A finiteness result
for
Siegel modular threefolds

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Zusammenfassung

In der vorliegenden Arbeit untersuchen wir die Kodairadimension von Siegelschen Modulvarietäten, die sich als Quotient des Siegelschen oberen Halbraumes $\mathcal{H}_g$ nach diskreten Untergruppen der symplektischen Gruppe $\mathrm{Sp}(2g, \mathbb{Z})$ darstellen lassen. Insbesondere geben wir eine explizite Beschreibung der nicht–kanonischen Singularitäten, die im Inneren dieser Varietäten liegen. Im Fall $g = 3$ liefert dies zusammen mit einer genauen Untersuchung der Geometrie ihrer Kompaktifizierungen das Hauptresultat dieser Arbeit, nämlich, dass es — abgesehen von einem technischen Detail — nur endlich viele Untergruppen von $\mathrm{Sp}(6, \mathbb{Z})$ gibt, für die die hierdurch gegebenen Siegelschen Modulvarietäten nicht von allgemeinem Typ sind. Dies verallgemeinert ein ähnliches Resultat von Borisov für den Fall $g = 2$, also für Untergruppen von $\mathrm{Sp}(4, \mathbb{Z})$.

Der Schlüssel zum Beweis dieses Theorems ist ein Resultat von Serre, welches besagt, dass für $g \geq 2$ jede Untergruppe von endlichem Index eine Hauptkongruenzuntergruppe $\Gamma_g(n)$ als Normalteiler enthält. Dies gestattet uns, jede Siegelsche Modulvarietät als Quotient von $\mathcal{A}_g(n)$, dem Modulraum der prinzipal–polarisierten abelschen Varietäten mit einer Level n–Struktur, nach einer Untergruppe der endlichen Gruppe $\mathrm{Sp}(2g, \mathbb{Z}/n\mathbb{Z})$ darzustellen. Dank der zahlreichen Resultate, die über diese Räume $\mathcal{A}_g(n)$ bekannt sind, können wir uns somit auf die Untersuchung der endlichen Quotientenabbildungen beschränken, welche durch die Untergruppen von $\mathrm{Sp}(2g, \mathbb{Z}/n\mathbb{Z})$ gegeben sind. Hierbei werden sowohl geometrische als auch algebraische Methoden benötigt, um die notwendigen Bedingungen zu bestimmen, die garantieren, dass die zugehörigen Siegelschen Modulvarietäten von allgemeinem Typ sind.

Schlagworte: Siegelsche Modulvarietät, Modulraum, abelsche Varietät, Kodairadimension

Abstract

In this thesis we study the Kodaira dimension of Siegel modular varieties which are obtained by taking quotients of the Siegel upper half space $\mathcal{H}_g$ by discrete subgroups of the symplectic group $\mathrm{Sp}(2g, \mathbb{Z})$. In particular, we give an explicit description of the non–canonical singularities in the interior of these varieties. For $g = 3$ this and a careful analysis of the geometry of their compactifications yield the main result of this thesis, namely that there are up to a technical issue only finitely many subgroups of $\mathrm{Sp}(6, \mathbb{Z})$ for which the corresponding Siegel modular variety is not of general type. This generalizes a similar finiteness result by Borisov for $g = 2$, i.e. for subgroups of $\mathrm{Sp}(4, \mathbb{Z})$.

The key to our proof is a result due to Serre, which shows that for $g \geq 2$ every subgroup of finite index contains a principal congruence subgroup $\Gamma_g(n)$ as a normal subgroup. This allows us to exhibit every Siegel modular variety as a quotient of $\mathcal{A}_g(n)$, the moduli space of principally polarized abelian varieties with a level n–structure, by a subgroup of the finite group $\mathrm{Sp}(2g, \mathbb{Z}/n\mathbb{Z})$. Thanks to various results on the well–studied spaces $\mathcal{A}_g(n)$, we can then confine ourselves to the study of the finite quotient maps given by the subgroups of $\mathrm{Sp}(2g, \mathbb{Z}/n\mathbb{Z})$. Here both geometric and algebraic techniques are required to determine the conditions to be imposed on these subgroups to ensure that the corresponding Siegel modular varieties are of general type.

Keywords: Siegel modular variety, moduli space, abelian variety, Kodaira dimension

Preface

Siegel modular varieties are obtained by taking quotients of the Siegel upper half space $\mathcal{H}_g$ by discrete subgroups of the symplectic group $\mathrm{Sp}(2g, \mathbb{R})$. They admit an interpretation as moduli spaces for abelian varieties with certain extra data, such as polarizations and level structures. These quasi–projective normal varieties have an analytic realization as locally symmetric varieties which allows them to be compactified in various ways, in particular by the method of toroidal compactification (cf. [AMRT]). One of the first questions to ask towards a classification of these varieties is the one for their Kodaira dimension, which is an important birational invariant.

The focus of this thesis lies on the Kodaira dimension of Siegel modular varieties defined by subgroups of $\mathrm{Sp}(2g, \mathbb{Z})$, i.e. the ones with integer coefficients. There are quite a number of results on the moduli space $\mathcal{A}_g$ of principally polarized abelian varieties which is defined by the full symplectic group $\mathrm{Sp}(2g, \mathbb{Z})$. It is rational for $g = 2$ and $g = 3$ (cf. [Igu1], [Kat]) and unirational for $g = 4$ and $g = 5$ (cf. [Cle], [Don], [MM]). On the other hand Tai showed in [Tai] that $\mathcal{A}_g$ is of general type for $g \geq 9$ which was improved later by Freitag (cf. [Fre] for $g \geq 8$) and Mumford to $g \geq 7$ (cf. [Mum2]). The only open case is $\mathcal{A}_6$, whose Kodaira dimension is still unknown.

Not nearly as much is known for subgroups of $\mathrm{Sp}(2g, \mathbb{Z})$. While there are some results for certain families of subgroups such as the one defining the moduli spaces $\mathcal{A}_g(n)$ parameterizing abelian varieties with level–n structure, there are almost no results for arbitrary subgroups. However, for $g = 2$, Borisov has shown that there are only finitely many subgroups of $\mathrm{Sp}(4, \mathbb{Z})$ of finite index such that the corresponding moduli spaces are not of general type (cf. [Bor]). His proof was inspired by the work of Thompson who proved a similar statement for arithmetic subgroups of $\mathrm{PSL}(2, \mathbb{R})$ (cf. [Tho]).

It is conjectured that Borisov's result can be generalized to arbitrary genus g. To do this, a good understanding of the geometry of the Siegel modular varieties and their compactifications is required. Furthermore, one also has to study closely the subgroups of $\mathrm{Sp}(2g, \mathbb{Z})$ to be able to analyze the singularities occurring both in the varieties themselves and in their compactifications. Thus both geometric and algebraic techniques are needed to acquire a complete picture of these varieties for arbitrary subgroups of $\mathrm{Sp}(2g, \mathbb{Z})$.

In this thesis we will mostly focus on the case $g = 3$, where we will apply these techniques to obtain a generalization of Borisov's result to subgroups of $\mathrm{Sp}(6, \mathbb{Z})$ up to a technical issue. However, quite a few of our intermediate results, which are of interest in their own right, are given for arbitrary g, e.g. the description of the non–canonical singularities lying in the interior, or can easily be generalized to higher genus.

