ .

Hence for each parameter in the complement of the hyperplanes  $\alpha_i = 0$  the Hessian of the corresponding function is of full rank, so all its critical points are of type  $A_1$  only.

Thus the parameter belongs to the bifurcation divisor if and only if the function maps at least two of its critical points to the same value.

The bifurcation locus outside the coordinate hyperplanes is therefore part of the Maxwell stratum and corresponds to the transversal intersection of several branches of the discriminant locus, the number of which is half the local degree of the bifurcation locus plus one.

On the other hand a parameter on a coordinate hyperplane is considered generic if it does not belong to any other hyperplane. Assume in this case w.l.o.g. that  $\alpha_n = 0$ , so the corresponding function is

$$F^\#(x, \alpha) = f(x) + \sum_{i=1}^{n-1} \alpha_i (l_i + 1) x_i.$$

At each point of its critical locus, the Hessian is of corank one, therefore the type of the critical point is easily seen to be  $A_{l_n}$ .

The generalization to arbitrary parameters on the coordinate hyperplanes can be checked to yield the second claim.  $\square$

**Lemma 3.8** *Suppose the exponents are in increasing order,  $l_1 \leq \dots \leq l_n$ , then the total degree of the Hefez Lazzeri polynomial  $p_B$  is*

$$\mu \left( \sum_{i=1}^n \left( \frac{l_i^2 - 1}{l_i} \prod_{i < j \leq n} l_j \right) \right).$$

*Proof:* In case  $n = 1$  the degree is  $l^2 - 1$  in accordance with the claim since

$$p_B = \prod_{\substack{1 \leq \nu, \eta \leq l \\ \nu \neq \eta}} \left( l \alpha^{\frac{l+1}{l}} (\xi^\nu - \xi^\eta) \right).$$

In case  $n > 1$  we argue by induction: In each formal factor of  $p_B$  only pure monomials occur, so if  $\alpha_1$  occurs, the corresponding monomial is the term of highest degree. The number of such factors is  $\mu^2(l_1 - 1)/(l_1)$ . The product of the other factors is nothing else but the leading coefficient with respect to the variable  $\alpha_1$ .

Hence by lemma 3.6 and the induction hypothesis the total degree is

$$\mu^2 \frac{l_1^2 - 1}{l_1^2} + \mu \left( \sum_{i=2}^n \left( \frac{l_i^2 - 1}{l_i} \prod_{i < j \leq n} l_j \right) \right).$$

But this is the claim, since  $\mu/l_1 = l_2 \cdots l_n$ .  $\square$

## 3.2 Hefez Lazzeri path system

In case of the Brieskorn Pham polynomials Hefez and Lazzeri described a method to define a path system using induction on the dimension for a regular fibre obtained by a linear perturbation under the sole assumption that the parameters  $\alpha_i$  are of quite distinctive magnitude.

So if  $\alpha_n \ll \dots \ll \alpha_2 \ll \alpha_1$  the critical values are distributed on circles of radius  $l_n \alpha_n^{\frac{l_n+1}{l_n}}$  centred at the critical values of the polynomial

$$x_1^{l_1+1} - \alpha_1(l_1 + 1)x_1 + \cdots + x_{n-1}^{l_{n-1}+1} - \alpha_{n-1}(l_{n-1} + 1)x_{n-1}$$

according to lemma 3.3.

Therefore we need to replace all paths by  $l_n$  copies and refine the system on these discs then.

Inductively we start with a path system of a first kind, as depicted in figure 1.

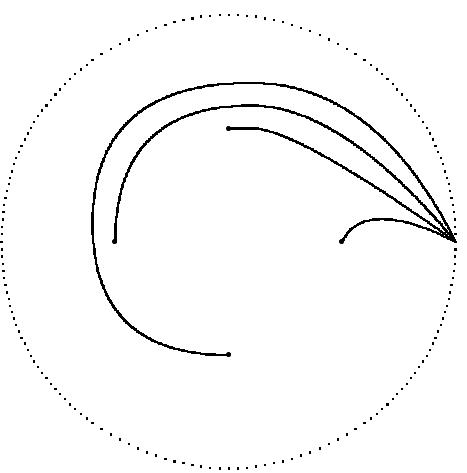


Figure 3.1: path system of the first kind,  $l = 4$

For the second step we replace it by the system of the second kind, which is given in the second figure, except for the fact that we did not care to distinguish the copies we take of each single path.

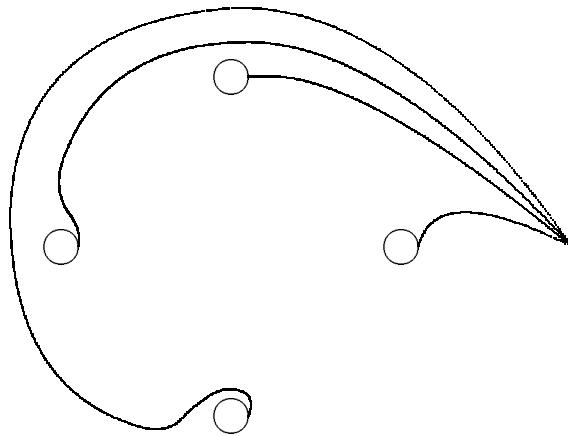


Figure 3.2: a path system of second kind in case  $l = 4$

Into each of the small discs we have to paste a path system of the first kind to get a path system in case  $n = 2$ .

Inductively then, we have to replace all systems of the first kind pasted in step  $n - 1$  by path systems of the second kind and to paste systems of the first kind into the new smaller discs left by the systems of the second kind.

These path systems are called *Hefez Lazzeri path systems*.

## Chapter 4

# singularities of type $A_n$

The simplest singular functions for which we have to determine the braid monodromy are those of type  $A_n$ . So let us investigate the holomorphic function  $f : x \mapsto x^{n+1}$ . A versal unfolding can be chosen to be quasi-homogeneous

$$F(x, u, z) = x^{n+1} + z + \sum_{i=1}^{n-1} u_i x^i,$$

while the associated truncated versal deformation is

$$F^\#(x, u) = x^{n+1} + \sum_{i=1}^{n-1} u_i x^i.$$

This case of functions of type  $A_n$  is special in some aspects. First, being quasi-homogeneous with good  $\mathbf{C}^*$ -action, we may work in the affine set up, 1.4.2, and avoid the cumbersome notation used for germs. So we consider the projection  $p$  from  $\mathbf{C}^n$  to  $\mathbf{C}^{n-1}$  of the base spaces of the versal resp. truncated versal unfoldings above and get the braid monodromy as the coefficient homomorphism for the bundle which we obtain by restriction of  $p$  to

$$p : \mathbf{C}^n \setminus \Delta \cup p^{-1}(\mathcal{B}) \longrightarrow \mathbf{C}^{n-1} \setminus \mathcal{B}.$$

Second, being of codimension one, there is another map besides  $p$  to which the theory of polynomial covers applies. Consider the projection  $\tilde{p}$  from  $\mathbf{C}^{n+1}$  to  $\mathbf{C}^n$  from the domain of the versal deformation  $F$  to its base and the zero set  $F^{-1}(0) \subset \mathbf{C}^{n+1}$ . Then there is a restriction, which again is a polynomial covering space, but of degree  $n+1$ :

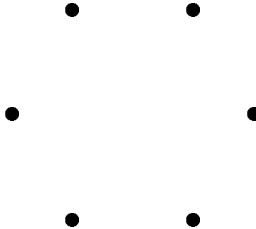
$$\tilde{p} : \mathbf{C}^{n+1} \setminus F^{-1}(0) \cup \tilde{p}^{-1}(\Delta) \longrightarrow \mathbf{C}^n \setminus \Delta.$$

Of course  $\tilde{p}$  is induced from the tautological bundle (1.1), p.11, we have seen before, but even more is true. It is a subbundle which is a strong deformation retract.

When we now attack the braid monodromy of  $f$  – recall this to be the braid monodromy of  $p$  – our argument is not straightforward, but relies heavily on properties of  $\tilde{p}$  and citations from the literature.

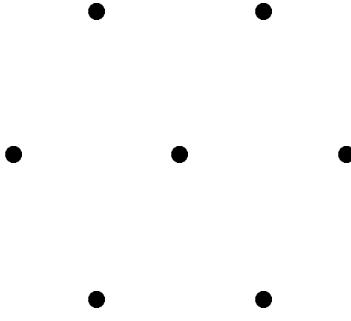
But setting out with the fibration  $p$ , let us first choose a base point, such that we can describe the corresponding fibre in more detail. To rely on previous results we

choose  $u = (0, \dots, 0, -\alpha(n+1)) \in \mathbf{C}^{n-1}$  from the base of the Hefez-Lazzeri unfolding with corresponding function  $f_\alpha = x^{n+1} - \alpha(n+1)x$ . The fibre of  $p$  at  $u$ , which we henceforth refer to by  $L_\alpha$ , falls into the topological class we denoted by  $\mathbf{C}_n$ , but we easily compute from  $\partial_x f_\alpha = 0$  that the critical locus is at the  $n^{\text{th}}$  roots of  $\alpha$  and that the fibre is punctured at the critical values which coincide with the  $n^{\text{th}}$  roots of  $\alpha^{n+1}$  multiplied by  $n$ , cf. lemma 3.3. Thus we get even the true geometric picture for  $L_\alpha$ :



This punctured line  $L_\alpha$  is a subset in the base of the bundle  $\tilde{p}$ , and as we said, an understanding of the bundle  $\tilde{p}$  will eventually lead to the braid monodromy of  $p$ .

Looking for a distinguished regular base point of the restricted bundle  $\tilde{p}_\alpha := \tilde{p}|_{L_\alpha}$  the origin is the obvious choice. The corresponding fibre is an affine line punctured at the zeroes of  $x^{n+1} - \alpha(n+1)x$ , i.e.. at 0 and the  $n^{\text{th}}$  roots of  $\alpha(n+1)$ .



But we need more information to compute the monodromy. Assuming  $\alpha$  to be real, we consider the punctures in the fibres of  $\tilde{p}$  over points  $\rho \geq 0$  of the positive real axis in the base  $L_\alpha \cong \mathbf{C}$ . We make the following observations:

- i) By the sign rule of Descartes the number of positive real zeroes of the polynomial  $x^{n+1} - \alpha(n+1)x + \rho$  is either two or none.
- ii) Let  $\rho_\alpha := n\sqrt[n]{\alpha^{n+1}}$  be a positive real by choosing  $\sqrt[n]{\alpha^{n+1}}$  the only positive real among the  $n^{\text{th}}$  roots of  $\alpha^{n+1}$ . Then  $\rho_\alpha$  is the base of the only singular fibre of  $\tilde{p}$  over the positive real axis, with a single ordinary double point  $x = \sqrt[n]{\alpha}$ .
- iii) Both facts together imply that the number of positive real zeroes along the real axis  $\rho \in \mathbf{R}^{\geq 0}$  is constantly two over  $[0, \rho_\alpha[$ , a zero of multiplicity two at  $\rho_\alpha$  and vanishes over  $]\rho_\alpha, \infty[$ .

We elaborate on these observations to get the proofs of the following two lemmas concerning the monodromy homomorphism to  $\Sigma_{n+1}$  for the cover associated to  $\tilde{p}$ , resp. the braid monodromy of  $\tilde{p}$  itself. Though the first would certainly suffice in the course of the argument, we couldn't help including the second, and if only to show the particular flavour of arguments applied in braid monodromy considerations.

**Lemma 4.1** *Given the simple branched cover of  $\mathbf{C}$  by  $\tilde{p}$  restricted to  $F^{-1}(0)$  over the complex line  $(0, -\alpha(n+1), z)$ ,  $\alpha \neq 0$  fix, in the base of the versal unfolding  $F$ . Suppose  $\alpha_n$  is any  $n^{\text{th}}$  root of  $\alpha$ , then the Hurwitz monodromy of the cover associates to the radial path from 0 around  $n\alpha\alpha_n$  the transposition of the points 0 and  $\alpha_n \sqrt[n]{n+1}$ .*

*Proof:* We just considered the radial path in the special case of  $\alpha, \alpha_n \in \mathbf{R}^{>0}$ . In that case the local monodromy is the transposition of the merging points which originate in the said points 0 and  $\alpha_n \sqrt[n]{n+1}$  of the fibre at the origin.

The setting is quasi-homogeneous with respect to weights  $wt(x) = 1$ ,  $wt(\alpha) = n$ ,  $wt(z) = n+1$ . That's why multiplication by a suitable factor maps the special case bijectively to any other and so the proof is obtained.  $\square$

**Lemma 4.2** *Consider the cover  $\tilde{p}$  restricted to the complex line  $(0, -\alpha(n+1), z)$ ,  $\alpha \neq 0$  fix, in the base of the versal unfolding  $F$ . Suppose  $\alpha_n$  is any  $n^{\text{th}}$  root of  $\alpha$ , then the braid monodromy associates to the radial path from 0 around  $n\alpha\alpha_n$  in the base the twist on the radial arc from 0 to  $\alpha_n \sqrt[n]{n+1}$  in the fibre.*

*Proof:* By quasi-homogeneity it suffices to prove the special case of  $\alpha \in \mathbf{R}^{>0}$ ,  $\alpha_n \in \mathbf{R}^{>0}$ . In that case the local monodromy is a twist since the degeneration is an ordinary double point. In a fibre sufficiently close to the degenerate fibre this is a twist on the straight arc between the merging points.

We choose the nearby fibre at a positive real base parameter and get an arc supported on the positive real axis in the fibre.

In the description of the general strategy for the computation of braid monodromies we have mused for some time on the fact that it suffices to understand the parallel transport of this arc to the reference fibre, which is the fibre at the origin in the case at hand.

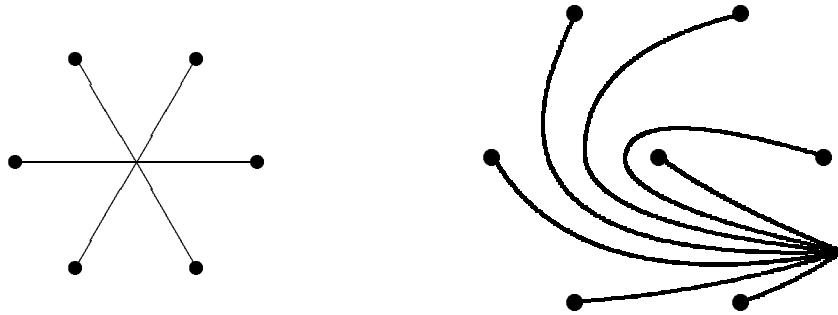
So we have to prove that the arc in a close by fibre can be transported to the radial arc from 0 to  $\alpha_n \sqrt[n]{n+1}$  in the fibre at the origin. In fact we can do so by choosing a suitable vector field. Since the punctures have to be transported along integral curves, the vector field is determined at the punctures, but otherwise we may smoothly interpolate as we need. At the two punctures moving on the positive real axis the vector field is parallel to the axis. Moreover no other puncture hits the positive real axis, as we observed above, due to the sign rule of Descartes. So we interpolate by a vector field on the ray parallel to the axis, such that the arc in the nearby fibre is stretched to the claimed interval.  $\square$

As was noted previously, 1.3.1, the fundamental group  $\pi_1(\mathbf{C}^n - \Delta)$  is a quotient of  $\pi_1(\mathbf{C}_n)$  such that the braid automorphisms of  $\pi_1(\mathbf{C}_n)$  descend to trivial automorphisms of  $\pi_1(\mathbf{C}^n - \Delta)$ .

So the braid monodromy homomorphism of  $p$  with range considered as automorphism of  $\pi_1(\mathbf{C}_n)$  may map only to elements  $a_*(\beta)$  which stabilize the coefficient homomorphism  $\tilde{a}_*$  for  $\tilde{p}$  followed by the natural homomorphism  $\pi : \text{Br}_{n+1} \rightarrow \mathcal{S}_{n+1}$ :

$$\begin{array}{ccc} \pi_1(\mathbf{C}_n) & \xrightarrow{\pi \circ \tilde{a}_*} & \mathcal{S}_{n+1} \\ a_*(\beta) \downarrow & & \downarrow \text{id} \\ \pi_1(\mathbf{C}_n) & \xrightarrow{\pi \circ \tilde{a}_*} & \mathcal{S}_{n+1} \end{array}$$

To make the argument more explicit we introduce geometric bases in the two punctured lines. If we do so for the fibre of  $p$  punctured at the  $n^{th}$  roots of  $\alpha^{n+1}$  dilated by  $n$  in the way as depicted on the left and for the fibre of  $\tilde{p}$  punctured at zero and the  $n^{th}$  roots of  $\alpha(n+1)$  as shown on the right



then the mapping class group is identified with  $\text{Br}_{n+1} = \langle \sigma_i | \dots \rangle$  in such a way that the twists on the radial arcs considered in the lemma above correspond to  $\sigma_{1,2}, \sigma_{1,3}, \dots, \sigma_{1,l+1}$ .

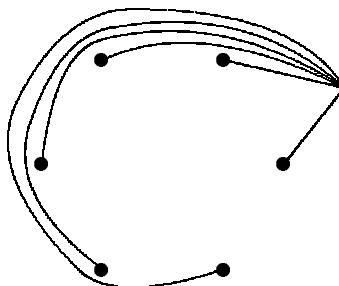
The coefficient homomorphism  $a_*$  is given on the generator system  $\langle t'_1, \dots, t'_n | \rangle$  of  $F_n$  thus determined by

$$t'_i \mapsto \sigma_{1,i+1}.$$

If we work instead with a geometric basis given by the next figure, the map  $a_*$  is given on the generator set  $t_1, \dots, t_n$  thus defined by

$$F_n \rightarrow \text{Br}_{n+1}, \quad t_i \mapsto \sigma_i \quad \left( \xrightarrow{\pi} (i \ i+1) \right)$$

as can be checked inductively.



Not least do we prefer the second choice since the corresponding group of stabilizing automorphisms in the image of the Artin homomorphism  $\text{Br}_n \rightarrow \text{Aut}(F_n)$  has been investigated in several aspects which – when brought together – are sufficient to yield our goal.

Let us first cite from the results of Catanese/Wajnryb [9], Kluitmann [22], and Dörner [12], who gave the stabilizer group for the naturally induced homomorphism to the symmetric group  $\mathcal{S}_{n+1}$  in terms of generators as follows.

**Lemma 4.3** *Suppose  $h : \langle t_1, \dots, t_n | \rangle \rightarrow \mathcal{S}_{n+1}$  is given on generators by  $t_i \mapsto (i \ i+1)$  and  $\alpha : \text{Br}_n \rightarrow \text{Aut}(\langle t_1, \dots, t_n | \rangle)$  is the Artin homomorphism, then the set of braids  $\beta \in \text{Br}_n$  such that  $h \circ \alpha(\beta) = h$  is a subgroup, called the braid stabilizer group of  $h$ , which is generated by*

$$\delta_2^3, \dots, \delta_n^{n+1},$$

or equivalently by

$$\sigma_i^3, \sigma_{i,j}^2, |i - j| \geq 2.$$

*Proof:* We refer to [9] and give only the relations between the two sets of generators which can be proved inductively as in [22], [12]:

$$\delta_i^{i+1} = (\sigma_1^3(\sigma_{1,3}^2 \cdots \sigma_{1,i}^2))(\sigma_2^3(\sigma_{2,4}^2 \cdots \sigma_{2,i}) \cdots (\sigma_{i-1}^3)$$

respectively for the other direction:

$$\begin{aligned} \sigma_1^3 &= \delta_2^3 \\ \sigma_{1,3}^2 &= \delta_2^{-3}(\delta_3^4)^2\delta_2^{-3}\delta_3^{-4} \\ \sigma_{1,i+1}^2 &= \delta_i^{-i-1}\delta_{i+1}^{i+2}\delta_{i+1}^{i+2}\delta_i^{-i-1}\delta_{i+1}^{-i-2} \\ \sigma_i^3 &= (\delta_n^{n+1})^i\sigma_1^3(\delta_n^{n+1})^{-i} \\ \sigma_{i+1,j}^2 &= (\delta_n^{n+1})^i\sigma_{1,j-i}^2(\delta_n^{n+1})^{-i} \end{aligned}$$

□

The stabilizer of the braid monodromy homomorphism with respect to the same geometric bases can be deduced now. Of course it must be a subgroup of the stabilizer group just computed and a direct check shows that it is even the whole group.

**Lemma 4.4** *Suppose  $h : \langle t_1, \dots, t_n | \rangle \rightarrow \text{Br}_{n+1}$  is given on generators by  $t_i \mapsto \sigma_i$  and  $\alpha : \text{Br}_n \rightarrow \text{Aut}(\langle t_1, \dots, t_n | \rangle)$  is the Artin homomorphism, then the set of braids  $\beta \in \text{Br}_n$  such that  $h \circ \alpha(\beta) = h$  is a subgroup, called the braid stabilizer group of  $h$ , which is generated by*

$$\delta_1^3, \dots, \delta_n^{n+1},$$

or equivalently by

$$\sigma_i^3, \sigma_{i,j}^2, |i - j| \geq 2.$$

*Proof:* We show the stabilizing property for the second set of generators. To do so we recall the automorphisms associated to  $\sigma_i \in \text{Br}_n$ :

$$\sigma_i(t_k) = \begin{cases} t_i t_{i+1} t_i^{-1} & \text{if } k = i, \\ t_i & \text{if } k = i+1, \\ t_k & \text{otherwise.} \end{cases}$$

and compute then the automorphisms associated to  $\sigma_i^3, \sigma_{i,j}^2$ :

$$\begin{aligned} \sigma_i^3(t_k) &= \begin{cases} t_i t_{i+1} t_i t_{i+1} t_i^{-1} t_{i+1}^{-1} t_i^{-1} & \text{if } k = i, \\ t_i t_{i+1} t_i t_{i+1}^{-1} t_i^{-1} & \text{if } k = i+1, \\ t_k & \text{otherwise.} \end{cases} \\ \sigma_{i,j}(t_k) &= \begin{cases} t_i t_j t_i^{-1} & \text{if } k = i, \\ t_i & \text{if } k = j, \\ t_i t_j^{-1} t_k t_j t_i^{-1} & \text{if } i < k < j, \\ t_k & \text{otherwise.} \end{cases} \\ \sigma_{i,j}^2(t_k) &= \begin{cases} t_i t_j t_i t_j^{-1} t_i^{-1} & \text{if } k = i, \\ t_i t_j t_i^{-1} & \text{if } k = j, \\ t_i t_j t_i^{-1} t_j^{-1} t_k (t_i t_j t_i^{-1} t_j^{-1})^{-1} & \text{if } i < k < j, \\ t_k & \text{otherwise.} \end{cases} \end{aligned}$$

So we are left checking for  $i < n - 1$  and  $i < k < j$ :

$$\begin{aligned} \sigma_i &= \sigma_i \sigma_{i+1} \sigma_i \sigma_{i+1} \sigma_i^{-1} \sigma_{i+1}^{-1} \sigma_i^{-1} \\ \sigma_{i+1} &= \sigma_i \sigma_{i+1} \sigma_i \sigma_{i+1}^{-1} \sigma_i^{-1} \\ \sigma_i &= \sigma_i \sigma_j \sigma_i \sigma_j^{-1} \sigma_i^{-1} \\ \sigma_j &= \sigma_i \sigma_j \sigma_i^{-1} \\ \sigma_k &= \sigma_i \sigma_j \sigma_i^{-1} \sigma_j^{-1} \sigma_k (\sigma_i \sigma_j \sigma_i^{-1} \sigma_j^{-1})^{-1} \end{aligned}$$

which follow easily from the braid relations.  $\square$

The counterpart of these results is provided by the paper of Looijenga, [27]. He already investigated – in different terminology – the coefficient homomorphism of the projection  $p$  and he observed:

**Lemma 4.5** *The image of the coefficient homomorphism of the projection  $p$  coincides with the stabilizer of the monodromy homomorphism of the finite cover associated to  $\tilde{p}$ .*

*Proof:* We have seen before that there is an inclusion of these groups. So the seemingly weaker result of Looijenga that both groups are conjugate subgroups of  $\text{Br}_n$  immediately implies the stronger form.  $\square$

We must admit, that we fall short of finding the braid monodromy for the projection  $p$ , but what we have determined is the braid monodromy group of  $p$  with respect to the base choices specified above.

**Lemma 4.6** *The braid monodromy group of the singular function  $x \mapsto x^{n+1}$  is identified by the geometric bases above with the subgroup of  $\text{Br}_n$  generated by*

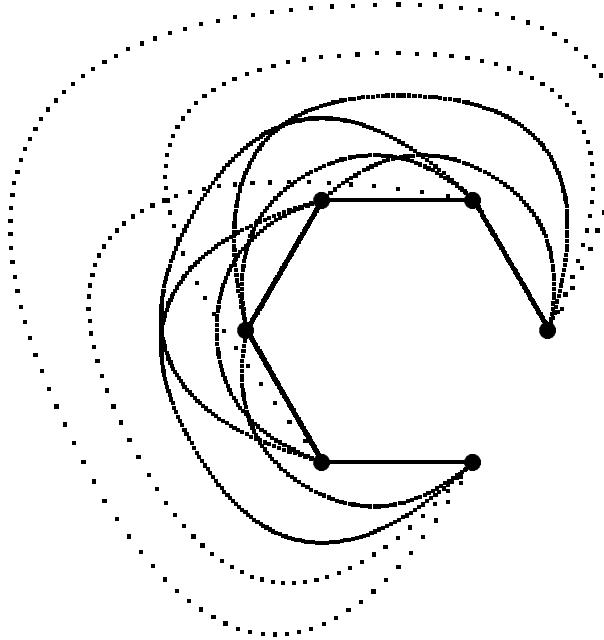
$$\sigma_i^3, \sigma_{i,j}^2, |i - j| \geq 2.$$

*Proof:* The braid monodromy group is the image of the coefficient homomorphism. By the result of Looijenga this coincides with the stabilizer, so the claim follows.  $\square$

For later application we translate this result back into a statement on mapping classes of the reference fibre at a positive real  $\alpha$ .

**Lemma 4.7** *A set of mapping classes which generate the braid monodromy in the fibre at  $(0, \dots, 0, -\alpha(n+1))$  punctured at the roots  $n\alpha^{\frac{n+1}{n}}$  is given by*

- i) *the  $\frac{3}{2}$ -twists on the straight arcs joining consecutive punctures,*
- ii) *the full twists on arcs joining non-consecutive punctures in the complement of the inscribed polygon and the open cone defined by the 1 and the last puncture.*



*Proof:* We have to show that the generators in lemma 4.6 and the arcs described in the assertion are related as claimed by the geometric basis chosen above.

More precisely, the generators with exponent 3 correspond to  $\frac{3}{2}$ -twists on arcs obtained by joining consecutive paths of the geometric bases up to isotopy, the generators with exponent 2 correspond to full twists on arcs obtained by joining non-consecutive paths.

The arcs thus obtained can be characterized as in the assertion due to the fact that the complement of the inscribed polygon and the given cone is simply connected and contains the geometric basis.  $\square$

Having thus computed our first braid monodromy group, we can deduce by 2.25 a presentation for the fundamental group of the discriminant complement of the  $A_n$  singularity, which of course is well known since long:

**Corollary 4.8** *The fundamental group of the complement of the discriminant for  $f(x) = x^{n+1}$  is isomorphic to the braid group on  $n+1$  strands given by the presentation*

$$\langle t_1, \dots, t_n | t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1}, t_i t_j = t_j t_i \text{ if } |i - j| > 1 \rangle.$$

The results of this chapter should also be regarded as a tool to compute braid monodromy groups of complicated singularities. In our present set up, we start with a generic one parameter family of functions induced from the base of the truncated versal unfolding, compute the local monodromy and its parallel transport to a common reference fibre.

In the next chapter we prove that we may consider instead special families to compute the braid monodromy group. We will then have to compute not the monodromy but the monodromy group of each degeneration – what we can do now, if the degeneration is of type  $A_n$ . Parallel transport has still to be performed, but in a much simpler family.

In fact it will be families induced from the Hefez-Lazzeri unfolding.

# Chapter 5

## results of Zariski type

Having defined the braid monodromy group of a singular function we only succeeded to compute it for functions of type  $A_n$  by means of strong results cited from the literature. To proceed we have to develop powerful methods for the computations of braid monodromy groups.

Generally speaking, in this chapter we will link the braid monodromy of a singular function to the braid monodromy of adjacent functions. We will actually determine the braid monodromy of a function from a *tame  $\ell$ -perturbation*, a suitably defined unfolding over a two-dimensional base, using the degeneration properties over the conflict divisor and braid monodromy of adjacent singularities in this family only. In fact this method can be applied in such a way that – in principal – the braid monodromy group of a Brieskorn-Pham polynomial can be computed from its Hefez-Lazzeri unfolding and the monodromy groups of adjacent Brieskorn Pham polynomials.

The actual execution of this computation and the set up of the necessary induction are topics of subsequent chapters.

### 5.1 generalization of Morsification

A Morsification of a singular function  $f \in \mathcal{O}_n$  is usually defined as a map representing an unfolding of  $f$

$$\begin{aligned} \mathbf{C}^n \times \mathbf{C} &\longrightarrow \mathbf{C} \\ x, \lambda &\mapsto f_\lambda(x) \end{aligned}$$

such that for generic  $\lambda$  the function  $f_\lambda$  is a Morse function.

Given any versal unfolding of  $f$  represented by a map

$$\begin{aligned} \mathbf{C}^n \times \mathbf{C}^k &\longrightarrow \mathbf{C} \\ x, u &\mapsto f(x) - u_0 + \sum u_i b_i \end{aligned}$$

with  $b_i \in \mathbf{m}_n$ , cf. 2.7, then a Morsification as above can also be understood as an unfolding

$$\begin{aligned} \mathbf{C}^n \times \mathbf{C}^2 &\longrightarrow \mathbf{C} \\ x, \lambda, u &\mapsto f_\lambda(x) - u \end{aligned}$$

which is induced by a map  $\mathbf{C}^2 \rightarrow \mathbf{C}^k$  such that the restriction to a line with  $\lambda$  equal to a generic constant maps onto a line transversal to the discriminant.

In fact we get a pencil of lines in the base of the versal unfolding parameterized by  $\lambda$ , such that all lines sufficiently close to the line  $\lambda = 0$  are transversal to the discriminant.

We want to have a notion which generalizes this property to the case of truncated versal unfoldings.

However we do not have a preferred element like the constant  $1 \in \mathcal{O}_n$  anymore, but have to choose among the elements of  $\mathbf{m}_n$  which are not in the Jacobi ideal  $J_f$ . In fact we will allow any choice among the linear functions yielding Morsifications - as given by the usual existence proof for Morsifications, [13].

As long as this choice is unspecified we denote it by  $\ell$ , otherwise its place in the subsequent definitions can be taken by the polynomial actually chosen.

We consider now two parameter unfoldings of a singular function  $f$  in the maximal ideal  $\mathbf{m}_n$ ,

$$\begin{aligned} \mathbf{C}^n \times \mathbf{C}^2 &\longrightarrow \mathbf{C} \\ x, \lambda, u &\mapsto f_\lambda(x) + u\ell(x) \end{aligned}$$

which are of course induced from any truncated versal deformation of  $f$ . So we may also consider the associated pencil of lines parameterized by  $\lambda$  in the base of the truncated versal unfolding.

**Definition 5.1:** A two parameter unfolding as above is called  *$\ell$ -perturbation*, if all lines of the pencil sufficiently close to the origin meet the bifurcation set in isolated points only.

By assumption on  $\ell$  the line through the origin is not contained in the bifurcation set.

**Definition 5.2:** A two parameter unfolding as above is called  *$\ell$ -generification*, if all lines of the pencil sufficiently close to the origin are transversal to the bifurcation set. (In particular they meet the bifurcation set in generic points only corresponding to functions which have non-degenerate critical points only with distinct critical values except for a unique critical point of type  $A_2$  or a pair with conflicting values).

Any  $\ell$ -generification or  $\ell$ -perturbation of a function of Milnor number  $\mu$  determines a  $\mathbf{C}_\mu$ -bundle over a multiply punctured disc, well defined up to fibration isomorphism, since the lines of the pencil close to the origin are transverse to the bifurcation set in a uniform way.

As a Morsification may serve for the computation of monodromy groups so a generification of a function  $f$  can replace its truncated versal unfolding:

**Lemma 5.3** *The braid monodromy group of a  $\ell$ -generification is equal to the braid monodromy group.*

*Proof:* It just suffices to point to the analogous argument in the case of a Morification. The important point to note is, that the lines of the pencil are generic with respect to the bifurcation set and therefore the induced map on fundamental groups surjects.  $\square$

## 5.2 versal braid monodromy group

Given a one-parameter family of monic polynomials we have formerly divided the computation of the braid monodromy group into two steps. First we assign the local monodromy generator to a local Milnor fibre, second we use parallel transport to get mapping classes in just one regular fibre. Upon the choice of a geometrically distinguished system of paths in that fibre, the subgroup generated by the transported classes is identified with the braid monodromy group.

In a similar approach, we will assign a group instead of a generator to local Milnor fibres close to each singular fibre, and we will then use parallel transport of the group elements to get mapping classes again in a single regular fibre.

It is the local assignment which we have to define carefully to get a sensible additional notion of braid monodromy.

In fact it will only be defined for one parameter families of monic polynomials associated to a family of functions on which – for technical simplicity only – we impose the further restriction of *tameness*.

**Definition 5.4:** A one parameter family of functions is called *tame* if of each function only non-degenerate critical points may have conflicting values.

Given a tame family of functions then locally at a singular function  $f$  the associated family of monic polynomials  $p_\lambda$  is parameterized by  $\lambda$  in a disc such that the coefficient map is holomorphic and such that the polynomial is a simple Weierstrass polynomial for  $\lambda \neq 0$ :

$$p_\lambda : (\lambda, x) \mapsto x^n + \sum_{i=0}^{n-1} a_i(\lambda) x^i.$$

Suppose now  $p_0$  not to be simple with roots denoted by  $v_j$ . Then for  $\varepsilon$  and  $\delta = \delta(\varepsilon)$  sufficiently small, the local family

$$Y := \mathbf{C} \times B_\delta \setminus p_\lambda^{-1}(0)$$

is trivializable over  $B_\delta$  in the complement of  $\cup_j B_\varepsilon(v_j)$ , cf. fig. 5.1.

We conclude that all mapping classes of the local family  $Y$  can be given with support on the intersection  $\cup_j D_j$  of a local Milnor fibre with  $\cup_j B_\varepsilon(v_j)$ . Moreover we notice that the restriction to a disc  $D_j$  yields the braid monodromy transformations of the local discriminant divisor at  $v_j$ .

We finally assign a group of mapping classes to  $p_\lambda$  by a choice of a group of mapping classes supported on  $D_j$  for each root  $v_j$  of  $p_0$ .

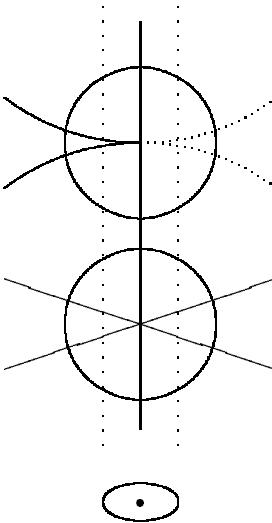


Figure 5.1: Milnor fibration

Consider first the case that  $v_j$  is a multiple root of  $p_0$ , which is the image of a single critical point  $c_j$  of  $f$ . Then for the germ of  $f$  at  $c_j$  braid monodromy yields a well defined group of mapping classes supported on  $D_j$ .

In case  $v_j$  is the image of several non-degenerate critical points of  $f$ , we simply choose the group of mapping classes of  $D_j$  which fix the punctures and which therefore correspond to pure braids.

By tameness these are the only possible cases. The generalization to other families is conceptionally straight forward, but notationally a mess, so we decided to skip it here.

Anyway we should sum up our definition:

**Definition 5.5:** Given a tame family of functions we assign a group of mapping classes to a Milnor fibre of each singular fibre of the associated family. The generators are given by mapping classes in the Milnor fibres  $D_j$  of the multiple roots  $v_j$ :

- i) all mapping classes which fix the punctures in case  $v_j$  is the image of non-degenerate critical points only,
- ii) the mapping classes provided by braid monodromy of a critical point  $c_j$  in case  $c_j$  is the only critical point which maps to  $v_j$ .

The group of mapping classes in a regular fibre obtained by parallel transport is called the *versal braid monodromy group* of the tame family of functions.

Of course this determines a well defined conjugacy class of subgroups of the braid group  $\text{Br}_n$  upon the choice of a geometrically distinguished system of paths in a regular fibre.

**Remark 5.6:** We note that a given  $\ell$ -perturbation of a singular function determines a one-parameter family of functions up to smooth fibration isomorphism, hence

in case the family is tame we may also speak of the *versal braid monodromy* of the  $\ell$ -perturbation.

Moreover we may extend the notion of *versal braid monodromy* to the case where several tame families are given which have a regular fibre in common: Then it denotes the group of mapping classes of the common fibre which is generated by the subgroups which are the versal braid monodromies of the separate families.

By the definition we have to consider all possible transports to the reference fibre, but in fact we can restrict the computation to the transport along the paths of a distinguished system.

**Lemma 5.7** *The versal braid monodromy of a family is obtained if the locally assigned groups are transported along the paths of a geometrically distinguished system.*

*Proof:* The key observation is, that the local braid monodromy transformation belongs to the locally assigned group. We can thus conclude as in the classical case.  $\square$

### 5.3 comparison of braid monodromies

In this section we relate the versal braid monodromy of a tame  $\ell$ -perturbation to the braid monodromy of a versal family, which will eventually justify the name.

**Proposition 5.8** *The braid monodromy group of a function  $f$  is equal to the versal braid monodromy group of any of its  $\ell$ -perturbation which is tame.*

*Proof:* Suppose the  $\ell$ -perturbation is represented by a map

$$\begin{aligned} \mathbf{C}^n \times \mathbf{C}^2 &\longrightarrow \mathbf{C} \\ x, \lambda, u &\mapsto f_\lambda(x) + u\ell(x). \end{aligned}$$

We extend it to an unfolding with base of dimension 3

$$\begin{aligned} \mathbf{C}^n \times \mathbf{C}^3 &\longrightarrow \mathbf{C} \\ x, \lambda, u_1, u &\mapsto f_\lambda(x) + u_1 b_1(x) + u\ell(x), \end{aligned}$$

such that the restriction to  $\lambda = 0$  is a  $\ell$ -generification. We can check that the bifurcation set is of codimension one and its singular locus of codimension two in this base. This is immediate from our assumptions, since a non empty part of a generification is induced from the complement of the bifurcation set and another non empty part from the complement of its singular locus.

The conclusion still holds for the dominantly induced unfolding

$$\begin{aligned} \mathbf{C}^n \times \mathbf{C}^3 &\longrightarrow \mathbf{C} \\ x, \lambda, \lambda_1, u &\mapsto f_\lambda(x) + \lambda\lambda_1 b_1(x) + u\ell(x), \end{aligned}$$

Restricted to  $\lambda_1 = 0$  this is the  $\ell$ -perturbation we started with. Restricted to some fixed  $\lambda_1 \neq 0$  and sufficiently small it is still a  $\ell$ -perturbation of  $f$ , but its base only

meets the singular locus of the bifurcation set in isolated points. Hence we get even a  $\ell$ -generification of  $f$ .

For the rest of the proof we fix  $\lambda$  at a sufficiently small non-zero value and consider various restrictions of the family

$$\begin{aligned} F : \mathbf{C}^n \times \mathbf{C}^2 &\longrightarrow \mathbf{C} \\ x, \lambda_1, u &\mapsto f_\lambda(x) + \lambda\lambda_1 b_1(x) + u\ell(x). \end{aligned}$$

For example  $F|_{\lambda_1=0}$  is the tame family of which we want to understand the versal braid monodromy and for sufficiently small  $\eta$  we get a tame family  $F|_{\lambda_1=\eta}$  which has braid monodromy equal to that of  $f$  by lemma 5.3.

For each critical parameter  $y_i$  in the bifurcation set on the line  $\lambda_1 = 0$  we choose a local ball  $U_i$  in the base of  $F$  centered at  $y_i$ . We fix  $\eta$  sufficiently small and the tubular neighbourhood  $N_\eta$  of the line  $\lambda_1 = 0$  of radius  $\eta$ , such that the bifurcation set of  $F|_{N_\eta}$  is contained in the union of the  $U_i$  and its singular locus is a subset of the  $y_i$ .

The braid monodromy of  $F|_{N_\eta}$  is then equal to the braid monodromy of  $f$ , since it contains the family  $F|_{\lambda_1=\eta}$ . On the other hand it is generated by the braid monodromies of the  $F|_{U_i}$  and parallel transport over the complement of the  $U_i$ .

This should be compared to the fact that the versal braid monodromy of  $F|_{\lambda_1=0}$  is generated by the versal braid monodromies of  $F|_{E_i}$  – where  $E_i$  denotes the intersection of  $\lambda_1 = 0$  with  $U_i$  – and parallel transport over the complement of the  $E_i$ .

We have therefore accomplished a major reduction step in the proof:

It suffices to prove that the versal braid monodromy of  $F|_{E_i}$  is equal to the braid monodromy of  $F|_{U_i}$  for each  $i$ , since the complement of the  $U_i$  in  $N_\eta$  retracts onto the complement of the  $E_i$  on  $\lambda_1 = 0$ .

We move thus our attention to the discriminant family of  $F$  restricted to a single ball  $U$  of the base. The restriction to  $E$  yields a discriminant family with a single singular fibre for which we gave a local description in section 5.2 already.

In fact this description extends to  $U$  if it is chosen appropriately: The complement  $Y$  of the discriminant in  $\mathbf{C} \times U$  is trivializable over  $U$  in the complement of balls  $B_\epsilon(v_j)$  centered at the roots  $v_j$  on the fibre over  $y$ .

The braid monodromy of  $F|_U$  and the versal braid monodromy of  $F|_E$  can thus be considered as a group of mapping classes which are supported on the intersection  $\cup_j D_j$  of a local Milnor fibre with  $\cup_j B_\epsilon(v_j)$ .

According to the decomposition of the discriminant into connected components  $\mathcal{D}_j$  over  $U$ , the bifurcation divisor decomposes as  $\mathcal{B} = \cup_j \mathcal{B}_j$ , such that each divisor  $\mathcal{B}_j$  is the branch locus of the finite map of  $\mathcal{D}_j$  onto  $U$ .

As the  $B_\epsilon(v_j)$  are disjoint the braid monodromy transformations along simple geometric elements associated to different parts of  $\mathcal{B}$  commute.

In particular the braid monodromy transformation along a simple geometric element based at the chosen Milnor fibre and associated to  $\mathcal{B}_j$  can be chosen with support in  $D_j$ .

Now let us consider first a multiple root  $v_j$  of the critical fibre at  $y$  which is the image of several non-degenerate critical points of the corresponding function. Then the corresponding local discriminant divisor  $\mathcal{D}_j$  in  $B_\varepsilon(v_j)$  has irreducible components in bijection to the preimages. Hence all mapping classes in the braid monodromy of  $F|_U$  restrict to mapping classes of  $D_j$  which fix the punctures pointwise.

On the other hand  $E' := U \cap \{\lambda_1 = \eta\}$  is transversal to the bifurcation set, so the divisorial discriminant components in  $B_\varepsilon(v_j)$  meet pairwise, transversally, and over distinct points of the bifurcation set  $\mathcal{B}_j \cap E'$ . This implies that the braid monodromy of  $F|_{E'}$  contains all pure mapping classes of  $D_j$ , i.e. the group of mapping classes which are supported on  $D_j$  and fix the punctures pointwise. Hence this braid monodromy contains all mapping classes we assign to  $v_j$  to get the versal braid monodromy group of  $F|_E$ .

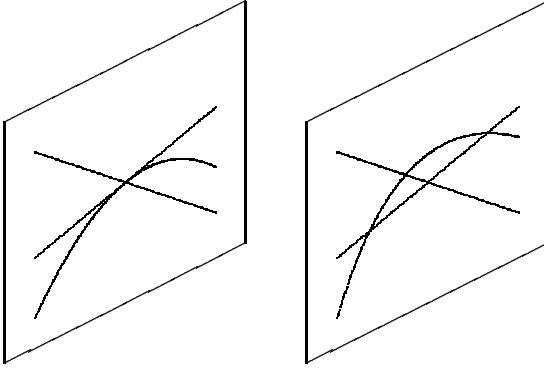


Figure 5.2: generification of smooth branches

Similarly we argue in case the multiple root  $v_j$  is the image of a unique critical point  $c_j$ . Then  $B_\varepsilon(v_j)$  can be considered as a discriminant family induced from the base of a versal truncated unfolding of the function at  $c_j$ .

It is in fact a generification, since its bifurcation set  $\mathcal{B}_j$  is met by  $E$  in a single point only and transversally by  $E'$ .

Hence the braid monodromy of  $F|_U$  contains the braid monodromy of the function at  $c_j$  considered as mapping classes on  $D_j$  extended by the identity to the Milnor fibre of  $F|_U$ , which is just what we assigned to  $v_j$  to get the versal braid monodromy group.

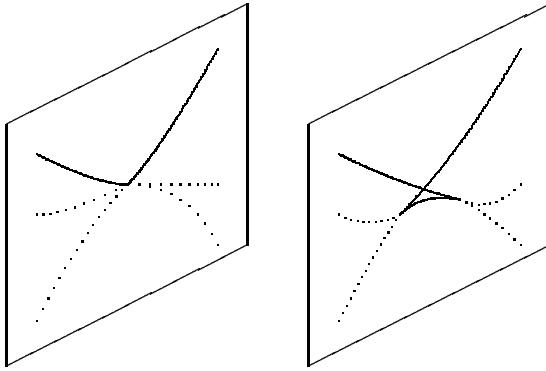


Figure 5.3: generification of irreducible branch

So we have shown, that the versal braid monodromy of  $F|_E$  is contained in the braid monodromy of  $F|_U$ .

For the reverse inclusion it suffices to argue that all braid monodromy transformations along simple geometric elements belong to the versal braid monodromy of  $F|_E$ . Suppose the element is associated to a component of  $\mathcal{B}_j$ , then the corresponding monodromy transformation is supported on the intersection with  $B_\varepsilon(v_j)$ , and therefore it is equal to its restriction to  $D_j$  extended by the identity.

If  $v_j$  is of the first kind considered above, then we noted that the restriction to  $D_j$  of any monodromy transformation belongs to the pure mapping classes of  $D_j$ . Their extension by the identity are thus elements of the versal braid monodromy group of  $F|_E$  assigned to  $v_j$ .

If  $v_j$  is of the second kind, then the restriction to  $D_j$  must be an element of the braid monodromy of the discriminant family given by the restriction to  $B_\varepsilon(v_j)$ . Again the extension by the identity is an element of the versal braid monodromy group of  $F|_E$  assigned to  $v_j$ .  $\square$

This result can be generalized to arbitrary perturbations, but we need only this form and have therefore preferred to avoid the bulk of technicalities involved in the general case.

## 5.4 Hefez-Lazzeri base

In the case of a Brieskorn-Pham polynomial we want finally reduce the computation of the braid monodromy group to the computation of versal braid monodromy groups of families which are monomial perturbations induced from the Hefez-Lazzeri unfolding.

$$\begin{array}{ccc} \mathbf{C}^n \times \mathbf{C}^n & \longrightarrow & \mathbf{C} \\ x, u & \mapsto & \sum_i (x_i^{l_i+1} + u_i x_i) \end{array}$$

**Notation 5.9:** We introduce the following shorthand notation for families parameterized by  $\alpha$  with fixed real constants  $\varepsilon_i > 0$ :

By  $f_\alpha(x_1, x_2)$  denote the  $\alpha$ -family

$$x_1^{l_1+1} - \alpha(l_1+1)x_1 + x_2^{l_2+1} - \varepsilon_2(l_2+1)x_2.$$

By  $g_\alpha(x_1, x_2)$  denote the  $\alpha$ -family

$$x_1^{l_1+1} - (l_1+1)x_1 + x_2^{l_2+1} - \alpha\varepsilon_2(l_2+1)x_2.$$

By  $f_\alpha(x_1, \dots, x_n)$  denote the  $\alpha$ -family

$$x_1^{l_1+1} - \alpha(l_1+1)x_1 + x_2^{l_2+1} - \varepsilon_2(l_2+1)x_2 + \dots + x_n^{l_n+1} - \varepsilon_n(l_n+1)x_n.$$

By  $g_\alpha(x_1, \dots, x_n)$  denote the  $\alpha$ -family

$$x_1^{l_1+1} - (l_1+1)x_1 + x_2^{l_2+1} - \alpha\varepsilon_2(l_2+1)x_2 + \dots + x_n^{l_n+1} - \alpha\varepsilon_n(l_n+1)x_n.$$

As a first step we have to show that at least for suitable choices of constants the versal braid monodromy is defined for the given families.

**Lemma 5.10** *The families  $g_\alpha(x_1, \dots, x_n)$  are tame.*

*Proof:* A degenerate critical point may only occur if the Hessian determinant vanishes. But for any family induced from the Hefez Lazzeri unfolding this determinant is constantly equal to

$$x_1^{l_1-1} \cdot \dots \cdot x_n^{l_n-1} \prod_i l_i(l_i+1).$$

Hence at least one coordinate  $x_i$  of a degenerate critical point must vanish. The vanishing of the gradient then implies via

$$\partial_i f(x, u) = (l_i+1)x_i^{l_i} + u_i$$

that the corresponding parameter vanishes too. So in the case of the induced family  $g_\alpha$  we conclude that  $\alpha = 0$  if a degenerate critical point occurs.

In that case the critical points are determined by

$$x_i^{l_i} = 1 \wedge x_2 = \dots = x_n = 0$$

and we get a bijection between the set of critical points and their values:

$$\{(\xi_1^k, 0, \dots, 0)\} \xleftrightarrow{1:1} \{-l_1\xi_1^k\}, \quad \xi_1^{l_1} = 1.$$

Hence the families  $g_\alpha$  are indeed tame.  $\square$

**Lemma 5.11** *The family  $f_\alpha(x_1, \dots, x_n)$  is tame if the function  $f_0|_{x_1=0}$  is a Morse function.*

*Proof:* By the preceding proof we need to worry only about the function for  $\alpha = 0$ , since otherwise all critical points are non-degenerate.

On the other hand any critical point of  $f_0$  is situated on the hyperplane  $x_1 = 0$  due to

$$\partial_1 f_0(x) = (l_1 + 1)x_1^{l_1} = 0.$$

So each must also be a critical point of  $f_0|_{x_1=0}$ . If now  $f_0|_{x_1=0}$  is a Morse function then it maps the set of critical points bijectively onto the set of critical values, hence no pair of critical points of  $f_0$  may map to the same critical value.  $\square$

Indeed we can deduce from the cases considered in the preceding proofs the following criterion for the tameness of a family induced from the Hefez Lazzeri base.

**Lemma 5.12** *A one-parameter family of functions induced from the Hefez Lazzeri base is tame, if any function is one of the following list:*

- i) *a function induced from the complement of the coordinate hyperplanes,*
- ii) *a function induced from a coordinate axis,*
- iii) *a function induced from a point on just one coordinate hyperplane  $\alpha_i = 0$ , such that its restriction to  $x_i = 0$  is a Morse function.*

We call a positive real constant  $\varepsilon_2$  resp. a tuple  $\varepsilon_2, \dots, \varepsilon_n$  of positive real constants admissible, if the fibre corresponding to  $g_1$  in the discriminant family associated to  $g_\alpha$  is regular. The condition is met if and only if  $g_1$  is a Morse function.

**Lemma 5.13** *If positive real constants  $\varepsilon_2, \dots, \varepsilon_n$  are chosen generically, then they are admissible and  $g_1(x_1, \dots, x_n)$  is a Morse function.*

*Proof:* There is a Zariski open set of complex constants, such that  $g_1$  is a Morse function. Furthermore we note that  $g_1$  is not a Morse function for all choices of complex constants if and only if the defining polynomial of the bifurcation divisor given in 3.4 vanishes on the hyperplane  $\alpha_1 = 1$ . Since that is not the case, the Zariski set above is non-empty.

But then it must contain a dense subset of all tuples of positive real constants, too.  $\square$

We can now consider the case  $n = 2$ . Then  $f_0|_{x_1=0}$  is a Morse function, thus for an admissible choice of  $\varepsilon_2 > 0$  the versal braid monodromy of the families  $f_\alpha$  and  $g_\alpha$  is defined and we can establish the following result:

**Proposition 5.14** *For an admissible choice  $\varepsilon_2 > 0$  the braid monodromy of the function  $f = x_1^{l_1+1} + x_2^{l_2+1}$  is given by the versal braid monodromy of the families  $f_{\alpha, \varepsilon}(x_1, x_2)$  and  $g_\alpha(x_1, x_2)$ , where the parameter of the second family may be restricted to the unit disc.*

*Proof:* By the result of the last section it suffices to show that the versal braid monodromy of the two families is equal to the versal braid monodromy of some  $\ell$ -perturbation of the function  $f$ .

So let us first pick an  $\ell$ -perturbation which we want to compare to the families  $f_\alpha$  and  $g_\alpha$ :

Since the bifurcation set in the Hefez Lazzeri base is a divisor, there is a transverse line and the corresponding linear polynomial  $\ell$  may even be assumed to be different from  $x_1$  by genericity. Hence the  $\ell$ -perturbation

$$x, \alpha, \lambda \mapsto f(x) + \alpha x_1 + \lambda \ell$$

is just induced from the Hefez Lazzeri unfolding by a change of parameters. Denote by  $h_\lambda$  the family of functions which is induced from an affine line  $L_h$  in the Hefez Lazzeri base parallel and sufficiently close to the line given by  $\ell$ . Moreover from the first two criteria of lemma 5.12 it is immediate that this family is tame.

Note that incidentally the families  $f_1$  and  $g_1$  are induced from affine lines  $L_f$  and  $L_g$  in the Hefez Lazzeri base parallel to the  $u_2$ -axis and the  $u_1$ -axis respectively.

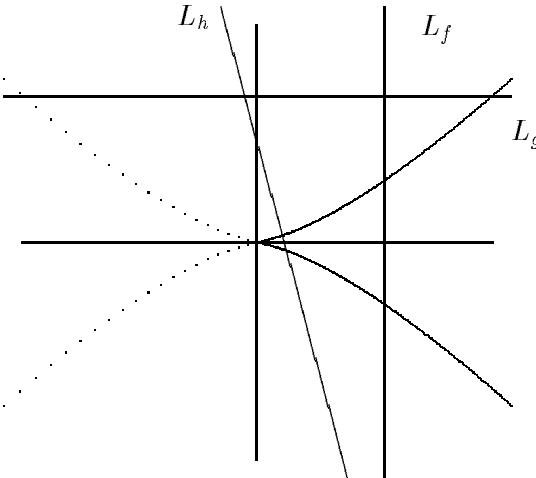


Figure 5.4: Hefez Lazzeri base

Next we make some observations regarding the fundamental group of the bifurcation complement:

Let  $L'$  denote the intersection of a line  $L$  with the complement of  $\mathcal{B}$ . We first note that  $\mathcal{B}$  is a curve without vertical components with respect to the linear projection parallel to  $L_h$ . So as we remarked before, lemma 1.14 and its proof imply

$$\pi_1(L'_h) \twoheadrightarrow \pi_1(\mathbf{C}^2 \setminus \mathcal{B}).$$

Moreover we extended this result to handle the projection parallel to  $L_f$ . Then the  $u_1$ -axis is the unique vertical component of  $\mathcal{B}$  and by lemma 1.17 we get

$$\pi_1(L'_f) * \mathbf{Z} \twoheadrightarrow \pi_1(\mathbf{C}^2 \setminus \mathcal{B}),$$

where the free generator is given by a geometric element supported on  $L_g$  and associated to  $\alpha_1 = 0$ . Therefore the braid monodromies of the families  $h_\lambda$  resp.  $f_\alpha$  and

$g_\alpha|$ , the restriction of  $g_\alpha$  to any base containing the unit disc, are both given by the braid monodromy of the Hefez Lazzeri unfolding.

A further ingredient of the argument is the geometry of the bifurcation divisor with respect to the lines  $L_f$  and  $L_g$ :

The Hefez Lazzeri base is of dimension two and the bifurcation divisor is quasi homogeneous with respect to a good  $\mathbf{C}^*$ -action. The line  $L_f$  has a common point with every  $\mathbf{C}^*$  orbit except the  $u_2$ -axis, so we may conclude that each component of the bifurcation divisor, which obviously is the closure of a  $\mathbf{C}^*$ -orbit, meets  $L_f$  in at least one point or  $L_g$  at its intersection with  $u_1 = 0$ .

Finally we bring back our attention to the versal braid monodromies associated to the families  $h_\lambda$  resp.  $f_\alpha$  and  $g_\alpha|$ :

To compare them we chose a path  $p_1$  in the Hefez Lazzeri base which connects the base points of the respective reference fibres and which is disjoint from the bifurcation divisor.

As we noted in the proof of lemma 5.7 the versal braid monodromy of a tame family contains its braid monodromy. Hence it suffices by the surjectivity result above to show that every generator of the versal braid monodromy on one hand is equal to a generator on the other hand transported along  $p_1$  up to the braid monodromy of the families.

The set of generators of the versal braid monodromy of the family  $h_\lambda$  can be chosen among parallel transports of generators in the local groups for  $h_\lambda$ . Suppose  $\beta$  is such a generator associated to a degeneration of  $h_\lambda$ , i.e. to a multiple root  $v_j$  in a singular fibre of the corresponding discriminant family. Let  $\mathcal{B}_0$  denote the component of the bifurcation divisor in the Hefez Lazzeri base, from which this singular fibre is induced.

Then the same component meets the base of  $f_\alpha$  or  $g_\alpha|$  as we noted above. So we may find a path  $q_2$  in the smooth locus of  $\mathcal{B}_0$  which connects the singular fibres. By equisingularity along the smooth part of  $\mathcal{B}_0$  or by the explicit equations we get a Whitney stratification for the discriminant family over a neighbourhood of  $q_2$ , where the components of the smooth locus of the discriminant are the strata of codimension one, and the components of its singular locus are the strata of codimension two.

A lift  $\hat{q}_2$  of  $q_2$  to the codimension two stratum which contains  $v_j$  ends at some multiple root  $v_j$  in the singular fibre at the end of  $q_2$ . Topological triviality along  $\hat{q}_2$  implies that the local families over  $L_f$  or  $L_g$  and  $L_h$  restricted to neighbourhoods of  $v_j$  resp.  $v'_j$  are topologically identified under parallel transport. Hence  $\beta$  transported along a path  $p_2$  in the complement of the bifurcation divisor but sufficiently close to  $q_2$  yields a mapping class  $\beta'$  which belongs to the local group of  $f_\alpha$  or  $g_\alpha|$ .

Therefore the parallel transport of  $\beta$  along any path  $p_h$  in  $L'_h$  and the parallel transport of  $\beta'$  along a path  $p_{fg}$  in  $L'_h \cup L'_g$  yield mapping classes which are equal up to parallel transport along the closed path obtained as the union of  $p_1, p_2, p_h$  and  $p_{fg}$ .

But then by the surjectivity on fundamental groups the same class is represented by a path in the base of  $f_\alpha$  and  $g_\alpha$ , hence any parallel transport of  $\beta$  in the base of  $h_\lambda$  is equal to some parallel transport of  $\beta'$  in the base of  $f_\alpha$  and  $g_\alpha$ .

The argument can also be read with  $h_\lambda$  taking the role of  $f_\alpha$  and  $g_\alpha|$  and vice versa, and then yields the reverse implication.  $\square$

We have to extend the result to the higher dimensional case. In principal the argument is the same but now some particular steps become more involved since the components of the bifurcation divisor are no longer smooth outside the origin. So to work on the smooth locus we have to add some more genericity assumptions

**Lemma 5.15** *For a generic choice of admissible constants  $\varepsilon_2, \dots, \varepsilon_n$ , the base of  $f_\alpha$  meets the reduced bifurcation divisor in regular points only.*

*Proof:* The claim can be deduced by a dimension count. The set of singular points of the reduced bifurcation divisor is of codimension two in the Hefez Lazzeri base.

On the other hand the complex constants  $\varepsilon_2, \dots, \varepsilon_n$  parameterize the parallels to the  $u_1$ -axis in the base. A generic line of this family does not meet the singular set above, so there is a non-empty Zariski open set of constants, such that the corresponding line is disjoint from the singular set.

But then even most tuples of positive real constants must belong to this set.  $\square$

**Proposition 5.16** *For a generic choice of admissible constants  $\varepsilon_2, \dots, \varepsilon_n > 0$  the braid monodromy of the function  $x_1^{l_1+1} + \dots + x_n^{l_n+1}$  is given by the versal braid monodromy of the families  $f_\alpha(x_1, \dots, x_n)$  and  $g_\alpha(x_1, \dots, x_n)$  where the parameter of the second family may be restricted to the unit disc.*

*Proof:* Due to lemma 5.8 again it suffices to show that the versal braid monodromy of the families  $f_\alpha$  and  $g_\alpha|$  is equal to the versal braid monodromy of some tame  $\ell$ -perturbation of the function  $f = x_1^{l_1+1} + \dots + x_n^{l_n+1}$ .

We assume that admissible constants  $\varepsilon_i > 0$  are chosen in such way, that the base  $L_f$  of  $f_\alpha$  meets the reduced bifurcation divisor only in regular points, cf. lemma 5.15. By assumption  $f_0|_{x_1=0} = g_1|_{x_1=0}$  is a Morse function. But then so is  $g_\alpha|_{x_1=0}$  for almost all values of the parameter  $\alpha$ .

We define  $\ell = cx_1 + \varepsilon_2(l_2 + 1)x_2 + \dots + \varepsilon_n(l_n + 1)x_n$  for a generic constant  $c \neq 0$  and an unfolding with two-dimensional base  $E$

$$x, u, \lambda \mapsto f(x) + ux_1 + \lambda\ell.$$

Then let  $h_\lambda$  be the family induced for  $u = \varepsilon_1$  constant and sufficiently close to 0. Furthermore note that  $f_\alpha$  is induced from  $E$  for  $\lambda = -1$  and  $u = -\alpha(l_1 + 1) - c$  while  $g_\alpha$  is induced for  $\lambda = -\alpha$ ,  $u = \alpha c$ .

Since the only functions in  $h_\lambda$  induced from the coordinate hyperplanes of the Hefez Lazzeri base are  $h_0$  and  $h_{-\varepsilon_1/c}$  we conclude that the family  $h_\lambda$  is tame by 5.12:

- i)  $E$  considered as a plane in the Hefez Lazzeri base is not contained in the coordinate hyperplanes, hence each function  $h_\lambda$ ,  $\lambda \neq 0, -\varepsilon_1/c$  is of the first kind considered in 5.12,

- ii)  $h_0(x) = f(x) + \varepsilon_1 x_1$  is induced from a coordinate axis, so it is of the second kind considered in 5.12,
- iii)  $h_{-\varepsilon_1/c}|_{x_1=0} = g_{-\varepsilon_1/c}|_{x_1=0}$  is a Morse function for  $\varepsilon_1$  sufficiently small, so  $h_{-\varepsilon_1/c}$  is of the third kind considered in 5.12.

Next we make some observations on the bifurcation locus restricted to  $E$  considered as a plane in the Hefez Lazzeri base.

By assumption  $g_1$  is a Morse function, hence the bifurcation locus is of codimension one in  $E$ . The restriction of the Hefez Lazzeri base to  $E$  is obtained by imposing some linear relations on the base coordinates. From the defining equation 3.4 of the bifurcation divisor  $\mathcal{B}$  we can read off the fact, that each component of the restriction  $\mathcal{B}_E = \mathcal{B} \cap E$  contains the origin. Therefore each component different from the  $u_1$ -axis meets  $L_f$ .

This situation is very similar to that encountered in the case  $n = 2$ . Since  $L_h$  is a general line in  $E$  with respect to  $\mathcal{B}_E$  and since by assumption  $L_f$  meets  $\mathcal{B}_E$  in its regular locus only, we may conclude as before:

$$\begin{aligned}\pi_1(L'_h) &\twoheadrightarrow \pi_1(E \setminus \mathcal{B}_E), \\ \pi_1(L'_f) * \mathbf{Z} &\twoheadrightarrow \pi_1(E \setminus \mathcal{B}_E),\end{aligned}$$

where the free generator maps to any geometric element supported on  $L_g$  associated to the  $u_1$ -axis.

Since  $\pi_1(E \setminus \mathcal{B}_E) \cong \pi_1(\mathbf{C}^n \setminus \mathcal{B})$ , the braid monodromies of the families  $h_\lambda$  resp.  $f_\alpha$  and  $g_\alpha|$  are both given by the braid monodromy of the Hefez Lazzeri unfolding.

The argument then proceeds as in the  $n = 2$  case. □

## Chapter 6

# braid monodromy of plane curve families

This chapter can be considered the central one. Admittedly it is devoted to a partial computation only of the braid monodromy group of a plane Brieskorn Pham polynomial. Nevertheless it will be the essential ingredient for the computation of braid monodromy groups associated to arbitrary Brieskorn Pham polynomials which will be the topic of the next chapter.

Given a plane Brieskorn Pham polynomial  $x_1^{l_1+1} + x_2^{l_2+1}$ , we consider for generic small  $\varepsilon_2 > 0$  the one parameter family of functions defined in the last chapter:

$$f_\alpha(x_1, x_2) = x_1^{l_1+1} - \alpha(l_1 + 1)x_1 + x_2^{l_2+1} - \varepsilon_2(l_2 + 1)x_2.$$

It corresponds to a line in the Hefez Lazzeri base which in turn can be considered as a plane in the base of a suitable truncated miniversal unfolding. We have already obtained the necessary formulae for the singular value divisor in the third chapter which are of course needed for further calculations.

Using the preceding chapter we will actually compute the versal braid monodromy of the family  $f_\alpha$ . So remember that we have two tasks. For each singular function we have to determine the local group which is a group of mapping classes in a regular fibre close to the corresponding fibre in the discriminant family. We have then to find the group of mapping classes in a distinguished reference fibre generated by translates of the elements of the local groups obtained by parallel transport.

Let us muse a moment on what we have to do: We are given a one dimensional base of a family with a lot of degeneration points. Close to these points we may choose local Milnor fibres and are then given mapping classes in these Milnor fibres. Next we have to choose a geometrically distinguished system of paths joining them to a global reference fibre. Finally we transport the local mapping classes to the reference fibre and determine the subgroup of braids they generate under the identification of the mapping class group with the braid group given by the Hefez Lazzeri system of paths in the fibre.

All steps will be addressed in this chapter, though not in the given order. Moreover we need to modify this approach, so let us point out some details:

Instead of parallel transport in the discriminant family we will consider parallel transport in a closely related family from the first section on. Only in section 6.8 we will return to the proper discriminant family and exploit the relation of both families to transfer the result to the family where we actually need them.

The computation of the local monodromy groups is postponed to section 6.7, so we can concentrate on the parallel transport in the first sections. Note that we won't transport general classes but exploit the bijection between twist classes and isotopy classes of arcs which join the two punctures twisted around each other. In fact it suffices to consider the parallel transport of a specific sort of arcs only since we will prove in section 6.7, that all local mapping classes form a group generated by twists on such arcs.

The most difficult task is the parallel transport of mapping classes. As said we will in fact transport the arcs such that the corresponding twists generate the mapping class groups.

Since all the paths we use decompose into radial and circular segments, we study the differential flow over such segments and determine the parallel transport of arcs along radial segments in the first section. It will be possible in subsequent sections to determine the parallel transport of arcs, since we arrange our arcs to be determined by geometric data which are preserved and some geometric datum which changes in a way we can actually measure.

## 6.1 parallel transport in the model family

In this section we consider the parallel transport in a punctured disc bundle associated to the discriminant bundle of the function families  $f_\alpha$ . Recall that we have to transport the local groups to a global reference fibre along a geometrically distinguished system of paths. Such a system can be chosen to consist of paths which are obtained by unions of radial segments with circular segments on circles centred at the origin.

So we find suitable vector fields and differentiable flows along such segments. Thereby we get representatives of mapping classes which in turn will be used to get representatives of parallel transport of braid mapping classes.

Let us consider then the punctured disc bundle associated to the line arrangement

$$\prod_{\xi_1^{l_1}, \xi_2^{l_2}=1} (z - \lambda \xi_1 - \eta_2 \xi_2) = 0.$$

which we call the *model discriminant family* associated to  $l_1, l_2, \eta_2 \ll 1$ .

In section 3.2 we have indexed the paths of the Hefez Lazzeri path system for the fibre at  $\lambda = 1$  by elements of the set  $\{i_1 i_2 \mid 1 \leq i_1 \leq l_1, 1 \leq i_2 \leq l_2\}$ . We assign indices accordingly to the punctures in the fibre at  $\lambda = 1$ , to the lines of the arrangement and hence to any puncture in any fibre.

Our aim is to describe the parallel transport along radial paths and along circle segments with radius 1 or close to 0. We will find appropriate diffeomorphisms and obtain transported arcs.

**Notation 6.1:** We introduce polar coordinates  $\lambda = te_\vartheta$ ,  $e_\vartheta := e^{i\vartheta}$  of unit absolute value and  $t \in \mathbf{R}^{\geq 0}$ .

**Definition 6.2:** A parameter  $te_\vartheta$  is called *critical*, if there is a pair  $i_1i_2, j_1j_2$  of indices such that the corresponding lines meet at  $te_\vartheta$ .

The pair may be specified and  $te_\vartheta$  called *critical for the pair*  $i_1i_2, j_1j_2$ .

Let us first outline our general approach. For a family we first give a vector field on its total space. Next we check that the punctures form integral curves, so the corresponding flow preserves the punctures. Then we obtain some of the properties of the induced diffeomorphisms, to get finally the parallel transport of some geometric objects.

As most important technical tool we employ *bump functions*:

**Notation 6.3:** We introduce smooth functions  $\chi, \chi_\varepsilon : \mathbf{C} \rightarrow \mathbf{R}$  for any real  $\varepsilon > 0$ :

$$\begin{aligned}\chi &: 0 \leq \chi(z) = \chi(|z|) \leq 1, \chi(z) = 0 \text{ if } |z| \geq 1, \chi(z) = 1 \text{ if } |z| \leq \frac{1}{2}, \\ \chi_\varepsilon &: \chi_\varepsilon(z) = \chi(z/\varepsilon),\end{aligned}$$

with support contained in the unit disc, resp. the disc of radius  $\varepsilon$ .

For a start we consider the local situation, i.e. the family  $z - t = 0$ ,  $|z| \leq 1$ ,  $t \in [-\frac{1}{2}, \frac{1}{2}]$ .

**Lemma 6.4** *On the unit disc  $|z| \leq 1$  there is the vector field*

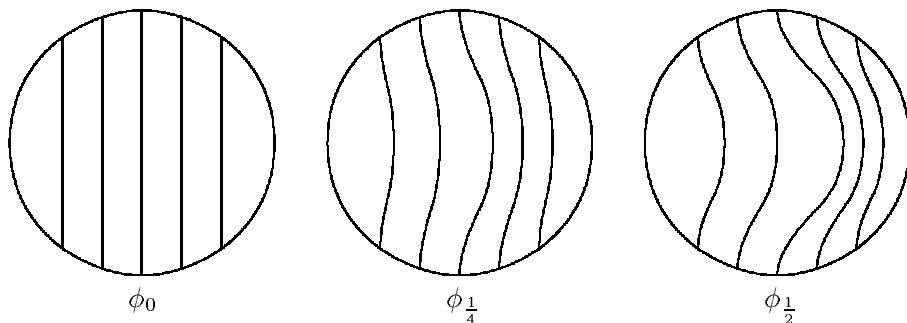
$$v(z) = \chi(z),$$

*which induces a family  $\phi_t$ ,  $t \in [-\frac{1}{2}, \frac{1}{2}]$  of diffeomorphisms of the disc such that*

- i)  $\phi_t$  preserves horizontal lines,
- ii)  $\operatorname{Re} \phi_{t_2}(z) \geq \operatorname{Re} \phi_{t_1}(z)$  if  $t_2 \geq t_1$ , i.e.  $\phi_t$  propagates on horizontal lines,
- iii)  $\phi_t(0) = t$ , i.e.  $z = t$  is an integral curve.

*Proof:* The first and second property are consequences of the vector field directing parallel to the positive real numbers. The last is due to the vector field being the derivative of the function  $z \mapsto \operatorname{Re} z$  in the disc of radius  $\frac{1}{2}$ .  $\square$

**Example 6.5:** As an illustration we sketch the images of some vertical lines under  $\phi_0, \phi_{\frac{1}{4}}$  and  $\phi_{\frac{1}{2}}$ :



So we may infer the following result.

**Lemma 6.6** *The family of diffeomorphisms represents the diffeotopy class associated to the family of punctured discs given by the function  $f_t(z) = z - t$ .*

### radial families

Now we investigate the model discriminant family restricted to a radial path  $te_{\vartheta_0}$ ,  $t \in [t_0, 1]$ . We will consider the case only when this restriction has constant number of punctures, in which case we call it a *regular family*. Since all punctures depend affine linearly on the parameter without ever meeting, the local situation is modeled on the case considered first.

When passing to a global view, we want to understand the corresponding parallel transport diffeomorphism mapping the initial fibre to the terminal fibre. Considered as an endomorphism of the plane it is seen to be supported on the set of points which are close enough to some puncture at some parameter, i.e. close enough to the union of their *traces*:

**Definition 6.7:** The *trace* of index  $i_1 i_2$  in a family is the set of points  $z$  in the plane  $\mathbf{C}$  such that  $z$  is a puncture of index  $i_1 i_2$  for some parameter of the family base.

This we can make explicit with a quick check:

**Lemma 6.8** *Let  $\varepsilon > 0$  be bounded from above by half the minimal distance between punctures in the fibres of the regular family over  $te_{\vartheta_0}$ ,  $t \in [t_0, 1]$ . Then the punctures form integral curves for the vector field*

$$v_\varepsilon(z, t) = \sum_{\xi_1^{l_1}, \xi_2^{l_2}=1} \chi_\varepsilon(z - te_{\vartheta_0}\xi_1 - \eta_2\xi_2) e_{\vartheta_0}\xi_1,$$

and the corresponding diffeomorphisms are supported on the  $\varepsilon$ -neighbourhood of the union of all traces.

Hence parallel transport only affects small neighbourhoods of the punctures. Any arc will be changed only due to the movements of its endpoints and of the critical values which come close enough, to distances less than  $\varepsilon$  in fact. So we can imagine what happens to a given arc in the fibre at  $t_0$ :

Let the arc be a piece of rope. As the parameter  $t$  increases additional rope is laid out on the traces of both the critical values which form the ends of the arc. A critical value about to cross the arc will push it ahead and lay out a double rope behind forming a loop around its trace.

Likewise any time a critical value crosses a trace along which a multiple rope has previously laid down, it picks this rope up and pushes a multiple loop into it along its own trace.

So in the end the rope is lain down in an arbitrary small neighbourhood of the union of all traces, in fact the union can be restricted to that part of each trace traced after the corresponding critical value picked up rope for the first time.

We want to apply parallel transport to a very restricted set of arcs:

**Definition 6.9:** Given a critical parameter  $t_0 e_{\vartheta_0}$  for the index pair  $i_1 i_2, j_1 j_2$ , an arc between the corresponding critical points in the fibre at  $t_1 e_{\vartheta_1}$  is called *local v-arc* if

- i) it is supported on the corresponding traces,
- ii) the difference  $t_1 - t_0$  is positive and small compared to the distances of critical parameters.

In case of  $j_1 - i_1 = l_1/2$ ,  $i_2 \neq j_2$ , the traces of the corresponding critical points in a fibre  $t_1 e_{\vartheta_1}$  meet only if  $\vartheta_1 = \vartheta_0$ . In this case we allow  $\vartheta_1 \neq \vartheta_0$  nevertheless and concede that the local v-arc are supported on the traces except for a small part to join them.

**Definition 6.10:** Parallel transport of a local v-arc in a radial family by the differentiable flow to radius  $t = 1$  yields an arc called *tangled v-arc*.

**Definition 6.11:** An arc in the fibre at  $t_1 e_{\vartheta_1}$  is called *local w-arc* if

- i) it connects punctures of indices  $i_1 i_2, j_1 j_2$ ,  $i_1^+ < j_1, i_2 = j_2$ , by four line segments,  $i_1^+ = i_1 + 1$ ,
- ii) two segments are supported on the traces of the two punctures,
- iii) the central pair forms a sharp wedge over the trace of the puncture of index  $i_1^+ i_2$ ,
- iv) its length and  $t_1$  are small compared to the distance of critical parameters.

**Definition 6.12:** Parallel transport of a local w-arc in a radial family by the differentiable flow to radius  $t = 1$  yields an arc called *tangled w-arc*.

To describe the local situation at a crossing of two or more critical points, we consider *tangled tails* of punctures. These one should imagine just as a piece of rope laid out by a critical point on its trace and tangled by subsequent critical points. Looking locally at the tail implies that it may decompose into several pieces.

**Example 6.13:** Imagine a crossing of just two traces, then the tangled tails look locally like



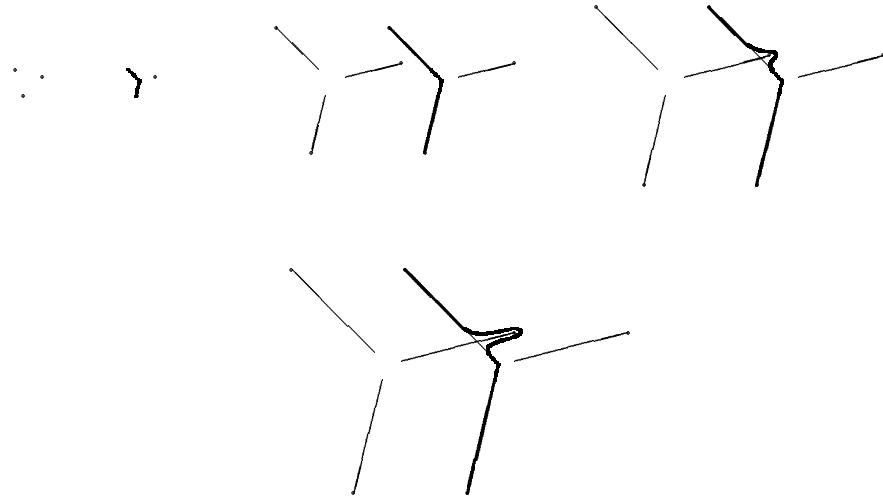
(The critical points pass from bottom to top, the first from left to right, the second from right to left.)

By construction a local v-arc is approximately supported on tails hence so is the transported arc throughout the radial family. In fact more is true. At each crossing of critical points, to which the transported arc comes close, it is approximately supported on the tails of the crossing punctures:

**Lemma 6.14** *Locally at a crossing  $P$  all local components of a tangled v-arc can be assumed to be arbitrarily close approximations to one of the tangled tails of the punctures passing through  $P$ .*

*Proof:* All local components are laid out by a critical point which pushes them through  $P$ . Hence the smaller  $\varepsilon$  is, the better the approximation will be.  $\square$

**Example 6.15:** For the family with  $l_1 = 3, l_2 = 2$  and  $\vartheta = \frac{\pi}{12}$  we have the sketches of a local v-arc at  $.12e_\vartheta$  and its parallel transports at  $.56e_\vartheta, .78e_\vartheta$  and  $e_\vartheta$  together with the traces of all critical values.



In this example only one additional critical value is entangled.

Finally we observe, that almost all radial families are regular.

**Definition 6.16:** The angle  $\vartheta_1$  is called *regular* if for all  $t_1 > 0$  the family over the line segment from  $t_1 e_{\vartheta_1}$  to  $e_{\vartheta_1}$

- i) is a regular family, i.e. the segment does not pass a critical parameter,
- ii) has no pair of distinct traces having more than one point in common,

Note that distinct traces have at most one point in common, if and only if no trace contains a point  $\eta_2 \xi_2, \xi_2^{l_2} = 1$ . So we can show:

**Lemma 6.17** *Given a critical parameter  $t_0 e_{\vartheta_0}$ , then there is a regular angle  $\vartheta_1$  arbitrarily close to  $\vartheta_0$ .*

*Proof:* Both regularity conditions are open. Hence it suffices to prove that for each condition there is an angle arbitrarily close to  $\vartheta_0$  such that the condition is met. For both conditions this is easy to see, since in both cases a finite set is to be avoided by the traces and the traces of one puncture along different radial paths have no point in common.  $\square$

## circular families

Similar to the case of radial families we can get hold of a diffeomorphism which represents the parallel transport over circular segments in the base. On particular subsets the map is in fact quite easily described.

**Lemma 6.18** *Given the vector field*

$$v(z, \vartheta) = i \left( z + \sum_{\xi_1^{l_1}=1} \chi_{2\eta_2}(z - e_\vartheta \xi_1)(e_\vartheta \xi_1 - z) \right)$$

then

- i) the punctures of the model family over the circle of radius  $t = 1$  form integral curves,
- ii) supposing  $|z_0 - e_\vartheta \xi_1| \leq 2\eta_2$  the flow of  $v$  preserves the distance of  $z_0(\vartheta)$  and  $e_\vartheta \xi_1$ .
- iii) supposing  $|z_0 - \xi_1| \geq 2\eta_2$  for all  $\xi_1, \xi_1^{l_1} = 1$ ,  $z_0(\vartheta) = z_0 e_\vartheta$  is an integral curve.

*Proof:* i) Each puncture forms a curve  $e_\vartheta \xi_1 + \eta_2 \xi_2$ ,  $\xi_1^{l_1}, \xi_2^{l_2} = 1$ , for which we can check the integrality condition:

$$\frac{d}{d\vartheta}(e_\vartheta \xi_1 + \eta_2 \xi_2) = ie_\vartheta \xi_1 = i(e_\vartheta \xi_1 + \eta_2 \xi_2 - \eta_2 \xi_2) = v(e_\vartheta \xi_1 + \eta_2 \xi_2, \vartheta).$$

ii) We have to show that the following complex numbers considered as real vectors are perpendicular for all  $\vartheta$ :

$$(e_\vartheta \xi_1 - z_0(\vartheta)) \cdot \frac{d}{d\vartheta}(e_\vartheta \xi_1 - z_0(\vartheta)).$$

Both points move along integral curves, hence  $\frac{d}{d\vartheta} e_\vartheta \xi_1 = ie_\vartheta \xi_1$  and

$$\begin{aligned} \frac{d}{d\vartheta} z_0(\vartheta) &= v(z_0(\vartheta), \vartheta) \\ &= iz_0(\vartheta) + i\chi_{2\eta_2}(z_0(\vartheta) - e_\vartheta \xi_1)(e_\vartheta \xi_1 - z_0(\vartheta)). \end{aligned}$$

Since  $\chi$  is a real valued function, the second function is a purely imaginary multiple of the first, hence they are orthogonal at all  $\vartheta$ .

iii) Again we have only to check an integrality condition

$$\frac{d}{d\vartheta} z_0 e_\vartheta = iz_0 e_\vartheta = v(z_0 e_\vartheta, \vartheta).$$

□

Let us rephrase the result of the lemma in more geometrical terms:

- i) the flow realises parallel transport in the model family over circle segments of radius  $t = 1$ ,

- ii) the  $2\eta_2$ -discs at points  $\xi_1, \xi_1^{l_2} = 1$  are mapped bijectively to  $2\eta_2$ -discs of the transported points preserving the distance,
- iii) points outside these discs are mapped by a rigid rotation around the origin.

**Lemma 6.19** *Given the vector field for  $\varepsilon << \eta_2$*

$$v(z) = i \sum_{\xi_2^{l_2}=1} \chi_{4\varepsilon}(z - \eta_2\xi_2)(z - \eta_2\xi_2)$$

*then*

- i) the punctures of the model family over the circle of radius  $t = \varepsilon$  form integral curves,
- ii) suppose  $|z_0 - \eta_2\xi_2| \leq 2\varepsilon$ ,  $\xi_2^{l_2} = 1$  then the curves  $z_0(\vartheta) = (z_0 - \eta_2\xi_2)e_\vartheta + \varepsilon_1\xi_2$  are integral for the flow of  $v$ ,
- iii) suppose  $|z_0 - \eta_2\xi_2| \geq 4\varepsilon$  for all  $\xi_2, \xi_2^{l_2} = 1$ , then  $z_0(\vartheta) = z_0$  is an integral curve.

*Proof:* i) Since each puncture is on a curve  $\varepsilon e_\vartheta \xi_1 + \eta_2 \xi_2$ , the assertion follows from case ii).

ii) We check the integrality condition:

$$\frac{d}{d\vartheta}((z_0 - \eta_2\xi_2)e_\vartheta + \eta_2\xi_2) = i(z_0 - \eta_2\xi_2)e_\vartheta = v((z_0 - \eta_2\xi_2)e_\vartheta + \eta_2\xi_2), \vartheta).$$

iii) Since the vector field vanishes at these points constant curves are integral curves.  $\square$

Again we restate these results in geometrical terms:

- i) the flow realises parallel transport in the model family over circle segments of radius  $t = \varepsilon$ ,
- ii) the  $2\varepsilon$ -discs at points  $\eta_2\xi_2, \xi_2^{l_2} = 1$ , are rotated rigidly under parallel transport,
- iii) points outside  $4\varepsilon$ -discs of these points stay fix.

## 6.2 from tangled v-arcs to isosceles arcs

In this section we consider two different kinds of mapping classes in a fibre of large radius. Both kinds are twists on embedded arcs. So we may equally well investigate these arcs. Arcs of the first kind are called tangled v-arcs, they are obtained from local v-arcs by parallel transport along a radial path using the differentiable flow of the preceding section.

Arcs of the second kind are called *isosceles arcs*. They are supported on traces of two punctures and form the two sides of an approximate isosceles triangle. Again the degenerate case requires extra care. If two traces are parallel but close, an arc which is supported on these traces except for a small join between them is called a *straight isosceles arc*.

An isosceles arc is said to correspond to a tangled v-arc if it connects the same punctures. In general these two arcs are not isotopic. But we will define a group of mapping classes such that they belong to one orbit. In fact we will give some arcs, such that the group generated by the full twists on these arcs will do. They will be called *bisceles arcs* for the reason that they are supported on segments of two traces not necessarily of similar length.

Note that by this definition all isosceles arcs are subsumed under the notion of bisceles arcs except for the straight isosceles arcs.