As already mentioned in the abstract, the key idea of the proof of the main result involves a result of Serre (cf. [BLS]), which shows that for $g \geq 2$ every subgroup of finite index in $\mathrm{Sp}(2g, \mathbb{Z})$ is in fact a congruence subgroup, i.e. it contains a principal congruence subgroup $\Gamma_g(n)$ for some level n. Thus Siegel modular varieties can be studied by considering quotients of the moduli space $\mathcal{A}_g(n)$ of principally polarized abelian varieties with a level n–structure by suitable subgroups of $\mathrm{Sp}(2g, \mathbb{Z}/n\mathbb{Z})$. Since the spaces $\mathcal{A}_g(n)$ are known to be of general type for sufficiently big n (cf. Theorem 2.9 for a precise bound), it suffices to study the action of $\mathrm{Sp}(2g, \mathbb{Z}/n\mathbb{Z})$ on them to determine the necessary conditions to be imposed on its subgroups such that the quotients are of general type, too.

In the first chapter we will provide the necessary background and tools for studying Siegel modular varieties and their Kodaira dimension. In particular, we will present the method of toroidal compactification which can be used to compactify these quasi–projective varieties. Moreover, we will introduce modular forms which play a key role when one wants to study pluricanonical forms on these compactifications to determine their Kodaira dimension.

The reader familiar with these topics might want to skip this chapter and immediately jump to Chapter 2, where we will state the main result of this thesis and give a rough outline of its proof which will be carried out in the following chapters.

Chapter 3 provides a characterization of the boundary components of the so–called Voronoi compactification $\mathcal{A}_g^{\mathrm{Vor}}(n)$ of $\mathcal{A}_g(n)$ in terms of isotropic submodules of $(\mathbb{Z}/n\mathbb{Z})^{2g}$.

The following three chapters each contain a different part of the proof as described in the outline in Chapter 2. While Chapter 4 studies the ramification divisors of the quotient maps on $\mathcal{A}_g^{\mathrm{Vor}}(n)$ defined by the subgroups of $\mathrm{Sp}(2g, \mathbb{Z}/n\mathbb{Z})$, chapters 5 and 6 mostly apply algebraic techniques to get a description of the elements in $\mathrm{Sp}(6, \mathbb{Z}/n\mathbb{Z})$ which cause non–canonical singularities in the modular varieties and in the boundary of their compactifications respectively.

In the last chapter, Chapter 7, we put all the results of the previous chapters together to form the proof of the main result.

Throughout this thesis we will use the following notations and conventions:

- We work over the field of the complex numbers $\mathbb{C}$. In particular, all varieties are defined over $\mathbb{C}$.
- The $k \times k$ unit matrix is denoted by $\mathbb{1}_k$. If the dimensions are clear, we also sometimes write just $\mathbb{1}$.
- When it is not ambiguous, we will often omit the pullback map when considering the pullback of a divisor. Thus, if D is a divisor on a variety X, π is a map from $\widetilde{X}$ to X, and E is a divisor on $\widetilde{X}$, we abuse notation and write $D + E$ for the sum $\pi^*D + E$ on $\widetilde{X}$.
- When considering the ideal sheaf $\mathcal{J}_X$ of a subscheme X, we write $\mathcal{J}_X^k$ for its k–th multiplicative power rather than its k–th tensor power. Thus $\mathcal{F} \otimes \mathcal{J}_X^k$ is the sheaf of sections of $\mathcal{F}$ which vanish on X of order at least k.
- For any vector bundle $\mathcal{E}$, the bundle $\mathbb{P}(\mathcal{E})$ denotes the projectivized bundle of *lines* in $\mathcal{E}$, i.e. the geometric projective bundle.
- If p and q are two polynomials of the same degree in the variable n which differ only by a polynomial of strictly smaller degree, we say that p grows as fast as q as n tends to infinity and write $p(n) \sim q(n)$.
- Furthermore, when comparing the growth of two polynomials, we write $p(n) \lesssim q(n)$ or also $p(n) \preceq q(n)$, if there is a polynomial r with $p(n) \leq r(n)$ for all sufficiently big n satisfying $r(n) \sim q(n)$.

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Contents

Chapter 1

Preliminaries

In this introductory chapter we will provide the foundations for studying Siegel modular varieties and their Kodaira dimension. We will give the basic definitions and notations as well as the important tools needed in the following chapters. In doing so, some of the topics will be presented in more generality than actually needed for our purposes to give the reader an overview of the theory of moduli spaces.

We will start in Section 1.1 by introducing a generalization of elliptic curves in higher dimension, the so–called abelian varieties.
In Section 1.2 we will present moduli spaces of abelian varieties which are an example of Siegel modular varieties. Furthermore, we will give the general definition of these Siegel modular varieties which are the main objects studied in this thesis.
The Kodaira dimension is an important birational invariant which will be discussed in Section 1.3, in particular with respect to Siegel modular varieties.
These modular varieties are quasi–projective, but in general not projective. In Section 1.4 we will present the method of toroidal compactification which can be used to compactify them using toric varieties.
The compactified modular varieties are in general not smooth, but have singularities. However, the singularities that occur are all so–called quotient singularities. In Section 1.5 we will provide tools for studying these quotient singularities and will give conditions which guarantee the extensibility of pluricanonical forms over them.
We conclude this chapter in Section 1.6 by introducing modular forms which will be used in the following chapters to determine the Kodaira dimension of the Siegel modular varieties.

The reader familiar with these topics might want to skip this chapter and go directly to Chapter 2 where we present the main result of this thesis and give an outline of the proof which will then be carried out in the following chapters. We will there also repeat the notations as introduced in this chapter whenever needed.

1.1 Abelian varieties

As a generalization of elliptic curves we will introduce in this section abelian varieties and collect some elementary results on them. A more detailed description can be found in [LB].

Let Λ be a rank $2g$ lattice in $\mathbb{C}^g$. This defines a *complex g-dimensional torus* $A := \mathbb{C}^g/\Lambda$.

Definition 1.1 *A g–dimensional complex torus A is called an* abelian variety *if it is a projective variety, i.e. if it can be embedded into some projective space $\mathbb{P}^n$.*

Whereas every 1–dimensional torus is in fact an algebraic curve and thus projective, this is no longer true for $g \geq 2$. To give a criterion for the projectivity of the torus we need the notion of Riemann forms.

Definition 1.2 *A* Riemann form *on $\mathbb{C}^g$ with respect to Λ is a positive semi–definite hermitian form $H : \mathbb{C}^g \times \mathbb{C}^g \to \mathbb{C}$ with the property that its imaginary part* $\mathrm{Im}(H)$ *is integer–valued on Λ.*

We can now state the following characterization of abelian varieties:

Theorem 1.3 *A complex torus $A := \mathbb{C}^g/\Lambda$ is an abelian variety if and only if there exist a Riemann form H on $\mathbb{C}^g$ with respect to Λ. The form H is then called* polarization *of A.*

Proof. [Mum1, p. 35] □

Let H be a Riemann form on $\mathbb{C}^g$. Then $H' =: \mathrm{Im}(H) : \Lambda \otimes \Lambda \to \mathbb{Z}$ is an alternating bilinear form on Λ which admits an $\mathbb{R}$–linear extension to $\mathbb{C}^g$. We have then by [Igu3, p. 65] the identity

$$H(z, w) = H'(iz, w) + iH'(z, w) .$$

This implies that each of the two forms H and H' uniquely determines the other. The form H is positive definite if and only if H' is non–degenerate (cf. [Igu3, Chapter II, 3, Lemma 2]). Because of this we will sometimes also say that H' is a polarization of an abelian variety provided H' is non–degenerate. Using the elementary divisor theorem (cf. [Fro]) we can represent H' in this case with respect to a suitable $\mathbb{Z}$–basis of Λ by a skew–symmetric matrix of the form

$$F := \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix} , \tag{1}$$

where E is a diagonal matrix with the positive integers $e_1, \ldots, e_g$ satisfying $e_1 \,|\, e_2 \,|\, \ldots \,|\, e_g$. The integers $e_1, \ldots, e_g$ are uniquely determined by the polarization H resp. H'. In this context we call the $\mathbb{Z}$–basis of Λ a *symplectic basis* and the tuple $(e_1, \ldots, e_g)$ (or equivalently the diagonal matrix E) the *type of the polarization* H. If all e_i are equal to 1, we say that H is a *principal polarization*.