We want to encode the isotopy class of a tangled v-arc into a planar diagram in the fibre at  $e_\vartheta$ . This diagram will consist of all the traces each of which is directed from its *source point* – which is one of  $\xi_2$ ,  $\xi_2^{l_2} = 1$  – to its puncture.

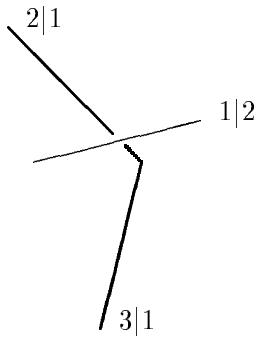
Apart from the source points, there are only ordinary crossings, which are given by the mutual transversal intersection of several traces.

Crossings which are sufficiently close to the tangled v-arc are called *vertices* of the diagram. The segments of traces close to the tangled v-arc are called *essential traces*, they connect a vertex to a puncture.

At each vertex we put an order on the essential traces. The first or *dominant trace* is the one which passed last, which is incidentally the one such that the puncture end is closest. The other follow according to increasing distance to their puncture end. The order can be made explicit by labels assigned to the essential traces at each vertex. We can also make the dominant trace pass over by replacing the other traces by broken lines. Finally the lines are labeled at their ends by the index of the corresponding puncture.

We define the *essential diagram* to be obtained by discarding all lines except the essential traces and we notice that the tangled v-arc is still determined by this datum.

**Example 6.20:** From the tangled v-arc of the previous example, we get the following diagram for  $l_1 = 3, l_2 = 2, \vartheta = \frac{\pi}{12}$ , in which we have discarded all traces which do not pass a vertex.



**Definition 6.21:** No essential diagram contains a directed cycle, hence the *height* function on vertices is well-defined by

$$\text{ht}(P) = \max_{P' < P} (\text{ht}(P'), 0) + 1.$$

where the maximum is taken over all vertices  $P'$  between  $P$  and a puncture on an essential trace. Each such vertex is called *subordinate* to  $P$ .

Given an essential diagram we consider *simple transformations* at vertices. We may change the crossing order at a vertex  $P$  if and only if all traces through  $P$  are dominant at each subordinate vertex. Note that on transformed diagrams we have to make the order explicit, since it can no longer be read off the distances to the punctures.

The first observation is that we can change an essential diagram by simple transformations only to get a diagram in which the traces of the v-arc punctures are dominant at all vertices they cross.

**Lemma 6.22** *Given any vertex there is a composition of simple transformations which changes the crossing order at this vertex but nowhere else.*

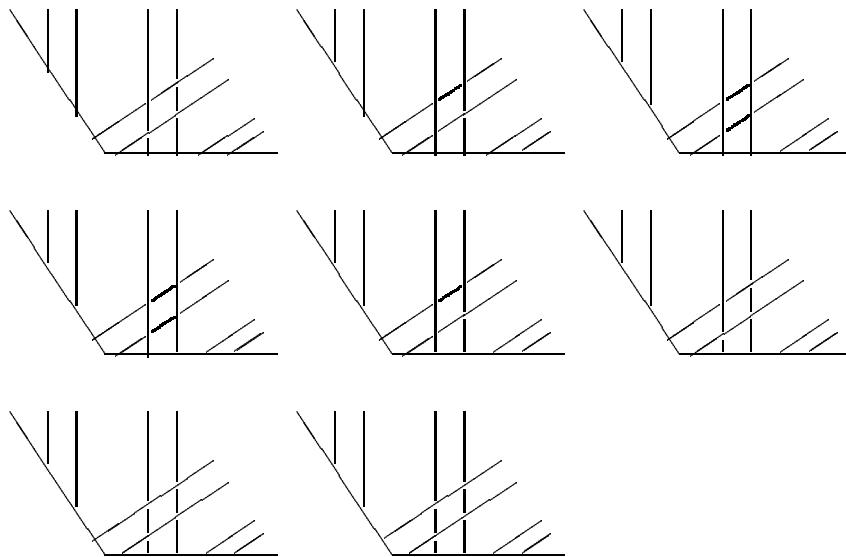
*Proof:* If the vertex is of height one we can change it by a simple transformation. If not, a simple transformation can only be performed if the essential traces are dominant on subordinate vertices. But then we can argue inductively on the height of the vertex. All subordinate vertices are of less height, so by induction we may assume the existence of a composite transformation which makes the traces under consideration dominant there.

Then we can perform the simple transformation to change the local order. Finally we invoke the inverse of the composite transformation to put all other transformed vertices back to their initial state.  $\square$

In particular, a series of simple transformations can be found such that the traces of the v-arc punctures become dominant.

**Example 6.23:** We illustrate this procedure in the following sequence of diagrams.

In each step we perform a simple transformation on some vertices which do not share subordinate vertices.



The important step is to see, that for any simple transformation at a vertex  $P$  there is a choice of a mapping class such that

- i) the mapping class is given by a product of full twists on bisceles arcs supported on the essential traces through  $P$ ,
- ii) a diagram transformed by a sequence of simple transformations encodes the isotopy class of the tangled v-arc transformed by the composition of the chosen mapping classes.

For the induction in the proof of the following lemma we need also a relation between tails at a vertex.

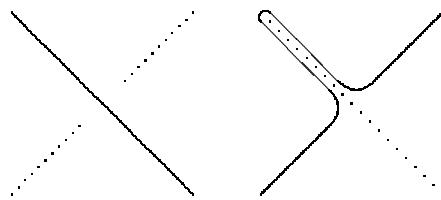
**Definition 6.24:** At a vertex a tail *dominates* another one, if it is isotopic to its trace up to an isotopy fixing the endpoints of both tails but not necessarily the punctures not involved.

**Lemma 6.25** *Given a diagram with orders at its vertices which are obtained by a composition of simple transformations from those of the essential diagram of a tangled v-arc. Then there is a diffeomorphism such that*

- i) *it represents a mapping class which is a product of full twists on bisceles arcs supported on essential traces,*
- ii) *it is supported close to the essential traces,*
- iii) *locally at every vertex the dominant trace is close to the image of the corresponding tail.*

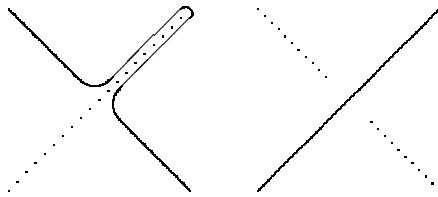
*Proof:* We assume in addition that each simple transformation reverses the order of consecutive traces and start an induction on the number of such transformations in the composite transformation.

So we consider a simple transformation. For simplicity we first assume that the vertex at which the order is changed is met by only two essential traces. By assumption these traces are dominant at subordinate vertices, hence we can depict the tangled tails of the two punctures involved as follows:



(The critical points pass from bottom to top, the first from left to right, the second from right to left.)

Now a full twist on the bisceles arc with the appropriate choice of orientation can be performed close to these traces to yield:



Hence our claim is true in this case.

The same applies if there are more essential traces and we want to reverse the order of the first two, since the corresponding tails are not effected by tails of lower order.

The situation changes drastically if our simple transformation reverses the order of traces none of which is dominant. Then the picture is modified by the essential traces of larger order pushing loops into the depicted tails.

But on the same time they push loops into the bisceles arc and hence into the support of the diffeomorphism we want to perform. Hence we need only to show that this pushed diffeomorphism will do.

Of course it has the second property. It also has the first property since the full twist on the modified bisceles arc is isotopic to the full twist on the bisceles arc conjugated by full twists on bisceles arcs with apex in the same vertex.

The third property is given, since the dominant traces and the corresponding tails are locally not changed except for the explicit case considered first, where the property can be simply checked.

Moreover for the induction process we should notice that any of our diffeomorphisms preserves domination of a tail over another one, except that it exchanges the role of the tails corresponding to the traces of which the order has been reversed.

To proceed our induction the first two properties are no obstacle. But we have to prove that the third property is preserved when performing an additional transformation.

If the additional transformation does not affect a dominant trace, then neither does the diffeomorphism we perform. Since it also preserves the corresponding tail, we are done in this case.

So let us assume the additional transformation affects a dominant trace. Then the diffeomorphism we choose also affects both the trace and the tail. What we have to show is that the image of the tail which was second before and is first now has the claimed property.

By assumption this tail is only tangled along the essential traces through the vertex under consideration. Moreover we may assume that it dominates all tails through this vertex apart from the dominant one. Hence it is only tangled by the dominant trace and our diffeomorphism can be chosen to map it close to its trace as in the case depicted above.  $\square$

**Lemma 6.26** *Given a tangled v-arc there is a mapping class given by a composition of full twists on bisceles arcs supported on essential traces which maps the tangled v-arc to the isotopy class of the corresponding isosceles arc.*

*Proof:* By lemma 6.22 there is a composition of simple transformation changing vertex orders of the essential diagram of the given tangled v-arc in such a way that the traces of both puncture ends are dominant at each vertex.

Then by lemma 6.25 there is a diffeomorphism representing a mapping class as in the claim, which maps the tangled tails in such a way that locally at each vertex the dominant trace is close to its tail.

Thus the images of the tangled tails of both puncture ends may no longer deviate from the traces at any vertex. So they are isotopic to the traces and we conclude that the image arc is isotopic to the corresponding isosceles arc.  $\square$

We did not bother to adjust our arguments explicitly for  $j_1 - i_1 = l_1/2$ , since we can choose  $0 < \varepsilon \ll t_0|\vartheta_0 - \vartheta_1|$  small in comparison with the minimal diameter of local neighbourhoods of vertices.

We close this section with two observations, which will be used later:

**Remark 6.27:** All bisceles arcs supported on essential traces are – apart from the obvious one – not isosceles arcs, since one critical point has to pass after the other.

For the same reason, the *length* of each bisceles arc supported on essential traces of a tangled v-arc is bounded by the length of the corresponding isosceles arc.

The *length* is defined to be the maximum of the lengths of the two sides.

### 6.3 from bisceles arcs to coiled isosceles arcs

We stay in the same fibre as before, so we work in the same group of mapping classes. And we are still interested into orbits of subgroups generated by full twists on bisceles arcs.

We have accomplished so far, that we can express a tangled v-arc by means of an isosceles arc and twists on bisceles arcs. Now in a similar way we want to relate bisceles arcs and straight isosceles arcs to a third kind of arcs called *coiled isosceles arcs*. With straight isosceles arcs we will deal only at the end of the section.

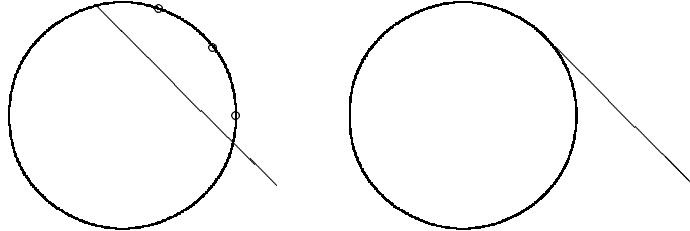
Again a bisceles arc and the associated coiled isosceles arc connect the same pair of punctures and – though not isotopic in general – belong to one orbit of a group generated by twists on specific bisceles arcs.

To make these statements precise, we first need to introduce some more geometric notions.

**Definition 6.28:** The *central core* is the disc of radius  $\eta_2$  at the origin with all source points distributed on its boundary circle.

**Definition 6.29:** The *peripheral cores* are the discs of radius  $\eta_2$  centred at the points  $\xi_1 e_\vartheta, \xi_1^{l_1} = 1$ . All critical points for  $\lambda = e_\vartheta$  are distributed on their boundaries, the *peripheral circles*.

By looking at the following sketches we notice that a bisceles arc can take essentially two different positions relative to a peripheral core which contains one of its punctures.



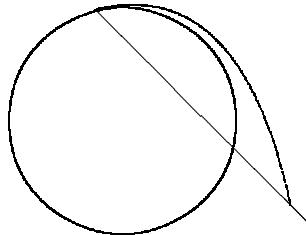
**Definition 6.30:** A bisceles arc is called *unobstructed* if it is isotopic to some arc supported outside the peripheral cores. It is called *obstructed* otherwise.

A bisceles arc of index pair  $i_1 i_2, j_1 j_2$  is said to be *obstructed on the  $i$ -side*, if punctures of index  $i_1 i'_2$  are obstacles to unobstructedness.

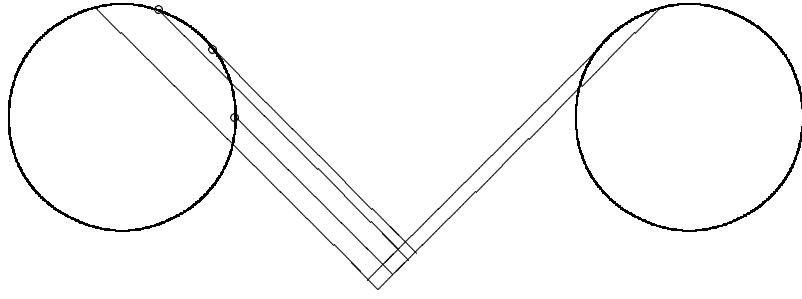
If a bisceles arc is obstructed then at least one side cuts through the corresponding peripheral circle and thus divides the set of critical points on the circle into two subsets.

**Definition 6.31:** If a bisceles arc is obstructed, then a set of critical points is called *obstructing set*, if the bisceles arc is unobstructed in the complement of the other punctures, i.e. isotopic to some arc supported outside the peripheral cores.

Since we may not isotopy arcs through punctures, we have to resort to changing the isotopy class by means of full twists on some suitable bisceles arcs. This has to be done in such a way, that up to isotopy the terminal part of the obstructed bisceles arc is simply replaced by a spiral segment coiled around the peripheral core.



To do so properly we choose a suitable obstructing set and employ twists on arcs which are supported on pairs of parallels to the sides of the bisceles arc and which connect a point of the obstructing set to another one or to a puncture of the bisceles arc.



By construction a bisceles arc bounds a well defined convex cone which we call the *inner cone* of the bisceles arc.

Thus given an obstructed bisceles arc, the critical points on its peripheral circles in the inner cone form a natural obstructing set and the parallels for this obstructing set are naturally called either *inner parallels* or *obstructing parallels* of the bisceles arc.

Next we choose a topological disc, which contains the obstructed bisceles arc and its inner parallels, but no further critical point. There is a natural way to identify the mapping class group of this disc with an abstract braid group:

Number all traces from left to right – supposing the cone opens upwards as in the sketch above. Let  $k'$  be the number of traces parallel to the first and let  $k$  be the total number of traces. If  $\sigma_{i,j}$ ,  $1 \leq i \leq k' < j \leq k$  is the class of the half twist on the parallel supported on the  $i^{\text{th}}$  and  $j^{\text{th}}$  trace, then we put

$$\begin{aligned}\sigma_{i,j} &= \sigma_{j,k} \sigma_{i,k} \sigma_{j,k}^{-1} && \text{if } 1 \leq i < j \leq k', \\ \sigma_{i,j} &= \sigma_{1,i} \sigma_{1,j} \sigma_{1,i}^{-1} && \text{if } k' < i < j \leq k.\end{aligned}$$

Then considering the elements  $\sigma_{i,i+1}$  as the Artin generators of an abstract braid group yields the isomorphism, since it can be checked that arcs for the  $\sigma_{i,i+1}$  can be chosen in such a way that they are disjoint outside the punctures.

Under this identification the full twists on obstructing parallels are given by

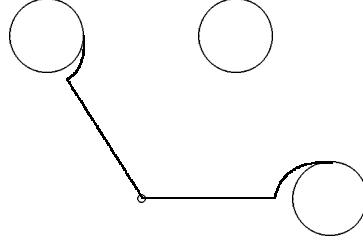
$$\sigma_{i,j}^2, \quad i \leq k' < j \leq k, (i,j) \neq (1,k).$$

We can now prove the result concerning the new kind of arcs we want to consider:

**Definition 6.32:** Any arc supported on two radial rays and two spiral segments in the  $\eta_2$ -neighbourhoods of peripheral cores is called a *coiled isosceles arc*.

Given a bisceles arc it is called the *associated coiled isosceles arc*, if both are isotopic to each other up to full twists on inner parallels.

**Example 6.33:** Naturally we imagine a coiled arc to spiral monotonously towards the peripheral cores. For  $l_2 = 6$  and  $l_2 = 4$  the given arc is a coiled isosceles arc.



**Remark 6.34:** By this definition an unobstructed bisceles arc is its own associated coiled isosceles arc.

**Lemma 6.35** *Given a bisceles arc, there is an associated coiled isosceles arc unique up to isotopy.*

*Proof:* Due to the remark above in case of unobstructed bisceles arcs there is nothing to prove, because there are no inner parallels.

Otherwise, given a bisceles arc connecting punctures of indices  $i_1 i_2, j_1 j_2$ , an associated coiled isosceles arc – if it exists – must be isotopic to an arc supported in the topological disc considered above. But up to isotopy there is a unique arc in this disc which is supported in the complement of the peripheral cores and which connects the same pair of punctures. Hence the uniqueness claim is proved.

Then we consider the half twists corresponding to the bisceles arc and the arc just considered. They are identified with  $\sigma_{1,k}$  and  $\check{\sigma}_{i,k}$  (as defined on page 148). Since by A.7 they belong to an orbit under conjugation by full twists on the inner parallels, so do the corresponding arcs and existence of an associated coiled isosceles arc is shown.  $\square$

From the simple observation that any side of a bisceles arc may only cut through either the central core or a peripheral core we can conclude – relying on some results from the appendix on geometry – that obstructing parallels are in fact bisceles arcs.

**Lemma 6.36** *If a bisceles arc is obstructed then its apex is outside the central core.*

*Proof:* Suppose the apex of a bisceles arc is inside the central core, then both its sides pass through the central core. But then they are both disjoint to the peripheral cores, so the bisceles arc is unobstructed.  $\square$

**Lemma 6.37** *Each obstructing parallel is a bisceles arc.*

*Proof:* Since an obstructing parallel is supported on the lines given by a pair of non parallel sides, a parallel is not a bisceles arc if and only if the intersection point fails to belong to both sides.

Let us assume now that an obstructing parallel to the bisceles arc is not a bisceles arc itself. Hence by the previous considerations there must be a source point in the closed cone defined by the bisceles arc which does not contain its source points. With lemma 6.86 we conclude that the apex of the bisceles arc is contained inside the circle through the three source points under consideration, that is in the central core. This is a contradiction by lemma 6.36.  $\square$

We prefer to rephrase lemma 6.35 using lemma 6.37.

**Proposition 6.38** *The class of a bisceles arc and the associated coiled isosceles arc belong to the same orbit for the action of full twists on bisceles arcs which are inner parallels.*

For the closing remark we come back to the topic of straight isosceles.

**Remark 6.39:** A straight isosceles arc only occurs for  $j_1 - i_1 = l_1/2$  and by a short check we see, that the corresponding traces are directing in opposite ways. So they come close only if they pass the central core. Immediately we deduce, that a straight isosceles arc is isotopic to its associated coiled isosceles arc.

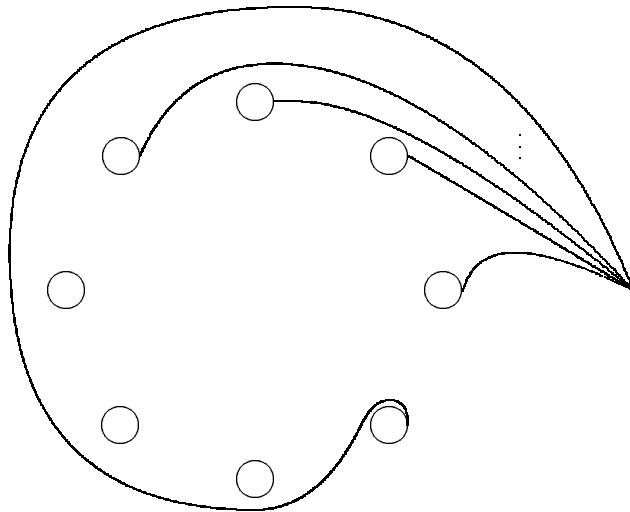
## 6.4 from coiled isosceles arcs to coiled twists

The aim of this section is to identify the isotopy class of the transported arc at  $\lambda = 1$  in terms of the Hefez Lazzeri system of paths. In fact this system yields a well-defined identification of the mapping class group of the corresponding fibre with the abstract braid group, so we finally can even identify the twists on transported arcs with abstract braids.

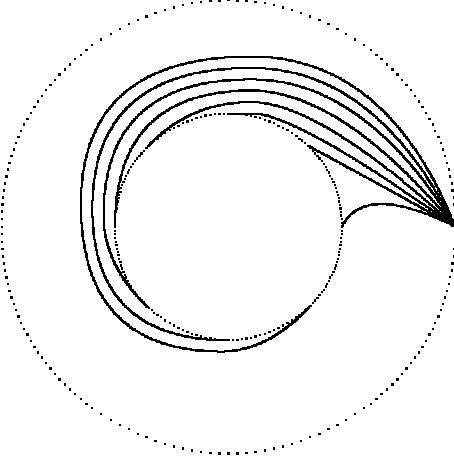
We will see that a coiled isosceles arc transported along a circular segment at radius  $t = 1$  is a coiled isosceles arc again, so we have to introduce notations and definitions in such a way, that we get hold of those geometric properties which eventually determine the braid associated to a coiled isosceles arc.

The fibre at  $\lambda = 1$ , equal to the fibre at  $\alpha = 1$ , is a punctured disc for which Hefez and Lazzeri have given a strongly distinguished system of paths, of which we should remind ourselves, 3.2.

If paths  $\omega_{i_1 i_2}$ ,  $1 \leq i_1 \leq l_1, 1 \leq i_2 \leq l_2$ , ordered lexicographically, form the Hefez Lazzeri system, then up to isotopy they can be obtained from two figures like the following in case  $l_1 = l_2 = 8$ .



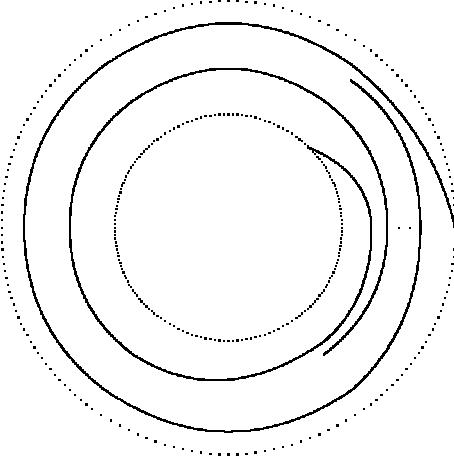
In the first figure a path has to be selected according to  $i_1$ . It terminates at a disc (an  $\eta_2$ -neighbourhood of a peripheral core), which should be replaced by the second figure.



The path selected in the second figure according to  $i_2$  can be joint to the first, so they represent the isotopy class of  $\omega_{i_1 i_2}$ .

**Notation 6.40:** Denote by  $\omega^{(i_1)}$  the positive loop around all  $\omega_{i_1 i_2}$ ,  $1 \leq i_2 \leq l_2$ .

Instead of a path segment as given in the second figure we may also join a path which spirals around the core  $n$  full times and then down to a puncture



Such a path is naturally selected by an index  $i'_2 = i_2 + nl_2$  and the notation  $\omega_{i_1 i_2}$  is naturally extended to indices  $i_1 i_2$  with  $i_2$  an arbitrary integer.

**Remark 6.41:** For  $i_1 \neq j_1$  the paths  $\omega_{i_1 i_2}, \omega_{j_1 j_2}$  do not intersect, whatever the integers  $i_2, j_2$  are.

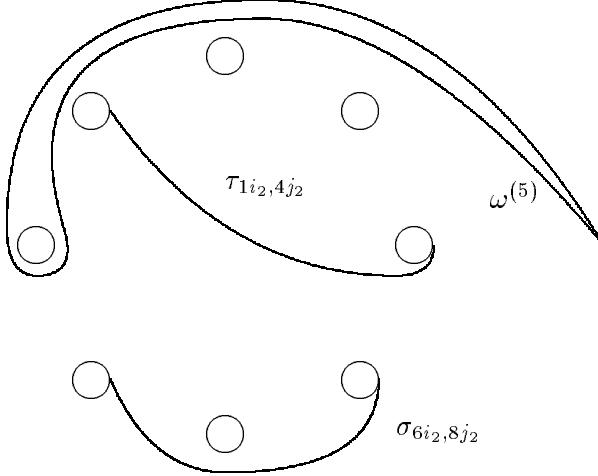
Moreover the loops  $\omega^{(k_1)}$  can be chosen disjoint from both.

Now we can introduce twist braids corresponding to arcs which are determined by suitable joins of paths.

**Notation 6.42:**  $\sigma_{i_1 i_2, j_1 j_2}$  is the  $\frac{1}{2}$ -twist on the union of  $\omega_{i_1 i_2}$  with  $\omega_{j_1 j_2}$ .

**Notation 6.43:**  $\tau_{i_1 i_2, j_1 j_2}$  is the  $\frac{1}{2}$ -twist on the union of  $\omega_{i_1 i_2}$  and  $\omega_{j_1 j_2}$  with the  $\omega^{(k_1)}$ ,  $i_1 \leq k_1 < j_1$  in between.

**Example 6.44:**



So far we have dwelled on the topology of the fibre at  $\lambda = 1$ . Now we extract the characteristic properties of the coiled isosceles.

**Definition 6.45:** The *winding angle* of a directed arc  $\Gamma$  in the plane of complex numbers with respect to a disjoint point  $z_0$  is – in generalization of the winding number of a closed curve – defined by

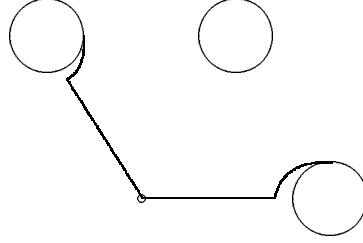
$$\vartheta_\Gamma := \int_\Gamma \frac{-idz}{z - z_0}, \quad (i^2 = -1).$$

**Notation 6.46:** We introduce notation for some characteristic angles:

- i)  $\vartheta_1, \vartheta_2$ , the angles between consecutive  $l_1^{th}$ , resp.  $l_2^{th}$  roots of unity,
- ii)  $\vartheta^\circ := (j_1 - i_1)\vartheta_1$ , the angle at the apex of the coiled isosceles arc with index pair  $i_1 i_2, j_1 j_2$ , note that  $0 < \vartheta^\circ < 2\pi$ ,
- iii)  $\vartheta_i, \vartheta_j$ , the winding angle of the i-side, resp. j-side starting at the apex, with respect to the center of the core of the corresponding peripheral circle,
- iv)  $\vartheta_j^\circ := \vartheta_j + \vartheta^\circ$ , a useful shorthand.

The winding angle of a spiral is positive if it turns positively when approaching the peripheral core.

**Example 6.47:** In the example considered before, suppose the horizontal line supports the i-side then  $\vartheta_i = -\frac{\pi}{2}, \vartheta_j = \frac{\pi}{3}$ , otherwise  $\vartheta_i = \frac{\pi}{3}, \vartheta_j = \frac{\pi}{6}$ .



We want now to pin down some geometric properties shared by the coiled isosceles arcs associated to bisceles arcs or straight isosceles arcs.

**Lemma 6.48** *The winding angle of a side of a coiled isosceles arc is in the open interval  $] -\frac{3\pi}{2}, \frac{3\pi}{2}[$ .*

*Proof:* The side of the bisceles arc is parallel to a side of the associated coiled isosceles arc. If the endpoint is on the half of the peripheral circle facing the origin, then the side is unobstructed and the winding angle is therefore in the range  $[-\frac{\pi}{2}, \frac{\pi}{2}]$ .

Otherwise it may be obstructed and there are two ways to make it unobstructed depending on the other side. But in any case the absolute value of the winding angle does not exceed  $\frac{3\pi}{2}$ .  $\square$

**Lemma 6.49** *The following inclusions hold:*

$$\begin{aligned} \text{if } \vartheta^o \leq \pi : \quad & \vartheta_i \in ] -\frac{3\pi}{2}, \frac{\pi}{2}], \\ & \vartheta_j \in [ -\frac{\pi}{2}, \frac{3\pi}{2}[; \\ \text{if } \vartheta^o > \pi : \quad & \vartheta_i \in [ -\frac{\pi}{2}, \frac{3\pi}{2}[, \\ & \vartheta_j \in ] -\frac{3\pi}{2}, \frac{\pi}{2}[. \end{aligned}$$

*Proof:* If the endpoint of the  $i$ -side is on the half circle facing the origin, then its winding angle is in  $[-\frac{\pi}{2}, \frac{\pi}{2}]$ .

If the endpoint is on the opposite half circle, then the winding angle is in either  $] -\frac{3\pi}{2}, -\frac{\pi}{2}[$  or  $]\frac{\pi}{2}, \frac{3\pi}{2}[$  and the sign depends on the second endpoint. The sign is that of  $\pi - \vartheta^o$  for the  $i$ -side and the opposite for the  $j$ -side.  $\square$

Moreover the considerations of this proof immediately yield the observation:

**Lemma 6.50** *The  $j$ -side of the bisceles arc is disjoint from the central core if and only if*

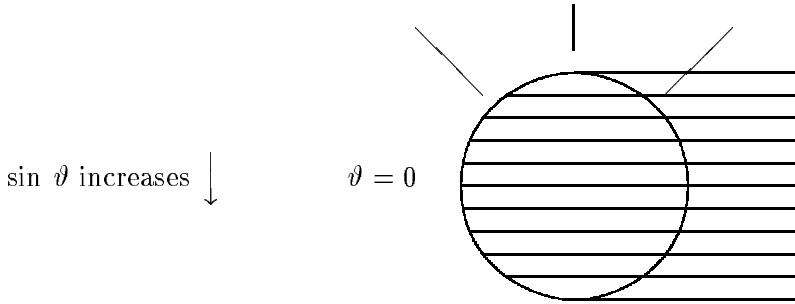
$$\begin{aligned} \text{either } \quad & \vartheta^o \leq \pi \quad \text{and} \quad \vartheta_j \in ] \frac{\pi}{2}, \frac{3\pi}{2}[, \\ \text{or } \quad & \vartheta^o > \pi \quad \text{and} \quad \vartheta_j \in ] -\frac{3\pi}{2}, -\frac{\pi}{2}[. \end{aligned}$$

The next thing we have to exploit is the fact that at a parameter  $\lambda = e_\vartheta$  bisceles do not exists for all index pairs.

**Lemma 6.51** *A bisceles arc with index pair  $i_1i_2, j_1j_2$  exists at  $\lambda = e_\vartheta$  only if*

- i)  $i_2 = j_2$ ,
- ii)  $\vartheta^o \leq \pi$  and  $\sin \vartheta_i \leq \sin \vartheta_j^o$ ,  $\vartheta_j^o < \frac{3\pi}{2}$ ,
- or
- iii)  $\vartheta^o > \pi$  and  $\sin \vartheta_i \geq \sin \vartheta_j^o$ ,  $\vartheta_j^o > \frac{\pi}{2}$ .

*Proof:* In the first case the claim is obvious since the traces have their source points in common. So from now on we assume that the source points are distinct. Let us consider the case  $\vartheta^o \leq \pi$  next. Then the possible traces for the index  $i_1i_2$  are sketched near to the central core as well as the direction of possible traces with index  $j_1j_2$ . The second inequality is now read off easily, since  $\sin \vartheta_i$  is the vertical component of the  $i$ -side and  $\sin \vartheta_j^o$  the maximal vertical component of the  $j$ -side.



Suppose now  $\vartheta_j^o$  exceeds  $\frac{3\pi}{2}$ , then  $\vartheta_j$  exceeds  $\frac{\pi}{2}$  and by 6.50 the  $j$ -side does not pass the central core. To be cut properly by the trace of the  $i$ -side its source point must be on the right hand half of the circle. But the horizontal component of  $\vartheta_j^o$  is  $\cos \vartheta_j^o$  which is not positive for  $\vartheta_j^o \in [\frac{3\pi}{2}, \vartheta^o + \frac{3\pi}{2}]$ .

The final case  $\vartheta^o > \pi$  can be handled in strict analogy.  $\square$

Since these better bounds hold obviously in the case of straight isosceles we get an improvement on the assertion of 6.49:

**Lemma 6.52** *The following inclusions hold:*

$$\begin{aligned} \text{if } \vartheta^o \leq \pi : \quad & \vartheta_i \in \left] -\frac{3\pi}{2}, \frac{\pi}{2} \right], \\ & \vartheta_j \in \left[ -\frac{\pi}{2}, \frac{3\pi}{2} - \vartheta^o \right]; \\ \text{if } \vartheta^o > \pi : \quad & \vartheta_i \in \left[ -\frac{\pi}{2}, \frac{3\pi}{2} \right[, \\ & \vartheta_j \in \left] \frac{\pi}{2} - \vartheta^o, \frac{\pi}{2} \right]. \end{aligned}$$

Now we combine the results to obtain a relation between the winding angles.

**Lemma 6.53** *The winding angles are subject to*

$$\vartheta_i \leq \vartheta_j^o \leq \vartheta_i + 2\pi.$$

*Proof:* Suppose  $\vartheta^o > \pi$ . Then by lemma 6.52  $\vartheta_j^o \in ]\frac{\pi}{2}, \frac{\pi}{2} + \vartheta^o]$  so

$$i) \quad \vartheta_i \leq \vartheta_j^o \quad \text{or} \quad ii) \quad \vartheta_i, \vartheta_j^o \in ]\frac{\pi}{2}, \frac{3\pi}{2}[.$$

Also the conditions in the latter case imply  $\vartheta_i \leq \vartheta_j^o$ , since  $\sin \vartheta_i \geq \sin \vartheta_j^o$  by lemma 6.51 and the sine function is decreasing in  $] \frac{\pi}{2}, \frac{3\pi}{2}[$ .

On the other hand by lemma 6.52

$$i) \quad \vartheta_i + 2\pi \geq \vartheta_j^o \quad \text{or} \quad ii) \quad \vartheta_i + 2\pi, \vartheta_j^o \in ]\frac{3\pi}{2}, \frac{5\pi}{2}[.$$

and again the second case is a subcase of the first, since the sine function is increasing on  $] \frac{3\pi}{2}, \frac{5\pi}{2}[$ .

The case  $\vartheta^o \leq \pi$  is done analogously.  $\square$

Next we investigate the impact of parallel transport.

**Lemma 6.54** *Under parallel transport along a circle segment of winding angle  $\vartheta$  at radius  $t = 1$  a coiled isosceles arc is mapped up to isotopy to a coiled isosceles arc with winding angles changed by  $-\vartheta$ .*

*Proof:* The line segments of the isosceles arc belong to the part which is rotated rigidly by the flow of the vector field in 6.18. The total rotation is of angle  $\vartheta$ . On the other hand the spirals are wound resp. unwound, since the endpoints are fixed relative to their peripheral centres, while the points on the boundary of the  $2\eta_2$ -discs are relatively rotated in opposite direction, hence the amount and sign of the change in the winding angles.  $\square$

**Remark 6.55:** If we introduce  $\vartheta'_i := \vartheta_i - \vartheta$ , (similarly  $\vartheta'_j := \vartheta_j - \vartheta$ ), then  $\vartheta_i$  is the  $i$ -side winding angle of the isosceles arc transported from angular parameter  $\vartheta$  to  $\lambda = 1$ .

**Lemma 6.56** *Suppose a coiled isosceles arc is associated to a bisceles arc or to an isosceles arc with index pair  $i_1 i_2, j_1 j_2$ ,  $i_1 < j_1$ ,  $i_2 \neq j_2$ , then the full twist on any of its parallel transports to  $\lambda = 1$  along a circle segment of radius  $t = 1$  is identified with one of the abstract braid elements*

$$\tau_{i_1 i'_2, j_1 j'_2}^2, \quad 1 \leq i_1 < j_1 \leq l_1, \quad 1 \leq j'_2 - i'_2 < l_2.$$

*Proof:* The transported coiled isosceles arc at  $\lambda = 1$  can be represented in a unique way by the join of loops  $\omega^{(k_1)}$ ,  $i_1 \leq k_1 < j_1$  and two paths  $\omega_{i_1 i'_2}$ ,  $\omega_{j_1 j'_2}$  with  $i'_2, j'_2$  suitable chosen. Hence the corresponding half twist is identified with the abstract braid element  $\tau_{i_1 i'_2, j_1 j'_2}$ .

We note further that

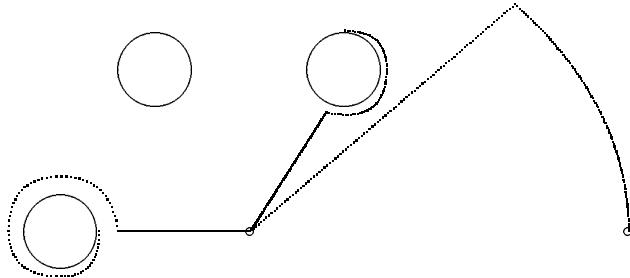
$$\begin{aligned} (j'_2 - 1)\vartheta_2 &= \vartheta'_j - \pi + (j_1 - 1)\vartheta_1, \\ (i'_2 - 1)\vartheta_2 &= \vartheta'_i - \pi + (i_1 - 1)\vartheta_1. \end{aligned}$$

Computing the difference using  $\vartheta'_j - \vartheta'_i = \vartheta_j - \vartheta_i$  and  $(j_1 - i_1)\vartheta_1 = \vartheta^o$  we get:

$$(j'_2 - i'_2)\vartheta_2 = \vartheta_j - \vartheta_i + \vartheta^o = \vartheta_j^o - \vartheta_i.$$

In case  $j'_2 - i'_2 \leq 0$  this implies  $\vartheta_j^o - \vartheta_i \leq 0$ , in case  $j'_2 - i'_2 \geq l_2$  we conclude  $2\pi \leq \vartheta_j^o - \vartheta_i$ , so both these cases contradict the assertion of lemma 6.53, since neither  $\vartheta_j^o = \vartheta_i$  nor  $\vartheta_j^o = \vartheta_i + 2\pi$  is possible under the assumption  $i_2 \neq j_2$ . Therefore we get  $1 \leq j'_2 - i'_2 < l_2$ , as claimed.  $\square$

**Example 6.57:** Suppose the example from page 77 has been transported by an angle  $\vartheta = \frac{5\pi}{3}$  along the circle arc of radius  $t = 1$ . Following the recipe above we get:



Hence assuming  $l_1 = 6, l_2 = 4$  the associated abstract braid is  $\tau_{22,45}$ .

Let us call a coiled isosceles arc in the fibre at  $\lambda = 1$  *associated* to a local v-arc, if it is obtained from the local v-arc by parallel transport along a radial segment, a transformation by full twists to get the associated coiled isosceles arc and parallel transport along a circle segment at  $t = 1$ .

We note then the following converse to 6.56:

**Lemma 6.58** *Each element  $\tau_{i_1 i_2, j_1 j_2}^2$ ,  $1 \leq i_1 < j_1 \leq l_1$ ,  $1 \leq j_2 - i_2 < l_2$ , is the full twist on a coiled isosceles arc associated to a local v-arc.*

*Proof:* There is a local v-arc for each index pair  $i_1 i_2, j_1 j_2$ ,  $1 \leq i_1 < j_1 \leq l_1$ ,  $1 \leq i_2, j_2 \leq l_2$ . For each such local v-arc there is an associated coiled isosceles arc in the fibre at  $\lambda = 1$ , which determines some  $\tau$  as above by 6.56. All others are then obtained by changing the winding angle of the circular path by suitable multiples of  $2\pi$ .  $\square$

The case  $i_2 = j_2$  requires extra care. We have analogues to lemma 6.56 and lemma 6.58.

**Lemma 6.59** *Suppose a coiled isosceles arc is associated to a bisceles arc or to an isosceles arc with index pair  $i_1 i_2, j_1 j_2$ ,  $i_1 < j_1$ ,  $i_2 = j_2$ , then the full twist on any of its parallel transports to  $\lambda = 1$  along a circle segment of radius  $t = 1$  is identified with one of the abstract braid elements*

$$\tau_{i_1 i_2, j_1 j_2}^2, 1 \leq i_1 < j_1 \leq l_1, j'_2 - i'_2 \in \{0, l_2\}.$$

*Proof:* With the same argument as in the proof of lemma 6.56 we can exclude the cases  $j'_2 - i'_2 < 0$  and  $j'_2 - i'_2 > l_2$ . Since  $i_2, i'_2$  and  $j_2, j'_2$  may only differ by multiples of  $l_2$  we are left with the two possibilities of the claim.  $\square$

**Lemma 6.60** *Given an index pair  $i_1 i_2, j_1 j_2$ ,  $1 \leq i_1 < j_1 \leq l_1$ ,  $j_2 - i_2 = l_2$ , at least one of  $\tau_{i_1 i_2, j_1 i_2}, \tau_{i_1 i_2, j_1 j_2}$  is the half twist on a coiled isosceles arc associated to a local v-arc.*

*Proof:* There is an local v-arc for the index pair  $i_1 i_2, j_1 i_2$ ,  $1 \leq i_1 < j_1 \leq l_1$ ,  $1 \leq i_2 \leq l_2$ . We know that for each such local v-arc there is an associated coiled isosceles arc in the fibre at  $\lambda = 1$ , which determines one of  $\tau_{i_1 i_2', j_1 i_2'}, \tau_{i_1 i_2', j_1 j_2'}$  as above by 6.59. All others are then obtained by changing the winding angle of the circular path by suitable multiples of  $2\pi$ .  $\square$

We close this section by identifying the twists unambiguously under special geometric assumptions.

**Lemma 6.61** *Given any coiled isosceles arc with punctures of indices  $i_1 i_2, j_1 j_2$ ,  $i_1 < j_1$ ,  $i_2 = j_2$ , facing the origin, then the twist on any of its parallel transports to  $\lambda = 1$  along a circle segment of radius  $t = 1$  is identified with one of the abstract braid elements*

$$\tau_{i_1 i_2', j_1 j_2'}, 1 \leq i_1 < j_1 \leq l_1,$$

with  $i_2' = j_2'$  if  $\vartheta^o \leq \pi$  and  $i_2' + l_2 = j_2'$  if  $\vartheta^o > \pi$ .

*Proof:* We run through the same consideration as in 6.56. But in the final step we are stuck since now  $\vartheta_i = \vartheta_j^o \bmod 2\pi$ , so  $\vartheta_i = \vartheta_j^o$  and  $\vartheta_i = \vartheta_j^o - 2\pi$  are possible by 6.53. Since both punctures face the origin by hypothesis,

$$\vartheta_i \in [-\frac{\pi}{2}, \frac{\pi}{2}], \vartheta_j \in [-\frac{\pi}{2}, \frac{\pi}{2}].$$

In case  $\vartheta_i = \vartheta_j^o$  which corresponds to  $i_2' = j_2'$  we must have  $\vartheta^o = \vartheta_i - \vartheta_j \leq \pi$ . In case  $\vartheta_i + 2\pi = \vartheta_j^o$  corresponding to  $i_2' + l_2 = j_2'$  we conclude that  $\vartheta^o = 2\pi + \vartheta_i - \vartheta_j \geq \pi$ .  $\square$

**Lemma 6.62** *Given any coiled isosceles arc associated to a bisceles arc with punctures of indices  $i_1 i_2, j_1 j_2$ ,  $i_1 < j_1$ ,  $i_2 = j_2$ , one of which exactly facing the origin, then the half twist on any of its parallel transports to  $\lambda = 1$  along a circle segment of radius  $t = 1$  is identified with one of the abstract braid elements*

$$\tau_{i_1 i_2', j_1 j_2'}, 1 \leq i_1 < j_1 \leq l_1, i_2' + l_2 = j_2',$$

under the assumption that  $\vartheta^o \leq \pi$ .

*Proof:* The claim is secured by similar considerations as in the proof of 6.61. We know that  $\vartheta_i = \vartheta_j^o$  or  $\vartheta_i = \vartheta_j^o + 2\pi$  and we imposed  $\vartheta^o \leq \pi$ .

If the puncture of index  $i_1 i_2$  faces the origin, then from 6.49 and 6.50:

$$\vartheta_i \in [-\frac{\pi}{2}, \frac{\pi}{2}], \vartheta_j \in [\frac{\pi}{2}, \frac{3\pi}{2}[.$$

If the puncture of index  $i_1 i_2$  does not face the origin, then we get:

$$\vartheta_i \in ]-\frac{3\pi}{2}, -\frac{\pi}{2}[, \vartheta_j \in [-\frac{\pi}{2}, \frac{\pi}{2}].$$

In either case we can check that we are left with the possibility  $\vartheta_i = \vartheta_j^o - 2\pi$ . Hence  $j_2' = i_2' + l_2$  holds in the index pair of the corresponding abstract braid.  $\square$

## 6.5 from local w-arcs to coiled twists

Having understood the parallel transport of local v-arcs sufficiently well, we can now consider the parallel transport of local w-arcs. They are only considered close to the degeneration at  $\lambda = 0$  with  $i_2 = j_2$ .

Let us first look at parallel transport along circular segments of very small radius.

**Lemma 6.63** *Under parallel transport along circle segments of radius  $\varepsilon \ll \eta_2$  local w-arcs are mapped to local w-arcs.*

*Proof:* This is immediate, for the parallel transport can be realised by the flow of the vector field in 6.19, which is rigid rotation for the support of the local w-arcs.  $\square$

Next local w-arcs are transported along a radial segment. We get then tangled w-arcs in the fibre  $e_\vartheta$ . The simpler arcs, to which we want to compare them, are called *isosceles w-arcs* and they relate to isosceles arcs as local w-arcs relate to local v-arcs.

The isosceles w-arc of index pair  $i_1 i_2, j_1 i_2$  can be best understood from the isosceles arcs of index pairs  $i_1 i_2, i_1^+ i_2$  and  $i_1^+ i_2, j_1 i_2$ , which are called its *constituents*. It is isotopic to the first constituent acted on by a positive half twist on the second. It can be chosen to be composed of four line segments, two of which are supported on the traces of the punctures  $i_1 i_2$  and  $j_1 i_2$ , while the middle pair forms a sharp wedge over the trace of the puncture  $i_1^+ i_2$ , cf. the example below.

An isosceles w-arc is called *corresponding* to a given tangled w-arc, if both connect the same pair of punctures.

The same methods as in the case of tangled v-arcs can now be employed to relate tangled and isosceles w-arcs.

**Lemma 6.64** *Up to conjugation by full twists on bisceles arcs of shorter length a tangled w-arc is isotopic to the corresponding isosceles w-arc.*

We now make an observation which will help us to be concerned mostly with isosceles w-arcs which are supported outside the peripheral circles except for an arbitrarily small neighbourhood of the puncture  $i_1^+ i_2$ . They shall be referred to as *unobstructed isosceles w-arcs*.

**Lemma 6.65** *Any local w-arc for the index pair  $i_1 i_2, j_1 i_2$  can be transported to radius  $t = 1$  along a circle segment of radius  $t = \varepsilon$  and a radial segment such that the corresponding isosceles w-arc is unobstructed, except in case of  $j_1 - i_1 = (l_1 + 1)/2$ .*

*Proof:* In the cases under consideration either  $j_1 - i_1 \leq l_1/2$  or  $j_1 - i_1^+ > l_1/2$ . We choose  $\vartheta = (i_2 - 1)\vartheta_2 - (j_1 - 1)\vartheta_1 \mp \frac{\pi}{2}$  respectively, so we get

$$|(i_2 - 1)\vartheta_2 - (k_1 - 1)\vartheta_1 - \vartheta| \geq \frac{\pi}{2} \quad \text{for } k_1 = i_1, i_1^+, j_1.$$

Therefore at  $\lambda = e_\vartheta$  the punctures with index pairs  $i_1 i_2, i_1^+ i_2, j_1 i_2$  are all situated on the halves of their peripheral circles facing the origin. Accordingly the isosceles w-arc corresponding to the transported local w-arc is unobstructed.  $\square$

In the remaining case we can only arrange that the punctures of the  $i$ -side and of the  $j$ -side face the origin.

**Lemma 6.66** *If  $j_1 - i_1 = (l_1 + 1)/2$ , then the local w-arc can be transported to radius  $t = 1$  along a circle segment of radius  $t = \varepsilon$  and a radial segment such that*

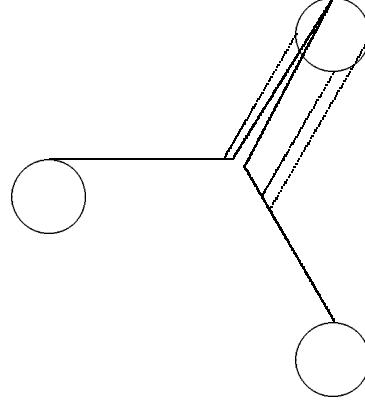
- i) *only the wedge of the corresponding isosceles w-arc is obstructed,*
- ii) *every critical point of its peripheral circle belongs to the inner cone of either of the constituents or is of index  $i_1^+ i_2$ .*

*Proof:* We choose  $\vartheta = (i_2 - 1)\vartheta_2 - (j_1 - 1)\vartheta_1 + \frac{\pi}{2}$  and get

$$\begin{aligned} |(i_2 - 1)\vartheta_2 - (k_1 - 1)\vartheta_1 - \vartheta| &\geq \frac{\pi}{2} \quad \text{for } k_1 = i_1, j_1, \\ |(i_2 - 1)\vartheta_2 - (i_1^+ - 1)\vartheta_1 - \vartheta| &< \frac{\pi}{2}. \end{aligned}$$

So the punctures of the  $i$ -side and of the  $j$ -side of the corresponding isosceles w-arc face the origin as before, but the wedge is obstructed. Since both inner angles are less than  $\pi$ , all critical points on the corresponding peripheral circle belong to an inner cone, except the puncture of index  $i_1^+ i_2$ .  $\square$

**Example 6.67:** An isosceles w-arc with obstructed wedge is obtained in case of  $l_1 = 3, l_2 = 4, i_2 = 2$ :



The final parallel transport of an isosceles w-arc along a circular segment at radius  $t = 1$  can be understood using its constituents.

**Lemma 6.68** *Parallel transport along a circle segment of radius  $t = 1$  of an unobstructed isosceles w-arc yields an arc isotopic to the parallel transport of its first constituent acted upon by a positive half twist on the parallel transport of its second constituent.*

*Proof:* The relation between an isosceles w-arc and its constituents is preserved under parallel transport.  $\square$

**Lemma 6.69** *Up to full twists on bisceles arcs of shorter length an isosceles w-arc obstructed on its wedge only is isotopic to the coiled isosceles arc associated to its first constituent acted upon by a positive half twist on the coiled isosceles arc associated to its second constituent.*

*Proof:* The same full twists on inner parallels which map the constituents to the isotopy classes of their associated coiled isosceles arcs also maps the isosceles w-arc to the isotopy class of the arc obtained from the associated coiled isosceles arcs.  $\square$

Let us finally summarize the results of this section:

**Lemma 6.70** *Local w-arcs in a fibre close to the origin and twists among the following elements with  $1 \leq i_1 < j_1 \leq l_1$ ,  $i'_2 = i_2 - l_2$ ,*

$$\begin{aligned} \tau_{i_1 i'_2, i_1^+ i'_2}^{-1} \tau_{i_1^+ i'_2, j_1 i'_2}^2 \tau_{i_1 i'_2, i_1^+ i'_2}, & \quad j_1 - i_1 \leq l_1/2, \\ \tau_{i_1 i'_2, i_1^+ i'_2}^{-1} \tau_{i_1^+ i'_2, j_1 i_2}^2 \tau_{i_1 i'_2, i_1^+ i'_2}, & \quad j_1 - i_1 \geq l_1/2 + 1, \\ \tau_{i_1 i'_2, i_1^+ i_2}^{-1} \tau_{i_1^+ i'_2, j_1 i_2}^2 \tau_{i_1 i'_2, i_1^+ i_2}, & \quad j_1 - i_1 = l_1/2 + 1/2. \end{aligned}$$

correspond in such a way that

- i) each local w-arc can be transported along a circle arc of radius  $\varepsilon$  and a radial segment to  $t = 1$ , such that the twist on the corresponding isosceles w-arc transports to  $\lambda = 1$  along the circle of radius  $t = 1$  to yield one of the given twists up to conjugation by full twists on obstructing parallels to its constituents.
- ii) each given twist can be obtained from a local w-arc as in i).

*Proof:* If  $j_1 - i_1 \leq l_1/2$  then  $(j_1 - i_1^+) \vartheta_1, (i_1^+ - i_1) \vartheta_1 \leq \pi$ , hence by 6.61 and 6.68 we can get an element of the first row, since by the braid relation it does not matter if we transform the first constituent by a positive full twist on the second or if we transform the second by a negative full twist on the first.

Similarly if  $j_1 - i_1^+ > l_1/2$  then  $(i_1^+ - i_1) \vartheta_1 \leq \pi$  but  $(j_1 - i_1^+) \vartheta_1 > \pi$ , so we get a twist of the second row, again with 6.61 and 6.68.

In the final case we argue along the same line with 6.62 and 6.69, so also in case  $j_1 - i_1 = l_1/2 + 1/2$  we get twists among the given ones.

As in the similar cases proved before, we get all twist this way as we can transport around the circle at  $t = 1$  as many times as necessary.  $\square$

## 6.6 the length of bisceles arcs

In this section we want to compare the length of bisceles arcs to a real number we assign to index pairs.

**Definition 6.71:** The modulus of a pair  $i_1 i_2, j_1 j_2$  of indices is given by

$$\eta_2 \left| \frac{\sin(\pi \frac{i_2 - j_2}{l_2})}{\sin(\pi \frac{i_1 - j_1}{l_1})} \right|.$$

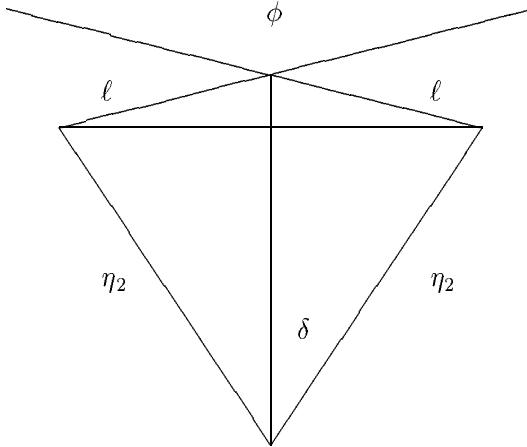
In this way a modulus is assigned to all objects with an index pair.

Since modulus is in some way complementary to length, we introduce it also for bisceles arcs.

**Definition 6.72:** The modulus of a bisceles arc is the shorter of the two distances from the apex to both source points.

**Lemma 6.73** *The modulus  $t_0$  of a critical parameter  $t_0 e_{\vartheta_0}$  for the pair  $i_1 i_2, j_1 j_2$  coincides with the modulus for that index pair.*

*Proof:* Given the traces at angle  $\vartheta_0$  the pair corresponding to  $i_1 i_2, j_1 j_2$  meet at an apex which forms an isosceles triangle with both source points on the circle of radius  $\eta_2$ . So with  $\delta = \pm \pi \frac{i_2 - j_2}{l_2}$ ,  $\phi = \pm 2\pi \frac{i_1 - j_1}{l_1}$  the length of the sides equals the modulus as can be seen from the following sketch.

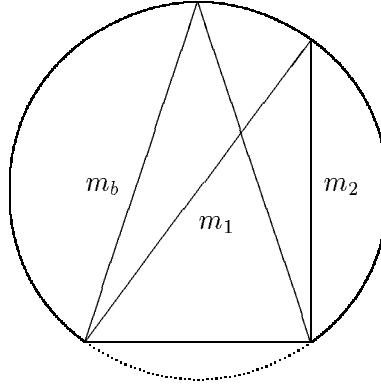


□

**Lemma 6.74** *The modulus of a bisceles arc bounds the modulus of the corresponding index pair from below. Equality holds only in the case that the bisceles arc is an isosceles arc.*

*Proof:* The apex of the bisceles arc which depends on the parameter angle  $\vartheta$  determines a triangle over the base given by the two source points. The base and the angle over it are independent of  $\vartheta$ , whereas the length of the shorter side is the bisceles arc modulus. The modulus of the pair is the length of a side if both sides are equal which happens for a specific  $\vartheta$ .

The claim is now obvious from the following sketch,  $m_b \geq m_2 = \min(m_1, m_2)$ :



The algebraic argument reads as follows: By the cosine formula

$$m_1^2 + m_2^2 - 2m_1m_2 \cos(\text{apex}) = 2m_b^2 - 2m_b^2 \cos(\text{apex}).$$

We can get a lower bound for the l.h.s. assuming w.l.o.g.  $m_1 \leq m_2$ :

$$\begin{aligned} m_1^2 + m_2^2 - 2m_1m_2 \cos(\text{apex}) &= (m_1 - m_2)^2 + m_1m_2(2 - 2 \cos(\text{apex})) \\ &\geq m_1^2(2 - 2 \cos(\text{apex})) \end{aligned}$$

Then the conclusion  $m_b \geq m_1$  is immediate.  $\square$

Now we compare the modulus of arcs we encountered in preceding sections.

**Lemma 6.75** *The modulus of a bisceles arc supported on the essential traces of a tangled v-arc is strictly larger than the modulus of the corresponding isosceles arc.*

*Proof:* This claim follows from lemma 6.74 above and the remark on page 75.  $\square$

**Lemma 6.76** *An obstructing parallel to a bisceles arc is of strictly larger modulus.*

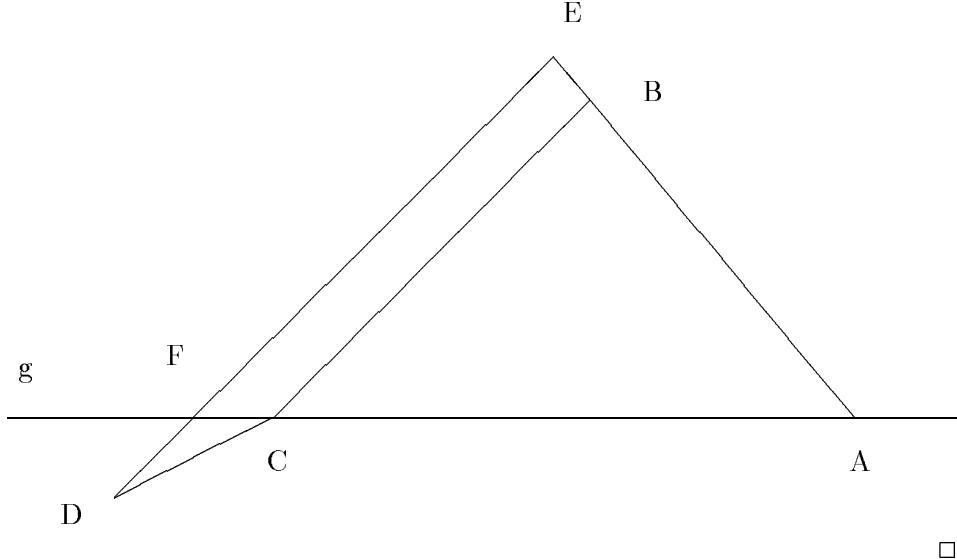
*Proof:* Let us consider first the case that the obstructing parallel has a side in common with the obstructed bisceles arc:

We have thus a triangle  $ABC$  formed by the source points  $A, C$  of the traces of obstructed bisceles arc and its apex  $B$ . Similarly we have a triangle  $AED$  formed by the source points  $A, D$  of the obstructing parallel and its apex  $E$ . We have  $g_{AE} = g_{AB}$  and  $B \in \overline{AE}$ . Moreover  $g_{DE} \parallel g_{BC}$  and  $D$  is separated from  $A$  by  $g_{BC}$ . Denote by  $F$  the intersection of  $g_{DE}$  and  $g_{AC}$ . Then  $g_{DE}$  is divided into rays bounded by  $E$  resp.  $F$  and the finite segment  $\overline{EF}$ .

Now  $D$  may not be on the ray bounded by  $E$ , since then  $B$  is in the interior of  $ACD$ , hence in the central core contrary to the assumption on obstructedness. Neither may  $D$  belong to  $\overline{EF}$  since otherwise  $\overline{BD}$  cuts  $\overline{BC}$  which is impossible since  $\overline{BC}$  is on the obstructed side of the bisceles arc and may hence not be cut by the chord  $\overline{BD}$  of the central core.

So finally  $F$  is on  $\overline{DE}$  and we conclude with lemma 6.88.

Suppose now that the obstructing parallel has no side in common with the obstructed bisceles arc, then there is an intermediate obstructing parallel which has a side in common with each. So the full result is obtained in two steps as above.



□

**Lemma 6.77** *The modulus of a bisceles arc supported on the essential traces of a tangled  $v$ -arc or a tangled  $w$ -arc and the modulus of any of its obstructing parallels is strictly larger than the modulus of the corresponding isosceles arc.*

*Proof:* Thanks to 6.75 we need only to argue for the obstructing parallels, but their modulus is strictly bounded from below by the modulus of the obstructed bisceles arc by lemma 6.76. □

**Lemma 6.78** *The full twist on a bisceles arc which is not an isosceles arc transported along  $t = 1$  to  $\lambda = 1$  is in the group generated by all twists  $\tau^2$  of modulus larger than the modulus of the bisceles arc.*

*Proof:* The obstructing parallels are bisceles arcs of strictly larger modulus. Hence we may as well assume the bisceles arc to be unobstructed. Its parallel transport is then isotopic to an arc defining some  $\tau$  of larger modulus, which is strictly larger in case the bisceles arc is no isosceles arc. □

## 6.7 the discriminant family

In this section we will work with the discriminant family of the families of function we consider. In order to compute the versal braid monodromy in the next section, we have to find the locally assigned groups. Moreover we need to compare the parallel transport in the discriminant family to parallel transport in the model family.

**Lemma 6.79** *The discriminant and the model discriminant family over the punctured parameter bases have a common unramified cover.*

*Proof:* The equation for the discriminant family has a formal factorisation

$$\prod_{\xi_1^{l_1}, \xi_2^{l_2}=1} (z - \alpha^{\frac{l_1+1}{l_1}} \xi_1 - \varepsilon_2^{\frac{l_2+1}{l_2}} \xi_2) = \prod_{\xi_1^{l_1}=1} ((z - \alpha^{\frac{l_1+1}{l_1}} \xi_1)^{l_2} - \varepsilon_2^{l_2+1}) = 0.$$

as opposed to the equation for the model discriminant family:

$$\prod_{\xi_1^{l_1}, \xi_2^{l_2}=1} (z - \lambda \xi_1 - \eta_2 \xi_2) = \prod_{\xi_1^{l_1}=1} ((z - \lambda \xi_1)^{l_2} - \eta_2^{l_2}) = 0.$$

These equations coincide for  $\eta_2^{l_2} = \varepsilon_2^{l_2+1}$  and  $\lambda^{l_1} = \alpha^{l_1+1}$ . Hence the family parameterized by  $\beta$

$$\prod_{\xi_1^{l_1}=1} ((z - \beta^{l_1+1} \xi_1)^{l_2} - \varepsilon_2^{l_2+1}) = 0.$$

is isomorphic to the pull backs of the discriminant family and the model discriminant family by the covering map  $\beta \mapsto \alpha = \beta^{l_1}$  resp.  $\beta \mapsto \lambda = \beta^{l_1+1}$ , if  $\varepsilon_2 l_1 l_2 + 1 = \eta_2^{l_2}$ .

In this way we can understand polar coordinates of the bases of the two discriminant families as different coordinates of the universal cover of the bases punctured at the origin.

So with polar coordinates  $r$  and  $\theta$  in the base of the discriminant family we can immediately compare parallel transport in the two families:

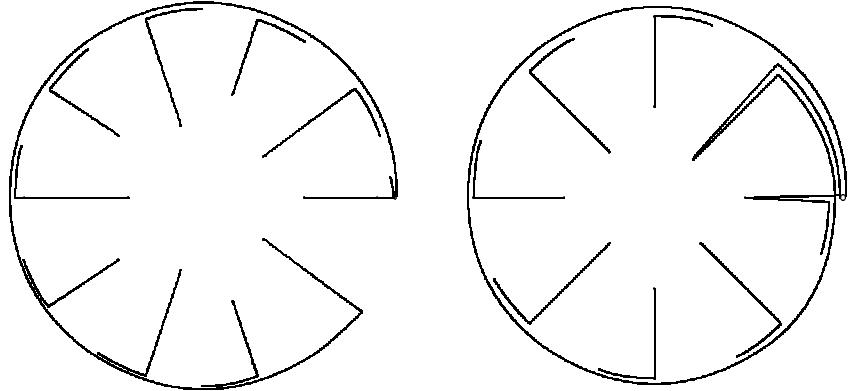
**Lemma 6.80** *Parallel transport in the discriminant family and in the model discriminant family coincides if  $r_0^{l_1+1} = t_0^{l_1}$  and  $\theta(l_1 + 1) = \vartheta l_1$*

- i) along radial paths  $te_\theta$ ,  $t \in [t_0, 1]$  and  $re_\theta$ ,  $r \in [r_0, 1]$ ,
- ii) along circular paths of radius 1 of winding angles  $\vartheta$  and  $\theta$  respectively.

We can now define *standard paths* in the bases of both families by asking them to be supported on radial segments and circular segments as in the lemma.

And for each standard path in one base we get another one in the other with the same parallel transport.

**Example 6.81:** A system of standard paths for the discriminant family associated to  $l_1 = 4, l_2 = 2$  is thus related to standard paths in the base of the model discriminant family:



To get the versal braid monodromy of the discriminant family, we therefore need to transfer the locally assigned groups from local Milnor fibres of the discriminant family to local Milnor fibres of the model discriminant family and transport them along all possible standard paths.

We assign a group to a local Milnor fibre in the model discriminant using the fact that the fibre is isomorphic to a local Milnor fibre in the discriminant family by way of the two finite covering maps.

**Lemma 6.82** *The group assigned to a Milnor fibre at a regular parameter  $t_1 e_{\vartheta_1}$ , sufficiently close to a singular parameter  $t_0 e_{\vartheta_0} \neq 0$  with  $t_1 - t_0 > 0$ , is generated by full twists on local v-arcs.*

*Proof:* The singular fibre corresponds to a function with non-degenerate critical points only, cf. the proof of the lemmas 5.10, 5.11. So by definition the locally assigned group is generated by mapping classes fixing all punctures and supported on small discs each of which is a Milnor fibre for just one multiple puncture.

By close inspection we can see that the local v-arcs are supported on such discs and the full twists on local v-arcs generate the group of all mapping classes of each disc which preserve the punctures.  $\square$

**Lemma 6.83** *The group assigned to a Milnor fibre at a regular parameter  $t_1 e_{\vartheta_1}$ , sufficiently close to a singular parameter  $\lambda = 0$ , is generated by full twists on local w-arcs and  $\frac{3}{2}$ -twists on local v-arcs with index pair  $i_1 i_2, i_1^+ i_2$ .*

*Proof:* The singular fibre corresponds to a function which has  $l_2$  critical points of type  $A_{l_1}$  with distinct critical values. So by definition the group locally assigned to each disc, which is a local Milnor fibre of a multiple puncture, is generated by the mapping classes of the braid monodromy of the singular function germ it corresponds to.

Each of the critical points of type  $A_{l_1}$  is unfolded linearly, so the local Milnor fibre can be naturally identified with the Milnor fibre encountered in lemma 4.7.

And in combination with lemma 4.6 we conclude that local generators are given by the  $\frac{3}{2}$ -twists on v-arcs with index pairs  $i_1 i_2, i_1^+ i_2$  and full twists on arcs winding positively from a puncture of index  $i_1 i_2$  to a puncture of index  $j_1 i_2, j_2 > i_1^+ = i_1 + 1$ , around all v-arcs.

By lemma A.4 we can see that instead we can use the twists of the claim to generate the same group.  $\square$

To summarize the preceding discussion we should note:

**Remark 6.84:** The versal braid monodromy of the family of functions

$$x_1^{l_1+1} - \alpha(l_1 + 1)x_1 + x_2^{l_2+1} - \varepsilon_2(l_2 + 1)x_2$$

is generated by the parallel transport of the appropriate twists as given by lemma 6.82 and lemma 6.83 along all standard paths in the model discriminant family.

## 6.8 conclusion

Finally we keep our promise and give braid elements which generate the versal braid monodromy:

**Proposition 6.85** *The versal braid monodromy the family of functions*

$$x_1^{l_1+1} - \alpha(l_1 + 1)x_1 + x_2^{l_2+1} - \varepsilon_2(l_2 + 1)x_2$$

*is generated by twists ( $i_1^+ = i_1 + 1$ ,  $i_2' = i_2 - l_2$ ):*

$$\begin{aligned} \tau_{i_1 i_2, j_1 j_2}^2, & \quad 1 \leq i_1 < j_1 \leq l_1, 1 \leq j_2 - i_2 < l_2, \\ \tau_{i_1 i_2', i_1^+ i_2}^3, & \quad 1 \leq i_1 < i_1^+ \leq l_1, 1 \leq i_2 \leq l_2, \\ \tau_{i_1 i_2', i_1^+ i_2}^{-1} \tau_{i_1^+ i_2', j_1 j_2}^2 \tau_{i_1 i_2', i_1^+ i_2}, & \quad 1 < i_1^+ < j_1 \leq l_1, 1 \leq i_2 \leq l_2. \end{aligned}$$

*Proof:* The versal braid monodromy of a one parameter family can by definition be computed from their locally assigned groups of mapping classes and the parallel transport of these groups along a distinguished system of paths in the associated discriminant family, cf. lemma 5.7

The locally assigned groups in the discriminant family were given in lemma 6.82 and lemma 6.83 to be twists on local v-arcs and local w-arcs.

So by the closing remark of the last section parallel transport of local v-arcs and local w-arcs along all possible standard paths in the base of the model discriminant family generate the versal braid monodromy.

Note that the length of the circular part is not necessarily restricted to  $[0, 2\pi[$ .

We denote by  $T$  the set of braid generators obtained by parallel transport and identification using the Hefez Lazzeri path system in the fibre at  $\lambda = 1$ .

$T$  is divided into subsets according to the index pair of the punctures connected by the corresponding arc.

The given set  $S$  of braid group elements is also divided into subsets according to the modulus the index pairs of each element, which is unambiguous since we note immediately that the modulus of all index pairs occurring in the second and third row is zero.

Since the moduli of elements in  $T$  and  $S$  form a finite descending sequence  $m_1 > \dots > m_n = 0$ , we can impose finite filtrations

$$T_k := \{\tau \in T \mid m(\tau) \geq m_k\}, \quad S_k := \{\tau \in S \mid m(\tau) \geq m_k\}.$$

To prove our claim, we are thus left to check the hypotheses of lemma A.17:

Since  $l_2 > 1$ , the maximal modulus  $m_1$  is positive. Hence  $S_1$  only contains twists on parallel transports of local v-arcs. The local v-arcs of highest modulus get not tangled when transported along a radial arc, since entangling bisceles arcs have to be of larger modulus 6.75. The isosceles thus obtained are unobstructed, since obstructing parallels would be of larger modulus, 6.76. By 6.56 each element of  $T_1$  is in  $S_1$ . Conversely by 6.58, each element in  $S_1$  is an element in  $T$  of equal modulus, hence in  $T_1$ .

Given an element in  $T$  of modulus  $m_k > 0$ , which is the parallel transport of an local v-arc, then there is an element in  $S$  obtained from the same local v-arc transported along the same path, but conjugated by twists which are the parallel transports of bisceles arcs of strictly larger modulus, 6.75, 6.77, 6.76. So the second hypothesis of lemma A.17 holds for elements in  $T_k - T_{k-1}$  of positive modulus.

Conversely each full twist in  $S$  of positive modulus is obtained by parallel transport from an local v-arc of equal modulus up to twists by entangled and obstructing bisceles arcs, 6.58. So due to 6.75, 6.77, 6.76 again the third hypothesis holds for the twists obtained from local v-arcs of positive modulus.

We are left with elements of modulus  $m_n = 0$  and consider local w-arcs first. Though we have to transport along a standard path to get full twist elements in  $T_n - T_{n-1}$ , we have to rely on a result which makes use of a different kind of paths. We recall that in section 6.5 we transported a local w-arc along a circular arc of small radius, then along a radial segment and finally along a circular segment of radius  $t = 1$ .

Each standard path can be coupled with a path of the second kind in such a way, that the closed path obtained as their join has winding number zero with respect to the origin. Hence parallel transport from  $\lambda = 1$  along this closed path amounts to conjugation by a composition of full twists of positive modulus in  $T_{n-1}$ .

We deduce that the elements of  $T$  obtained by parallel transport along standard paths yield the elements given in lemma 6.70 up to conjugation by full twists in  $T_{n-1}$  and  $S_{n-1}$ . By lemma A.9 they are even conjugate to the full twists in  $S_n - S_{n-1}$ .

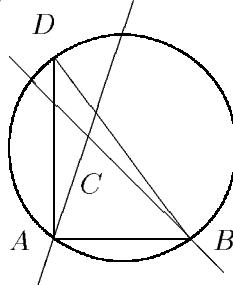
This relation can obviously be reversed in the sense, that for each full twist in  $S_n$  of modulus zero we have an element in  $T$  equal up to conjugation by elements in  $S_{n-1}$  and  $T_{n-1}$ .

Finally we have to address the  $\frac{3}{2}$ -twists in  $T$  and  $S$ . A  $\frac{3}{2}$ -twist in  $T$  is obtained by the parallel transport of a local v-arc with index pair  $i_1 i_2, i_1^+ i_2$ . We conclude that up to full twists elements in  $S_{n-1}$  and  $T_{n-1}$  the  $\frac{3}{2}$ -twists in  $T$  are among the elements given in lemma 6.59. By lemma A.8 they are among the elements in  $S$  even up to full twists in  $S_{n-1}$  and  $T_{n-1}$ .

Conversely the pairs of twists considered in lemma 6.60 correspond bijectively to the  $\frac{3}{2}$ -twists of  $S$  and are both equal up to conjugation by full twists of positive modulus by lemma A.8. We deduce that also each  $\frac{3}{2}$ -twist of  $S$  is an element of  $T$  up to conjugation by full twists in  $S_{n-1}$  and  $T_{n-1}$ .  $\square$

## 6.9 appendix on plane elementary geometry

**Lemma 6.86** *Given a proper triangle  $ABC$  and lines through  $A, C$  resp.  $B, C$ . If a fourth point  $D$  is in the opposite cone to  $\overline{AB}$  at  $C$ , then  $C$  is a point in the disc with boundary through  $ABD$ .*



*Proof:* Denote the line through a pair of points by  $g$  indexed with the pair. Then the assumptions can be stated as:

- i)  $C, D$  are on the same side of the line  $g_{AB}$ ,
- ii)  $A, D$  are on different sides of  $g_{BC}$ ,
- iii)  $B, D$  are on different sides of  $g_{AC}$ .

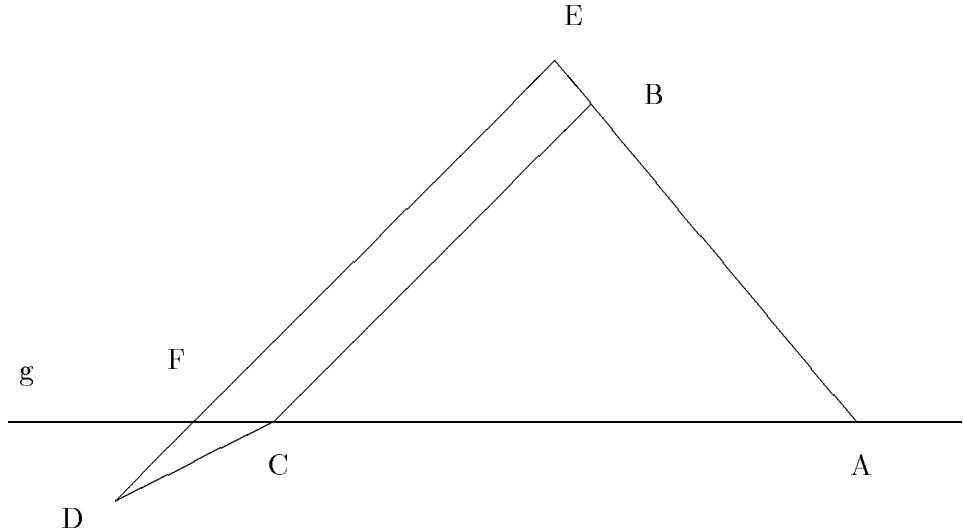
These imply

- iv)  $A, C$  are on the same side of  $g_{BD}$  by i) and ii),
- v)  $B, C$  are on the same side of  $g_{AD}$  by i) and iii).

But i), iv) and v) together form the assertion of the lemma.  $\square$

**Lemma 6.87** *Suppose in a quadrilateral  $ABCD$  the points  $B, D$  are on opposite sides of the diagonal  $d$  through  $A$  and  $C$ . Let  $E$  be the intersection of the line through  $A$  and  $B$  and the parallel to  $BC$  through  $D$ . If  $|AB| < |BC|$  then  $|AE| < |DE|$ .*

*Proof:* By hypothesis there is an intersection point  $F$  on  $\overline{DE}$  with the diagonal  $d$ , so  $|EF| < |DE|$ . Hence the claim follows since by proportionality  $|AB| < |BC| \Rightarrow |AE| < |EF|$ .



□

**Lemma 6.88** Suppose in a quadrilateral  $ABCD$  the points  $B, D$  are on opposite sides of the diagonal  $d_{AC}$  through  $A$  and  $C$  and the points  $A, D$  are on opposite sides of the line through  $B$  and  $C$ . If  $E$  is the intersection of the line through  $A$  and  $B$  and the parallel to  $BC$  through  $D$  then

$$\min(|DE|, |AE|) > \min(|AB|, |BC|).$$

*Proof:* By hypothesis  $E$  and  $A$  are on opposite sides of the line through  $B$  and  $C$ , hence  $|AB| < |AE|$  and by proportionality  $|EF| > |BC|$  which implies  $|DE| > |BC|$ .

So with the result of lemma 6.87 we get in case  $|AB| = \min(|AB|, |BC|)$  that  $|DE| > |AE| > |AB|$ . In case  $|BC| = \min(|AB|, |BC|)$  we get the claim since  $|AE| > |AB| \geq |BC|$  and  $|DE| > |BC|$ . □

## Chapter 7

# braid monodromy induction to higher dimension

In the previous chapter we determined the braid monodromy of a one parameter family of polynomials on the plane. Due to the results of chapter 5 this yields the main contribution to the braid monodromy of a plane Brieskorn Pham polynomial.

In the present chapter we exploit the results of both chapters to set up an induction for the computation of the braid monodromy for Brieskorn Pham polynomials in arbitrary dimensions. Since we have to relate the monodromies of families of different numbers of variables and deal with an arbitrary number in the general case, we made some efforts to chose our notation. We devote the first section to introduce this notations, quite a few new definitions and to rephrase results of the preceding chapters in a unified way.

In the second and third section the induction argument is given by way of considering the versal braid monodromies of families of functions

$$\begin{aligned} f_\alpha & : x_1^{l_1+1} - \alpha(l_1 + 1)x_1 + \sum_{i=2}^n \left( x_i^{l_i+1} - \varepsilon_i(l_i + 1)x_i \right), \\ g_\alpha & : x_1^{l_1+1} - (l_1 + 1)x_1 + \sum_{i=2}^n \left( x_i^{l_i+1} - \alpha\varepsilon_i(l_i + 1)x_i \right), \end{aligned}$$

which were introduced in chapter 5.

To merge the various groups we have to choose the generating sets in many different ways. Even though many of the braid computations have been put into an appendix, the computational load is quite high.

The actual geometric argument which makes induction possible is presented in the fourth and fifth section. We present a prove of a result which connects the braid monodromy of a discriminant family to a family which is obtained by replacing the divisor by a number of parallel copies. This  $l$ -companion family is studied to the details to relate the associated versal monodromies.

Both sections should be regarded as a sort of appendix to which we may resort on need.

## 7.1 preliminaries

The choice of a Hefez Lazzeri base in the reference fibre provides a natural bijection of the punctures with a multiindex set associated appropriately to the given Brieskorn Pham polynomial.

**Notation 7.1:** Given a finite sequence  $l_1, \dots, l_n$  of positive integers, define the multiindex set  $I_n = I_n(l_1, \dots, l_n)$  to be

$$I_n := \{i_1 \dots i_n \mid 1 \leq i_\nu \leq l_\nu, 1 \leq \nu \leq n\}$$

equipped with the natural lexicographical order.

While  $i$  denotes an element  $i_1 \dots i_n$ , we will use  $i^+$  for its immediate successor and  $i' := i_1 \dots i_{n-1}$  for the naturally associated element in  $I_n(l_1, \dots, l_{n-1})$ .

Whether the following property is given or not determines to some extend the role of index pairs and index triples.

**Definition 7.2:** Multiindices  $i, j \in I_n$  are called *correlated* or a *correlated pair*, if  $i < j$  and  $j_\nu \in \{i_\nu, i_\nu + 1\}, 1 \leq \nu \leq n$ .

Multiindices  $i, j, k \in I_n$  are called *correlated* or a *correlated triple*, if  $i < j, i < k, j < k$  are correlated.

A quadruple of indices is called *correlated* if each pair is correlated.

For the induction we need several homomorphisms between braid groups of different numbers of strings.

**Definition 7.3:** Given a multiindex set  $\{i_1 \dots i_n \mid 1 \leq i_\nu \leq l_\nu, 1 \leq \nu \leq n\}$  the *primary homomorphisms* are defined for  $1 \leq i_1 \leq l_1$ :

$$\begin{aligned} \phi_{i_1} : \quad \text{Br}_{l_2 \dots l_n} &\longrightarrow \text{Br}_{l_1 \dots l_n} \\ \sigma_{i_2 \dots i_n, j_2 \dots j_n} &\mapsto \sigma_{i_1 i_2 \dots i_n, i_1 j_2 \dots j_n} \end{aligned}$$

the *secondary homomorphisms* are defined for  $1 \leq i_n \leq l_n$ :

$$\begin{aligned} \psi_{i_n} : \quad \text{Br}_{l_1 \dots l_{n-1}} &\longrightarrow \text{Br}_{l_1 \dots l_n} \\ \sigma_{i_1 \dots i_{n-1}, j_1 \dots j_{n-1}} &\mapsto \sigma_{i_1 \dots i_n, j_1 \dots j_{n-1} i_n} \end{aligned}$$

the  $l_n$ -band homomorphism

$$\eta_l : \quad \text{Br}_{l_1 \dots l_{n-1}} \longrightarrow \text{Br}_{l_1 \dots l_n}$$

which assigns to a braid a braid of  $l_n$ -times as many strand by replacing each strand with a ribbon of  $l_n$  strands.

This definition should be understood from the following picture: To a twist, i.e. an exchange of two strands, we associate a band or ribbon twist, i.e. an exchange of two bands or ribbons into which  $l$  strands each have been assembled:

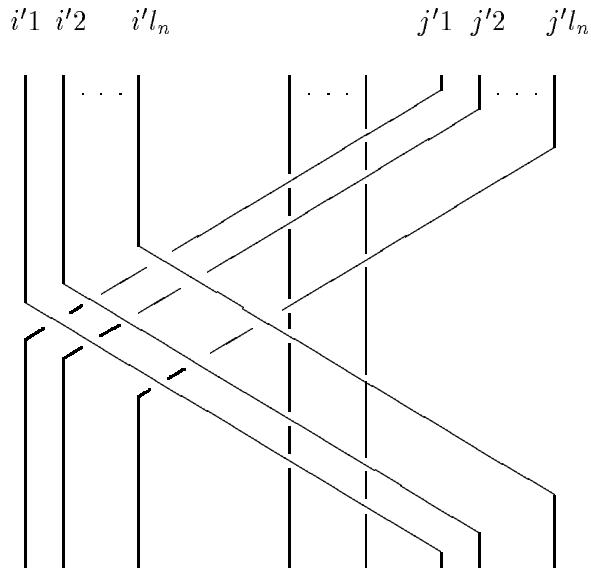


Figure 7.1: band twist, the image of a half twist under  $\eta_{l_n}$

**Definition 7.4:** The *level* of an index pair is the difference of the leading components.

The *level* of a braid  $\sigma_{i,k}$  or  $\tau_{i,k}$  is the level of its index pair.

**Definition 7.5:** The element  $\delta_n := \sigma_{1,2} \cdots \sigma_{n-1,n}$  is called the *fundamental element* of the BKL presentation, cf. [6] of the braid group  $\text{Br}_n$ ,  $\delta^n$  is the full twist on the disc, i.e. a generator of the center of  $\text{Br}_n$ .

**Definition 7.6:** Given a pair of indices  $i, j$  the associated *subcable twist* is defined as the BKL fundamental word on the braid subgroup on the punctures with index  $k, i \leq k \leq j$ , it can be given as

$$\delta_{i,j} := \prod_{i \leq k < j} \sigma_{k,k+1}.$$

**Definition 7.7:** The  $l_n$ -cable twist in  $\text{Br}_{l_1 \dots l_n}$  is defined to be the element

$$\delta_{\phi,n} := \prod_{i' \in I_{n-1}} \delta_{i'1,i'l_n}.$$

In the disguise of turning the peripheral circles we already considered this cable twist, so in case  $n = 2$  conjugation by the cable twist  $\delta_{j_1 1, j_1 l_2}$  yields maps

$$\begin{aligned} \tau_{i_1 i_2, j_1 j_2} &\mapsto \tau_{i_1 i_2, j_1 j_2^+}, \\ \tau_{j_1 j_2, k_1 k_2} &\mapsto \tau_{j_1 j_2^+, k_1 k_2}, \\ \sigma_{i_1 i_2, j_1 j_2} &\mapsto \sigma_{i_1 i_2, j_1 j_2^+}, \\ \sigma_{j_1 j_2, k_1 k_2} &\mapsto \sigma_{j_1 j_2^+, k_1 k_2}, \end{aligned}$$

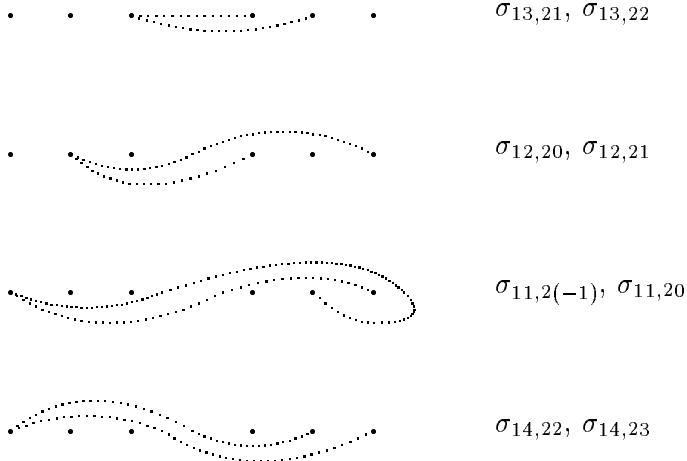
where as usual we assume  $i < j < k$ .