Definition 1.4 *A* polarized abelian variety of type $(e_1, \ldots, e_g)$ *is a pair* (A, H)*, where* A *is an abelian variety and* H *is a polarization of* A *of type* $(e_1, \ldots, e_g)$*.*

Assume we are given a complex torus $A := \mathbb{C}^g/\Lambda$ together with a symplectic basis of Λ and a fixed type of polarization E. Let $\Omega \in \mathrm{Mat}(2g \times g, \mathbb{C})$ be the matrix whose rows consists of the elements of the symplectic basis of Λ expressed as row vectors with respect to the standard basis of $\mathbb{C}^g$. The matrix Ω is called a *period matrix* of A. Note that the type of the polarization E determines the matrix F in (1) uniquely. Together with the period matrix Ω (or equivalently the symplectic basis of Λ) this defines the forms H' and H. We can now ask whether this form H is a positive definite Riemann form, i.e. whether the pair (A, H) defines a polarized abelian variety of given type E. This question is answered by the following theorem using the so–called *Riemann bilinear relations.*

Theorem 1.5 *Let* $A := \mathbb{C}^g/\Lambda$ *be a complex* g*–dimensional torus,* Ω *a period matrix of* A *and* E *the type of a polarization. The form* H *defined by these data is hermitian and positive definite if and only if the* Riemann bilinear relations

$$\Omega^T F^{-1} \Omega = 0 \quad \textit{and} \quad i\,\Omega^T F^{-1} \bar{\Omega} > 0 \tag{2}$$

are satisfied. In this case the pair (A, H) *is an* E*–polarized abelian variety.*

Proof. [LB, Chapter 4, Theorem 2.1] and Theorem 1.3 □

We will conclude this section by introducing an extra structure on polarized abelian varieties. Let $(\mathbb{C}^g/\Lambda, H)$ be a polarized abelian variety of type E. We define a lattice $\Lambda(H)$ in $\mathbb{C}^g$ by

$$\Lambda(H) := \{z \in \mathbb{C}^g ;\ \mathrm{Im}(H)(z, w) \in \mathbb{Z} \text{ for all } w \in \Lambda\}$$

and write $K(H)$ for the finite group $\Lambda(H)/H$. On $K(H)$ we define an alternating bilinear form e^H by

$$\begin{aligned} e^H : K(H) \times K(H) &\to \mathbb{C}^* \\ (\bar{z}, \bar{w}) &\mapsto e^{-2\pi i\, \mathrm{Im}(H)(z,w)} , \end{aligned}$$

where $\bar{z}$ and $\bar{w}$ denote the images of $z, w \in \Lambda(H)$ in $K(H)$ (cf. [LB, Chapter 6, Proposition 3.1]).

Furthermore, we define the group $K(E) := \left(\mathbb{Z}_{e_1} \times \mathbb{Z}_{e_g}\right)^2$ and an alternating bilinear form e^E on $K(E)$ as follows. Let F_0 be the matrix

$$F_0 := \begin{pmatrix} 0 & E^{-1} \\ -E^{-1} & 0 \end{pmatrix} .$$

We define

$$\begin{aligned} e^E : K(E) \times K(E) &\to \mathbb{C}^* \\ (x, y) &\mapsto e^{-2\pi i x F_0 y^T} , \end{aligned}$$

where we interpret $x, y \in K(E)$ with respect to the standard generators of $K(E)$ as row vectors in $\mathbb{Z}^{2g}$. We are now ready to introduce the notion of a level structure.

Definition 1.6 *(i) A* canonical level structure *on an E–polarized abelian variety (A, H) is a symplectic isomorphism $\alpha : K(H) \to K(E)$, i.e. $\alpha^* e^E = e^H$ holds.*

(ii) A level–n structure *on a principally polarized abelian variety (A, H) is a canonical level structure on A with respect to the polarization given by nH.*

1.2 Siegel modular varieties

In this section we will briefly recall the construction of the modular curve $X^\circ(1)$ parameterizing elliptic curves and generalize it to abelian varieties of arbitrary dimension. In this way we will obtain moduli spaces of abelian varieties which are examples of Siegel modular varieties. We will conclude this section by giving the definition of these modular varieties which are the main objects studied in this thesis.

Let τ be a point of the complex upper half plane $\mathcal{H}_1 := \{\tau \in \mathbb{C}; \operatorname{Im}(\tau) > 0\}$. It defines a lattice Λ_τ in $\mathbb{C}$ given by $\Lambda_\tau := \mathbb{Z}\tau + \mathbb{Z}$ and thus an elliptic curve $E_\tau := \mathbb{C}/\Lambda_\tau$. Since, up to isomorphism, every elliptic curve can be constructed in this manner, we obtain a surjective map

$$\mathcal{H}_1 \to \{\text{elliptic curves}\} / \sim$$

where $E_1 \sim E_2$ holds for two elliptic curves E_1 and E_2 if they are isomorphic.

On $\mathcal{H}_1$ we have the action of the group $\mathrm{SL}(2,\mathbb{Z})$ given by linear fractional transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : \tau \mapsto \frac{a\tau + b}{c\tau + d} .$$

The elliptic curves E_τ and $E_{\tau'}$ are isomorphic if and only if the points $\tau, \tau' \in \mathcal{H}_1$ are identified under the action of $\mathrm{SL}(2,\mathbb{Z})$. This means that the mapping

$$\mathcal{H}_1 / \mathrm{SL}(2,\mathbb{Z}) \to \{\text{elliptic curves}\} / \sim$$

is bijective. In this sense the modular curve $X^\circ(1) := \mathcal{H}_1 / \mathrm{SL}(2,\mathbb{Z})$ parameterizes elliptic curves.

We will now generalize this construction to obtain a moduli space parameterizing abelian varieties. We start by giving a generalization of the Siegel upper half plane.

Definition 1.7 *Let $\mathcal{H}_g$ denote the* Siegel upper half space of degree g

$$\mathcal{H}_g := \{\tau \in \mathrm{Sym}(g,\mathbb{C});\ \mathrm{Im}(\tau) > 0\} .$$

This space carries the structure of an open submanifold of the vector space $\mathrm{Sym}(g,\mathbb{C})$ of dimension $g(g+1)/2$. It is related to polarized abelian varieties as follow. We fix a type of polarization E and associate to every $\tau \in \mathcal{H}_g$ the lattice Λ_τ spanned by the rows of the *normalized period matrix* Ω_τ given by

$$\Omega_\tau := \begin{pmatrix} \tau \\ E \end{pmatrix} . \tag{3}$$

We thus obtain a complex torus $A_\tau := \mathbb{C}^g/\Lambda_\tau$. Furthermore, we define a Riemann form H_τ on $\mathbb{C}^g$ via

$$H_\tau(z,w) := z\,\mathrm{Im}(\tau)^{-1}\bar{w}^T$$

which is positive definite since $\mathrm{Im}(\tau) > 0$. With respect to the symplectic basis of Λ_τ given by the rows of Ω_τ the non–degenerate bilinear form $H'_\tau := \mathrm{Im}(H_\tau)$ is represented by the matrix

$$F = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix} .$$

This means that the torus A_τ carries the structure of an E–polarized abelian variety.