We extend the range of possible indices in higher dimensions accordingly.

**Definition 7.8:** Suppose  $i'_n - i_n =: m_i, j'_n - j_n =: m_j$  then define

$$\sigma_{i'i'_n, j'j'_n} := \delta_{j'1, j'l_n}^{-m_j} \delta_{i'1, i'l_n}^{-m_i} \sigma_{i_1 i_2, j_1 j_2} \delta_{i'1, i'l_n}^{m_i} \delta_{j'1, j'l_n}^{m_j}.$$

**Example 7.9:** In case  $n = 2, i_1 = 1, j_1 = 2$  there are the following examples:



**Lemma 7.10** Conjugation by the cable twist  $\delta_\phi$  induces a level preserving bijection of braids  $\tau_{i_1 i_2, j_1 j_2}^2$  with  $1 \leq i_1 < j_1 \leq l_1, 1 \leq j_2 - i_2 < l_2$  and of braids  $\sigma_{i_1 i_2, j_1 j_2}^2$  with  $1 \leq i_1 < j_1 \leq l_1, 1 \leq j_2 - i_2 < l_2$  such that:

$$\begin{aligned} \tau_{i_1 i_2, j_1 j_2}^2 &\mapsto \tau_{i_1 i_2^+, j_1 j_2^+}^2, \\ \sigma_{i_1 i_2, j_1 j_2}^2 &\mapsto \sigma_{i_1 i_2^+, j_1 j_2^+}^2 \end{aligned}$$

**Notation 7.11:** By  $A(l_1)$  denote the singular function  $x^{l_1+1}$ . By  $BP(l_1, \dots, l_n)$  denote the singular function  $x_1^{l_1+1} + \dots + x_n^{l_n+1}$ .

With respect to the Hefez Lazzeri base these functions determine by braid monodromy well defined subgroups of the braid group  $\text{Br}_\mu$ , which are denoted by

$$\text{Br}_{A(l_1)} \quad \text{resp.} \quad \text{Br}_{BP(l_1, \dots, l_n)}.$$

### braid monodromy results

We have to review our main results for braid monodromy of Brieskorn Pham polynomials up to now. The first records a slight reformulation of theorem 6.85. The second is the previously noted result for singularities of type  $A_l$ .

We remark once and for all that the constants  $\varepsilon_1 = 1, \varepsilon_i > 0$  are chosen in such a way that  $\varepsilon_{i+1} \ll \varepsilon_i$ .

**Lemma 7.12** *The versal braid monodromy of the family*

$$x_1^{l_1+1} - \alpha(l_1 + 1)x_1 + x_2^{l_2+1} - \varepsilon_2(l_2 + 1)x_2$$

*is generated by twists*

$$\begin{aligned} \tau_{i_1 i_2, j_1 j_2}^2, & \quad 1 \leq i_1 < j_1 \leq l_1, 1 \leq j_2 - i_2 < l_2, \\ \sigma_{i_1 i_2, i_1^+ i_2}^3, & \quad 1 \leq i_1 < i_1^+ \leq l_1, 1 \leq i_2 \leq l_2, \\ \sigma_{i_1 i_2, j_1 i_2}^2, & \quad 1 < i_1^+ < j_1 \leq l_1, 1 \leq i_2 \leq l_2. \end{aligned}$$

*Proof:* By definition  $\tau_{i_1 i_2, i_1^+ i_2}^3 = \sigma_{i_1 i_2, i_1^+ i_2}^3$ ,  $1 \leq i_1 < i_1^+ \leq l_1, 1 \leq i_2 \leq l_2$ .

We then notice that  $\psi_{i_2}(\text{Br}(A_{l_1}))$  is generated by

$$\sigma_{i_1 i_2, i_1^+ i_2}^3, \sigma_{i_1 i_2, j_1 i_2}^2, \quad 1 \leq i_1 < i_1^+ < j_1 \leq l_1.$$

By lemma A.4 then  $\psi_{i_2}(\text{Br}(A_{l_1}))$  is also generated by

$$\sigma_{i_1 i_2, i_1^+ i_2}^3, \psi_{i_2}(\sigma_{i, i^+}^{-2} \check{\sigma}_{i_1, j_1}^2 \sigma_{i, i^+}^2), \quad 1 \leq i_1 < i_1^+ < j_1 \leq l_1.$$

Since the latter elements coincide with  $\tau_{i_1 i_2, i_1^+ i_2}^{-1} \tau_{i_1^+ i_2, j_1 i_2}^2 \tau_{i_1 i_2, i_1^+ i_2}^2$  we are done.  $\square$

In this proof we used the braid monodromy of  $A_l$ -singularities, which we recall from 4.6 for convenience.

**Lemma 7.13** *The braid monodromy of the function  $x^{l+1}$  with respect to the Hefez Lazzeri system of paths is given by*

$$\sigma_{i, i^+}^3, 1 \leq i < l, \quad \sigma_{i, j}^2, 1 < i^+ < j < l.$$

## 7.2 families of type $g_\alpha$

The topic of this section are the versal braid monodromies of families of functions  $g_\alpha$ . We refer by  $G_2, G_3$  and  $G_n$  to the groups associated to polynomials of two, three resp.  $n$  variables.

Let us remark that by lemma 5.16 the groups determined in this section are in a sense the smaller complement to the the groups for the families of type  $f$ , which have been investigated in the plane case in the last chapter and will also be the topic of the next section.

The choice of the constants  $\varepsilon_i$  is tacitly assumed to be made in such a way, that the unit parameter disc of the families  $g_\alpha$  contains no singular parameter apart from the origin.

**Lemma 7.14** *The versal braid monodromy  $G_2$  of the family  $g_\alpha(x_1, x_2)$  restricted to the unit disc is generated by the homomorphic images in  $\text{Br}_{l_1 l_2}$  of the braid monodromy groups  $\text{Br}_{A(l_2)} \subset \text{Br}_{l_2}$  under the primary homomorphisms  $\phi_{i_1}, 1 \leq i_1 \leq l_1$ .*

*Proof:* The only critical parameter in the disc  $|\alpha| \leq 1$  is  $\alpha = 0$ . The corresponding critical function is  $x_1^{l_1+1} - (l_1 + 1)x_1 + x_2^{l_2+1}$ , which has  $l_1$  critical points of type  $A(l_2)$  with distinct critical values.

The bifurcation divisors of the families of functions parameterized by  $\alpha$ ,

$$-l_1\xi + x_2^{l_2+1} + \varepsilon_2\alpha(l_2 + 1)x_2, \quad \xi^{l_1} = 1,$$

embed into the bifurcation divisor of  $g_\alpha$ , and the corresponding embeddings of punctured discs induce embeddings of mapping class groups which correspond to the embeddings  $\phi_{i_1}$  under the standard identifications with the braid groups  $\text{Br}_{l_1 l_2}$  and  $\text{Br}_{l_2}$  by the Hefez Lazzeri choice of a strongly distinguished system of paths. The versal braid monodromies of the families above can be identified with the braid monodromy of lemma 7.13 and yield then the versal braid monodromy of the family  $g_\alpha$  as claimed.  $\square$

**Lemma 7.15** *The group  $G_2 \subset \text{Br}_{l_1 l_2}$  is generated by the elements*

$$\begin{aligned} \sigma_{i_1 i_2, i_1 j_2}^3, & \quad 1 \leq i_2 = j_2 - 1 < l_2, 1 \leq i_1 \leq l_1, \\ \sigma_{i_1 i_2, i_1 j_2}^2, & \quad 1 \leq i_2 < j_2 - 1 < l_2, 1 \leq i_1 \leq l_1. \end{aligned}$$

*Proof:* The group  $\text{Br}_{A(l_2)}$  is generated by elements

$$\sigma_{i_2, i_2}^3, 1 \leq i_2 < l_2, \quad \sigma_{i_2, j_2}^2, 1 \leq i_2 < j_2 - 1 < l_2,$$

so the claim holds by lemma 7.14 since their images under the primary homomorphisms  $\phi_{i_1}$  are the elements of the assertion.  $\square$

**Lemma 7.16** *The versal braid monodromy  $G_3$  of the family  $g_\alpha(x_1, x_2, x_3)$  restricted to the unit disc is the subgroup of  $\text{Br}_{l_1 l_2 l_3}$  generated by the homomorphic images of the braid monodromy groups  $\text{Br}_{BP(l_2, l_3)} \subset \text{Br}_{l_2 l_3}$  under the primary homomorphisms  $\phi_{i_1}, 1 \leq i_1 \leq l_1$ .*

*Proof:* We give the proof stressing the analogy to the case  $n = 2$  of 7.14: The only critical parameter in the disc  $|\alpha| \leq 1$  is  $\alpha = 0$ . The corresponding critical function is  $x_1^{l_1+1} - (l_1 + 1)x_1 + x_2^{l_2+1} + x_3^{l_3+1}$ , which has  $l_1$  critical points of type  $BP(l_2, l_3)$  with distinct critical values.

The bifurcation divisors of the families of functions

$$-l_1\xi + x_2^{l_2+1} + \varepsilon_2\alpha(l_2 + 1)x_2 + x_3^{l_3+1} + \varepsilon_3\alpha(l_3 + 1)x_3, \quad \xi^{l_1} = 1,$$

embed into the bifurcation divisor of  $g_\alpha$ , and the corresponding embeddings of punctured discs induce embeddings of mapping class groups which correspond to the embeddings  $\phi_{i_1}$  under the Hefez Lazzeri identifications with the braid groups  $\text{Br}_{l_1 l_2 l_3}$  and  $\text{Br}_{l_2 l_3}$ . The versal braid monodromies of the families above, which are given by the braid monodromy of polynomials of type  $BP(l_2, l_3)$ , then yield the claim.  $\square$

For the next lemma we have to resort for the first time to the forthcoming sections on  $l$ -companion families.

**Lemma 7.17** *The group  $G_3$  contains the image of  $G_2$  under the  $l_3$ -band homomorphism.*

*Proof:* The group  $G_2$  is generated by the braid monodromy of singular polynomials of type  $A_{l_1}$  under the primary homomorphisms. The  $l_3$ -companion family of the discriminant family of  $g_\alpha(x_1, x_2)$  is the discriminant family of:

$$x_1^{l_1+1} - (l_1 + 1)x_1 + x_2^{l_2+1} + \varepsilon_2\alpha(l_2 + 1)x_2 + x_3^{l_3+1} - \varepsilon_3(l_3 + 1)x_3.$$

Similarly its versal braid monodromy group is generated by the images under the primary homomorphisms of the versal braid monodromy for the  $l_3$ -companion of the discriminant family of the families of functions

$$-l_1\xi + x_2^{l_2+1} + \varepsilon_2\alpha(l_2 + 1)x_2, \quad \xi^{l_1} = 1.$$

Their versal braid monodromy coincides with the braid monodromy of the versal unfolding of a singular function of type  $A_{l_2}$ . So by lemma 7.50 we conclude that the versal braid monodromy of the family of functions

$$x_1^{l_1+1} - (l_1 + 1)x_1 + x_2^{l_2+1} + \varepsilon_2\alpha(l_2 + 1)x_2 + x_3^{l_3+1} - \varepsilon_3(l_3 + 1)x_3$$

contains the images under the  $\phi_{i_1}$  and  $\eta_{l_3}$  of the braid monodromy of the function

$$-l_1\xi + x_2^{l_2+1} + \varepsilon_2\alpha(l_2 + 1)x_2, \quad \xi^{l_1} = 1.$$

Hence also  $\eta_{l_3}(G_2)$  is contained. But then it must be a subgroup of the versal braid monodromy of the family of functions

$$x_1^{l_1+1} - (l_1 + 1)x_1 + x_2^{l_2+1} + \varepsilon_2\alpha(l_2 + 1)x_2 + x_3^{l_3+1} - \varepsilon_3\alpha(l_3 + 1)x_3$$

at  $\alpha = 0$ , which is  $G_3$ .  $\square$

Analogous to the cases  $n = 2, 3$  dealt with in 7.14, 7.16 we obtain the generalisation to arbitrary  $n$ .

**Lemma 7.18** *The versal braid monodromy  $G_n$  of the family  $g_\alpha(x_1, \dots, x_n)$  restricted to the unit disc is generated by the homomorphic images of the braid monodromy groups  $\text{Br}_{BP(l_2, \dots, l_n)} \subset \text{Br}_{l_2 \dots l_n}$  under the primary homomorphisms  $\phi_{i_1}, 1 \leq i_1 \leq l_1$ .*

Similarly we can extend the assertion of lemma 7.17 for the pair 2, 3 to arbitrary pairs  $n - 1, n$ .

**Lemma 7.19** *The group  $G_n$  contains the image of  $G_{n-1}$  under the  $l_n$ -band homomorphism.*

Since the generators given by Catanese and Wajnryb [9], cf. 4.3, are cable twists we can guess that there are many cable twists in the groups considered here.

**Lemma 7.20** *Suppose  $i'i_n < i'j_n$  is a pair of indices, then  $\delta_{i,j}^{j_n-i_n+2} \in G_n$ , especially  $\delta_\phi^{l_n+1} \in G_n$ .*

*Proof:* If we consider the family of functions

$$x_1^{l_1+1} - (l_1 + 1)x_1 + \left( \sum_{\nu=2}^{n-1} x_\nu^{l_\nu+1} - \varepsilon_\nu(l_\nu + 1)x_\nu \right) + x_n^{l_n+1} - \alpha\varepsilon_n(l_n + 1)x_n,$$

we see that its monodromy at  $\alpha = 0$  is contained in the monodromy at  $\alpha = 0$  of the family  $g_\alpha(x_1, \dots, x_n)$ . Similarly to 7.14 the family can be shown to have  $l_1 \dots l_{n-1}$  critical points of type  $A(l_n)$ . Their local monodromies embed via compositions  $\phi_{i'} := \phi_{i_1} \circ \dots \circ \phi_{i_{n-1}}$  of primary homomorphisms into  $\text{Br}_{BP(l_1 \dots l_n)}$ , so  $\delta_{i,j}^{j_n-i_n+2}$  is in  $G_n$  if  $\delta_{i_n,j_n}^{j_n-i_n+2}$  is in  $\text{Br}_{A(l_n)}$ .

Since the classical geometric monodromy of the  $A_n$  singularity is

$$\delta_\phi^{n+1} = \sigma_{1,2}^3 \sigma_{1,3}^2 \dots \sigma_{1,n}^2 \sigma_{2,3}^3 \sigma_{2,4}^2 \dots \sigma_{2,n}^2 \dots \sigma_{n-1,n}^3,$$

we can deduce in fact

$$\delta_{i_n,j_n}^{j_2-i_2+2} = \prod_{i \leq k < j} (\sigma_{k,k+}^3 \prod_{k+ < k' \leq j} \sigma_{k,k'}^2) \in \text{Br}_{A(l_n)}.$$

The additional claim follows from  $\delta_\phi^{l_2+1} = \prod_{i'} \delta_{i'_1,i'_n}^{l_n+1}$ .  $\square$

**Lemma 7.21** *For given  $i' < j'$  the same braid subgroup is generated by the elements*

$$\sigma_{i'_1 i_n, j'_1 j_n}^2, \quad 1 \leq i_n - j_n < l_n,$$

and by suitably chosen  $G_n$ -conjugates of elements

$$\begin{aligned} \sigma_{i'_1 i_n, j'_1 j_n}^2 & \quad 1 \leq i_n, j_n \leq l_n, i_n, i_n^+ \neq j_n, \\ \sigma_{i'_1 i_n, j'_1 i_n}^2 \sigma_{i'_1 i_n, j'_1 j_n}^2 \sigma_{i'_1 i_n, j'_1 i_n}^{-2} & \quad 1 < i_n^+ = j_n \leq l_n, \\ \sigma_{i'_1 i_n, i'_1 j_n}^2 \sigma_{i'_1 i_n, j'_1 j_n}^2 \sigma_{i'_1 i_n, i'_1 j_n}^{-2} & \quad 1 < i_n^+ = j_n \leq l_n. \end{aligned}$$

*Proof:* We introduce filtrations  $T = T_3 \supset T_2$  and  $S = S_3 \supset S_2 \supset S_1$  on the two sets of elements by

$$\begin{aligned} S_1 &= \{\sigma_{i'_1 i_n, j'_1 j_n}^2 \mid 1 \leq j_n < i_n \leq l_n\} \\ S_2 &= S_1 \cup \{\sigma_{i'_1 i_n, j'_1 j_n}^2 \mid 1 \leq i_n < i_n^+ < j_n \leq l_n\} \\ T_2 &= T_1 \cup \{\sigma_{i'_1 i_n, j'_1 j_n}^2 \mid i_n, j_n \not\equiv 0 \pmod{l_n+1}\} \end{aligned}$$

Then it suffices to show that for each  $s \in S_2$  there is a  $t \in T_2$  with  $t$  equal to some  $G_n$ -conjugate of  $s$  and that for each  $s \in S_3 - S_2$  there is a  $t \in T_3 - T_2$  with a  $G_n$ -conjugate of  $s$  equal to  $t$  conjugated by some elements of  $T_2$  and vice versa, cf. lemma A.17.

Since  $\delta_\phi^{l_n+1} \in G_n$  by 7.20, we may conjugate the elements of  $S_1$  by all powers  $\delta_\phi^{m(l_n+1)}$ . Similarly  $\delta_{j'_1, j'_n}^{l_n+1}$  is an element of  $G_n$  by 7.20, hence all elements

$$\delta_\phi^{m(l_n+1)} (\delta_{j'_1, j'_n}^{l_n+1} \sigma_{i'_1 i_n, j'_1 j_n}^2 \delta_{j'_1, j'_n}^{-l_n-1}) \delta_\phi^{-m(l_n+1)} = \delta_\phi^{m(l_n+1)} \sigma_{i'_1 i_n, j'_1 j_n}^2 \delta_\phi^{-m(l_n+1)},$$

where  $j'_n = j_n - l_n - 1$ , are  $G_n$ -conjugates of elements in  $S_2 - S_1$ . In fact the elements thus obtained are just all the elements in  $T_2$ .

Finally we observe that the braids  $\delta_{j'l_n^+, j'l_n}^{l_n-i_n+1}$ , resp.  $\delta_{i'1, i'j_n^-}^{j_n}$  are elements of  $G_n$  due to 7.20 again. Hence we may invoke A.16 to show that the elements in  $S_3 - S_2$  have  $G_n$ -conjugates which are equal up to conjugation by elements in  $T_2$  to elements

$$\sigma_{i'i_n, j'0}^2, \sigma_{i'l_n^+, j'j_n}^2, \quad 1 \leq i_n < l_n, 1 < j_n \leq l_n,$$

which in turn are contained in  $T_3 - T_2$ .

Because conjugation by powers  $\delta_\phi^{m(l_n+1)}$  yields all elements of  $T_3 - T_2$  and preserves the set  $T_2$ , all elements in  $T_3 - T_2$  up to conjugation by elements in  $T_2$  are  $G_n$ -conjugates of elements in  $S_3 - S_2$ , so we are done.  $\square$

### 7.3 families of type $f_\alpha$

We turn our attention now to the families of type  $f_\alpha$ , for which the versal braid monodromy has to be computed not only locally. But in fact all geometric insight is in the case  $n = 2$  dealt with in chapter 6 and the notion of  $l$ -companion families, which will be exploited in the next two section. So here we mainly have to translate between various results and to organize them in such a way we need for the induction.

**Lemma 7.22** *The versal braid monodromy of a family of functions  $f_\alpha(x_1, x_2)$  is generated by the elements*

$$\begin{aligned} \sigma_{i_1 i_2, j_1 j_2}^3, & \quad i_2 = j_2, i_1 < j_1 \text{ correlated}, \\ \sigma_{i_1 i_2, j_1 j_2}^2, & \quad i_2 = j_2, i_1 < j_1 \text{ not correlated}, \\ \sigma_{i_1 i_2, j_1 j_2}^2 & \quad \text{with } 1 \leq i_1 < j_1 \leq l_1, 1 \leq i_2 - j_2 < l_2. \end{aligned}$$

*Proof:* We have to show that the elements in 7.22 and those in 7.12 generate the same subgroup of  $\text{Br}_{l_1 l_2}$ . Since both generator sets have the elements with equal second index component in common, it suffices to prove that the remaining elements of each set generate the same braid subgroup.

Notice that both sets are filtered by level – which is underlined – with

$$\begin{aligned} S_1 &= \{\sigma_{i_1 i_2, j_1 j_2}^2 \mid 1 \leq i_1 < j_1 \leq l_1, \underline{j_1 - i_1} = 1, 1 \leq i_2 - j_2 < l_2\} \\ S_2 &= S_1 \cup \{\sigma_{i_1 i_2, j_1 j_2}^2 \mid 1 \leq i_1 < j_1 \leq l_1, \underline{j_1 - i_1} = 2, 1 \leq i_2 - j_2 < l_2\} \\ &\vdots \\ S_{l_1} &= \{\sigma_{i_1 i_2, j_1 j_2}^2 \mid 1 \leq i_1 < j_1 \leq l_1, 1 \leq i_2 - j_2 < l_2\} \\ T_1 &= \{\tau_{i_1 i_2, j_1 j_2}^2 \mid 1 \leq i_1 < j_1 \leq l_1, \underline{j_1 - i_1} = 1, 1 < j_2 - i_2 \leq l_2\} \\ T_2 &= T_1 \cup \{\tau_{i_1 i_2, j_1 j_2}^2 \mid 1 \leq i_1 < j_1 \leq l_1, \underline{j_1 - i_1} = 2, 1 < j_2 - i_2 \leq l_2\} \\ &\vdots \\ T_{l_1} &= \{\tau_{i_1 i_2, j_1 j_2}^2 \mid 1 \leq i_1 < j_1 \leq l_1, 1 < j_2 - i_2 \leq l_2\}. \end{aligned}$$

For the proof we need therefore to check the hypotheses of A.17 only: The first,  $S_1 = T_1$ , is immediate, since elements of level one coincide almost by definition

$$\sigma_{i_1 i_2, i_1^+ j_2} = \tau_{i_1 i_2', i_1^+ j_2}, \quad i_2' + l_2 = i_2.$$

For the inductive hypothesis lemma A.10 yields, that elements  $\tau_{i_1 0, k_1 k_2}^2$  and  $\sigma_{i_1 l_2, k_1 k_2}^2$  with  $1 \leq i_1 < k_1 \leq l_1, 1 \leq k_2 < l_2$  are equal up to conjugation by elements in  $S_{k_1-i_1-1} \cup T_{k_1-i_1-1}$ , i.e. by elements of smaller level.

To extend this result to the remaining elements we consider the action of overall conjugation by  $\delta_\phi$ . Since this conjugation is level preserving, we get, that

$$\sigma_{i_1 i_2, k_1 j_2}^2 = \delta_\phi^{i_2'} \sigma_{i_1 l_2, k_1 k_2}^2 \delta_\phi^{-i_2'}, \quad \tau_{i_1 i_2', k_1 j_2}^2 = \delta_\phi^{i_2'} \tau_{i_1 0, k_1 k_2}^2 \delta_\phi^{-i_2'}$$

are equal up to conjugation by elements of smaller level since  $\sigma_{i_1 l_2, k_1 k_2}^2, \tau_{i_1 0, k_1 k_2}^2$  are. The hypotheses are hence met, for each generator  $\sigma_{i_1 i_2, j_1 j_2}^2$  or  $\tau_{i_1 i_2, j_1 j_2}^2$  is in the conjugation orbit of a  $\sigma_{i_1 l_2, k_1 k_2}^2$  resp.  $\tau_{i_1 0, k_1 k_2}^2$  by  $\delta_\phi$ .  $\square$

The generator set has still the draw back that it is not finite. Though this could be amended we even take a step further and proceed to a generator set which generates the same group only up to the subgroup  $G_2$ .

**Lemma 7.23** *The versal braid monodromy of  $f_\alpha(x_1, x_2)$  is generated by elements  $G_2$ -conjugate to*

$$\begin{aligned} \sigma_{i_1 i_2, j_1 j_2}^3 &: i_2 = j_2, i_1 < j_1 \text{ correlated,} \\ \sigma_{i_1 i_2, j_1 j_2}^2 &: i_2 = j_2, i_1 < j_1 \text{ not correlated,} \\ \sigma_{i_1 i_2, j_1 j_2}^2 &: i_1 < j_1, i_2 \neq j_2, i < j \text{ not correlated,} \\ \sigma_{i_1 i_2, j_1 j_2}^2 \sigma_{i_1 i_2, k_1 k_2}^2 \sigma_{i_1 i_2, j_1 j_2}^{-2} &: i_2 + 1 = k_2, i < j < k \text{ correlated.} \end{aligned}$$

*Proof:* We have to show, that we can assign  $G_2$  conjugates to the given elements, such that these conjugates generate the same braid subgroup as the elements of 7.22.

The first two rows of elements obviously coincide. If we apply the case  $n = 2$  of lemma 7.21 to the last row of 7.22, then we get the bottom rows here.  $\square$

**Lemma 7.24** *The versal braid monodromy of the family of functions*

$$x_1^3 - 3\alpha x_1 + x_2^{l_2+1} - \varepsilon_2(l_2+1)x_2$$

*is generated by elements*

$$\begin{aligned} \sigma_{1 i_2, 2 j_2}^3, \quad &i_2 = j_2, \\ \sigma_{1 i_2, 2 j_2}^2 \quad &\text{with } 1 \leq i_2 - j_2 < l_2. \end{aligned}$$

*Proof:* This is the special case  $l_1 = 2$  of lemma 7.22.  $\square$

Again we have to use the results of the next two sections.

**Lemma 7.25** *The versal braid monodromy of  $f_\alpha(x_1, x_2, x_3)$  is generated by elements  $\eta(G_2)$ -conjugate to*

$$\begin{aligned} \sigma_{i_1 i_2 i_3, j_1 j_2 j_3}^3 &: i_1 = j_1 - 1, i_2 = j_2, i_3 = j_3, \\ \sigma_{i_1 i_2 i_3, j_1 j_2 j_3}^2 &: i_1^+ = j_1, i_2 = j_2, 1 \leq i_3 - j_3 < l_3 \\ \sigma_{i_1 i_2 i_3, j_1 j_2 j_3}^2 &: i_1 < j_1 - 1, i_2 = j_2, \\ \sigma_{i_1 i_2 i_3, j_1 j_2 j_3}^2 &: i_1 < j_1, i_2 \neq j_2, i_1 i_2 < j_1 j_2 \text{ not correlated}, \\ \eta(\sigma_{i,j}^2) \sigma_{i i_3, k k_3}^2 \eta(\sigma_{i,j}^{-2}) &: i_1 i_2 < j_1 j_2 < k_1 k_2 \text{ correlated}. \end{aligned}$$

*Proof:* First note that the discriminant family associated to  $f_\alpha(x_1, x_2, x_3)$  is the  $l_3$ -companion family of the discriminant family associated to  $f_\alpha(x_1, x_2)$ . So by lemma 7.49 the versal braid monodromy is generated by the  $l_3$ -companions of the generators given in lemma 7.22.

But instead of the  $l_3$ -companions of the generators in 7.22 we take the  $l_3$ -companions of the elements in 7.23. The definition implies that up to conjugation by elements of  $\eta(G_2)$  they generate the same group, which suffices for our claim.

So we need only to run the list of 7.23 through the procedure given in 7.38 to get the list of the claim.  $\square$

**Lemma 7.26** *The versal braid monodromy of the family  $f_\alpha(x_1, x_2, x_3)$  is generated by a set of braids  $G_3$ -conjugate to elements of*

$$\begin{aligned} \sigma_{i,j}^3 &: i_1^+ = j_1, i_2 = j_2, i_3 = j_3, \\ \sigma_{i,k}^2 &: i_1 < k_1, i < k \text{ not correlated}, \\ \sigma_{i,j}^2 \sigma_{i,k}^2 \sigma_{i,j}^{-2} &: i_1 < k_1, i < k \text{ correlated}, j \in \{i_1^+ i_2 i_3, i_1 k_2 k_3\}. \end{aligned}$$

*Proof:* Since  $\eta(G_2) \subset G_3$  by 7.17 it suffices to show that the elements in 7.25 are  $G_3$ -conjugates of those given here. Our method of proof consists in replacing elements of a row by others obtained through conjugation with elements of previous rows and elements of  $G_3$ , since under such transformations the group generated by  $G_3$ -conjugates does not change.

Let us start with the fifth row. We have there the  $2(l_1 - 1)(l_2 - 1)l_3^2$  elements

$$\begin{aligned} \eta(\sigma_{i_1 i_2, j_1 j_2}^2) \sigma_{i_1 i_2 i_3, k_1 k_2 k_3}^2 \eta(\sigma_{i_1 i_2, j_1 j_2}^{-2}) & 1 < i_1^+ = k_1 \leq l_1, 1 < i_2^+ = k_2 \leq l_2, \\ & 1 \leq i_3, k_3 \leq l_3, j_1 j_2 \in \{i_1 k_2, k_1^+ i_2\}. \end{aligned}$$

By A.14 those with  $j_1 j_2 = i_1^+ i_2$  equal  $\sigma_{i_1 j_1 j_2 i_3}^2 \sigma_{i,k}^2 \sigma_{i_1 j_2 i_3}^{-2} = \sigma_{j_1 j_2 i_3, k}^{-2} \sigma_{i,k}^2 \sigma_{j_1 j_2 i_3, k}^2$  up to conjugation by elements of the second row. Since  $j_1 = k_1$ , the twist  $\sigma_{j_1 j_2 i_3, k}^2$  is in  $G_3$ , if  $\sigma_{j_1 j_2 i_3, k}^2$  is in the braid monodromy group  $\text{Br}_{BP(l_2, l_3)}$ , cf. 7.16.

If  $i < k$  are not correlated, then  $j_2 i_3 < k_2 k_3$  aren't either, with  $j_2 < k_2$ . We deduce with 7.23 that  $\sigma_{j_1 j_2 i_3, k}^2$  then is in  $G_3$ . Hence the elements of the fifth row with  $j_1 j_2 = i_1^+ i_2$  are equal to

i)  $\sigma_{i_1 i_2 i_3, j_1 j_2 i_3}^2 \sigma_{i_1 i_2 i_3, k_1 k_2 k_3}^2 \sigma_{i_1 i_2 i_3, j_1 j_2 i_3}^{-2}$  if  $i < k$  are correlated,

ii)  $\sigma_{i_1 i_2 i_3, k_1 k_2 k_3}$  if  $i < k$  are not correlated,

up to conjugation by elements of the second row and of  $G_3$ .

Similarly by A.15 those with  $j_1 j_2 = i_1 k_2$  are equal to

$$\text{i'}) \sigma_{i_1 i_2 i_3, j_1 j_2 k_3}^2 \sigma_{i_1 i_2 i_3, k_1 k_2 k_3}^2 \sigma_{i_1 i_2 i_3, j_1 j_2 k_3}^{-2} \text{ if } i < k \text{ are correlated,}$$

$$\text{ii'}) \sigma_{i_1 i_2 i_3, k_1 k_2 k_3} \text{ if } i < k \text{ are not correlated,}$$

up to conjugation by elements of the second row and of  $G_3$ .

Note that *ii*) and *ii'*) yield the same elements and that we have the following restrictions on the indices,  $1 \leq i_\nu, j_\nu, k_\nu \leq l_\nu, \nu = 1, 2, 3$ , assumed in all cases:

$$\text{i}) i_1^+ i_2^+ i_3, i_1^+ i_2^+ i_3^+ = k_1 k_2 k_3, j_1 j_2 j_3 = i_1^+ i_2 i_3,$$

$$\text{ii}) i_1^+ i_2^+ = k_1 k_2, i_3, i_3^+ \neq k_3,$$

$$\text{i'}) i_1^+ i_2^+ i_3, i_1^+ i_2^+ i_3^+ = k_1 k_2 k_3, j_1 j_2 j_3 = i_1 k_2 k_3.$$

Second we want to replace the elements of the second row. Lemma 7.21 implies in case  $i' = i_1 i_2, j' = j_1 j_2, n = 3$  that the elements

$$\sigma_{i_1 i_2 i_3, j_1 j_2 j_3}^2, \quad i_1^+ i_2 = j_1 j_2, 1 \leq i_n - j_n < l_n,$$

are  $G_3$ -conjugates of elements

$$\text{iii}) \sigma_{i_1 i_2 i_3, i_1^+ i_2 k_3}^2, i_3, i_3^+ \neq k_3,$$

$$\text{iv}) \sigma_{i_1 i_2 i_3, i_1^+ i_2 i_3}^2 \sigma_{i_1 i_2 i_3, i_1^+ i_2 i_3^+}^2 \sigma_{i_1 i_2 i_3, i_1^+ i_2 i_3}^{-2},$$

$$\text{v}) \sigma_{i_1 i_2 i_3, i_1 i_2 i_3^+}^2 \sigma_{i_1 i_2 i_3, i_1^+ i_2 i_3^+}^2 \sigma_{i_1 i_2 i_3, i_1 i_2 i_3^+}^{-2}.$$

We reassemble the elements thus obtained according to the form in which they are given, twists of exponent 3, twists of exponent 2 given as  $\sigma_{i,k}^2$  and twists of exponent 2 given as  $\sigma_{i,j}^2 \sigma_{i,k}^2 \sigma_{i,j}^{-2}$ . The two final steps then consists in collecting the corresponding sets of indices.

For the elements which are full twists of the form  $\sigma_{i,k}^2$  we get:

$$\begin{aligned} & \{i, k | i_1^+ i_2 = k_1 k_2, i_3, i_3^+ \neq k_3\} & \text{iii)} \\ \cup & \{i, k | i_1^+ < k_1, i_2 = k_2\} & 3^{rd} \text{ row of 7.25} \\ \cup & \{i, k | i_1 < k_1, i_2 \neq k_2, i_1^+ i_2^+ \neq k_1 k_2\} & 4^{th} \text{ row of 7.25} \\ \cup & \{i, k | i_1^+ i_2^+ = k_1 k_2, i_3, i_3^+ \neq k_3\} & \text{ii)} \\ = & \{i, k | i_1 < k_1, i < k \text{ not correlated}\} & 2^{nd} \text{ row of 7.26} \end{aligned}$$

For the elements which are full twists of the form  $\sigma_{i,j}^2 \sigma_{i,k}^2 \sigma_{i,j}^{-2}$  we get:

$$\begin{aligned} & \{i, j, k | j_1 j_2 j_3 = i_1^+ i_2 i_3, k_1 k_2 k_3 \in \{i_1^+ i_2^+ i_3, i_1^+ i_2^+ i_3^+\}\} & \text{i)} \\ \cup & \{i, j, k | k_1 k_2 k_3 \in \{i_1^+ i_2^+ i_3, i_1^+ i_2^+ i_3^+\}, j_1 j_2 j_3 = i_1 k_2 k_3\} & \text{i')} \\ \cup & \{i, j, k | j_1 j_2 j_3 = i_1^+ i_2 i_3, k_1 k_2 k_3 = i_1^+ i_2 i_3^+\} & \text{iv)} \\ \cup & \{i, j, k | j_1 j_2 j_3 = i_1 i_2 i_3^+, k_1 k_2 k_3 = i_1^+ i_2 i_3^+\} & \text{v)} \\ = & \{i, j, k | i_1 < k_1, i < k \text{ correlated}, j \in \{i_1^+ i_2 i_3, i_1 k_2 k_3\}\} \end{aligned}$$

So the set we obtained replacing the elements of 7.25 by  $G_3$ -conjugates of another generating set is exactly that of our claim, and we are done.  $\square$

**Lemma 7.27** *The versal braid monodromy of the family  $f_\alpha(x_1, \dots, x_n)$  is generated by a set of  $G_n$ -conjugates of braids*

$$\begin{aligned} \sigma_{i,j}^3 &: i_1^+ = j_1, i_2 \dots i_n = j_2 \dots j_n, \\ \sigma_{i,k}^2 &: i_1 < k_1, i < k \text{ not correlated}, \\ \sigma_{i,j}^2 \sigma_{i,k}^2 \sigma_{i,j}^{-2} &: i_1^+ = k_1, i < k \text{ correlated}, j \in \{i_1^+, i_2 \dots i_n, i_1 k_2 \dots k_n\}. \end{aligned}$$

*Proof:* This can be proved by induction on  $n$ , with the cases  $n = 2, 3$  already done, 7.23, 7.26. So let us assume the claim holds for  $n - 1$ .

By 7.49 generators are given by the  $l_n$ -companions of generators for the braid monodromy group of  $f_\alpha(x_1, \dots, x_{n-1})$ .

If such generators are given up to conjugation by elements of  $G_{n-1}$  only, then their  $l_n$ -companions are known up to conjugation by elements of  $\eta(G_{n-1})$  due to the definition. This is fine here, since we are interested in generators up to conjugation by elements of  $G_n$  which is even coarser because  $\eta(G_{n-1}) \subset G_n$  by 7.19.

Hence we conclude that up to conjugation by elements of  $G_n$  generators for the braid monodromy of  $f_\alpha(x_1, \dots, x_n)$  are obtained from the list of  $G_{n-1}$ -conjugates of generators for  $f_\alpha(x_1, \dots, x_{n-1})$  by taking the  $l_n$ -companions according to the rule 7.38:

$$\begin{aligned} \sigma_{i',k'}^3 &: \sigma_{i,k}^3 & i_1^+ i_2 \dots i_n = k_1 k_2 \dots k_n, \\ \sigma_{i',k'}^2, & \sigma_{i,k}^2, & i_1^+ i_2 \dots i_{n-1} = k_1 k_2 \dots k_{n-1}, \\ & & 1 \leq i_n - k_n < l_n \\ \sigma_{i',k'}^2 &: \sigma_{i,k}^2, & i_1 < k_1, i' < k' \text{ not correlated}, \\ \sigma_{i',j'}^2 \sigma_{i',k'}^2 \sigma_{i',j'}^{-2} &: \eta(\sigma_{i',j'}^2) \sigma_{i,k}^2 \eta(\sigma_{i',j'}^{-2}) & i_1 < k_1, i' < k' \text{ correlated}, \\ & & j' \in \{i_1^+ i_2 \dots i_{n-1}, i_1 k_2 \dots k_{n-1}\}. \end{aligned}$$

We proceed in strict analogy to the proof of 7.26. Without changing the group which  $G_n$ -conjugates generate, we replace the elements of the fourth row by

- i)  $\sigma_{i,j'i_n}^2 \sigma_{i,k}^2 \sigma_{i,j'i_n}^{-2}$ , if  $i < k$  correlated,  $j' = i_1^+ i_2 \dots i_{n-1}$ ,
- i')  $\sigma_{i,j'k_n}^2 \sigma_{i,k}^2 \sigma_{i,j'k_n}^{-2}$ , if  $i < k$  correlated,  $j' = i_1 k_2 \dots k_{n-1}$ ,
- ii)  $\sigma_{i,k}^2$  if  $i < k$  not correlated,  $i_1^+ = k_1, i' < k'$  correlated.

We go on and replace the elements of the second row by elements

- iii)  $\sigma_{i_1 \dots i_n, i_1^+ i_2 \dots i_{n-1} k_n}^2$ ,  $i_n, i_n^+ \neq k_n$ ,
- iv)  $\sigma_{i,j}^2 \sigma_{i,k}^2 \sigma_{i,j}^{-2}$ ,  $i_2 \dots i_{n-1} = j_2 \dots j_{n-1} = k_2 \dots k_{n-1}$ ,  $i_1^+ i_n^+ = j_1 j_n^+ = k_1 k_n$ ,
- v)  $\sigma_{i,j}^2 \sigma_{i,k}^2 \sigma_{i,j}^{-2}$ ,  $i_2 \dots i_{n-1} = j_2 \dots j_{n-1} = k_2 \dots k_{n-1}$ ,  $i_1^+ i_n^+ = j_1^+ j_n = k_1 k_n$ .

A final check that the lists of index pairs and triples for the generators thus obtained and the elements of the claim coincide completes the proof.  $\square$

**Theorem 7.28** *The braid monodromy of the function  $x_1^{l_1+1} + \cdots + x_n^{l_n+1}$  is generated by*

$$S = \left\{ \begin{array}{lll} \sigma_{i,k}^3 & i < k & \text{correlated}, \\ \sigma_{i,k}^2 & i < k & \text{not correlated}, \\ \sigma_{i,j}^2 \sigma_{i,k}^2 \sigma_{i,j}^{-2} & i < j < k & \text{correlated} \end{array} \right\}$$

*Proof:* The proof can be obtained by induction on  $n$ . Since the claim has been proved already for  $n = 1, 2$  we may suppose  $n > 2$  and that the case  $n - 1$  is already known.

So we deduce from 7.18 that the braid monodromy group  $G_n$  is generated by

$$\begin{aligned} \sigma_{i,k}^3 & i_1 = k_1, i < k \text{ correlated}, \\ \sigma_{i,k}^2 & i_1 = k_1, i < k \text{ not correlated}, \\ \sigma_{i,j}^2 \sigma_{i,k}^2 \sigma_{i,j}^{-2} & i_1 = k_1, i < j < k \text{ correlated} \end{aligned}$$

To get generators for the total braid monodromy we have – by 5.16 – to add the elements of 7.27.

$$\begin{aligned} \sigma_{i,j}^3 & i_1^+ = j_1, i_2 \dots i_n = j_2 \dots j_n, \\ \sigma_{i,k}^2, & i_1 < k_1, i < k \text{ not correlated}, \\ \sigma_{i,j}^2 \sigma_{i,k}^2 \sigma_{i,j}^{-2} & i_1^+ = k_1, i < k \text{ correlated}, j \in \{i_1^+, i_2 \dots i_n, i_1 k_2 \dots k_n\}. \end{aligned}$$

By a check on the indices occurring in these two sets, we see, that in order to get the claim we have to add elements

$$\begin{aligned} \sigma_{i,j}^3 & i_1^+ = j_1, i_2 \dots i_n \neq j_2 \dots j_n, \\ \sigma_{i,j}^2 \sigma_{i,k}^2 \sigma_{i,j}^{-2} & i_1^+ = k_1, i < j < k \text{ correlated}, j \notin \{i_1^+ i_2 \dots i_n, i_1 k_2 \dots k_n\}. \end{aligned}$$

Of course they may be added without harm if and only if they are elements of the braid monodromy.

So let us first consider triples of correlated indices  $i < j < k$  with  $j_1 = i_1^+$ . Then  $i < i_1^+ i_2 \dots i_n < j < k$  is a correlated quadruple such that the full twists associated to the correlated triples of  $i < k$ ,  $i < j$  and  $j < k$  with  $i_1^+ i_2 \dots i_n$  are among the given generators of the braid monodromy. Hence we may conclude with A.12 that all elements  $\sigma_{i,j}^2 \sigma_{i,k}^2 \sigma_{i,j}^{-2}$  with  $j_1 = i_1^+$ ,  $i < j < k$  correlated, are in the braid monodromy.

Similarly we argue for correlated index triples  $i < j < k$  with  $j_1 = i_1$ . Since  $i < j < i_1 k_2 \dots k_n < k$  is a correlated quadruple then, we may conclude as above that all elements  $\sigma_{i,j}^2 \sigma_{i,k}^2 \sigma_{i,j}^{-2}$  with  $j_1 = i_1$ ,  $i < j < k$  correlated, are in the braid monodromy.

Finally let  $i < k$  be any pair of correlated indices. If  $\sigma_{i,k}^3$  is not among the generators given for the braid monodromy, set  $j := i_1^+ i_2 \dots i_n$ . Then  $i < j < k$  is a correlated triple such that  $\sigma_{i,j}^3$  and  $\sigma_{j,k}^3$  are among the given generators of the braid monodromy. Since so is  $\sigma_{i,j}^2 \sigma_{i,k}^2 \sigma_{i,j}^{-2}$ , we conclude with A.11 that also  $\sigma_{i,k}^3$  is an element of the braid monodromy.  $\square$

## 7.4 $l$ -companion models

We introduce in this section the notion of a companion family, which is associated to a discriminant family by replacing the singular value divisor by a number of parallel copies. If the discriminant family arises from a family of functions, so does the companion family, hence we may try to relate not only the braid monodromy but also the versal braid monodromies of both families.

In this section we have a closer look at the companion families of the simplest discriminant families, the ordinary node and the ordinary cusp, and provide the necessary definitions and arguments to get first results on these companion families and their monodromy.

**Definition 7.29:** Given a discriminant family  $\mathcal{E}$  defined by a polynomial  $d(z, \alpha)$ , a discriminant family  $\mathcal{E}^{(l)}$  defined by a polynomial  $d^{(l)}$  is called  *$l$ -companion family* if

$$d^{(l)}(z, \alpha) = \prod_{\xi^l=1} d(z - \xi \varepsilon, \alpha)$$

for some  $0 < \varepsilon \ll 1$ .

So for example we get an  $l$ -companion family of the ordinary node  $z^2 - \alpha^2$  defined by the equation

$$\prod_{\xi^l=1} ((z - \xi \varepsilon)^2 - \alpha^2) = 0.$$

The close relation between the two equations should – and shall – result in more ties between the two families  $\mathcal{E}$  and  $\mathcal{E}^{(l)}$ .

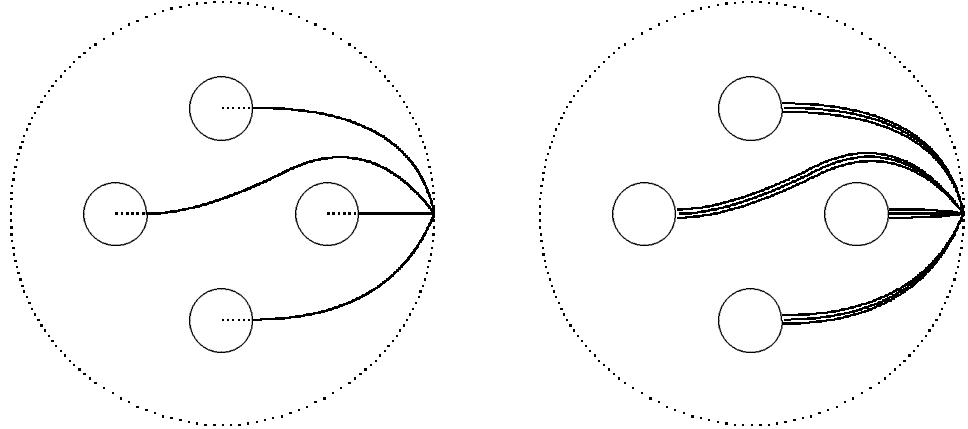
First note that there is a natural way to pass from a distinguished system of paths for a regular fibre of  $\mathcal{E}$  at a parameter  $\alpha$  to a system of paths for the fibre of  $\mathcal{E}^{(l)}$  at  $\alpha$ .

Of course we have to impose  $\varepsilon$  to be small enough to get again a regular fibre. For a suitable choice of  $\varepsilon$  then the given system of paths for  $\mathcal{E}$  meets the boundary of the  $2\varepsilon$ -discs at the punctures once only. By changing the system in its isotopy class only, we may even assume that this boundary point corresponds to puncture translated by  $2\varepsilon$ .

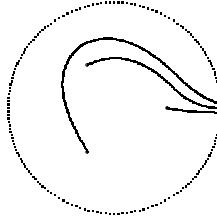
We then split each path into a bunch of  $l$  paths and connect it with the punctures of the  $\mathcal{E}^{(l)}$  fibre in the order of increasing angle  $\arg(\xi)$ .

The system of paths thus obtained we call the *Hefez-Lazzeri refinement*, as the procedure mimics the iteration step given in the Hefez-Lazzeri article for a special system of paths.

**Example 7.30:** Cut discs of radius  $2\varepsilon$  off the original fibre and replace the truncated paths by  $l$ -bunches.



Glue in copies of a  $2\varepsilon$  disc punctured at points of absolute value  $\varepsilon$  and argument  $\xi, \xi^l = 1$ , and provided with a standard system of paths.



So from now on given a system of paths for a fibre of  $\mathcal{E}$ , we may tacitly assume that a system of paths in the fibre of  $\mathcal{E}^{(l)}$  is given by the Hefez-Lazzeri refinement. We will still sometimes say so explicitly but even if we won't this should be understood without mentioning.

In particular this means that given the braid monodromy group of a family as a subgroup of an abstract braid group  $\text{Br}_l$ , which involves an implicit choice of a system of paths, we may consider the braid monodromy group of a  $l$ -companion family to be a subgroup of  $\text{Br}_{l_1}$  since we may make the necessary identification by implicit use of the Hefez-Lazzeri refined system of paths.

Now let us have some exercise with the simplest discriminant families:

**Lemma 7.31** *The versal braid monodromy of the  $l$ -companion family associated to the discriminant family defined by  $z^2 - \alpha^2$  with defining equation*

$$\prod_{\xi^l=1} ((z - \xi \varepsilon)^2 - \alpha^2) = 0$$

*is generated by*

$$\sigma_{1i,2j}^2, \quad 1 \leq i, j \leq l.$$

*and is isomorphic to  $\ker(\text{PBr}_{2l} \rightarrow \text{PBr}_l \times \text{PBr}_{l_2})$ .*

*Proof:* The  $l$ -companion family in this case coincides with the model discriminant family for the family of function  $f_\alpha(x_1, x_2)$  in case  $l_1 = 2, l_2 = l$ . Hence the versal braid monodromy is almost that of the discriminant family of  $f_\alpha(x_1, x_2)$ . We need only remark that the group locally assigned to the singular fibre at the origin is generated by full twists on v-arcs as in the case of all other fibres. Then the same methods show then that

$$\tau_{1i,2j}, \quad 1 \leq j - i \leq l_2$$

generate the braid monodromy. But this is just the claim up to inner conjugations.  $\square$

**Lemma 7.32** *The versal braid monodromy of the  $l$ -companion family associated to the discriminant family defined by  $z^2 - \alpha^3$  is equal to the subgroup of  $\text{Br}_{BP(2,l)}$  generated by*

$$\begin{aligned} \sigma_{1i_2,2j_2}^3, & \quad i_2 = j_2, \\ \sigma_{1i_2,2j_2}^2 & \quad \text{with } 1 \leq i_2 - j_2 < l_2. \end{aligned}$$

*Proof:* The  $l$ -companion family associated to the discriminant family defined by  $z^2 - \alpha^3$  coincides with the discriminant family of the family of functions

$$x_1^3 - 3\alpha x_1 + x_2^{l_2+1} - \varepsilon_2(l_2+1)x_2.$$

The versal braid monodromy has be computed in 7.24.  $\square$

In order to avoid giving lists of elements as in the preceding lemmas, we introduce some new notions.

**Definition 7.33:** An element  $g \in \text{Br}_n$  is called *positive twist element of exponent*  $k \in \mathbb{N}$ , if there is  $h \in \text{Br}_n$  such that  $hgh^{-1} = \sigma_1^k$ .

**Definition 7.34:** The  $l$ -companions in  $\text{Br}_{l'l}$  of  $\sigma_1^2 \in \text{Br}_{l'}$  are the elements

$$\sigma_{1i,2j}^2, \quad 1 \leq i, j \leq l,$$

the  $l$ -companions in  $\text{Br}_{l'l}$  of  $\sigma_1^3 \in \text{Br}_l$  are the elements

$$\begin{aligned} \sigma_{1i,2i}^3, & \quad 1 \leq i \leq l, \\ \sigma_{1i,2j}^2, & \quad 1 \leq i - j < l. \end{aligned}$$

**Definition 7.35:** If  $g \in \text{Br}_{l'}$  is a twist of exponent 2 and  $hgh^{-1} = \sigma_1^2$ , then the  $l$ -companions in  $\text{Br}_{l'l}$  of  $g \in \text{Br}_{l'}$  are the elements

$$\eta(h^{-1})\sigma_{1i,2j}^2\eta(h), \quad 1 \leq i, j \leq l.$$

**Definition 7.36:** If  $g \in \text{Br}_{l'}$  is a twist of exponent 3 and  $hgh^{-1} = \sigma_1^3$ , then the  $l$ -companions in  $\text{Br}_{l'l}$  of  $g \in \text{Br}_{l'}$  are the elements

$$\begin{aligned} \eta(h^{-1})\sigma_{1i,2i}^3\eta(h), & \quad 1 \leq i \leq l, \\ \eta(h^{-1})\sigma_{1i,2j}^2\eta(h), & \quad 1 \leq i - j < l. \end{aligned}$$

Of course in the last two definitions there arises the question of well-definedness. But this is easily taken care of:

**Lemma 7.37** *The definition of  $l$ -companions is independent on the choice of  $h$ .*

*Proof:* If we replace  $h$  by  $h'h$  with  $h'\sigma_1 = \sigma_1 h'$ , i.e.  $h'$  in the stabilizer of  $\sigma_1$ , then  $\eta_l(h')$  commutes with  $\sigma_{1i,2j}$ .  $\square$

With these definitions we can easily determine  $l$ -companions of various elements:

**Lemma 7.38** *The following table lists the  $l$ -companions of the elements given in the left column:*

$$\begin{aligned} \sigma_{i',j'}^2 &: & \sigma_{i'i_n,j'j_n}^2, & i_n, j_n \leq l, \\ \sigma_{i',j'}^3 &: & \sigma_{i'i_n,j'j_n}^3, & i_n = j_n < l, \\ & & \sigma_{i'i_n,j'j_n}^2, & 1 \leq i_n - j_n < l, \\ \sigma_{i',j'}^2 \sigma_{i',k'}^2 \sigma_{i',j'}^{-2} &: & \eta(\sigma_{i',j'}^2) \sigma_{i'i_n,k'k_n}^2 \eta(\sigma_{i',j'}^{-2}), & i_n, j_n \leq l. \end{aligned}$$

**Lemma 7.39** *The versal braid monodromy of the  $l$ -companion family associated to the discriminant family defined by a homogeneous polynomial  $d(z, \alpha)$  of degree  $l_1$  with defining equation*

$$\prod_{\xi^{l_2}=1} d(z - \xi \varepsilon, \alpha) = 0$$

is generated by

$$\sigma_{i_1 i_2, j_1 j_2}^2, \quad 1 \leq i_1, j_1 \leq l, 1 \leq i_2, j_2 \leq 2.$$

and is isomorphic to  $\ker(\text{PBr}_{l_1 l_2} \rightarrow \times_{l_1} \text{PBr}_l)$ .

*Proof:* Without loss of generality we may assume  $d(z, \alpha) = z^{l_1} - \alpha^{l_1}$ . The  $l_2$ -companion family in this case coincides with the model discriminant family for the family of function  $f_\alpha(x_1, x_2)$ . Hence the versal braid monodromy is almost that of the discriminant family of  $f_\alpha(x_1, x_2)$ . We need only remark that the group locally assigned to the singular fibre at the origin is generated by full twists on v-arcs as in the case of all other fibres. Then the same methods show then that

$$\tau_{i_1 i_2, j_1 j_2}, \quad 1 \leq i_1, i_2 \leq l_1, 1 \leq j_2 - i_2 \leq l_2$$

generate the braid monodromy. But this is just the claim up to inner conjugations.  $\square$

**Lemma 7.40** *The versal braid monodromy of the  $l_2$ -companion family associated to the discriminant family of the families of functions*

$$x^{l_1+1} - \alpha(l_1 + 1)x$$

is equal to the subgroup of  $\text{Br}_{l_1 l_2}$  generated by the  $l_2$ -companions of  $\sigma_{i_1, i_1^+}^3$  and  $\sigma_{i_1, j_1}^2$ ,  $1 < i_1^+ < j_1 \leq l_1$ .

*Proof:* The  $l_2$ -companion family coincides with the discriminant family of the family of functions

$$x_1^{l_1+1} - \alpha(l_1 + 1)x_1 + x_2^{l_2+1} - \varepsilon_2(l_2 + 1)x_2.$$

A set of generators for the versal braid monodromy has been given in lemma 7.22. By close inspection these are the  $l_2$ -companions of the given elements.  $\square$

## 7.5 $l$ -companion monodromy

The aim of this section is to compute the versal braid monodromy of the  $l$ -companion family  $\mathcal{E}^{(l)}$  associated to a family  $\mathcal{E}$  in terms of the braid monodromy data given for  $\mathcal{E}$ .

So we suppose  $\mathcal{E}$  is a discriminant family with critical parameters confined to the interior of the unit disc  $\mathbf{E}$ .

Moreover we suppose that suitable strongly distinguished systems of paths are given in the parameter disc with base point at  $\alpha = 1$  and in the corresponding fibre.

The first provides us with a basis for the fundamental group of the base of the punctured disc bundle associated to  $\mathcal{E}$ , the second is used to provide an isomorphism between the mapping class group of the punctured fibre at  $\alpha = 1$  with the abstract braid group  $\text{Br}_{l'}$  on  $l'$  strands in bijection to the punctures.

With these data given, the versal braid monodromy is the subgroup of  $\text{Br}_{l'}$  generated by the parallel transport of generators of the groups locally assigned to regular fibres close to singular ones.

From now on we pursue the strategy to build up the monodromy for  $\mathcal{E}^{(l)}$  from small pieces related to the critical parameters of  $\mathcal{E}$  in order to obtain a close relation between the monodromies over these pieces in terms of the results of the previous section. To do so we actually need only to choose  $\varepsilon$  sufficiently small, but to make this precise we start with the following technical tool:

**Definition 7.41:** Given a monic polynomial  $p$  in  $\mathbf{C}[z]$ , define the *root distance* by

$$\delta(p) = \min \{|z_1 - z_2| \mid (z - z_1)(z - z_2) \text{ divides } p\}.$$

**Definition 7.42:** Given a discriminant polynomial  $p \in \mathbf{C}[z, \alpha]$  which is monic in the variable  $z$ . Define the *discriminant root distance* to be the function

$$\delta(\alpha) := \delta(p_\alpha),$$

which is a continuous non-negative function.

**Lemma 7.43 ('minimum principle')** *If  $\delta$  has a strictly positive minimum on a bounded domain, then it attains the minimum on the boundary.*

*Proof:* Assume, that  $\delta$  attains the minimum at an interior point  $\alpha_0$ . Since by assumption  $\delta(\alpha_0) > 0$ , there is a neighbourhood  $U$  of  $\alpha_0$  such that  $p(z, \alpha)$  is given over  $U$  as the union of graphs of holomorphic functions  $\phi_i : U \rightarrow \mathbf{C}$ . Hence  $\delta$  is given over  $U$  by  $\min_{i \neq j} |\phi_i(\alpha) - \phi_j(\alpha)|$ . By the minimum principle for non-vanishing holomorphic functions, we conclude, that  $\delta$  is constant on  $U$ . But we can extend this argument to the whole closure of our domain, so  $\delta$  attains the minimum on the boundary, too.  $\square$

Suppose the critical points of  $\mathcal{E}$  are  $z_{i,\nu}, \alpha_i$ , which are the critical points for the projection to the parameter space of the zero set of the defining polynomial  $p = p(z, \alpha)$ .

We then want to introduce small discs  $U_i := B_\eta(\alpha_i)$  in the parameter base subjected to a lot of properties which actually all hold if we choose  $\eta$  appropriately small enough.

- i) Since the  $\alpha_i$  are distinct and in the interior of  $\mathbf{E}$ , we may impose  $U_i \subset \mathbf{E}$ , and  $U_i \cap U_{i'} = \{\}$  if  $i \neq i'$ .
- ii) The distinguished system of paths intersects the boundary of each  $U_i$  in a unique point  $\alpha'_i$ . So its restriction to the complement of the  $U_i$  is a distinguished system of paths for the  $U_i$ .
- iii) At a critical parameter  $\alpha_i$  the polynomial  $p_{\alpha_i}$  of a generic discriminant family factors as

$$\prod_{\nu} (z - z_{i,\nu})^{m_{i,\nu}}$$

according to the multiplicities  $m_{i,\nu}$  of the punctures  $z_{i,\nu}$ . So Hensel's lemma yields –  $\eta$  again assumed small enough – a factorization over  $U_i$

$$p = \prod q_{i,\nu}, \text{ with } q_{i,\nu}(z, \alpha_i) = (z - z_{i,\nu})^{m_{i,\nu}}.$$

- iv) There is a positive constant  $\delta_0$  such that for  $\alpha \in U_i$ :

$$q_{i,\nu}(z, \alpha) = 0 \implies z \in B_{\frac{1}{2}\delta_0}(z_{i,\nu})$$

and roots of different factors are at least  $2\delta_0$  apart from each other.

Next we introduce  $\mathbf{E}' := \mathbf{E} - \bigcup U_i$  and show that the discriminant root distance is bounded away from 0 on  $\mathbf{E}'$ .

**Lemma 7.44** *There is a positive constant  $\delta_\eta$  such that for the defining polynomial  $p$  of the discriminant family  $\mathcal{E}$*

$$\delta(\alpha) > \delta_\eta, \quad \forall \alpha \in \mathbf{E}'.$$

*Proof:* The set  $\bar{\mathbf{E}'}$  is a domain on which  $\delta$  is strictly positive. Moreover by the minimum principle for  $\delta$  the function attains its minimum on  $\partial \mathbf{E}'$  which we may take as  $2\delta_\eta$  to get the claim.  $\square$

We can now prove a kind of continuity of critical parameters.

**Lemma 7.45** *There is a locally generic perturbation  $\mathcal{E}^{(l)}$  of an  $l$ -companion family of  $\mathcal{E}$ , such that all critical parameters in  $\mathbf{E}$  are confined to  $\bigcup U_i$ .*

*Proof:* First we prove the claim for the  $l$ -companion family itself, which is defined by a polynomial  $p^{(l)} := \prod p(z - \xi_\varepsilon, \alpha)$ . Since a parameter is critical if and only if it is in the zero locus of the discriminant root distance  $\delta$ , it suffices to show that  $\delta$  does not vanish on  $\mathbf{E}'$ .

For  $\alpha \in \mathbf{E}'$  by lemma 7.44 the distance of roots of  $p_\alpha$  is bounded from below by some positive number  $\delta_\eta$ . If  $z_\kappa$  are the roots of  $p_\alpha$ , then  $z_\kappa + \xi_\nu \varepsilon$  are those of  $p_\alpha^{(l)}$ . The distance of a pair of roots is then

$$\begin{aligned} |z_{\kappa_1} + \xi_{\nu_1} \varepsilon - z_{\kappa_2} - \xi_{\nu_2} \varepsilon| &= |\xi_{\nu_1} \varepsilon - \xi_{\nu_2} \varepsilon| \neq 0, \quad \text{in case } \kappa_1 = \kappa_2, \nu_1 \neq \nu_2, \\ |z_{\kappa_1} + \xi_{\nu_1} \varepsilon - z_{\kappa_2} - \xi_{\nu_2} \varepsilon| &\geq |z_{\kappa_1} - z_{\kappa_2}| - |\xi_{\nu_1} \varepsilon - \xi_{\nu_2} \varepsilon| \\ &> \delta_\eta - 2\varepsilon, \quad \text{in case } \kappa_1 \neq \kappa_2. \end{aligned}$$

So if we impose  $\varepsilon < \frac{1}{2}\delta_\eta$ , we get the claim for  $\mathcal{E}^{(l)}$ .  $\square$

Now that we got little discs  $U_i$  carrying all the degenerations and hence the local contributions to the monodromy, we may take the next step and compute the monodromy over  $U_i$  for a local generic perturbation of  $\mathcal{E}^{(l)}|_{U_i}$ .

Let us first have a look at the monodromy of  $\mathcal{E}|_{U_i}$  in the fibre over  $\alpha'_i \in \partial U_i$ . The fibre contains the discs  $B_{\delta_0}(z_{i,\nu})$ . Outside polydiscs  $U_i \times B_{\delta_0}(z_{i,\nu})$  the fibration is locally trivial, therefore all mapping classes of the monodromy can be realized by diffeomorphisms supported on  $B_{\delta_0}(z_{i,\nu})$ . In fact they are given by the locally assigned group.

Let us choose now a strongly distinguished system of paths in the fibre of  $\mathcal{E}$  at  $\alpha'_i$ . We refine this system of paths to a system for the fibre of  $\mathcal{E}^{(l)}$  following the Hefez-Lazzeri construction. Note that the local triviality in the complement of  $U_i \times B_{\delta_0}(z_i)$  carries over to  $\mathcal{E}^{(l)}$ .

In the final step we want to obtain generators for the braid monodromy group of the  $l$ -companion family, which we identify with a subgroup of  $\mathrm{Br}_{ll}$  by the choice of the Hefez-Lazzeri refined system of paths in the fibre at  $\alpha = 1$ .

Since we have got a system of paths for the  $U_i$ , it suffices to understand parallel transport to  $\alpha = 1$  to get generators for the global monodromy group from local ones.

The chosen system of paths in the fibre at  $\alpha'_i$  for  $\mathcal{E}$  and its Hefez-Lazzeri refinement for  $\mathcal{E}^{(l)}$  yield system of paths in the fibres at  $\alpha = 1$  under parallel transport along the truncated system of paths in  $\mathbf{E}'$ . This has to be subjected to a thorough investigation in order to determine properties which are preserved:

**Remark 7.46:** Recall the existence of smooth 'bump' functions  $\chi, \chi_\varepsilon : \mathbf{C} \rightarrow \mathbf{R}$  for any real  $\varepsilon > 0$ :

$$\begin{aligned} \chi &: 0 \leq \chi(z) = \chi(|z|) \leq 1, \chi(z) = 0 \text{ if } |z| \geq 1, \chi(z) = 1 \text{ if } |z| \leq \frac{1}{2}, \\ \chi_\varepsilon &: \chi_\varepsilon(z) = \chi(z/\varepsilon), \end{aligned}$$

with support contained in the unit disc, resp. the disc of radius  $\varepsilon$ .

**Lemma 7.47** Suppose  $\gamma$  parameterizes a path in  $\mathbf{E}'$ . Given a vector field on  $\gamma^*\mathcal{E}$  such that the punctures form integral curves, then for  $\varepsilon$  sufficiently small a vector field can be found such that all parallels of distance bounded by  $\varepsilon$  form integral curves.

*Proof:* Let  $v(z, t)$  be the given field,  $z_i(t)$  the punctures, then define

$$v_\varepsilon(z, t) := v\left(z - \sum_i \chi_{2\varepsilon}(z - z_i(t))(z - z_i(t)), t\right).$$

The parallels of punctures of distance bounded by  $\varepsilon$  are given by functions  $z_{i,c}(t) := z_i(t) + c$  with  $|c| \leq \varepsilon$ .

Since  $|z_{i,c}(t) - z_j(t)| \geq 2\varepsilon$  for  $i \neq j$  (given that  $|z_i - z_j| \geq 3\varepsilon$ ) all summands except for the  $i^{th}$  vanish at  $z_{i,c}$ , hence

$$v_\varepsilon(z_{i,c}(t), t) = v(z_{i,c} - \chi_{2\varepsilon}(c)c, t) = v(z_i, t).$$

The claim is therefore proved, because  $\frac{\partial}{\partial t} z_{i,c} = \frac{\partial}{\partial t} z_i = v(z_i, t)$  by hypothesis.  $\square$

**Lemma 7.48** If parallel transport in  $\mathcal{E}$  over a path in  $\mathbf{E}'$  is realized by a diffeomorphism  $\phi$ , then parallel transport in  $\mathcal{E}$  and  $\mathcal{E}^{(l)}$  is realized by a diffeomorphism defined by

$$\phi^{(l)}(z) := \phi(z) + \sum_i \chi_{2\varepsilon}(z - z_i)(\phi(z_i) - z_i - \phi(z) + z).$$

*Proof:* If  $\phi$  is the integrated flow of a vector field  $v$ , just take  $\phi^{(l)}$  to be the integrated flow of the vector field provided by 7.47.  $\square$

We conclude that parallel transport commutes with Hefez-Lazzeri refinement: The parallel transport of a Hefez-Lazzeri refinement is the Hefez-Lazzeri refinement of the parallel transport.

The change of path system from the transported one to the given one on the  $\alpha = 1$  fibre is realized by the action of a suitable diffeomorphism. This induces a map on the braid group given by conjugation with the corresponding mapping class.

The same considerations apply to the family  $\mathcal{E}^{(l)}$  and the corresponding mapping class is the image under the band homomorphism  $\eta_l$  of the former.

So now we can determine the generators transported from the fibre at  $\alpha'_i$ . We know the transported generator of  $\mathcal{E}$ . This determines the conjugating mapping class up to stabilizers. The image under  $\eta_l$  is the conjugating mapping class up to stabilizers for the local generators of  $\mathcal{E}^{(l)}$ , if these are only among the  $l$ -companions of the generators for  $\mathcal{E}$ .

As in the proof of 7.37 the stabilizers do not influence the conjugation of  $l$ -companions, and we may conclude, that the braid monodromy of  $\mathcal{E}^{(l)}$  is generated by the  $l$ -companions of generators of the monodromy of  $\mathcal{E}$ .

**Lemma 7.49** *For  $n \geq 2$  the versal braid monodromy of a family of functions  $f_\alpha(x_1, \dots, x_n)$ , is generated by the  $l_n$ -companions of generators of the versal braid monodromy of the family of functions  $f'_\alpha$  given by*

$$x_1^{l_1+1} - \alpha(l_1 + 1)x_1 + \sum_{i=2}^{n-1} \left( x_i^{l_i+1} - \varepsilon_i(l_i + 1)x_i \right)$$

*Proof:* In the family  $f'_\alpha$  there are only ordinary multiple points and multiple points which are the images of critical points of type  $A_{l_n}$ . In fact the local pieces are isomorphic to the cases given in lemma 7.39 and lemma 7.40. So versal braid monodromy over the  $U_i$  is given by the  $l_n$ -companions of the generators of the locally assigned groups for  $f'_\alpha$ . Hence the claim follows by the discussion above.  $\square$

**Lemma 7.50** *The versal braid monodromy of the  $l$ -companion of the discriminant family  $\mathcal{E}$  contains the image under the  $l$ -band homomorphism  $\eta_l$  of the braid monodromy of  $\mathcal{E}$ .*

*Proof:* For any given element of the braid monodromy of  $\mathcal{E}$  there is a path in the base to which it is assigned. This path can be taken outside the  $U_i$  so we can conclude that in the associated  $l$ -companion family the image under  $\eta_l$  is assigned to this path.  $\square$



## Chapter 8

# bifurcation braid monodromy of elliptic fibrations

Given a family over a base  $T$  of smooth regular elliptic surfaces with an elliptic fibration induced by a global map to  $\mathbf{P}^1$ . Suppose all surfaces have smooth fibres only except for  $k$  fibres of type  $I_1$ ,  $l$  of type  $I_0^*$ , the divisor of critical values defined in  $T \times \mathbf{P}^1$  is a finite cover of  $T$  of degree  $k + l$ .

The associated monodromy homomorphism takes values in the braid group of the sphere. We show that its image is contained up to conjugacy in a subgroup associated to a family  $\mathcal{X}_{k,l}$  of elliptic fibrations.

On the other hand a fibration preserving topological automorphism of an elliptic fibration induces an mapping class of the base  $\mathbf{P}^1$  punctured at the base points of the singular fibres. We give a topological characterization of a subgroup of induced mapping classes which we show to contain the image of the braid monodromy homomorphism and to coincide with the image in case of the families  $\mathcal{X}_{6,l}$ .

### 8.1 introduction

The monodromy problems we want to discuss fit quite nicely into the following general scheme: Given an algebraic object  $X$  consider an algebraic family  $g : \mathcal{X} \rightarrow T$  such that a fibre  $g^{-1}(t_0)$  is isomorphic to  $X$  and such that the restriction to a connected subfamily  $g| : \mathcal{X}' \rightarrow T'$  containing  $X$  is a locally trivial  $C^\infty$  fibre bundle. If  $G$  is the structure group of this bundle, the geometric monodromy is the natural homomorphism  $\rho : \pi_1(T', t_0) \rightarrow G$ . A monodromy map with values in a group  $A$  is obtained by composition with some representation  $G \rightarrow A$ .

In the standard setting  $X$  is a complex manifold, e.g. a smooth complex projective curve. In this case  $\mathcal{X}$  is a flat family of compact curves containing  $X$ , the subfamily  $\mathcal{X}'$  contains only the smooth curves and is a locally trivial bundle of  $C^\infty$  surfaces with structure group the mapping class group  $Map(X)$ . From the geometric monodromy one can obtain the algebraic monodromy by means of the natural representation  $Map(X) \rightarrow Aut(H_1(X))$ .

In the present paper we investigate the monodromy of regular elliptic fibrations.

So  $X$  is an elliptic surface with a map  $f : X \rightarrow \mathbf{P}^1$  onto the projective line. We consider families  $g : \mathcal{X} \rightarrow T$  of elliptic surfaces containing  $X$  with a map  $f_T : \mathcal{X} \rightarrow \mathbf{P}^1$  which extends  $f$  and induces an elliptic fibration on each surface. Subfamilies  $\mathcal{X}'$  are to be chosen as  $C^\infty$  fibre bundles with structure group  $\text{Diff}_f(X)$ , the group of isotopy classes of diffeomorphisms which commute with the fibration map up to a diffeomorphism of the base.

This structure group has a natural representation in the mapping class group of the base  $\mathbf{P}^1$  punctured at the singular values of the fibration map  $f$ . The corresponding monodromy homomorphism is called the braid monodromy of the family  $\mathcal{X}'$ , since the mapping class group of the punctured base is isomorphic to a braid group of the sphere.

$$\text{Br}_n^s = \left\langle \sigma_1, \dots, \sigma_{n-1} \mid \begin{array}{l} \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \\ \sigma_i \sigma_j = \sigma_j \sigma_i \text{ if } |i - j| \geq 2, \\ (\sigma_1 \cdot \dots \cdot \sigma_{n-1})^n = 1 \end{array} \right\rangle.$$

From all possible  $C^\infty$ -types of elliptic fibrations we choose the subset only of those represented by elliptic fibrations with singular fibres of types  $I_1, I_0^*$  only, which is significant for any elliptic fibrations without multiple fibres deforms to such a fibration. We call a subgroup  $E$  of a spherical braid group the *braid monodromy group* of a fibration  $X$ , if  $E$  is the smallest subgroup (w.r.t. inclusion) such that for all admissible  $\mathcal{X}$  the image of the braid monodromy is a subgroup of  $E$  up to conjugation and prove:

**Main Theorem** *The braid monodromy group of a regular elliptic fibration  $X$  with no singular fibres except  $6k$  fibres of type  $I_1$  and  $l$  fibres of type  $I_0^*$  is a subgroup of  $\text{Br}_{6k+l}^s$  representing the conjugacy class of*

$$E_{6k,l}^s := \left\langle \sigma_{ij}^{m_{ij}}, i < j \mid m_{ij} = \begin{cases} 1 & \text{if } i, j \leq l \vee i \equiv j \pmod{2}, i, j > l \\ 2 & \text{if } i \leq l < j \\ 3 & \text{if } i, j > l, i \not\equiv j \pmod{2} \end{cases} \right\rangle.$$

(Here  $\sigma_{i,i+1} := \sigma_i$ , while for  $j > i + 1$  we define  $\sigma_{i,j} := \sigma_{j-1} \cdots \sigma_{i+1} \sigma_i \sigma_{i+1}^{-1} \cdots \sigma_{j-1}^{-1}$ .)

Along with the proof of the theorem we will notice that each mapping class in the braid monodromy group  $E_n^s(X)$  is represented by a diffeomorphism which can be lifted to a diffeomorphism of  $X$  inducing the trivial mapping class on some generic fibre. Hence we ask for the converse:

Does every diffeomorphism of  $X$ , isotopic to the identity mapping on some generic fibre, induce a mapping class of the punctured base which is in the monodromy group of  $X$ ?

A positive answer would yield a topological characterisation of the braid monodromy group!

In fact we show that the group of mapping classes induced in the said way coincides with the stabiliser group of an appropriate Hurwitz action. Then we use [26] to give an affirmative answer to the question above in case the number of fibres of type  $I_1$  does not exceed 6.

**Theorem 8.1** *The braid monodromy group  $E_n^s$  of a regular elliptic fibration  $X$  with no singular fibres except 6 fibres of type  $I_1$  and  $l$  fibres of type  $I_0^*$  coincides with the group of mapping classes of the punctured base of  $X$  which are induced by diffeomorphisms of  $X$  which respect the fibration, preserve a fibre  $F$  and induce a map on  $F$  isotopic to the identity.*

## 8.2 bifurcation braid monodromy

With each locally trivial bundle one can associate the structure homomorphism defined on the fundamental group of the base with respect to any base point. It takes values in the mapping classes of the fibre over the base point.

Given a curve  $C$  in the affine plane we can take a projection to the affine line which restricts to a finite covering  $C \rightarrow \mathbf{C}$ . The complement of the curve and its vertical tangents is the total space of a punctured disc bundle over the complement of the branch points in the affine line.

The structure homomorphism of this bundle is called the braid monodromy of the plane curve with respect to the projection, and it can be naturally regarded as a homomorphism from the fundamental group of the branch point complement to the braid group, since the latter is naturally isomorphic to the mapping class group of the punctured disc.

This definition is readily generalized to the case of a divisor in the Cartesian product of the affine or projective line with an irreducible base  $T$ . Then the structure homomorphism takes values in a braid group, resp. in a mapping class group of a punctured sphere which is naturally isomorphic to the spherical braid group  $\text{Br}_n^s$ .

In situations as we are interested in, such a divisor is defined as the locus of critical values of a family of algebraic functions of constant bifurcation degree with values in  $\mathbf{L} \cong \mathbf{P}^1$  or  $\mathbf{C}$ . Thus we give the relevant definitions:

**Definition:** A flat family  $\mathcal{X} \rightarrow T$  with an algebraic morphism  $f : \mathcal{X} \rightarrow \mathbf{L}$  is called a *framed family of functions*  $(\mathcal{X}, T, f)$ .

**Definition:** The *bifurcation set* of a framed family of functions over  $T$  is the smallest Zariski closed subset  $\mathcal{B}$  in  $T \times \mathbf{L}$  such that the diagonal map  $\mathcal{X} \rightarrow T \times \mathbf{L}$  is smooth over the complement of  $\mathcal{B}$ .

**Definition:** The *discriminant set* of a framed family of functions over  $T$  is the divisor in  $T$  such that the bifurcation set  $\mathcal{B}$  is an unbranched cover over its complement by the restriction of the natural projection  $T \times \mathbf{L} \rightarrow T$ .

**Definition:** A framed family of functions is called *of constant bifurcation degree* if the bifurcation set is a finite cover of  $T$ .

**Definition:** The *bifurcation braid monodromy* of a framed family of functions with constant bifurcation degree over an irreducible base  $T$  is defined to be the braid monodromy of  $\mathcal{B}$  in  $T \times \mathbf{L}$  over  $T$ .

Note that this definition of braid monodromy differs slightly from the definition given in the introduction but that the resulting objects are the same.

### 8.3 families of divisors in Hirzebruch surfaces

Given a Hirzebruch surface  $\mathbf{F}_k$  with a unique section  $C_{-k}$  of selfintersection  $-k$ , we consider families of divisors on which the ruling of the Hirzebruch surface defines families of functions with constant bifurcation type.

We can pull back divisors from the base along the ruling to get divisors on  $\mathbf{F}_k$  which we call vertical, among others the fibre divisor  $L$ .

Consider now the family of divisors on  $\mathbf{F}_k$  which consist of a vertical part of degree  $l$  and a divisor in the complete linear system of  $\mathcal{O}_{\mathbf{F}_k}(4C_{-k} + 3kL)$  called the horizontal part. It is a family parameterized by  $T = PH^0(\mathcal{O}_{\mathbf{P}^1}(l)) \times PH^0(\mathcal{O}_{\mathbf{F}_k}(4C_{-k} + 3kL))$  with total space

$$\mathcal{D}_{k,l} = \{(x, t) \in \mathbf{F}_k \times T \mid x \in D_t \subset \mathbf{F}_k\}.$$

Let  $T'$  be the Zariski open subset of  $T$  which is the base of the family  $\mathcal{D}'_{k,l}$  of divisors in  $\mathcal{D}_{k,l}$  with reduced horizontal part.

**Lemma 8.2** *The ruling on  $\mathbf{F}_k$  defines a morphism  $\mathcal{D}'_{k,l} \rightarrow \mathbf{P}^1$  by which it becomes a framed family of functions of constant bifurcation degree.*

*Proof:* The critical value set of the vertical part of a divisor is the divisor of which it is the pull back, thus it is constant of degree  $l$ .

The assumption on reducedness forces the horizontal part to be without fibre components. We may even conclude that a reduced horizontal part consists of  $C_{-k}$  and a disjoint divisor which is a branched cover of the base of degree 3. The critical values set is therefore the branch set which is of constant degree  $6k$ , and we are done.  $\square$

Note that we can consider the abstract braid group presented as

$$\text{Br}_n = \langle \sigma_1, \dots, \sigma_{n-1} \mid \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \sigma_i \sigma_j = \sigma_j \sigma_i \text{ if } |i - j| \geq 2 \rangle.$$

resp.  $\text{Br}_n^s$  as presented in the introduction to be realised by the mapping class group of the punctured disc, resp. sphere. Such an identification is given if each  $\sigma_i$  is realised by the half-twist on an embedded arc  $a_i$  connecting two punctures provided that  $a_i \cap a_{i+1}$  is a single puncture and  $a_i \cap a_j$  is empty if  $|i - j| \geq 2$ .

**Proposition 8.3** *The image of the bifurcation braid monodromy homomorphism of the family  $\mathcal{D}'_{k,l}$  is conjugated to the subgroup of  $\text{Br}_{6k+l}^s$ :*

$$E_{6k,l}^s := \left\langle \sigma_{ij}^{m_{ij}}, i < j \mid m_{ij} = \begin{cases} 1 & \text{if } i, j \leq l \vee i \equiv j \pmod{2}, i, j > l \\ 2 & \text{if } i \leq l < j \\ 3 & \text{if } i, j > l, i \not\equiv j \pmod{2} \end{cases} \right\rangle.$$

The proof of this proposition and a couple of preparatory results will take the rest of the section.

First note that our whole concern lies in the understanding of the bifurcation set  $\mathcal{B}$  in  $T' \times \mathbf{P}^1$  with its projection to  $T'$ . As an approximation we will consider families of affine plane curves given by families of polynomials in affine coordinates  $x, y$  with the regular map induced by the affine projection  $(x, y) \mapsto x$ .

Their bifurcation sets are contained in the Cartesian product of the family bases with the affine line  $\mathbf{C}$ , and it will soon be shown that this pair can be induced from  $(T' \times \mathbf{P}^1, \mathcal{B})$ . Eventually we can extract all necessary information from such families to prove our claim.

**Lemma 8.4** *Consider  $y^3 - 3p(x)y + 2q(x)$  as a family of polynomial functions  $\mathbf{C}^2 \times T \rightarrow \mathbf{C}$  parametrised by a base  $T$  of pairs  $p, q$  of univariate polynomials. Then the bifurcation set is the zero set of  $g(x) := p^3(x) - q^2(x)$ , the discriminant set is the zero set of the discriminant of  $g$  with respect to  $x$ .*

*Proof:* The bifurcation divisor is cut out by the discriminant polynomial of  $y^3 - 3p(x)y + 2q(x)$  with respect to  $y$ . The first claim is then immediate since  $g$  is proportional to the corresponding Sylvester determinant:

$$\begin{vmatrix} 1 & 0 & -3p & 2q \\ & 1 & 0 & -3p & 2q \\ 3 & 0 & -3p \\ & 3 & 0 & -3p \\ & & 3 & 0 & -3p \end{vmatrix}$$

For the second claim we only note that a pair  $p, q$  belongs to the discriminant set if and only if  $p^3 - q^2$  has a multiple root hence this locus is cut out by the discriminant of  $g$  with respect to  $x$ .  $\square$

**Lemma 8.5** *The discriminant locus of a family  $y^3 + 3r(x)y^2 - 3p(x)y + 2q(x)$  is the union of the degeneration component of triples  $p, q, r$  defining singular curves and the cuspidal component of triples defining polynomial maps with a degenerate critical point.*

*Proof:* In general a branched cover of  $\mathbf{C}$  has not the maximal number of branch points only if the cover is singular, or the number of preimages of a branch point differs by more than one from the degree of the branching. The second alternative occurs only if there is a degenerate critical point in the preimage or if there are two critical points. Since the last case can not occur in a cover of degree only three we are done.  $\square$

**Lemma 8.6** *Given the family  $y^3 + 3r(x)y^2 - 3p(x)y + 2q(x)$  the cuspidal component of the discriminant is the zero set of the resultant of  $p(x) + r^2(x)$  and  $2q(x) - r^3(x)$  with respect to  $x$ .*

*Its equation - considered a polynomial in the variable  $\lambda_0$  - is of degree  $n$  with coprime coefficients if*

$$p(x) = \sum_{i=0}^d \lambda_i x^i, \quad q(x) = x^n + \sum_{i=0}^{n-1} \xi_i x^i, \quad r(x) = \sum_{i=0}^{\lfloor n/3 \rfloor} \zeta_i x^i.$$

*Proof:* The cuspidal discriminant is the locus of all parameters for which there is a common zero of  $f, \partial_y f, \partial_y^2 f$ . Since  $\partial_y^2 f = 0$  is linear in  $y$ , we can eliminate  $y$  and

get the resultant of  $p(x) + r^2(x)$  and  $2q(x) - r^3(x)$  with respect to  $x$ .