We will now show that conversely up to isomorphism every E–polarized abelian variety can be obtained in this way. Note that two complex tori $\mathbb{C}^g/\Lambda$ and $\mathbb{C}^g/\Lambda'$ are isomorphic if and only if there is a matrix in $\mathrm{GL}(g,\mathbb{C})$ mapping the lattice Λ onto Λ'. This means that given a basis of the lattice Λ we can transform it by a suitable matrix in $\mathrm{GL}(g,\mathbb{C})$ in such a way that the last g basis vectors are just $(e_1, 0, \ldots, 0), \ldots, (0, \ldots, 0, e_g)$, the rows of E. We thus obtain a normalized period matrix as given in (3). The Riemann relations (2) translate in this case to

$$\tau = \tau^T \quad \text{and} \quad \mathrm{Im}(\tau) > 0 \,,$$

which means that τ is an element of the Siegel upper half space $\mathcal{H}_g$. Analogous to the case of elliptic curves we thus obtain a surjective map

$$\mathcal{H}_g \to \{E\text{–polarized abelian varieties of dimension } g\} \,/ \sim \,,$$

where $\sim$ identifies isomorphic abelian varieties.

In order to construct a bijective correspondence, we need to introduce an operation on $\mathcal{H}_g$ which identifies exactly those points in $\mathcal{H}_g$ which correspond to isomorphic abelian varieties. For that we introduce the symplectic group.

Definition 1.8 *Let $\mathcal{R}$ be a commutative ring with* 1. *The* symplectic group $\mathrm{Sp}(2g, \mathcal{R})$ *is defined as*

$$\mathrm{Sp}(2g, \mathcal{R}) := \left\{ M \in \mathrm{GL}(2g, \mathcal{R});\ MJM^T = J \right\} \,,$$

where $J := \begin{pmatrix} 0 & \mathbb{1}_g \\ -\mathbb{1}_g & 0 \end{pmatrix}$ *is the* standard symplectic form.

There is the following characterization for symplectic matrices:

Lemma 1.9 *A matrix* $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ *with* $A, B, C, D \in \mathrm{Mat}(g, \mathcal{R})$ *is symplectic, i.e. an element of* $\mathrm{Sp}(2g, \mathcal{R})$, *if and only if the relations*

$$A^T D - C^T B = \mathbb{1},\ A^T C = C^T A,\ B^T D = D^T B$$

are satisfied.

Proof. [LB, Chapter 8, Lemma 2.1] □

The group $\mathrm{Sp}(2g, \mathbb{R})$ operates on the Siegel upper half space $\mathcal{H}_g$ as follows:

Proposition 1.10 *The group* $\mathrm{Sp}(2g, \mathbb{R})$ *operates on* $\mathcal{H}_g$ *biholomorphically and transitively via*

$$\tau \mapsto M \cdot \tau = (A\tau + B)(C\tau + D)^{-1}$$

for all $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2g, \mathbb{R})$.

Proof. [LB, Chapter 8, Proposition 2.2 and Proposition 2.3] □

The fact that the action of $\mathrm{Sp}(2g, \mathbb{R})$ is transitive means that all points in $\mathcal{H}_g$ are identified. But we are interested in identifying exactly those points in $\mathcal{H}_g$ which give isomorphic E–polarized abelian varieties. We thus consider the subgroup Γ_E of $\mathrm{Sp}(2g, \mathbb{R})$ given by

$$\Gamma_E := \left\{ M \in \mathrm{Sp}(2g, \mathbb{Q});\ M^T \Lambda_E \subset \Lambda_E \right\} ,$$

where Λ_E is the lattice which is defined by $\begin{pmatrix} \mathbb{1}_g & 0 \\ 0 & E \end{pmatrix} \mathbb{Z}^{2g}$. This group provides the desired identification as the following proposition shows:

Proposition 1.11 *Let $\tau, \tau' \in \mathcal{H}_g$. The E–polarized abelian varieties (A_τ, H_τ) and $(A_{\tau'}, H_{\tau'})$ are isomorphic if and only if there exists a matrix $M \in \Gamma_E$ such that $\tau' = M \cdot \tau$.*

Proof. [LB, Chapter 8, Proposition 1.3] □

Before we construct the moduli space of E–polarized abelian varieties by considering the quotient of $\mathcal{H}_g$ by the group Γ_E, we will collect some properties of Γ_E. For that we first define the following notions.

Definition 1.12 *(i) A group G operates* properly discontinuously *on a topological space X if for every pair of compact subsets K_1, K_2 of X the set*

$$\{g \in G;\ g \cdot K_1 \cap K_2 \neq \emptyset\}$$

is finite.

(ii) A subgroup $\Gamma < \mathrm{Sp}(2g, \mathbb{R})$ is called discrete *if for every compact set $K \subset \mathrm{Sp}(2g, \mathbb{R})$ the intersection with Γ is finite.*

There is the following correspondence between these two notions:

Lemma 1.13 *A subgroup $\Gamma < \mathrm{Sp}(2g, \mathbb{R})$ operates properly discontinuously on $\mathcal{H}_g$ if and only if it is discrete.*

Proof. [Fre, Kapitel I, Satz 1.10] □

The importance of these two notions is apparent from the following theorem:

Theorem 1.14 *Let X be a complex analytic space and G be a group operating properly discontinuously on X. Then the quotient X/G is also a complex analytic space. Furthermore, X/G is normal, if X is normal.*

Proof. [Car2] □

According to [LB, p. 219] Γ_E is a discrete subgroup of $\mathrm{Sp}(2g, \mathbb{R})$ and thus operates properly discontinuously by Lemma 1.13 on $\mathcal{H}_g$. We define the moduli space

$$\mathcal{A}_E := \mathcal{H}_g/\Gamma_E \ ,$$

which carries by the above theorem the structure of a normal complex analytic space of dimension $g(g+1)/2$. This gives us the bijective map

$$\mathcal{A}_E = \mathcal{H}_g/\Gamma_E \to \{E\text{–polarized abelian varieties of dimension } g\} / \sim \ .$$

We obtain the following result:

Theorem 1.15 *$\mathcal{A}_E$ is a coarse moduli space of g–dimensional abelian varieties with a polarization of type E.*

Proof. [LB, Chapter 8, Theorem 2.6] □

In the case where E is a principal polarization, the group Γ_E is just the subgroup of $\mathrm{Sp}(2g, \mathbb{R})$ consisting of all symplectic matrices with integer coefficients, i.e. the group

$$\mathrm{Sp}(2g, \mathbb{Z}) = \left\{ M \in \mathrm{GL}(2g, \mathbb{Z});\ MJM^T = J \right\}$$

with J given as in Definition 1.8. We will write $\mathcal{A}_g$ for the moduli space given by the quotient of $\mathcal{H}_g$ by this group.

To obtain a moduli space for principally polarized abelian varieties with a level–n structure as defined in Definition 1.6 (ii), we need to consider a certain subgroup of $\mathrm{Sp}(2g, \mathbb{Z})$, namely

$$\Gamma_g(n) := \{ M \in \mathrm{Sp}(2g, \mathbb{Z});\ M \equiv \mathbb{1} \pmod{n} \} \ ,$$

the so–called *principal congruence subgroup of level n*. For each n it is a normal subgroup of finite index in $\mathrm{Sp}(2g, \mathbb{Z})$ (cf. [LB, p. 222]). Note that for $n = 1$ the group $\Gamma_g(1)$ is just the full symplectic group $\mathrm{Sp}(2g, \mathbb{Z})$.

As subgroups of the discrete group $\mathrm{Sp}(2g, \mathbb{Z})$, the principal congruence subgroups $\Gamma_g(n)$ are also discrete and thus operate properly discontinuously on $\mathcal{H}_g$. We obtain:

Theorem 1.16 *The normal, complex, analytic space $\mathcal{A}_g(n) := \mathcal{H}_g/\Gamma_g(n)$ is a moduli space of principally polarized g–dimensional abelian varieties with a level–n structure. The inclusion $\Gamma_g(n) \subset \mathrm{Sp}(2g, \mathbb{Z})$ induces a holomorphic map $\mathcal{A}_g(n) \hookrightarrow \mathcal{A}_g$ of finite degree.*

Proof. [LB, Chapter 8, Theorem 3.1] □

We have the following result on the spaces we introduced in this section.

Theorem 1.17 *The spaces $\mathcal{A}_E$, $\mathcal{A}_g$ and $\mathcal{A}_g(n)$ are quasi–projective, $g(g+1)/2$–dimensional algebraic varieties.*

Proof. [LB, Chapter 8, Remark 10.4] □

We conclude this section by generalizing the construction as follows. Instead of the groups $\Gamma_g(n)$, we can take any subgroup Γ of $\mathrm{Sp}(2g,\mathbb{Z})$ of finite index and consider the quotient

$$\mathcal{A}_\Gamma := \mathcal{H}_g/\Gamma \ .$$

Since Γ is as a subgroup of $\mathrm{Sp}(2g,\mathbb{Z})$ discrete, $\mathcal{A}_\Gamma$ is by Theorem 1.14 a complex analytic space. The fact that Γ has finite index implies that $\mathcal{A}_\Gamma$ is $g(g+1)/2$–dimensional.

The special role of $\Gamma_g(n)$ among the subgroups of finite index in $\mathrm{Sp}(2g,\mathbb{Z})$ becomes clear by the following theorem due to Serre, Bass, Lazard and Milnor.

Theorem 1.18 *If $g \geq 2$, then every subgroup Γ of $\mathrm{Sp}(2g,\mathbb{Z})$ of finite index is a* congruence subgroup*, i.e. contains a principal congruence subgroup $\Gamma_g(n)$ as a normal subgroup of finite index.*

Proof. [BLS, Théorème 3] and [BMS, Theorem 12.4] □

As a consequence we can realize every quotient $\mathcal{A}_\Gamma$ as a quotient of $\mathcal{A}_g(n)$ by the finite group $\Gamma/\Gamma_g(n)$ for some level n. Thus $\mathcal{A}_\Gamma$ is by Theorem 1.17 also a quasi–projective, $g(g+1)/2$–dimensional algebraic variety.

Definition 1.19 *The varieties $\mathcal{A}_\Gamma$ obtained by taking quotients of $\mathcal{H}_g$ by subgroups Γ of $\mathrm{Sp}(2g,\mathbb{Z})$ of finite index are called* Siegel modular varieties of genus g defined over $\mathbb{Z}$.

These Siegel modular varieties have been the subject of many studies, especially in the cases where they parameterize certain geometric objects such as abelian varieties with extra structures. But also when there is no such geometric interpretation, these varieties possess a rich geometry and many interesting properties. As the main objects of this thesis, they deserve a separate chapter, in which we will give a short survey on the known results and a more detailed description of their geometry (cf. Chapter 2).

1.3 Kodaira dimension

The Kodaira dimension is one of the first birational invariants to ask for, if one wants to classify algebraic varieties. In this section we will give a definition of this invariant and collect some elementary results which will be useful in the following chapters.

We define the notion of the Kodaira dimension for smooth projective varieties first.

Definition 1.20 *Let X be a smooth projective variety and let K_X denote its canonical divisor. The* Kodaira dimension $\kappa(X)$ *of X is defined as*

$$\kappa(X) := \begin{cases} -\infty & \text{if } H^0(X, \mathcal{O}_X(mK_X)) = 0 \text{ for all } m > 0 \ , \\ 0 & \text{if } \dim H^0(X, \mathcal{O}_X(mK_X)) \text{ is bounded, but } \neq 0 \text{ for one } m > 0 \ , \\ k & \text{if } \dim H^0(X, \mathcal{O}_X(mK_X)) \text{ grows as } m^k \text{ for } k > 0 \ . \end{cases}$$

According to [Har2, p. 421] the Kodaira dimension of a variety X is a birational invariant which is bounded by the dimension of X.

By the results from the previous section the Siegel modular varieties $\mathcal{A}_\Gamma$ we want to consider are quasi–projective varieties which are in general not projective. This means that a priori the Kodaira dimension is not defined for them. To extend this notion also to arbitrary algebraic varieties X we have to consider a suitable member of the birational equivalence class of X, a so–called *model* of X.

Definition 1.21 *Let X be an algebraic variety over $\mathbb{C}$. The* Kodaira dimension *$\kappa(X)$ of X is defined as the Kodaira dimension of a smooth projective model $\widetilde{X}$ of X.*

Note that the existence of such a model is guaranteed due to results of Hironaka [Hir1] and that $\kappa(X)$ is independent of the model chosen since it is a birational invariant.

In the following chapters we will try to impose conditions on Γ which ensure that the Kodaira dimension of the corresponding moduli space $\mathcal{A}_\Gamma$ is maximal. We thus define

Definition 1.22 *An algebraic variety X is called a* variety of general type *if its Kodaira dimension coincides with its complex dimension, i.e. if $\kappa(X) = \dim X$.*

We will often encounter finite morphisms between our moduli spaces. The following useful result tells us something about their Kodaira dimensions.

Proposition 1.23 *Let $\mu : X \to Y$ be a finite morphism of algebraic varieties. If Y is of general type, then so is X.*

Proof. It suffices to note that we can extend μ to a surjective morphism of smooth projective models. We can then pull back pluricanonical forms (cf. [Bor, Proposition 7.8]). □

1.4 Toroidal Compactification

In this section we will present a general method for compactifying the Siegel modular varieties from Section 1.2. An important role is played by so–called toric varieties which we will introduce in the first subsection. The reader familiar with the theory of toric varieties may skip this part and can go directly to the second subsection, where we present the method of toroidal compactification.

1.4.1 Toric Varieties

In this subsection we want to give a brief introduction to the area of toric varieties. These are special algebraic varieties that contain an algebraic torus as an open and dense subset. Here we follow the book of Oda [Oda] and will refer to it for the proofs.