By the degree bound on  $q$  and  $r$  the discriminant equation is the resultant of a matrix in which the variable  $\lambda_0$  occurs exactly  $n$  times. Moreover the diagonal determines the coefficient of  $\lambda_0^n$  to be  $(1 - \zeta_{n/3}^3)^{\max(d, 2n/43)}$  resp. 1 depending on whether  $n/3 \in \mathbf{Z}$  or not. Even in the first case the coefficients are coprime since the resultant is not divisible by  $(1 - \zeta_{n/3}^3)$ .  $\square$

**Lemma 8.7** *For the family  $y^3 + 3r(x)y^2 - 3p(x)y + 2q(x)$  the degeneration component of the discriminant is the locus of triples for which there is a common zero in  $x, y$  of the polynomial and its two partial derivatives. Its equation - considered a polynomial in the variable  $\xi_0$  - is monic of degree  $2n - 2$  if*

$$p(x) = \sum_{i=0}^d \lambda_i x^i, \quad q(x) = x^n + \sum_{i=0}^{n-1} \xi_i x^i, \quad r(x) = \sum_{i=0}^{\lfloor n/3 \rfloor} \zeta_i x^i.$$

*Proof:* The degeneration locus is given by the Jacobian criterion as claimed. The equation of the discriminant of the subdiagonal unfolding of the quasihomogeneous singularity  $y^3 - x^n$  is known to be the quasihomogeneous and of degree  $2n - 2$  in  $\xi_0$ . Since the unfolding over the  $\xi_0$ -parameter is a Morsification the coefficient of  $\xi_0^{2n-2}$  must be constant.  $\square$

**Lemma 8.8** *The bifurcation braid monodromy of the family  $y^3 - 3\lambda y + 2(x^n + \xi)$  maps onto a subgroup of  $\text{Br}_{2n}$  which is conjugated to the subgroup generated by*

$$(\sigma_1 \cdots \sigma_{2n-1})^3, (\sigma_{2n-2, 2n} \cdots \sigma_{2, 4})^{n+1}, (\sigma_{2n-3, 2n-1} \cdots \sigma_{1, 3})^{n+1}.$$

*Proof:* As one can show with the help of the preceding lemmas, the discriminant locus is the union of the degeneration locus and the cuspidal component which are cut out respectively by the polynomials  $\lambda^3 - \xi^2$  and  $\lambda$ .

By Zariski/van Kampen the fundamental group of the complement with base point  $(\lambda, \xi) = (1, 0)$  is generated by the fundamental group of the complement restricted to the line  $\lambda = 1$  and the homotopy class of a loop which links the line  $\lambda = 0$  once.

For the pair  $(1, 0)$  the set of regular values of the polynomial consists of the affine line punctured at the  $(2n)$ -th roots of unity, which we number counterclockwise, 1 the first puncture. To express the bifurcation braid group in terms of abstract generators, we identify the elements  $\sigma_i$  with the half twist on the circle segment between the  $i$ -th and  $i + 1$ -st puncture.

For the line  $\lambda = 1$  the bifurcation locus is given by  $(x^n + \xi - 1)(x^n + \xi + 1)$ . This locus is smooth but branches of degree  $n$  over the base at  $\xi = \pm 1$ . The corresponding monodromy transformations are the second and third transformation given in the claim.

Associated to the degeneration path  $(\lambda, \xi) = (1 - t, \sqrt{-1}t)$ ,  $t \in [0, 1]$  there is a loop in the complex line  $\lambda = 1 + \sqrt{-1}\xi$  which links the line  $\lambda = 0$ . For this degeneration the bifurcation divisor is regular and contains points of common absolute value determined by  $t$  only, except for  $t = 1$  where it has  $n$  ordinary cusps with horizontal tangent cone. Since a cusp corresponds to a triple half twist and the

first and second puncture merge in the degeneration, the monodromy transformation for our loop is the first braid of the claim.  $\square$

**Lemma 8.9** *The bifurcation braid monodromy of the family  $y^3 - 3\lambda y + 2(x^n + \xi + \varepsilon x)$ ,  $\varepsilon$  small and fix, is in the conjugation class of the subgroup of the braid group generated by*

$$(\sigma_1 \sigma_3 \cdots \sigma_{2n-1})^3, \sigma_{i,i+2}, i = 1, \dots, 2n-2.$$

*Proof:* The discriminant locus in the  $\lambda, \xi$  parameter plane consists again of the cuspidal component  $\lambda = 0$  and the degeneration component.

Since the perturbation  $\varepsilon$  is arbitrarily small, some features of the family of lemma 8.8 are preserved. The conclusion of the Zariski/van Kampen argument still holds, each braid group generators  $\sigma_i$  is now realized as half twist on segments of a slightly distorted circle, and the loop linking  $\lambda = 0$  is only slightly perturbed. So the monodromy transformation associated to this loop is formally the same as before, the first braid in the claim.

The dramatic change occurs in the bifurcation curve over the line  $\lambda = 1$ . Now the bifurcation locus is the union of two disjoint smooth components each of which branches simply of degree  $n$  with all branch points near  $\xi = 1$ , resp.  $\xi = -1$ . Since the local model  $x^n + \varepsilon x$  has the full braid group as its monodromy group, the monodromy along  $\lambda = 1$  is generated by the elements  $\sigma_{i,i+2}$  as claimed.  $\square$

**Lemma 8.10** *The bifurcation braid monodromy of the family*

$$y^3 - 3(\lambda + \lambda_1 x)y + 2(x^n + \xi + \xi_1 x)$$

*is in the conjugation class of the subgroup generated by*

$$\sigma_i^3, i \equiv 1(2), \sigma_{i,i+2}, 0 < i < 2n-1.$$

*Proof:* The components of the discriminant are the degeneration component and the cuspidal component.

The line  $\lambda = 1, \lambda_1 = 0, \xi_1 = \varepsilon$  small and fix, is generic for the degeneration component and we may conclude from lemma 8.9 that there are elements in the fundamental group of the discriminant complement with respect to  $(\lambda, \lambda_1, \xi, \xi_1) = (1, 0, 0, 0)$  which map to  $\sigma_{i,i+2}$  as in lemma 8.9.

Since the line  $\xi = i, \xi_1 = \lambda_1 = 0$  is transversal for the cuspidal component so are parallel lines with  $\lambda_1 = \varepsilon'$  small and fix. The bifurcation curve is then given by  $(\lambda + \varepsilon' x)^3 = (x^n + \sqrt{-1})^2$ . For  $\lambda = 0$  the critical values are distributed in pairs along a circle in the affine line which merge pairwise for  $\lambda_1 \rightarrow 0$ .

But for  $\varepsilon'$  sufficiently small and by varying  $\lambda$  to an appropriate small extend such that the degeneration component is not met, there are  $n$  obvious degenerations when  $\lambda$  is of the same modulus as  $\varepsilon'$  and  $\lambda + \varepsilon' x$  is a factor of  $x^n + \sqrt{-1}$ . By the local nature of the degeneration a degree argument shows that these are all possibilities. Moreover one can easily see that the corresponding monodromy transformations are triple half twist for each one of the pairs.

Moreover these twists are transformed to the transformations  $\sigma_{i,i+1}^3$ ,  $i$  odd, when transported along  $(1-t, \sqrt{-1}t, 0, \varepsilon't)$ ,  $t \in [0, 1]$  to the chosen reference point.

The monodromy group is thus completely determined since the fundamental group is generated by elements which map to the given group under the monodromy homomorphism.  $\square$

**Lemma 8.11** *Let a family of plane polynomials be given which is of the form*

$$y^3 + 3\left(\sum_{i=0}^{d_r} \zeta_i x^i\right)y^2 - 3\left(\sum_{i=0}^{d_p} \lambda_i x^i\right)y + 2\left(x^n + \sum_{i=0}^{n-1} \xi_i x^i\right),$$

$$2 \leq n, 0 < 3d_p \leq 2n, 3d_r \leq n.$$

*Then the bifurcation braid monodromy group is in the conjugation class of the subgroup of the braid group generated by*

$$\sigma_i^3, i \equiv 1(2), \sigma_{i,i+2}, 0 < i < 2n-1.$$

*Proof:* Since the family considered in the previous lemma is a subfamily now and has the claimed monodromy, we have to show that the new family has no additional monodromy transformations.

In the proof above we have seen that the cuspidal component is cut in  $n$  points by a line in  $\lambda_0$  direction. The component is reduced since its multiplicity at the origin is  $n$ , too, by lemma 8.6. The degeneration component is reduced by the analogous argument relying on lemma 8.7 and transversally cut in  $2n-2$  points by a line in  $\xi_0$  direction. Hence by Zariski/van Kampen arguments as proved in [4] the fundamental group of the discriminant complement of the subfamily surjects onto the fundamental group of the family considered now.  $\square$

**Lemma 8.12** *Define a subgroup of the braid group  $\text{Br}_{2n+l}$ :*

$$E_{2n,l} := \left\langle \sigma_{i,j}^{m_{ij}}, i < j \mid m_{ij} = \begin{cases} 1 & \text{if } i, j \leq l \vee i \equiv j(2), i > l \\ 2 & \text{if } i \leq l < j \\ 3 & \text{if } i, j > l \wedge i \not\equiv j(2) \end{cases} \right\rangle.$$

*Then the same subgroup is generated also by*

$$\sigma_i, i < l, \sigma_{i,i+2}, l < i, \sigma_{i,j}^2, i \leq l < j, \sigma_i^3, i > l, i \not\equiv l(2).$$

*Proof:* We have to show that the redundant elements can be expressed in the elements of the bottom line. This is immediate from the following relations ( $i < j$ ):

$$\begin{aligned} \sigma_{i,j} &= \sigma_{j-1}^{-1} \cdots \sigma_{i+1}^{-1} \sigma_i \sigma_{i+1} \cdots \sigma_{j-1}, \quad j \leq l, \\ \sigma_{i,j} &= \sigma_{j-2,j}^{-1} \cdots \sigma_{i+2,i+4}^{-1} \sigma_{i,i+2} \sigma_{i+2,i+4} \cdots \sigma_{j-2,j}, \quad l < i, i \equiv j(2), \\ \sigma_{i,j}^3 &= \sigma_{j-2,2}^{-1} \cdots \sigma_{i+1,i+3}^{-1} \sigma_i^3 \sigma_{i+1,i+3} \cdots \sigma_{j-2,j}^{-1}, \quad l < i, i \not\equiv l, j(2), \\ \sigma_{i,j}^3 &= \sigma_{j-2,2}^{-1} \cdots \sigma_{i+1,i+3} \sigma_{i-1,i+1} \sigma_{i-1,i+1}^3 \sigma_{i-1,i+1} \sigma_{i+1,i+3} \cdots \sigma_{j-2,j}, \quad l < i, j \not\equiv l, i(2). \end{aligned}$$

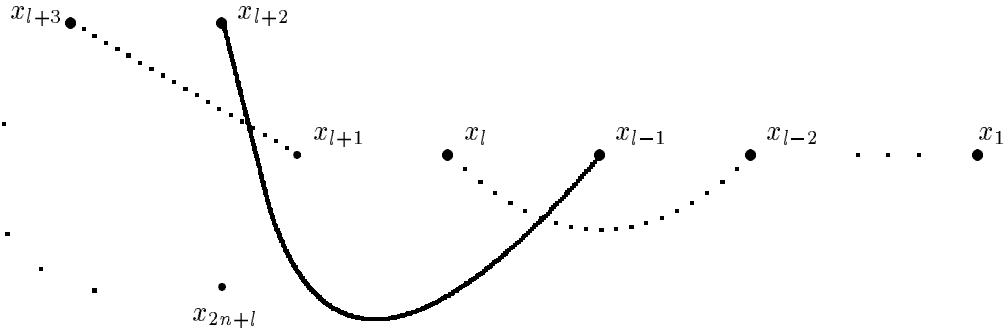
$\square$

**Lemma 8.13** Consider a family  $(y^3 - 3p(x)y + 2q(x)) a(x)$  parametrised by triples  $p, q, a$ , with  $p$  from the vector space of univariate polynomials of degree at most  $2n/3$ ,  $q, a$  from the affine space of monic polynomials of degree  $n$  and  $l$  respectively. Then the subgroup  $E_{2n,l}$  of  $\text{Br}_{2n+l}$  is conjugate to a subgroup of the image of the bifurcation braid monodromy.

*Proof:* We choose our reference divisor to be  $(y^3 - 3y + 2x^n) \prod_i^l (x - l - 2 + i)$  with corresponding bifurcation set  $x_i = l + 2 - i, i \leq l$  on the real axis and  $x_{l+1} = 1$  and the  $x_i, i > l + 1$  equal to the  $2k^{\text{th}}$ -roots of unity in counterclockwise numbering. We identify the elements  $\sigma_{i,j}$  of the braid group with the half twist on arcs between  $x_i, x_j$ , which are chosen to be

- i) a circle segment through the lower half plane, if  $i, j \leq l$ ,
- ii) a circle secant in the unit disc, if  $i, j > l$ ,
- iii) the union of a secant in the unit disc to a point on its boundary between  $x_{2n+l}$  and 1 with an arc through the lower half plane, if  $i \leq l < j$ .

(Of each kind we have depicted one in the following figure.)



Since keeping the horizontal part  $y^3 - 3y + 2x^n$  fix, the bifurcation divisor of the vertical is equivalent to that of the universal unfolding of the function  $x^l$  we have the elements  $\sigma_i, i < l$  in the braid monodromy. These elements are obtained for example in families

$$a(x) = ((x - l + i - 3/2)^2 + \lambda) \prod_{j \neq i, i+1}^l (x - l - 2 + j).$$

The second set of elements,  $\sigma_{i,j}^2, i \leq l < j$  is obtained by families of the kind

$$(y^3 - 3y + 2x^n)(x - l - 2 + i - \lambda) \prod_{j \neq i}^l (x - l - 2 + j)$$

since the zero  $l + 2 - i + \lambda$  may trace any given path in the range of the projection, in particular that around an arc on which the full twist  $\sigma_{i,j}^2$  is performed.

Finally varying the horizontal part as in lemma 8.10 while keeping the  $a(x)$  factor fix proves that the braid group elements  $\sigma_{i,i+2}, l < i$  and  $\sigma_i, l < i, i \not\equiv l(2)$  are in the image of the monodromy. So we may conclude that this image contains  $E_{2n,l}$  up to conjugacy.  $\square$

*Proof of prop. 8.3:* Denote by  $S$  the Zariski open subset of  $T'$  which parameterizes divisors of the family  $\mathcal{D}'_{k,l}$  which have no singular value at a point  $\infty \in \mathbf{P}^1$ . The corresponding family in  $\mathbf{F}_k \times S$  may then be restricted to a family  $\mathcal{F}_{k,l}$  in  $\mathbf{C} \times \mathbf{C} \times S$ , where  $\mathbf{F}_k$  is trivialized as  $\mathbf{C} \times \mathbf{C}$  in the complement of the negative section  $C_{-k}$  and the fibre over  $\infty$ . By construction  $\mathcal{F}_{k,l}$  has constant bifurcation degree.

Consider now the family of polynomials

$$(y^3 + 3r(x)y^2 - 3p(x)y + 2q(x)) a(x),$$

where  $r, p, q, a$  are taken from the family of all quadruples of polynomials in one variable subject to the conditions that

- i)  $r, p$  are of respective degrees  $k$  and  $2k$ ,
- ii)  $q$  is monic of degree  $3k$ ,  $a$  is monic of degree  $l$
- iii) the discriminant of  $y^3 + 3r(x)y^2 - 3p(x)y + 2q(x)$  is not identically zero.

This family can be naturally identified with  $\mathcal{F}_{k,l}$ . By lemma 8.13, up to conjugacy,  $E_{6k,l}$  is contained in the monodromy image  $\rho(\pi_1(S \setminus \text{Discr}(\mathcal{F}_{k,l})))$ .

For the converse we note that the bifurcation set of the family decomposes into the bifurcation sets  $\text{Bif}_h$  of the horizontal part  $y^3 + 3r(x)y^2 - 3p(x)y + 2q(x)$  and  $\text{Bif}_v$  of the vertical part  $a(x)$ . Hence the monodromy is contained in the subgroup  $\text{Br}_{(6k,l)}$  of braids which do not permute points belonging to different components.  $\text{Br}_{(6k,l)}$  has natural maps to  $\text{Br}_{6k}$  and  $\text{Br}_l$  which commute with the braid monodromies of both bifurcation set considered on their own.

The discriminant decomposes into the discriminants of  $\text{Bif}_h$ ,  $\text{Bif}_v$  and the divisor of parameters for which  $\text{Bif}_h \cap \text{Bif}_v$  not empty. They give rise in turn to braids which can be considered as elements in

$$\text{Br}_{6k}, \text{Br}_l \text{ resp. } \text{Br}_{(6k,l)}^{0,0} := \{\beta \in \text{Br}_{(6k,l)} \mid \beta \text{ trivial in } \text{Br}_{6k} \times \text{Br}_l\}.$$

Now with lemma 8.12 we can identify  $E_{6k,l}$  as the subgroup of  $\text{Br}_{6k+l}$  generated by  $E_{6k} \subset \text{Br}_{6k}$ ,  $\text{Br}_l$  and  $\text{Br}_{(6k,l)}^{0,0}$  which are generated in turn by the elements

$$\{\sigma_{i,i+2}, \sigma_i^3, l < i\}, \{\sigma_i, i < l\}, \{\sigma_{i,j}^2, i \leq l < j\} \text{ resp.}$$

And by lemma 8.11 the image can not contain more elements.

Since the bifurcation diagram of  $\mathcal{F}_{k,l}$  embeds in the bifurcation diagram of  $\mathcal{D}'_{k,l}$ , with complement of codimension one, there is a commutative diagram

$$\begin{array}{ccc} \pi_1(S \setminus \text{Discr}(\mathcal{F}_{k,l})) & \longrightarrow & \pi_1(T' \setminus \text{Discr}(\mathcal{D}'_{k,l})) \\ \downarrow & & \downarrow \\ E_{6k,l} & & \text{Br}_{6k+l}^s \end{array}$$

from which we read off our claim.  $\square$

**Corollary 8.14** *For any element  $\beta$  in the braid monodromy group of  $\mathcal{D}'_{k,l}$  there is a diffeomorphism of the base  $\mathbf{P}^1$  which fixes a neighbourhood of  $\infty \in \mathbf{P}^1$  and which represents the mapping class  $\beta$ .*

*Proof:* The element  $\beta$  is image of an element  $\beta'$  in the braid monodromy of the bifurcation diagram of  $\mathcal{F}_{k,l}$ . The bifurcation set does not meet the boundary so integration along a suitable vector field yields a realisation of  $\beta'$  as a diffeomorphism acting trivially on a neighbourhood of the boundary. Its trivial extension to the point  $\infty$  is the diffeomorphism sought for.  $\square$

## 8.4 families of elliptic surfaces

In this section we start investigating families of regular elliptic surfaces for which the type of singular fibres is restricted to  $I_1$  and  $I_0^*$ . We will go back and forth between a family of elliptic fibrations, its associated family of fibrations with a section and a corresponding Weierstrass model of the latter, so we note some of their properties:

**Proposition 8.15** *Given a family of elliptic fibrations with constant bifurcation type over an irreducible base  $T$ , there is a family of elliptic fibrations with a section, such that the bifurcation sets of both families coincide.*

*Proof:* Given a family as claimed there is the associated family of Jacobian fibrations, cf. [15, I.5.30]. The bifurcation sets of both families coincide.  $\square$

In turn, for each family of elliptic fibrations with a section there is a corresponding family of Weierstrass fibrations, cf. Seiler [39].

A regular Weierstrass fibration  $W$  is defined by an equation

$$wz^2 = 4y^3 - 3Pw^2y + 2Qw^3$$

in the projectivisation of the vector bundle  $\mathcal{O} \oplus \mathcal{O}(2\chi) \oplus \mathcal{O}(3\chi)$  over the projective line  $\mathbf{P}^1$  where  $\chi$  is the holomorphic Euler number of the fibration,  $w, y, z$  are ‘homogeneous coordinates’ of the bundle, and  $P, Q$  are sections of  $\mathcal{O}(4\chi), \mathcal{O}(6\chi)$  respectively.

So  $W$  is a double cover of the Hirzebruch surface  $\mathbf{F}_{2\chi} = \mathbf{P}(\mathcal{O} \oplus \mathcal{O}(2\chi))$  branched

along the section  $\sigma_{2\chi}$  and the divisor in its complement  $\mathcal{O}(2\chi)$  defined by the equation  $y^3 - 3Py + 2Q = 0$ .

A framed family of Weierstrass fibrations over a parameter space  $T$  is given by data as before where now  $P, Q$  are sections of the pull backs to  $T \times \mathbf{P}^1$  of  $\mathcal{O}(4\chi), \mathcal{O}(6\chi)$  such that for each parameter  $\lambda \in T$  they define a Weierstrass fibration. In the sequel  $P, Q$  are referred to as the coefficient data of the Weierstrass family.

**Lemma 8.16** *Let  $\mathcal{W}$  be the Weierstrass family associated to a framed family over  $T$  of regular elliptic fibrations in which all surfaces have no singular fibres except for  $l$  of type  $I_0^*$  and  $6k$  of  $I_1$  with coefficient data  $P, Q$ , then there are three families of sections  $a, p, q$  of  $\mathcal{O}(l), \mathcal{O}(2k), \mathcal{O}(3k)$  respectively, such that  $p, q$  have no common zero,*

$$p \cdot a^2 = P, \quad q \cdot a^3 = Q,$$

and the bifurcation set is given by

$$a(p^3 - q^2) = 0 \subset T \times \mathbf{P}^1.$$

*Proof:* By the classification of Kas [21] at base points of regular fibres the discriminant  $P^3 - Q^2$  does not vanish, at base points of fibres of type  $I_1$  the discriminant vanishes but neither  $P$  nor  $Q$  and at base points of fibres of type  $I_0^*$  the vanishing order of  $P$  is two, the vanishing order of  $Q$  is three.

Since by hypothesis the locus of base points of singular fibres of type  $I_0^*$  form a family of point divisors of degree  $l$  there is a section  $a$  of  $\mathcal{O}(l)$  such that  $P$  has a factor  $a^2$  and  $Q$  a factor  $a^3$ .

With  $\deg P = 2(l+k), \deg Q = 3(l+k)$  we get the other degree claims.

Finally the discriminant of the Weierstrass fibration is given by  $P^3 - Q^2$  which has – by the above – the same zero set as  $a(p^3 - q^2)$ .  $\square$

**Remark:** In the situation of the lemma, a family of divisors is given for  $\mathbf{F}_k$  by the equation  $a(y^3 - 3pw^2y + 2qw^3) = 0$ ,  $a$  cutting out the vertical part. The double cover along this divisor is a family of fibrations obtained from the original family by contracting all smooth rational curves of selfintersection  $-2$ , of which there are four for each fibre of type  $I_0^*$ .

We are now prepared to come back to the main theorem:

*Proof of the main theorem:* Given any framed family of regular elliptic fibrations containing  $X$  we consider a Weierstrass model  $\mathcal{W}$  of the associated Jacobian family. Since  $\mathcal{W}$  is again framed there is an induced family of divisors on a Hirzebruch surface obtained as before.

This family of divisors is a pull back from the space  $\mathcal{D}_{k,l}$  so the monodromy is a subgroup of the bifurcation monodromy of Hirzebruch divisors.

On the other hand for the family of triples of polynomials  $p(x), q(x), a(x)$  with  $p$  of degree at most  $2k$  and  $q, a$  monic of degree  $3k$  respectively  $l$ , we can form the family given by

$$z^2 = y^3 - 3p(x)a^2(x) + 2q(x)a^3(x),$$

which is Weierstrass in the complement of parameters where  $a(x)(p^3(x) - q^2(x))$  has a multiple root or vanishes identically. At least after suitable base change, cf. [15, p. 163], this Weierstrass family has a simultaneous resolution yielding a family  $\mathcal{X}_{k,l}$  of elliptic surfaces with a section.

The Jacobian of  $X$  is contained in  $\mathcal{X}_{k,l}$ , since its Weierstrass data consist of sections  $P, Q$  which are factorisable as  $a^2p, a^3q$  according to lemma 8.16 and after the choice of a suitable  $\infty$  this data can be identified with polynomials in this family.

The fibration  $X$  is deformation equivalent to its Jacobian with constant local analytic type, cf. [15, thm. I.5.13] and hence of constant fibre type. The monodromy group therefore contains the bifurcation monodromy group of divisors on Hirzebruch surfaces  $\mathcal{D}_{k,l}$  and so the two groups even coincide.  $\square$

Regarding elements in the braid monodromy as mapping classes again they can be shown to be induced by diffeomorphism of the elliptic fibration, but more is true in fact:

**Proposition 8.17** *For each braid  $\beta$  in the framed braid monodromy group there is a diffeomorphism of the elliptic fibration which preserves the fibration, induces  $\beta$  on the base and the trivial mapping class on some fibre.*

*Proof:* As we have seen in the corollary to prop. 8.3 we can find a representative  $\varphi$  for the braid  $\beta$  by careful integration of a suitable vector field such that  $\varphi$  is the identity next to a point  $\infty$ .

In [15, II.1.2] there is a proof for families of nodal elliptic fibrations and sufficient hints for more general families of constant singular fibre types, that a horizontal vector field on the total family can be found which fails to be a lift only in arbitrarily small neighbourhoods of singular points on singular fibres. Integration of such a vector field yields a diffeomorphism  $\Phi$  which is a lift of  $\varphi$ .

We have seen that the monodromy generators arising from the horizontal part can be realized over a suitable polydisc parameter space, cf. lemma 8.10. Since the vertical part as in lemma 8.13 does not have any effect on the fibre  $F_\infty$  over  $\infty$  we can conclude that this fibration family is the trivial family next to  $F_\infty$ . So we apply the argument above to get a lift  $\Phi$  which induces the trivial mapping class on  $F_\infty$ .  $\square$

## 8.5 Hurwitz stabilizer groups

In this section we determine the stabilizers of the action of the braid group  $\text{Br}_n$  on homomorphisms defined on the free group  $F_n$  generated by elements  $t_1, \dots, t_n$ . The action is given by precomposition with the Hurwitz automorphism of  $F_n$  associated to a braid in  $\text{Br}_n$ :

$$\text{Br}_n \rightarrow \text{Aut } F_n : \quad \sigma_{i,i+1} \mapsto \left( t_j \mapsto \begin{cases} t_j & j \neq i, i+1 \\ t_i t_j t_i^{-1} & j = i \\ t_i & j = i+1 \end{cases} \right).$$

We start with a result from [26]:

**Proposition 8.18** Let  $F_n := \langle t_i, 1 \leq i \leq n \mid \quad \rangle$  be the free group on  $n$  generators, define a homomorphism  $\phi_n : F_n \rightarrow \text{Br}_3 = \langle a, b \mid aba = bab \rangle$  by

$$\phi_n(t_i) = \begin{cases} a & i \text{ odd} \\ b & i \text{ even} \end{cases}$$

and let  $\text{Br}_n$  act on homomorphisms  $F_n \rightarrow \text{Br}_3$  by Hurwitz automorphisms of  $F_n$ . Then the stabilizer group  $\text{Stab}_{\phi_n}$  contains the braid subgroup

$$E_n = \langle \sigma_{i,j}^{m_{ij}} \mid m_{ij} = 1, 3 \text{ if } j \equiv i, \text{ resp. } i \not\equiv j \pmod{2} \rangle$$

with  $E_n = \text{Stab}_{\phi_n}$ , if  $n \leq 6$ .

Note that the action in [26] was defined on tuples  $(\phi_n(t_1), \dots, \phi_n(t_n))$  but that it is obviously equivalent to the action considered here.

This result can now be applied to find stabilizers of similar homomorphisms:

**Proposition 8.19** Let  $F_n := \langle t_i, 1 \leq i \leq n \mid \quad \rangle$  be the free group on  $n$  generators, define a homomorphism  $\psi_n : F_n \rightarrow \text{SL}_2 \mathbf{Z}$  by

$$\psi_n(t_i) = \begin{cases} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} & i \text{ odd} \\ \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix} & i \text{ even} \end{cases}$$

and let  $\text{Br}_n$  act on homomorphisms  $F_n \rightarrow \text{SL}_2 \mathbf{Z}$  by Hurwitz automorphisms of  $F_n$ . Then the stabilizer group  $\text{Stab}_{\psi_n}$  of  $\psi_n$  is equal to the stabilizer group  $\text{Stab}_{\phi_n}$  of  $\phi_n$ .

*Proof:* Both groups,  $\text{SL}_2 \mathbf{Z}$  and  $\text{Br}_3$ , are central extensions of  $\text{PSL}$ , and both  $\phi_n$  and  $\psi_n$  induce the same homomorphism  $\chi_n : F_n \rightarrow \text{PSL}$ . Of course  $\text{Stab}_{\chi_n}$  contains  $\text{Stab}_{\phi_n}$  and  $\text{Stab}_{\psi_n}$  and thus our claim is proved as soon as we can show the opposite inclusions.

First note that the braid action defined on homomorphisms as above is equivalent to the Hurwitz action on the tuples of images of the specified generators  $t_i \in F_n$ , hence the braid action will not change the conjugation class of these images.

Now let  $\beta$  be a braid in  $\text{Stab}_{\chi_n}$ . Then  $\phi \circ \beta(t_i) = (ab)^{3k_i} \phi(t_i)$  since  $(ab)^3$  is the fundamental element of  $\text{Br}_3$  which generates the center of  $\text{Br}_3$  and thus the kernel of the extension  $\text{Br}_3 \rightarrow \text{PSL}$ . The degree homomorphism  $d : \text{Br}_3 \rightarrow \mathbf{Z}$  is a class function with value one on all  $\phi(t_i)$ , hence  $d((ab)^{3k_i}) = 0$ . Since  $d(ab) = 2$  we conclude  $k_i = 0$  and  $\beta \in \text{Stab}_{\phi_n}$ .

Similarly we have  $\psi \circ \beta(t_i) = \pm \psi(t_j)$  for  $\beta \in \text{Stab}_{\chi_n}$ . Since the trace is a class function on  $\text{SL}_2 \mathbf{Z}$  which has value 2 on all  $\psi(t_i)$  while it is  $-2$  on  $-\psi(t_i)$ , we also get  $\beta \in \text{Stab}_{\psi_n}$ .  $\square$

**Proposition 8.20** Let  $F_n := \langle t_i, 1 \leq i \leq n \mid \quad \rangle$  be the free group on  $n = l + l'$  generators, define a homomorphism  $\psi_{l,l'} : F_n \rightarrow \text{SL}_2 \mathbf{Z}$  by

$$\psi_{l,l'}(t_i) = \begin{cases} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} & i > l, i \not\equiv l \pmod{2} \\ \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix} & i > l, i \equiv l \pmod{2} \\ \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} & i \leq l \end{cases}$$

and let  $\text{Br}_n$  act on homomorphisms  $F_n \rightarrow \text{SL}_2 \mathbf{Z}$  by Hurwitz automorphisms of  $F_n$ . Then the stabilizer group  $\text{Stab}_{\psi_{l,l'}}$  of  $\psi_{l,l'}$  is generated by the image of  $\text{Stab}_{\psi_l}$  under the inclusion  $\text{Br}_{l'} \hookrightarrow \text{Br}_n$  mapping to braids with only the last  $l'$  strands braided and

$$E_{l,l'} := \left\langle \sigma_{ij}^{m_{ij}}, 1 \leq i < j \leq n \mid m_{ij} = \begin{cases} 1 & \text{if } j \leq l \vee i \equiv j(2), i > l \\ 2 & \text{if } i \leq l < j \\ 3 & \text{if } i > l, i \not\equiv j(2) \end{cases} \right\rangle.$$

If  $l' \leq 6$  then even  $\text{Stab}_{\psi_{l,l'}} = E_{l,l'}$ .

*Proof:* Again we argue with the equivalent Hurwitz action on images of the generators. First we consider the induced action on conjugacy classes. On  $n$ -tuples of conjugacy classes the Hurwitz action induces an action of  $\text{Br}_n$  through the natural homomorphism  $\pi$  to the permutation group  $S_n$ . Since the tuple induced from  $\psi$  consists of  $l$  copies of the conjugacy class of  $-id$  followed by  $l'$  copies of the distinct conjugacy class of  $\psi(t_1)$ , the associated stabilizer group is  $\tilde{E} := \pi^{-1}(S_l \times S_{l'})$ , and as in [22] one can check that

$$\tilde{E} = \langle \sigma_{ij}, i < j \leq l \text{ or } l < i < j; \tau_{ij} := \sigma_{ij}^2, i \leq l < j \rangle.$$

So as a first step we have  $\text{Stab}_{\psi}$  contained in  $\tilde{E}$ .

Since  $-id$  is central it is the only element in its conjugacy class and we may conclude that the  $\tilde{E}$  orbit of  $\psi$  contains only homomorphisms which map the first  $l$  generators onto  $-id$ . With a short calculation using that  $-id$  is a central involution we can check that the  $\tau_{ij}$  act trivially on such elements:

$$\begin{aligned} & \tau_{ij}(-id, \dots, -id, M_{l+1}, \dots, M_n) \\ &= \sigma_{i+1}^{-1} \cdots \sigma_{j-1}^{-1} \sigma_j^2 \sigma_{j-1} \cdots \sigma_{i+1}(-id, \dots, -id, M_{l+1}, \dots, M_n) \\ &= \sigma_{i+1}^{-1} \cdots \sigma_{j-1}^{-1} \sigma_j^2(-id, \dots, -id, M_{l+1}, \dots, M_{j-1}, -id, M_j, \dots, M_n) \\ &= \sigma_{i+1}^{-1} \cdots \sigma_{j-1}^{-1}(-id, \dots, -id, M_{l+1}, \dots, M_{j-1}, -id, M_j, \dots, M_n) \\ &= (-id, \dots, -id, M_{l+1}, \dots, M_n) \end{aligned}$$

Therefore given  $\beta \in \tilde{E}$  as a word  $w$  in the generators  $\sigma_{ij}, \tau_{ij}$  of  $\tilde{E}$  the action of  $\beta$  on  $\psi$  is the same as that of  $\beta'$  where  $\beta'$  is given by a word  $w'$  obtained from  $w$  by dropping all letters  $\tau_{ij}$ . By the commutation relations of the  $\sigma_{ij}$  we may collect all letters  $\sigma_{ij}, i, j \leq l$  to the right of letters  $\sigma_{ij}, i, j > l$  without changing  $\beta'$  and get a factorization  $\beta' = \beta'_1 \beta'_2$  with  $\beta'_1 \in \text{Br}_l, \beta'_2 \in \text{Br}_{l'}$ .

Hence  $\beta \in \tilde{E}$  acts trivially on  $\psi$  if and only if  $\beta'_1 \beta'_2$  does so if and only if  $\beta'_2$  acts trivially on  $\psi_{l'}$ . Thus  $\text{Stab}_{\psi_{l,l'}}$  is generated by the  $\tau_{ij}$  the  $\sigma_{ij}, i, j \leq l$  and the  $\beta'_2 \in \text{Stab}_{\psi_{l'}}$ . Both conclusions of the proposition then follow since  $\sigma_{ij}, i, j > l$  are contained in  $\text{Stab}_{\psi_{l'}}$ , and since they are even generators if  $l' \leq 6$ , prop. 8.18.  $\square$

## 8.6 mapping class groups of elliptic fibrations

We return to elliptic fibrations and obtain some results concerning mapping classes of elliptic fibrations. We need in fact to enrich the structure a bit:

**Definition:** A *marked elliptic fibration* is an elliptic fibration with a distinguished regular fibre,  $f : X, F \rightarrow B, b_0$ , which can be thought of as given by a *marking*  $F \hookrightarrow E$ .

**Definition:** A *fibration preserving map* of a marked elliptic surface  $f : X, F \rightarrow B, b_0$  is a homeomorphism  $\Phi_X$  of  $X$  such that  $f \circ \Phi_X = \varphi_{B, b_0} \circ f$  for a homeomorphism  $\varphi_{B, b_0}$  of  $(B, b_0)$  and such that  $\Phi_X|_F$  is isotopic to the identity on  $F$ . The map  $\varphi_{B, b_0}$  is called the *induced base homeomorphism*.

An induced homeomorphism necessarily preserves the set  $\Delta(f)$  of singular values of the fibration map  $f$  and therefore can be regarded as a homeomorphism of the punctured base  $B, \Delta(f)$  preserving the base point.

On the other hand with each elliptic fibration  $f : X \rightarrow B$  we have a torus bundle over  $B - \Delta(f)$ . Its structure homomorphism is the natural map

$$\psi : \pi_1(B, b_0) \longrightarrow \text{Diff}(F)$$

to the group of isotopy classes of diffeomorphisms of the distinguished fibre.

**Proposition 8.21** *Given a marked elliptic fibration and a braid  $\beta$  representing an isotopy class of homeomorphisms of its punctured base, there is a fibration preserving map inducing  $\beta$  if and only if  $\beta$  stabilizes the structure map of the associated torus bundle.*

*Proof:* A fibration preserving homeomorphism  $\Phi$  of an unmarked elliptic surface induces a map  $\varphi_B$  of the punctured base. By the classification of torus bundles there exists then a commutative diagram

$$\begin{array}{ccc} \pi_1(B - \Delta(f), b_0) & \xrightarrow{(\varphi_B)_*} & \pi_1(B - \Delta(f), \varphi_B(b_0)) \\ \downarrow \psi_{b_0} & & \downarrow \psi_{\varphi(b_0)} \\ \text{Diff}(F) & \xrightarrow{(\Phi|_F)_*} & \text{Diff}(\Phi(F)) \end{array}$$

But the result of Moishezon [30, p. 169] implies that the reverse implication is true in the absence of multiple fibres.

If now  $\Phi$  is a fibration preserving homeomorphism of a marked elliptic surface then the bottom map is the identity and the claim is immediate.  $\square$

**Proposition 8.22** *Given a marked elliptic fibration with only fibres of types  $I_1, I_0^*$  there is a choice of free generators for  $\pi_1(B, b_0)$ , an isomorphism  $\text{Diff}(F) \cong \text{SL}_2 \mathbf{Z}$  and an isomorphism of the abstract braid group onto the mapping class group such that the structure homomorphism of the associated bundle is  $\psi_{6k,l}$  and such that the action of  $\text{Br}_n$  on  $F_n$  commutes with the action of  $\text{Diff}(B, \Delta)$  on  $\pi_1$ .*

*Proof:* The proof proceeds along the lines of Moishezon's proof, cf. [15], for the normal form of an elliptic surface with only fibres of type  $I_1$ . The same strategy leads to our claim since fibres of type  $I_0^*$  have local monodromy in the center of  $\text{SL}_2 \mathbf{Z}$ .  $\square$

By now we have finally got all necessary results to prove theorem 1 as stated in the introduction.

*Proof of theorem 8.1:* The mapping class group of the base punctured at the base points of singular fibres is isomorphic to the braid group  $\text{Br}_{6+l}^s$  of the sphere. We have previously shown that the mapping classes induced by fibration preserving maps are those acting trivially on the structure homomorphism of the torus bundle given with the elliptic fibration, prop. 8.21.

By prop. 8.22 and prop. 8.20 the corresponding group is conjugation equivalent to  $E_{6,l}$ . On the other hand so is the monodromy group by the main theorem. Moreover for each braid of the monodromy group there is by prop. 8.17 a fibration preserving diffeomorphism, so we get an inclusion and hence both groups coincide.  $\square$



# Chapter 9

## braid monodromy and fundamental groups

In this concluding chapter we want to address four topics. First we will deduce by the method of van Kampen a presentation for the fundamental group of the discriminant complement of a Brieskorn Pham polynomial. Second we want to relate the algebraic monodromy and the braid monodromy of Brieskorn Pham polynomials by means of Dynkin diagrams.

In a third section we give all corollaries for arbitrary singular functions which can be deduced immediately from our results and we finish in a four section with some conjectures and speculations.

### 9.1 fundamental groups

For convenience we restate our result on generators of the braid monodromy.

**Theorem 9.1** *The braid monodromy group of a Brieskorn-Pham polynomial  $x_1^{l_1+1} + \cdots + x_n^{l_n+1}$  is generated by the following twist powers:*

$$\begin{aligned}\sigma_{i,j}^3 &: i < j && \text{correlated} \\ \sigma_{i,j}^2 &: i < j && \text{not correlated} \\ \sigma_{j,k} \sigma_{i,j}^2 \sigma_{j,k}^{-1} &: i < j < k && \text{correlated}\end{aligned}$$

The most important corollary drawn from this theorem is a presentation of the fundamental group of the discriminant complement which can be computed by the Zariski van Kampen method.

**Theorem 9.2** *The fundamental group of the discriminant complement of a versal unfolding of a Brieskorn-Pham polynomial  $x_1^{l_1+1} + \cdots + x_n^{l_n+1}$  has a presentation given with respect to the multiindex set  $I = I(l_1, \dots, l_n)$  of cardinality  $\mu = l_1 \cdots l_n$ :*

$$\langle t_i, i \in I \mid \begin{aligned}t_i t_j t_i &= t_j t_i t_j, & i, j \in I, i < j & \text{correlated}, \\ t_i t_j &= t_j t_i, & i, j \in I, i < j & \text{not correlated}, \\ t_i t_j t_k t_i &= t_j t_k t_i t_j, & i, j, k \in I, i < j < k & \text{correlated}\end{aligned}\rangle$$

*Proof:* A presentation of the fundamental group can be obtained from generators of the braid monodromy according to 2.25.

So we have generators  $t_i$ ,  $i \in I_n$  in bijection to the critical points. We obtain the relations from the generators of the braid monodromy group, which are given in theorem 9.1. In fact a generator of the first two rows can be factored as  $\beta_0\sigma_1^2\beta_0^{-1}$  and  $\beta_0\sigma_1^3\beta_0^{-1}$  respectively in such a way that

$$\beta_0(t_1) = t_i, \quad \beta_0(t_2) = t_j.$$

and similarly generators of the last row can be conjugated such that

$$\beta_0(t_1) = t_i, \quad \beta_0(t_2) = t_j t_k t_j^{-1}.$$

We then compute a sufficient set of relators using lemma 2.27:

i) in case  $\sigma_{i,j}^3$ ,  $i < j$  correlated:

$$t_i^{-1} \beta(t_i) = t_i^{-1} t_j^{-1} t_i^{-1} t_j t_i t_j, \quad t_j^{-1} \beta(t_j) = t_j^{-1} t_j^{-1} t_i^{-1} t_j^{-1} t_i t_j t_i t_j.$$

ii) in case  $\beta = \sigma_{i,j}^2$ ,  $i < j$  not correlated:

$$t_i^{-1} \beta(t_i) = t_i^{-1} t_j^{-1} t_i t_j, \quad t_j^{-1} \beta(t_j) = t_j^{-1} t_i^{-1} t_i^{-1} t_j t_i t_j.$$

iii) in case  $\sigma_{j,k} \sigma_{i,j}^2 \sigma_{j,k}^{-1}$ ,  $i < j < k$  correlated:

$$\begin{aligned} t_i^{-1} \beta(t_i) &= t_i^{-1} (t_j t_k t_j^{-1})^{-1} t_i t_j t_k t_j^{-1} \\ &= t_i^{-1} t_j t_k^{-1} t_j^{-1} t_i t_j t_k t_j^{-1}, \\ (t_j t_k t_j^{-1})^{-1} \beta(t_j t_k t_j^{-1}) &= (t_j t_k t_j^{-1})^{-1} (t_j t_k t_j^{-1})^{-1} t_i^{-1} (t_j t_k t_j^{-1}) t_i (t_j t_k t_j^{-1}) \\ &= t_j t_k^{-1} t_j^{-1} t_i t_k^{-1} t_j^{-1} t_i^{-1} t_j t_k t_j^{-1} t_i t_j t_k t_j^{-1}. \end{aligned}$$

In all cases the relators are conjugate, so we can do with the following relations:

$$\begin{aligned} t_i t_j t_i &= t_j t_i t_j && \text{for } i, j \text{ correlated,} \\ t_i t_j &= t_j t_i && \text{for } i, j \text{ not correlated,} \\ t_i t_j t_k t_j^{-1} &= t_j t_k t_j^{-1} t_i && \text{for } i, j, k \text{ correlated.} \end{aligned}$$

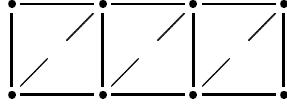
Any relation of the third kind we can multiply by  $t_j t_i$  on the right. Using a relation of the first kind and cancellation of inverse letters, we arrive at our claim:

$$\begin{aligned} t_i t_j t_k t_j^{-1} &= t_j t_k t_j^{-1} t_i \\ \iff t_i t_j t_k t_i &= t_j t_k t_j^{-1} t_i t_j t_i \\ \iff t_i t_j t_k t_i &= t_j t_k t_j^{-1} t_j t_i t_j \\ \iff t_i t_j t_k t_i &= t_j t_k t_i t_j. \end{aligned}$$

□

## 9.2 Dynkin diagrams

We want to interpret the theorems of the first section in terms of the Dynkin diagrams which Pham associated to the functions under consideration.



the Dynkin diagram of  $x_1^3 + x_2^5$  according to Pham

So let us recall the situation for the simple singularities. There we have the distinguished Dynkin diagram with no cycles and the corresponding generalised Artin groups. To these Brieskorn Artin groups we can associate the braid stabilizer subgroup, which in fact coincides with the braid subgroup generated by elements associated to edges and pairs without edge.