Let N be a free $\mathbb{Z}$–module of rank r over $\mathbb{Z}$ and $M := \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ its dual $\mathbb{Z}$–module. We denote the canonical pairing by $\langle \cdot, \cdot \rangle \ : M \times N \to \mathbb{Z}$. We now define an r–dimensional algebraic torus $T \cong (\mathbb{C}^*)^r$ by

$$T := T_N := \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{C}^*) \cong N \otimes_{\mathbb{Z}} \mathbb{C}^* \ . \tag{4}$$

Thereby we obtain the following interpretations:

For $n \in N$ we define the *one-parameter subgroup* γ_n as the homomorphism $\gamma_n : \mathbb{C}^* \to T$ that is given by the equation

$$\gamma_n(\lambda)(m) := \lambda^{\langle m,n \rangle} \quad \lambda \in \mathbb{C}^*, \ m \in M \ . \tag{5}$$

This gives us the identification $N \cong \mathrm{Hom}(\mathbb{C}^*, T)$, the *group of one-parameter subgroups of* T.

For $m \in M$ denote by $e(m)$ the homomorphism $e(m) : T \to \mathbb{C}^*$ defined by

$$e(m)(t) := t(m) \quad t \in T \ . \tag{6}$$

$e(m)$ is also called *character* and we obtain the description of $M \cong \mathrm{Hom}(T, \mathbb{C}^*)$ as the *group of characters of* T.

Let now $N_\mathbb{R} = N \otimes_\mathbb{Z} \mathbb{R}$ and $M_\mathbb{R} = M \otimes_\mathbb{Z} \mathbb{R}$. The canonical pairing can then be extended to $\langle \cdot, \cdot \rangle \, : M_\mathbb{R} \times N_\mathbb{R} \to \mathbb{R}$.

Definition 1.24 *(i) A subset* σ *of* $N_\mathbb{R}$ *is called a* strongly convex rational polyhedral cone *if* $\sigma \cap (-\sigma) = \{0\}$ *and if there exist elements* $n_1, \dots, n_s$ *of* N *such that*

$$\sigma = \mathbb{R}_{\geq 0}\, n_1 + \cdots + \mathbb{R}_{\geq 0}\, n_s \ .$$

(ii) σ *is called* nonsingular *(or* basic*) if there exist a* $\mathbb{Z}$*–base* $\{n_1, \dots, n_r\}$ *of* N *and an* $s \leq r$ *such that* $\sigma = \mathbb{R}_{\geq 0}\, n_1 + \cdots + \mathbb{R}_{\geq 0}\, n_s$.

(iii) The dimension *of* σ*, denoted by* $\dim \sigma$*, is the dimension of the smallest* $\mathbb{R}$*–subspace of* $N_\mathbb{R}$ *that contains* σ.

(iv) The cone $\sigma^\vee$ *in* $M_\mathbb{R}$ dual to σ *is defined as*

$$\sigma^\vee := \{x \in M_\mathbb{R} \, ; \ \langle x, y \rangle \geq 0 \ \forall y \in \sigma\} \ .$$

We now want to assign an affine algebraic variety that will later on form an open subset of our toric variety to every strongly convex rational polyhedral cone. For this purpose we will first construct a semigroup in M from such a cone with the help of the preceding definition.

Proposition 1.25 *Let* σ *be a strongly convex rational polyhedral cone in* $N_\mathbb{R}$.

(i) $M \cap \sigma^\vee$ *is a finitely generated additive semigroup with* 0.

(ii) $M \cap \sigma^\vee$ *generates* M *as a group, i.e.* $M = M \cap \sigma^\vee + (-(M \cap \sigma^\vee))$.

(iii) $M \cap \sigma^\vee$ *is saturated, i.e.* $cm \in M \cap \sigma^\vee$ *for an* $m \in M$ *and a positive integer* c *implies* $m \in M \cap \sigma^\vee$.

Proof. [Oda, Proposition 1.1] □

Using this we can define for a strongly convex rational polyhedral cone σ in $N_\mathbb{R}$ the finitely generated semigroup algebra $\mathbb{C}\,[M \cap \sigma^\vee]$ as

$$\mathbb{C}\,[M \cap \sigma^\vee] := \bigoplus_{m \in M \cap \sigma^\vee} \mathbb{C}\, e(m) \ , \tag{7}$$

with multiplication defined by $e(m)e(m') = e(m + m')$. This enables us to assign an affine variety to σ as follows:

Definition 1.26 *Let $\sigma \subset N_{\mathbb{R}}$ be a strongly convex rational polyhedral cone. The* affine torus embedding T_σ *is defined as*

$$T_\sigma := \{\varphi : \mathbb{C}\,[M \cap \sigma^\vee] \to \mathbb{C}\,;\ \varphi \text{ is an algebra homomorphism}\}\ .$$

According to [Oda, p. 5] T_σ may be interpreted as the set of $\mathbb{C}$–valued points of the affine scheme Spec $\mathbb{C}\,[M \cap \sigma^\vee]$. By this T_σ obtains the structure of an r–dimensional variety. As presented in [Oda, Proposition 1.2] it can be represented analytically by the affine embedding

$$\begin{aligned} T_\sigma \to & \mathbb{C}^b \\ \varphi \mapsto & (\varphi(e(m_1)), \ldots, \varphi(e(m_b)))\ , \end{aligned} \tag{8}$$

where $M \cap \sigma^\vee = \mathbb{Z}_{\geq 0}\, m_1 + \cdots + \mathbb{Z}_{\geq 0}\, m_b$. We now want to compare different cones and therefore introduce the following partial order:

Definition 1.27 *Let σ be a strongly convex rational polyhedral cone. A subset τ of σ is called a* face *of σ, written $\tau \prec \sigma$, if there exists $m_0 \in \sigma^\vee$ such that*

$$\tau = \sigma \cap \{m_0\}^\perp := \{y \in \sigma\,;\ \langle m_0, y\rangle = 0\}\ .$$

In particular $\{0\}$ is a face of σ.

According to [Oda, Proposition 1.3] for every face τ the above $m_0 \in \sigma^\vee$ can be chosen in such a way that $m_0 \in M \cap \sigma^\vee$ which shows that every face τ is again a strongly convex, rational, polyhedral cone.

This relation of two cones induces the following relationship of the corresponding affine torus embeddings:

Proposition 1.28 *For a strongly convex rational polyhedral cone σ and a face τ of σ the inclusion $M \cap \sigma^\vee \subset M \cap \tau^\vee$ induces an embedding $T_\tau \hookrightarrow T_\sigma$ via which T_τ may be interpreted as an open and dense subset of T_σ. In particular we obtain the embedding $T_{\{0\}} \cong T \hookrightarrow T_\sigma$ for $\tau = \{0\}$ such that every r–dimensional affine torus embedding contains an r–dimensional algebraic torus T as an open and dense subset.*

Proof. [Oda, Proposition 1.3] □

These embeddings allow us to "glue" open affine torus embeddings along common faces. We would like to choose a collection of cones in such a way that their affine torus embeddings can be used to obtain a toric variety. Doing so, in particular we have to make sure that the common faces of any two cones are contained within our collection such that we can glue the corresponding affine torus embeddings using the preceding proposition.

Definition 1.29 *(i) A* fan *for N is a non–empty family Σ of strongly convex rational polyhedral cones which satisfies the following two conditions:*

(a) $\sigma \in \Sigma, \tau \prec \sigma \implies \tau \in \Sigma$

(b) $\sigma_1, \sigma_2 \in \Sigma \implies \sigma_1 \cap \sigma_2 \prec \sigma_1, \sigma_2$

Sometimes we will also call the pair (N, Σ) a fan.

(ii) The union $|\Sigma| := \bigcup_{\sigma\in\Sigma} \sigma$ is called the support *of the fan Σ.*

(iii) A fan Σ for N is called nonsingular *if all $\sigma \in \Sigma$ are nonsingular.*

We can now construct the toric variety defined by a fan.

Definition 1.30 *Let Σ be a fan in $N_{\mathbb{R}}$. The* toric variety (torus embedding) T_Σ *defined by Σ is the identification space*

$$T_\Sigma := \coprod_{\sigma\in\Sigma} T_\sigma \Big/ \sim ,$$

where $\varphi_1 \sim \varphi_2$ for $\varphi_1 \in T_{\sigma_1}$ and $\varphi_2 \in T_{\sigma_2}$ with $\sigma_1, \sigma_2 \in \Sigma$ holds, if there exists a cone $\tau \in \Sigma$ with $\tau \subset \sigma_1 \cap \sigma_2$ such that $\varphi_1 = \varphi_2 \in T_\tau \subset T_{\sigma_1}, T_{\sigma_2}$.

T_Σ is a normal irreducible analytic Hausdorff space of dimension r which contains the torus $T = T_{\{0\}}$ as an open and dense subset. If we impose further conditions on the fan Σ we even obtain a stronger result:

Theorem 1.31 *The toric variety T_Σ defined by the fan Σ is smooth, i.e. a complex manifold, if and only if Σ is nonsingular.*

Proof. [Oda, Theorem 1.10] □

1.4.2 Toroidal Compactification

The method of toroidal compactification is due to Mumford and Hirzebruch (cf. [AMRT]). It uses toric varieties as introduced in the previous subsection to compactify locally symmetric varieties. Since we are only interested in obtaining compactifications for Siegel modular varieties, we will restrict ourselves to the results relevant to this task, although the method is much more general. Another useful source of information on this method in particular for Siegel modular threefolds is [HKW].

We will start by defining the boundary components of the Siegel upper half space $\mathcal{H}_g$ of degree g which we will compactify in a second step. To do this, we represent

$\mathcal{H}_g$ as a bounded homogeneous domain in $\mathbb{C}^{g(g+1)/2}$ which we can compactify by taking the topological closure. For that we use the *Cayley transformation* $\phi : \mathcal{H}_g \to \mathrm{Sym}(g, \mathbb{C})$ defined by

$$\phi(\tau) := (\tau - i\mathbb{1}_g)(\tau + i\mathbb{1}_g)^{-1} .$$

ϕ maps $\mathcal{H}_g$ biholomorphically to the $g(g+1)/2$–dimensional bounded domain

$$D_g := \left\{ Z \in \mathrm{Sym}(g, \mathbb{C}) \, ; \; \mathbb{1}_g - Z\overline{Z} > 0 \right\} .$$

The operation of $\mathrm{Sp}(2g, \mathbb{R})$ on $\mathcal{H}_g$ induces via the isomorphism ϕ an operation on D_g which extends to the topological closure $\overline{D_g}$. We can now decompose $\overline{D_g}$ into boundary components.

Definition 1.32 *A* boundary component F *of* D_g *is an equivalence class of points in* $\overline{D_g}$*. Two points* $Z_1, Z_2 \in \overline{D_g}$ *are equivalent if they can be connected by finitely many holomorphic curves, i.e. if there exist holomorphic maps* $\varphi_i : D_1 \to \overline{D_g}$, $i = 1, \ldots, n$ *from the open unit disc* $D_1 \subset \mathbb{C}$ *to* $\overline{D_g}$ *such that* $Z_1 \in \varphi_1(D_1)$, $Z_2 \in \varphi_n(D_1)$ *and* $\varphi_i(D_1) \cap \varphi_{i+1}(D_1) \neq \emptyset$ *for* $i = 1, \ldots, n-1$*. A boundary component is called* proper *if it is contained in* $\overline{D_g} \setminus D_g$*.*

It is easy to check that the operation of $\mathrm{Sp}(2g, \mathbb{R})$ leaves the set of boundary components of D_g invariant - in fact, each matrix $M \in \mathrm{Sp}(2g, \mathbb{R})$ induces a permutation of these components.

We will now establish a relation between boundary components of D_g and certain subspaces of $\mathbb{R}^{2g}$ and thus define:

Definition 1.33 *A subspace* $U \subset \mathbb{R}^{2g}$ *is called* J–isotropic*, if*

$$\langle u, v \rangle := u \, J \, v^T = 0$$

for all $u, v \in U$*, where* J *is the standard symplectic form from Definition 1.8.*

The relation is now given as follows. For $Z \in D_g$ let $U(Z)$ be the J–isotropic subspace of $\mathbb{R}^{2g}$ defined by $U(Z) := \ker(\psi_Z)$, where $\psi_Z : \mathbb{R}^{2g} \to \mathbb{C}^g$ is given by

$$\nu \mapsto \nu \cdot \begin{pmatrix} i(\mathbb{1}_g + Z) \\ \mathbb{1}_g - Z \end{pmatrix} .$$

This relation is invariant with respect to the boundary components of D_g. We can thus associate to a boundary component F of D_g a J–isotropic subspace $U(F) := U(Z), Z \in F$. A component F is called a *corank* i*–boundary component* if the dimension of $U(F)$ is equal to i.

Conversely, we can associate to every J–isotropic subspace a boundary component. In particular, for the following standard J–isotropic subspaces of $\mathbb{R}^{2g}$

$$U^{(k)} := (0, \ldots, 0, \underbrace{*, \ldots, *}_{g-k \text{ times}}) \subset \mathbb{R}^{2g} \qquad \text{for } k = 0, \ldots, g \tag{9}$$

we have that the corresponding boundary component is given by

$$F^{(k)} := \left\{ Z \in \overline{D_g}\,;\; U(Z) = U^{(k)} \right\} . \tag{10}$$

It is easy to check that

$$F^{(k)} := \left\{ \begin{pmatrix} Z & 0 \\ 0 & \mathbb{1}_{g-k} \end{pmatrix} ;\; Z \in D_k \right\} . \tag{11}$$

Using these standard components, we obtain the following description of $\overline{D_g}$:

Lemma 1.34 *$\overline{D_g}$ can be written as a union of boundary components as follows:*

$$\overline{D_g} := \bigcup_{\substack{M \in \mathrm{Sp}(2g,\mathbb{R}) \\ 0 \le k \le g}} M \cdot F^{(k)}$$

In particular, every boundary component of D_g is equivalent under the operation of $\mathrm{Sp}(2g, \mathbb{R})$ to a standard component $F^{(k)}$ for some $k \in \{0, \ldots, g\}$.

Proof. The proof given in [HKW, Proposition 3.12] can easily be generalized to arbitrary g. □

We now introduce the following relation on the set of boundary components of D_g:

Definition 1.35 *A boundary component F is said to be* adjacent *to a boundary component F' if $F \neq F'$ and $F \subset \overline{F'}$. In this case, we denote this relation by $F \prec F'$.*

In particular, every proper boundary component is adjacent to $F^{(g)} = D_g$. This adjacency relation on the boundary components induces a relation on the associated J–isotropic subspaces.

Proposition 1.36 *Let F, F' be two boundary components of D_g. We then have*

$$F \prec F' \Longleftrightarrow U(F') \subsetneq U(F)$$

Proof. Again the proof is completely analogous to the one given in [HKW, Proposition 3.16] for the $g = 2$–case. □

After we finished our description of the boundary components and their adjacency relations, we will turn our attention to the compactification in the direction of these components. However, we will not need to compactify all the components, but only the ones with a certain special property. We thus restrict ourselves to the following special case:

Definition 1.37 *A boundary component F of D_g is called* rational*, if its stabilizing subgroup*

$$\mathcal{P}(F) := \{M \in \mathrm{Sp}(2g, \mathbb{R})\,;\ M(F) = F\}$$

is defined over $\mathbb{Q}$ (cf. [HKW, Definition 3.17]).