We generalize this situation in the following way.

**Definition 9.3:** The braid subgroup associated to a Dynkin diagram obtained from a distinguished system of paths is defined as follows:

- To each edge of weight  $\pm 1$  associate the generator  $\sigma_{i,j}^3$ ,
- To each non-connected vertex pair associate the generator  $\sigma_{i,j}^2$ ,
- To each edge triangle of weight product  $-1$  associate the generator  $\sigma_{j,k}\sigma_{i,j}^2\sigma_{j,k}$ ,  $i < j < k$ .

In each case the indices are those of the vertices involved.

One can check that we can describe the Dynkin diagram obtained by Pham for the function

$$x_1^{l_1+1} + \cdots x_n^{l_n+1}$$

using the multiindex set  $I = I(l_1, \dots, l_n)$ . The pair of vertices of indices  $i = i_1 i_2 \dots i_n$  and  $j = j_1 j_2 \dots j_n$  are connected by an edge if  $i < j$  are correlated and we assign the weight

$$(-1)^{1 + \sum_\nu (j_\nu - i_\nu)}.$$

Then we can show the following relation.

**Lemma 9.4** *The braid monodromy group of a Brieskorn Pham polynomial in dimension  $n$ ,  $f = x_1^{l_1+1} + \cdots x_n^{l_n+1}$ , is given by the braid subgroup of  $\text{Br}_{l_1 \dots l_n}$  generated by the elements associated to the Dynkin diagram of  $f$ .*

$\sigma_{i,j}^3$	there is an edge between vertices $i, j$
$\sigma_{i,j}^2$	there is no edge between vertices $i, j$
$\sigma_{j,k}\sigma_{i,j}^2\sigma_{j,k}^{-1}$	there is a triangle of weight product $-1$ on the vertices $i, j, k$

*Proof:* The first two rows of generators in 9.1 are obviously associated to the diagram. Now in case there is any triangle of edges the indices  $i < j < k$  are correlated and the weight product is

$$(-1)^3 + \sum_{\nu} (j_{\nu} - i_{\nu}) + \sum_{\nu} (k_{\nu} - i_{\nu}) + \sum_{\nu} (k_{\nu} - j_{\nu}) = -1.$$

So there is a bijection between generators given in the theorem and the generators associated to the Dynkin diagram.  $\square$

In a similar way the presentation of the fundamental group of the discriminant complement can be expressed in terms of the Dynkin diagram.

### 9.3 other functions

Any versal family of functions is induced from the versal family of a Brieskorn Pham function, so we may draw some conclusions from this fact:

**Lemma 9.5** *Suppose a singular function  $f$  is adjacent to a Brieskorn Pham polynomial  $\tilde{f}$ , then its braid monodromy group is contained in the intersection of the braid monodromy group of  $\tilde{f}$  with some group in the conjugacy class of  $\text{Br}_{\mu} \subset \text{Br}_{\tilde{\mu}}$ .*

The reverse adjacency relation implies:

**Lemma 9.6** *Suppose a Brieskorn Pham polynomial  $g$  is adjacent to a singular function  $f$ , then the braid monodromy group of  $f$  contains a subgroup isomorphic to the braid monodromy group of  $g$ .*

The corollary concerning the fundamental group of the discriminant complements can be formulated as follows.

**Lemma 9.7** *Suppose a singular function  $f$  is adjacent to a Brieskorn Pham polynomial  $\tilde{f}$  with Milnor numbers  $\mu$  and  $\tilde{\mu}$  respectively, then the fundamental group of its discriminant complement fits into a commutative diagram*

$$\begin{array}{ccc} F_{\mu} & \hookrightarrow & F_{\tilde{\mu}} \\ \downarrow & & \downarrow \\ \pi_1(\mathbf{C}^{\mu} \setminus \mathcal{D}_f) & \longrightarrow & \pi_1(\mathbf{C}^{\tilde{\mu}} \setminus \mathcal{D}_{\tilde{f}}). \end{array}$$

### 9.4 conjectures and speculations

Now that we have defined and computed the braid monodromy groups of Brieskorn Pham polynomials we are in a position to make an educated guess what they are in case of general functions. We formulate two approaches both based on the fact that the braid monodromy determines the fundamental group of the discriminant complement, which in turn is restricted by the algebraic or geometric monodromy.

Since the braid monodromy acts by the Artin representation on the algebraic monodromy homomorphism, the immediate guess is:

**Conjecture 1** *The braid monodromy group is the stabilizer subgroup of the algebraic monodromy homomorphism.*

The other is much more explicit and relies on the fact that the algebraic monodromy is encoded into the Dynkin diagram. It presumes that the braid subgroups associated to Dynkin diagrams in an orbit under the braid group action belong to a single conjugacy class.

**Conjecture 2** *The braid monodromy group of a singular function coincides with the braid subgroup associated to a Dynkin diagram up to conjugacy.*

It leads immediately to a preferred finite presentation of the fundamental group of the discriminant complement.

**Conjecture 3** *The fundamental group of the discriminant complement of a versal unfolding of a singular function  $f$  has a presentation given with respect to a Dynkin diagram of  $f$  of vertex cardinality  $\mu$ :*

$$\langle t_i, 1 \leq i \leq \mu \mid \begin{aligned} t_i t_j t_i &= t_j t_i t_j, & i, j \in I, i < j \text{ joint by an edge of weight } \pm 1, \\ t_i t_j &= t_j t_i, & i, j \in I, i < j \text{ not joint by an edge}, \\ t_i t_j t_k t_i &= t_j t_k t_i t_j, & i, j, k \in I, i < j < k \text{ in a edge triangle} \\ && \text{of weight product } -1 \end{aligned} \rangle$$

In any case the braid monodromy group and the fundamental group would be shown to be topological invariants despite the observations by Brieskorn in [7] which show a dependence of the braid monodromy homomorphism on analytic invariants.

But we are not only interested into the fundamental groups themselves but also in their relations. Even if we knew that the fundamental groups are determined from the Dynkin diagrams, which implies that the braid monodromy groups embed for adjacent singular functions, it is by no means obvious what we speculate now:

**Conjecture 4** *Suppose  $f$  is adjacent to  $g$ , so a versal unfolding of  $g$  is a versal unfolding for  $f$ , then the natural map of fundamental groups of discriminant complements injects.*

$$\begin{array}{ccc} F_\mu & \hookrightarrow & F_{\tilde{\mu}} \\ \downarrow & & \downarrow \\ \pi_1(\mathbf{C}^\mu \setminus \mathcal{D}_f) & \longrightarrow & \pi_1(\mathbf{C}^{\tilde{\mu}} \setminus \mathcal{D}_{\tilde{f}}). \end{array}$$

The final speculation seems even further off. We have speculated on the image of the braid monodromy homomorphism. Now we consider the domain. It is the fundamental group of the bifurcation complement. So the question might be, how much of this fundamental group is captured by the braid monodromy group. Boldly put:

**Conjecture 5** *The braid monodromy group is the fundamental group of the complement of the bifurcation diagram.*

Here again one should be cautious, since the observation allude to above, implies that the generic plane section yields curves of different multiplicity in cases of equal topological invariants.



# Appendix A

## braid computations

This appendix is designed to serve several purposes. First the progress in the chapters is eased if some of the computational obstacles are hidden in this appendix. Second the arguments are often similar and it is easier to get used to them, if they are used in one place instead of being scattered throughout.

We distinguish the cases that the indices are just natural numbers, or pairs, or multiindices, and we add a final section, which contains a very helpful criterion to show, when two set of elements generate the same group.

We have to introduce some notation, which partly applies only for a single section. Since all index sets we use are ordered, we can always denote by  $i^+$  the immediate successor of  $i$  in some index set. The same notation applies also to single components of multiindices.

As before our most general index set is defined in terms of possible exponents of Brieskorn Pham polynomials.

**Notation A.1:** Given a finite sequence  $l_1, \dots, l_n$  of positive integers, define the multiindex set  $I_n = I_n(l_1, \dots, l_n)$  to be

$$I_n := \{i_1 \dots i_n \mid 1 \leq i_\nu \leq l_\nu, 1 \leq \nu \leq n\}$$

equipped with the natural lexicographical order.

Since we mostly deal with conjugates we underline a conjugated element to make the structure more obvious.

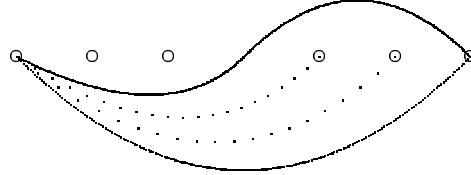
### simple indices

In this section  $i', j'$  etc. are used just to denote additional natural numbers.

**Remark A.2:** The twist  $\check{\sigma}_{i,k} := (\prod_{i < j < k} \sigma_{i,j}^2) \underline{\sigma_{i,k}} (\prod_{i < j < k} \sigma_{i,j}^2)^{-1}$  is the twist on the horizontal arc from  $i$  to  $k$  passing behind all intermediate punctures (as opposed to the arc of  $\sigma_{i,k}$  which passes in front).

**Remark A.3:** The half twist on the arc from  $i$  to  $k$  passing in front up to  $j$  and behind from  $j + 1$  onwards can be given as

$$\left( \prod_{j < j' < k} \sigma_{i,j'}^2 \right) \underline{\sigma_{i,k}} \left( \prod_{j < j' < k} \sigma_{i,j'}^2 \right)^{-1}.$$



**Lemma A.4** *The braid subgroup  $\text{Br}(A_n) \subset \text{Br}_n$  is generated by elements*

$$\begin{aligned} \sigma_{i,i+1}^3 &\quad 1 \leq i < n \\ \sigma_{i,i+1}^{-2} \check{\sigma}_{i,j}^2 \sigma_{i,i+1}^2 &\quad 1 < i, i+1 < j \leq n. \end{aligned}$$

*Proof:* Consider the following two filtered sets of elements of  $\text{Br}_n$ .

$$\begin{aligned} S_1 &:= \{\sigma_{i,i+1}^3\} \quad S_k := S_1 \cup \{\sigma_{i,j}^2 \mid 1 < j - i \leq k\}, \\ T_1 &:= \{\sigma_{i,i+1}^3\} \quad T_k := T_1 \cup \{\sigma_{i,i+1}^{-2} \check{\sigma}_{i,j}^2 \sigma_{i,i+1}^2 \mid 1 < j - i \leq k\} \end{aligned}$$

By the first remark we get the relation

$$\sigma_{i,i+1}^{-2} \check{\sigma}_{i,j}^2 \sigma_{i,i+1}^2 = \left( \prod_{i+1 < j' < j} \sigma_{i,j'}^2 \right) \underline{\sigma_{i,j}} \left( \prod_{i+1 < j' < j} \sigma_{i,j'}^2 \right)^{-1}.$$

so  $S_2 = T_2$  and the other hypotheses of lemma A.17 hold as well. Therefore the assertion is proved, since  $S_n$  is known to generate  $\text{Br}(A_n)$ .  $\square$

**Lemma A.5** *The subgroup  $\text{Br}_{A(l)}$  of  $\text{Br}_l$  is generated by*

- i)  $\sigma_{i,i+1}^3,$
- ii)  $\sigma_{i,i+2}^2,$
- iii)  $\left( \prod_{j=i+2}^{k-1} \sigma_{i,j}^2 \right) \underline{\sigma_{i,k}^2} \left( \prod_{j=i+2}^{k-1} \sigma_{i,j}^2 \right)^{-1}, \quad i+1 < k-1.$

*Proof:* This is the same result as A.4 just with different notation.  $\square$

**Lemma A.6** *Suppose in a punctured disc two otherwise distinct arcs meet in the punctures  $p, q$  thus defining a inner disc. If there is a system of arcs such that*

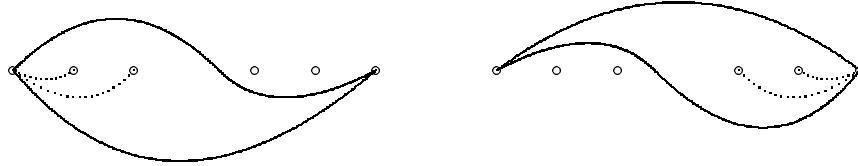
- i) *each puncture in the inner disc is connected by an arc with either  $p$  or  $q$ ,*

ii) apart from  $p, q$  all arcs have no points in common,

then the twist on the outer arcs are equal up to conjugation by full twists on the inner arcs.

*Proof:* We may identify the mapping class group of neighbourhood of the inner disc with the abstract braid group, such that the twists on inner arcs correspond to  $\sigma_{1,j}^2, 1 < j \leq m$  and  $\sigma_{j,n}^2, m < j < n$  and the twist on the outer arcs correspond to  $\sigma_{1,n}$  and  $\check{\sigma}_{1,n}(1)$ . The claim then follows from

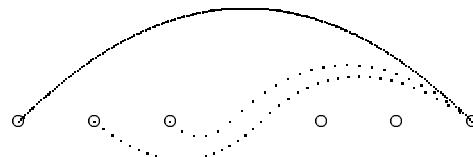
$$\left( \prod_{j=2}^m \sigma_{1,j}^2 \right) \underline{\sigma_{1,n}} \left( \prod_{j=2}^m \sigma_{1,j}^2 \right)^{-1} = \left( \prod_{j=m+1}^{n-1} \sigma_{j,n}^2 \right) \underline{\check{\sigma}_{1,n}} \left( \prod_{j=m+1}^{n-1} \sigma_{j,n}^2 \right)^{-1}.$$



□

**Lemma A.7** For any  $j, i < j \leq k$  the twist  $\check{\sigma}_{i,k}(i)$  can be given as  $\sigma_{i,k}$  suitably conjugated by braids  $\sigma_{i',j'}$ ,  $i \leq i' < j \leq j' \leq k$ .

*Proof:* First note that  $\sigma_{i,k}$  and  $\check{\sigma}_{i,k}$  are twists on arc which meet the initial hypothesis of lemma A.6. By the second remark above, the full twists on arcs from the puncture of index  $i'$  passing in front up to  $j-1$  and behind from  $j$  onwards is in the group generated by elements  $\sigma_{i',k}^2, \sigma_{i',j'}, i < i' < j \leq j' < k$ .



On the other hand these arcs and the arcs to which the  $\sigma_{i',j'}^2, j \leq j' < k$  are associated can be chosen simultaneously to meet the remaining hypotheses of lemma A.6. So we get our claim. □

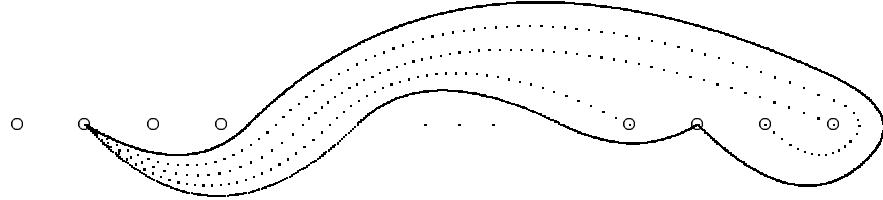
## pair indices

In this section we will not use primed integer variables except for  $i'_2$  which always is only a shorthand for  $i_2 - l_2$ .

**Lemma A.8** *The elements  $\tau_{i_1 i'_2, j_1 i'_2}$  and  $\tau_{i_1 i'_2, j_1 i_2}$  are equal up to conjugation by twists  $\tau_{i_1 i_2, j_1 j_2}^2$ ,  $1 \leq j_2 - i'_2 < l_2$  for all  $1 \leq i_1 < j_1 \leq l_1$ ,  $i'_2 = i_2 - l_2$ .*

*Proof:* In fact we have

$$\left( \prod_{i'_2 < j_2 < i_2} \tau_{i_1 i'_2, j_1 j_2}^2 \right)^{-1} \underline{\tau_{i_1 i'_2, j_1 i'_2}} \left( \prod_{i'_2 < j_2 < i_2} \tau_{i_1 i'_2, j_1 j_2}^2 \right) = \tau_{i_1 i'_2, j_1 i_2}.$$



Again the claim can also be proved by checking the hypotheses of lemma A.6.  $\square$

**Lemma A.9** *Up to conjugation by twists  $\tau_{i_1 i_2, j_1 j_2}^2$ ,  $1 \leq i_1 < j_1 \leq l_1$ ,  $1 \leq j_2 - i_2 < l_2$ , elements ( $i'_2 := i_2 - l_2$ )*

$$\begin{aligned} & \tau_{i_1 i'_2, i_1^+ i_2}^{-1} \tau_{i_1^+ i'_2, j_1 i'_2}^2 \tau_{i_1 i'_2, i_1^+ i'_2} \quad \text{if } j_1 - i_1 \leq l_1/2, \\ & \tau_{i_1 i'_2, i_1^+ i_2}^{-1} \tau_{i_1^+ i'_2, j_1 i_2}^2 \tau_{i_1 i'_2, i_1^+ i'_2} \quad \text{if } j_1 - i_1 \geq l_1/2 + 1, \\ & \tau_{i_1 i'_2, i_1^+ i_2}^{-1} \tau_{i_1^+ i'_2, j_1 i_2}^2 \tau_{i_1 i'_2, i_1^+ i_2} \quad \text{if } j_1 - i_1 = l_1/2 + 1/2. \end{aligned}$$

*correspond bijectively to the elements*

$$\tau_{i_1 i'_2, i_1^+ i_2}^{-1} \tau_{i_1^+ i'_2, j_1 i_2}^2 \tau_{i_1 i'_2, i_1^+ i_2} \quad 1 < i_1 + 1 = i_1^+ < j_1 \leq l_1, 1 \leq i_2 \leq l_2.$$

*Proof:* The elements of the third row already have the claimed factorisation. In the other cases we have to conjugate in such a way that central twist and conjugating twist are conjugated simultaneously to the claimed twists.

For elements of the second row, it is only the conjugating twist which has not the claimed form.

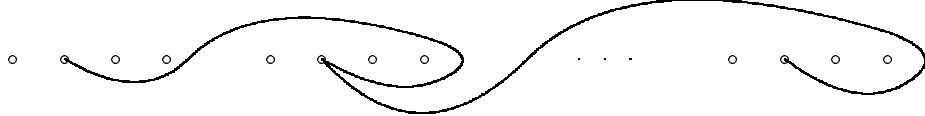


The proof of A.8 shows, that we can get

$$\left( \prod_{i'_2 < j_2 < i_2} \tau_{i_1 i'_2, j_1 j_2}^2 \right)^{-1} \underline{\tau_{i_1 i'_2, i_1^+ i'_2}} \left( \prod_{i'_2 < j_2 < i_2} \tau_{i_1 i'_2, j_1 j_2}^2 \right) = \tau_{i_1 i'_2, i_1^+ i_2}.$$

Moreover we check at once, that the conjugating factor commutes with  $\tau_{i_1^+ i_2', j_1 i_2}^+$ , hence an overall conjugation of  $\tau_{i_1 i_2', i_1^+ i_2'}^{-1} \tau_{i_1^+ i_2', j_1 i_2}^2 \tau_{i_1 i_2', i_1^+ i_2'}$  with  $(\prod_{i_2' < j_2 < i_2} \tau_{i_1 i_2', j_1 j_2}^2)$  yields  $\tau_{i_1 i_2', i_1^+ i_2}^{-1} \tau_{i_1^+ i_2', j_1 i_2}^2 \tau_{i_1 i_2', i_1^+ i_2'}$ .

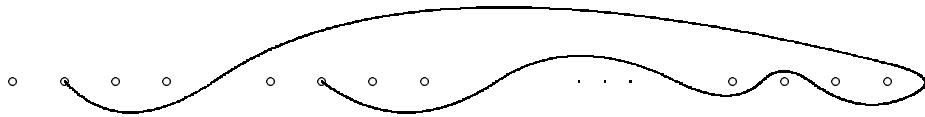
In case of elements of the first row, neither twist is in the right shape:



We first take care of the middle factor  $\tau_{i_1^+ i_2', j_1 i_2}^+$ , as before just by conjugation with  $(\prod_{i_2' < j_2 < i_2} \tau_{i_1^+ i_2', j_1 j_2}^2)$ .



But since we have to conjugate overall, also the conjugating factors are conjugated, and they are not unaffected:



But we can now conjugate by  $(\prod_{i_2' < j_2 < i_2} \tau_{i_2 + i_1', j_1 j_2}^2)$  and  $(\prod_{i_2' < j_2 < i_2} \tau_{i_1 i_2', i_1^+ j_2}^2)$ , which are twists on arcs disjoint to that of  $\tau_{i_1^+ i_2', j_1 i_2}^+$ , hence commuting with it.  $\square$

**Lemma A.10** *Given  $1 \leq i_1 < k_1 \leq l_1, 1 \leq k_2 < l_2$  the elements*

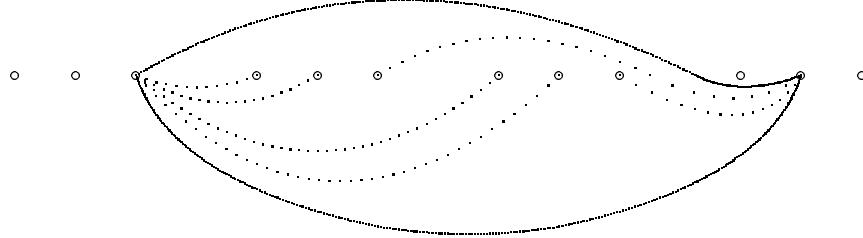
$$\tau_{i_1 0, k_1 k_2}^2 \quad \text{and} \quad \sigma_{i_1 l_2, k_1 k_2}^2$$

*coincide up to conjugation by elements*

- i)  $\sigma_{1 l_2, j_1 j_2}^2, i_1 < j_1 < k_1, 1 \leq j_2 < l_2,$
- ii)  $\tau_{j_1 0, k_1 k_2}^2, i_1 < j_1 < k_1.$

*Proof:* It suffices to check that arcs to which the given twists are associated can be chosen simultaneously in such a way that

- i) they are confined to the disc with boundary given by the arcs corresponding to  $\tau_{i_1 0, k_1 k_2}$  and  $\sigma_{i_1 l_2, k_1 k_2}$ ,
- ii) they are distinct outside the punctures of indices  $i_1 l_2$  and  $k_1 k_2$ ,
- iii) all punctures in the disc are joint by some arc with either the puncture of index  $i_1 l_2$  or that of index  $k_1 k_2$ .



So by lemma A.6 we may conclude that the assertion holds.  $\square$

## multiindices

In this section we reserve the notation  $i'$  etc. for the multiindex  $i' := i_1 \dots i_{n-1}$  in  $I_n(l_1, \dots, l_{n-1})$  naturally associated to  $i = i_1 \dots i_n$ .

**Lemma A.11** *Suppose the indices  $i, j, k$  form a correlated triple, then*

$$\sigma_{i,k}^3 = (\sigma_{j,k}^3 \sigma_{i,j}^2 \sigma_{i,k}^2 \sigma_{i,j}^{-2}) \underline{\sigma_{i,j}^3} (\sigma_{j,k}^3 \sigma_{i,j}^2 \sigma_{i,k}^2 \sigma_{i,j}^{-2})^{-1}.$$

*Proof:* Since we claim a conjugation relation it suffices to show that

$$\begin{aligned} \sigma_{i,k} &= \sigma_{j,k} \sigma_{i,k} \sigma_{j,k} \sigma_{i,k}^{-1} \sigma_{j,k}^{-1} \\ &= \sigma_{j,k} \sigma_{i,k}^2 \sigma_{j,k} \sigma_{i,k} \sigma_{j,k}^{-1} \sigma_{i,k}^{-2} \sigma_{j,k}^{-1} \\ &= \sigma_{j,k}^3 \sigma_{j,k}^{-2} \sigma_{i,k}^2 \sigma_{j,k}^2 \sigma_{i,j} \sigma_{j,k}^{-2} \sigma_{i,k}^{-2} \sigma_{j,k}^2 \sigma_{j,k}^{-1} \\ &= \sigma_{j,k}^3 (\sigma_{i,j}^2 \sigma_{i,k}^2 \sigma_{i,j}^{-2}) \sigma_{i,j} (\sigma_{i,j}^2 \sigma_{i,k}^{-2} \sigma_{i,j}^{-2}) \sigma_{j,k}^{-3} \end{aligned}$$

$\square$

**Lemma A.12** *Suppose the indices  $i, j, m, k$  form a correlated quadruple, then*

$$\sigma_{i,m}^2 \sigma_{i,k}^2 \sigma_{i,m}^{-2} = (\sigma_{j,m}^2 \sigma_{j,k}^2 \sigma_{j,m}^{-2} \sigma_{i,j}^2 \sigma_{i,m}^2 \sigma_{i,j}^{-2}) \sigma_{i,j}^2 \sigma_{i,k}^2 \sigma_{i,j}^{-2} (\sigma_{j,m}^2 \sigma_{j,k}^2 \sigma_{j,m}^{-2} \sigma_{i,j}^2 \sigma_{i,m}^2 \sigma_{i,j}^{-2})^{-1}.$$

*Proof:* It remains to prove that conjugation by  $\sigma_{j,m}^2 \sigma_{j,k}^2 \sigma_{j,m}^{-2} \sigma_{i,j}^2$  acts trivially on  $\sigma_{i,m}^2 \sigma_{i,k}^2 \sigma_{i,m}^{-2}$ .

Since the twists  $\sigma_{j,m}^2 \sigma_{j,k}^2 \sigma_{j,m}^{-2}$ ,  $\sigma_{i,j}^2$  and  $\sigma_{i,m}^2 \sigma_{i,k}^2 \sigma_{i,m}^{-2}$  correspond to three arcs which form a triangle with no punctures in its interior, our claim is the homomorphic image of the following claim in  $\text{Br}_3$ :

$$\sigma_{1,3}^2 = \sigma_{2,3}^2 \sigma_{1,2}^2 \sigma_{1,3}^2 \sigma_{1,2}^{-2} \sigma_{2,3}^{-2},$$

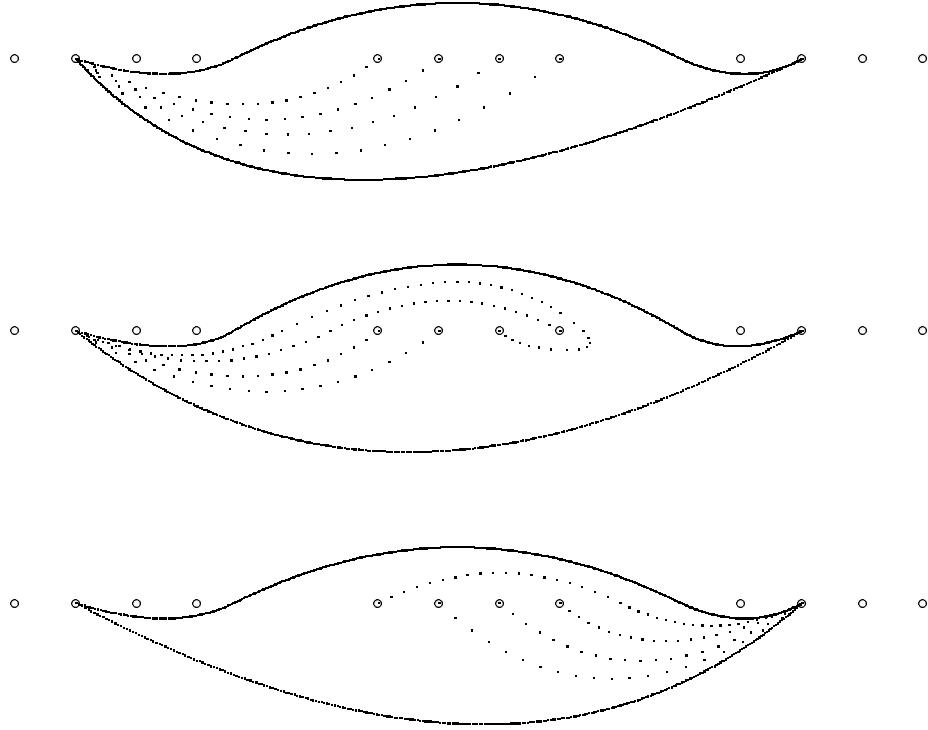
which is immediately seen to be true, because  $\sigma_{1,2}^2 \sigma_{1,3} \sigma_{1,2}^{-2} = \sigma_{2,3}^{-2} \sigma_{1,3} \sigma_{2,3}^2$ .  $\square$

**Lemma A.13** Given multiindices  $i < k$  and  $j'$  such that  $i' < j' < k'$ , then

$$\begin{aligned} \eta(\sigma_{i',j'}^2) \underline{\sigma_{i,k}} \eta(\sigma_{i',j'}^{-2}) &= \left( \prod_{1 \leq j_n \leq l_n} \sigma_{i,j'j_n}^2 \right) \underline{\sigma_{i,k}} \left( \prod_{1 \leq j_n \leq l_n} \sigma_{i,j'j_n}^2 \right)^{-1} \\ &= \left( \prod_{j_n=i_n-l_n+1}^{i_n} \sigma_{i,j'j_n}^2 \right) \underline{\sigma_{i,k}} \left( \prod_{j_n=i_n-l_n+1}^{i_n} \sigma_{i,j'j_n}^2 \right)^{-1} \\ &= \left( \prod_{j_n=k_n}^{k_n+l_n-1} \sigma_{i,j'j_n}^2 \right)^{-1} \underline{\sigma_{i,k}} \left( \prod_{j_n=k_n}^{k_n+l_n-1} \sigma_{i,j'j_n}^2 \right). \end{aligned}$$

*Proof:* We start with the observation that the arcs corresponding to the twists  $\sigma_{i,k}$  and  $\eta(\sigma_{i',j'}^2) \underline{\sigma_{i,k}} \eta(\sigma_{i',j'}^{-2})$  bound a disc which contains the punctures with indices  $j'j_n$ ,  $1 \leq j_n \leq l_n$  and only these.

The claim can now be deduced from A.6, since the conjugating braids correspond to arc systems which meet the hypotheses of A.6, cf. the following illustrations.



□

**Lemma A.14** Suppose  $i < k$  are multiindices in  $I_n = I_n(l_1, \dots, l_n)$  with  $i' < k'$  correlated and assume  $j' = i_1^+ i_2 \dots i_{n-1}$ . Then up to conjugation by elements  $\sigma_{i,i_n,j'j_n}^2$ ,  $1 \leq i_n - j_n < l_n$ ,

$$\eta(\sigma_{i',j'}^2) \underline{\sigma_{i,k}} \eta(\sigma_{i',j'}^{-2})$$

is equal to  $\sigma_{i,j'i_n}^2 \underline{\sigma_{i,k}} \sigma_{i,j'i_n}^{-2} = \sigma_{j'i_n,k}^{-2} \underline{\sigma_{i,k}} \sigma_{j'i_n,k}^2$ .

*Proof:* By A.13 the given elements are equal to

$$\left( \prod_{j_n=i_n-l_n+1}^{i_n} \sigma_{i,j'j_n}^2 \right) \underline{\sigma_{i,k}} \left( \prod_{j_n=i_n-l_n+1}^{i_n} \sigma_{i,j'j_n}^2 \right)^{-1},$$

hence they are equal to  $\sigma_{i,j'i_n}^2 \underline{\sigma_{i,k}} \sigma_{i,j'i_n}^{-2}$  up to conjugation by elements  $\sigma_{i,i_n,j'j_n}^2$ ,  $1 \leq i_n - j_n < l_n$ . The claim  $\sigma_{i,j'i_n}^2 \underline{\sigma_{i,k}} \sigma_{i,j'i_n}^{-2} = \sigma_{j'i_n,k}^{-2} \underline{\sigma_{i,k}} \sigma_{j'i_n,k}^2$  is immediate.  $\square$

**Lemma A.15** Suppose  $i < k$  are multiindices in  $I_n = I_n(l_1, \dots, l_n)$  with  $i' < k'$  correlated and assume  $j' = i_1 k_2 \dots k_{n-1}$ . Then up to conjugation by elements  $\sigma_{j'j_n,k'k_n}^2$ ,  $1 \leq j_n - k_n < l_n$ ,

$$\eta(\sigma_{i',j'}^2) \underline{\sigma_{i,k}} \eta(\sigma_{i',j'}^{-2})$$

is equal to  $\sigma_{i,j'k_n}^2 \underline{\sigma_{i,k}} \sigma_{i,j'k_n}^{-2} = \sigma_{j'k_n,k}^{-2} \underline{\sigma_{i,k}} \sigma_{j'k_n,k}^2$ .

*Proof:* This result is obtained using the last equation of A.13 along the lines of the proof above, A.14.  $\square$

**Lemma A.16** If  $1 \leq i_n < l_n$  and  $j_n = 0$  then

$$\begin{aligned} & \left( \prod_{k_n=1}^{i_n^-} \sigma_{i'i_n,j'k_n}^2 \right)^{-1} \left( \prod_{k_n=i_n^+ - l_n}^{-1} \sigma_{i'i_n,j'k_n}^2 \right) \underline{\sigma_{i'i_n,j'j_n}} \left( \prod_{k_n=i_n^+ - l_n}^{-1} \sigma_{i'i_n,j'k_n}^2 \right)^{-1} \left( \prod_{k_n=1}^{i_n^-} \sigma_{i'i_n,j'k_n}^2 \right), \\ & (\delta_{j'i_n^+, j'l_n}^{l_n - i_n + 1})^{-1} \sigma_{i'i_n,j'i_n}^2 \underline{\sigma_{i'i_n,j'i_n}^{-2}} \sigma_{i'i_n,j'i_n}^{-2} \delta_{j'i_n^+, j'l_n}^{l_n - i_n + 1}, \end{aligned}$$

are equal up to conjugation by twists  $\sigma_{i'i_n,j'k_n}$  with  $k_n \neq j_n$ ,  $i_n - l_n < k_n < i_n$ , and the sub-cable twist  $\delta_{j'i_n^+, j'l_n}^{l_n - i_n + 1}$ .

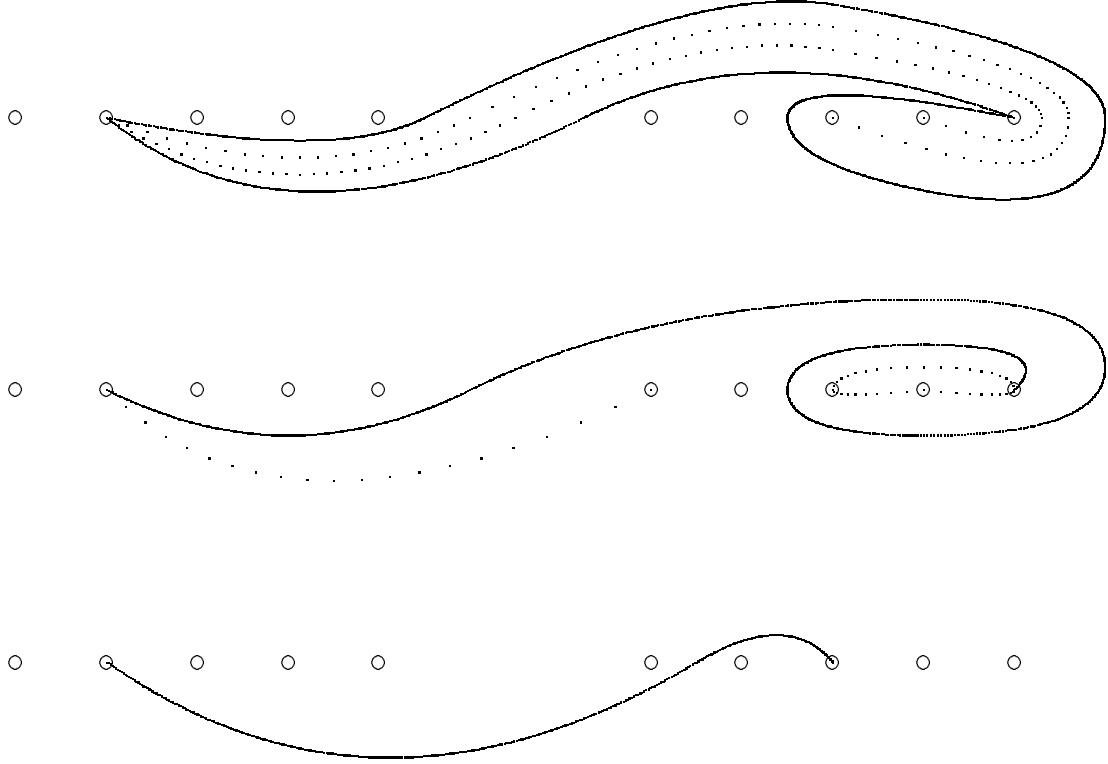
If  $i_n = l_n + 1$  and  $1 < j_n \leq l_n$  then

$$\begin{aligned} & \left( \prod_{k_n=j_n^+}^{l_n} \sigma_{i'k_n,j'j_n}^2 \right) \left( \prod_{k_n=i_n^+}^{j_n^- + l_n} \sigma_{i'i_n,j'k_n}^2 \right)^{-1} \underline{\sigma_{i'i_n,j'j_n}} \left( \prod_{k_n=i_n^+}^{j_n^- + l_n} \sigma_{i'i_n,j'k_n}^2 \right) \left( \prod_{k_n=j_n^+}^{l_n} \sigma_{i'k_n,j'j_n}^2 \right)^{-1}, \\ & \delta_{i'1, i'j_n^-}^{j_n} \sigma_{i'j_n^-, i'j_n}^2 \underline{\sigma_{i'j_n^-, j'j_n}^{-2}} \sigma_{i'j_n^-, i'j_n}^{-2} \delta_{i'1, i'j_n^-}^{-j_n}, \end{aligned}$$

are equal up to conjugation by twists  $\sigma_{i'k_n,j'j_n}$  with  $k_n \neq i_n$ ,  $j_n < k_n < j_n + l_n$ , and the sub-cable twist  $\delta_{i'1, i'j_n^-}^{j_n}$ .

*Proof:* The claims are symmetric to each other under the symmetry induced by the exchange of indices  $i' \leftrightarrow j'$ ,  $i_n \rightarrow l_n + 1 - j_n$ ,  $j_n \rightarrow l_n + 1 - i_n$ .

The claim is illustrated in the case of  $\sigma_{i'2,j'0}$  being conjugated to  $\sigma_{i'2,j'2}^2 \underline{\sigma_{i'2,j'3}} \sigma_{i'2,j'2}^{-2}$ .



The important point to note is the fact, that conjugation by  $(\prod_{k_n=i_n^+ - l_n}^{-1} \sigma_{i'i_n, j'k_n}^2)$  equals conjugation by  $(\delta_{j'i_n^+, j'l_n}^{l_n - i_n})$  and commutes with conjugation by  $(\prod_{k_n=1}^{i_n^-} \sigma_{i'i_n, j'k_n}^2)$  on the twists under consideration. Hence we are left to show that

$$\begin{aligned} & \left( \prod_{k_n=1}^{i_n^-} \sigma_{i'i_n, j'k_n}^2 \right)^{-1} (\delta_{j'i_n^+, j'l_n}) \underline{\sigma_{i'i_n, j'j_n}} (\delta_{j'i_n^+, j'l_n})^{-1} \left( \prod_{k_n=1}^{i_n^-} \sigma_{i'i_n, j'k_n}^2 \right) \\ & = \sigma_{i'i_n, j'i_n}^2 \underline{\sigma_{i'i_n, j'i_n^+}} \sigma_{i'i_n, j'i_n}^{-2} \end{aligned}$$

Since the arcs of  $(\delta_{j'i_n^+, j'l_n}) \underline{\sigma_{i'i_n, j'j_n}} (\delta_{j'i_n^+, j'l_n})^{-1}$  and  $\sigma_{i'i_n, j'i_n^+}$  can be chosen to bound a disc which contains the puncture of indices  $j'1$  to  $j'i_n$  we can conclude that

$$(\delta_{j'i_n^+, j'l_n}) \underline{\sigma_{i'i_n, j'j_n}} (\delta_{j'i_n^+, j'l_n})^{-1} = \left( \prod_{k_n=1}^{i_n} \sigma_{i'i_n, j'k_n}^2 \right) \underline{\sigma_{i'i_n, j'i_n^+}} \left( \prod_{k_n=1}^{i_n} \sigma_{i'i_n, j'k_n}^2 \right)^{-1}$$

from which we deduce the claim.  $\square$

## criterion for change of generators

**Lemma A.17** *Given two finitely filtered sets of elements of a group*

$$S = S_n \supset S_{n-1} \dots \supset S_1, \quad T_n \supset T_{n-1} \dots \supset T_1$$

Then  $S$  and  $T$  generate the same subgroup if

- i)  $T_1 = S_1$ ,
- ii) given  $t \in T_k - T_{k-1}$  there is  $s \in S_k$ , such that  $t$  is equal to  $s$  up to conjugation by elements in  $\langle S_{k-1} \rangle$ ,
- iii) given  $s \in S_k - S_{k-1}$  there is  $t \in T_k$ , such that  $s$  is equal to  $t$  up to conjugation by elements in  $\langle S_{k-1} \rangle$ .

The last hypothesis may be replaced by

- iii') given  $s \in S_k - S_{k-1}$  there is  $t \in T_k$  such that  $s$  is equal to  $t$  up to conjugation by elements in  $\langle T_{k-1}, S_{k-1} \rangle$ .

*Proof:* We show  $\langle T_k \rangle = \langle S_k \rangle$ . So i) starts the induction. Then  $\langle T_k \rangle \subset \langle S_k \rangle$  since by induction  $\langle T_{k-1} \rangle \subset \langle S_{k-1} \rangle \subset \langle S_k \rangle$  and by ii)  $t \in T_k - T_{k-1}$  implies  $t \in \langle S_k \rangle$ .

On the other hand by induction  $\langle S_{k-1} \rangle \subset \langle T_{k-1} \rangle$ , therefore  $s \in S_k - S_{k-1}$  implies  $s \in \langle T_k, S_{k-1} \rangle \subset \langle T_k \rangle$  if iii) holds resp. if iii') holds. In either case we get  $\langle S_k \rangle \subset \langle T_k \rangle$ .  $\square$

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