Rational boundary components behave in many respects like ordinary boundary components of D_g. This is due to the fact that the operation of $\mathrm{Sp}(2g, \mathbb{R})$ on the boundary components induces an operation of $\mathrm{Sp}(2g, \mathbb{Q})$ on the rational boundary components. For instance, we obtain a 1–to–1 correspondence between rational boundary components and J–isotropic subspaces of $\mathbb{Q}^{2g}$ and also the result of Proposition 1.36 has an analogue (cf. [HKW, Proposition 3.20]).

Let F be a rational boundary component of D_g. By [AMRT, III.3.2] its stabilizer $\mathcal{P}(F)$ is then a maximal parabolic subgroup of $\mathrm{Sp}(2g, \mathbb{R})$ defined over $\mathbb{Q}$. In particular, the stabilizer $\mathcal{P}(D_g)$ of D_g coincides with the full symplectic group $\mathrm{Sp}(2g, \mathbb{R})$. The rational parabolic subgroups are in 1–to–1 correspondence with flags of J–isotropic subspaces of $\mathbb{Q}^{2g}$ if we associate each flag to its stabilizing subgroup. Note that the maximal rational parabolic subgroups of $\mathrm{Sp}(2g, \mathbb{R})$ correspond to flags of length 1, i.e. the isotropic subspaces of $\mathbb{Q}^{2g}$ (cf. [HKW, Remark 3.45]).

So far we have described the boundary components with respect to the full symplectic group $\mathrm{Sp}(2g, \mathbb{R})$. Since we want to construct a compactification of a Siegel modular variety given by an arbitrary subgroup Γ of $\mathrm{Sp}(2g, \mathbb{Z})$, we need a description of the boundary components with respect to this group Γ. We will then give a partial compactification for each boundary component and "glue" them together in a second step. How this gluing takes place is determined by the equivalence classes of the boundary components and their adjacency relations with respect to Γ. To describe them we make use of the correspondences between boundary components and J–isotropic subspaces, and between flags of J–isotropic subspaces and maximal parabolic subgroups.

Definition 1.38 *Let Γ be a subgroup of $\mathrm{Sp}(2g, \mathbb{Z})$ of finite index. The* Tits building *of Γ is the set of Γ–equivalence classes of rational parabolic subgroups of $\mathrm{Sp}(2g, \mathbb{R})$ equipped with the partial order given by inclusion.*

Let F be a rational boundary component of D_g. Then $P(F) := \mathcal{P}(F) \cap \Gamma$ is the stabilizing subgroup of F with respect to Γ. There exists a $P(F)$–invariant interior neighborhood $N(F)$ of F in D_g such that the quotient map

$$p(F) : D_g/P(F) \to D_g/\Gamma \tag{12}$$

is an isomorphism when restricted to $N(F)$ (cf. [AMRT, III.1]). By an *interior neighborhood* of F we mean here the intersection of a neighborhood of F in the complex topology in $\overline{D_g}$ with D_g. In this sense we will consider a variety which contains $D_g/P(F)$ as an open dense subvariety as a partial compactification of D_g/Γ in the direction of F. We can therefore restrict to considering the elements in $P(F) \subset \Gamma$ when we want to study the structure of D_g/Γ in a neighborhood of F. To do this, we first decompose the stabilizer of F.

Definition 1.39 *Let F be a rational boundary component of D_g and let $\mathcal{P}'(F)$ be the center of the unipotent radical $R_{\mathrm{u}}(\mathcal{P}(F))$ of its stabilizing subgroup $\mathcal{P}(F)$. Let $\mathcal{P}''(F)$ denote the quotient $\mathcal{P}(F)/\mathcal{P}'(F)$. For a subgroup Γ of $\mathrm{Sp}(2g, \mathbb{Z})$ of finite index let $P(F) := \mathcal{P}(F) \cap \Gamma$, $P'(F) := \mathcal{P}'(F) \cap \Gamma$ and $P'' := P(F)/P'(F)$. The group $P(F)$ operates by conjugation on $P'(F)$. We can thus associate to each $g \in P''(F)$ the automorphism*

$$\mathrm{Ad}(g) : P'(F) \to P'(F),\ h \mapsto ghg^{-1}\ . \tag{13}$$

The group of all these automorphisms is denoted by $\bar{P}(F)$.

We can now define the *partial quotient map*

$$e(F) : D_g \to D_g/P'(F) \tag{14}$$

associated to $P'(F)$. We denote the image of D_g under this map by $X(F)$. As the center of a unipotent Lie group $\mathcal{P}'(F)$ is isomorphic to a real vector space. $P'(F)$ as a discrete subgroup defines a sublattice in this vector space (cf. [AMRT, III.5]). One can therefore hope to find a good description of the partial quotient $X(F) = D_g/P'(F)$. As a second step to obtain a description of $D_g/P(F)$, we also have to take the operation of $P''(F) = P(F)/P'(F)$ on $X(F)$ induced by the operation of $P(F)$ on D_g into account.

Theorem 1.40 *Let F be a rational boundary component of D_g. Then there exists a trivial torus bundle $\mathcal{X}(F)$ with fiber $T := P'(F) \otimes_{\mathbb{Z}} \mathbb{C}/P'(F) \cong (\mathbb{C}^*)^r$ and base $F \times V(F)$, where r denotes the rank of the lattice $P'(F)$ and $V(F)$ stands for the complex vector space $R_{\mathrm{u}}(\mathcal{P}(F))/\mathcal{P}'(F)$. The image $X(F)$ under the partial quotient map $e(F)$ is isomorphic to an open subset of $\mathcal{X}(F)$. The operation of $P''(F)$ on $X(F)$ can be extended to an operation on $\mathcal{X}(F)$.*

Proof. [AMRT, III.4] □

We will now choose a fan Σ in the real vector space $\mathcal{P}'(F) \cong P(F) \otimes_{\mathbb{Z}} \mathbb{R}$ and thus construct a trivial fiber bundle $\mathcal{X}_\Sigma(F)$, whose fibers are just the toric variety T_Σ defined by Σ and which contains the toric bundle $\mathcal{X}(F)$. To do this, we have to impose certain conditions on the fan Σ, in particular, it has to have sufficiently big support. We define the following spaces:

Definition 1.41 *(i) For a standard component $F^{(k)}$, $k = 0, \dots, g$ we define the open homogeneous cone $C(F^{(k)})$ in $\mathcal{P}(F^{(k)})$ by*

$$C(F^{(k)}) := \left\{ M \begin{pmatrix} \mathbb{1}_g & S_{(k)} \\ 0_g & \mathbb{1}_g \end{pmatrix} M^{-1} ;\ M \in \mathcal{P}(F^{(k)}) \right\} ,$$

where $S_{(k)}$ is the matrix given by $S_{(k)} := \begin{pmatrix} 0_k & 0 \\ 0 & \mathbb{1}_{g-k} \end{pmatrix}$ (cf. [AMRT, Theorem III.4.1]).

(ii) For a rational boundary component F of D_g with $F = M \cdot F^{(k)}$ for an $M \in \mathrm{Sp}(2g, \mathbb{Q})$ and some $k \in \{0, \dots, g\}$ we define the open homogeneous cone $C(F)$ in $\mathcal{P}(F)$ by $C(F) := M \cdot C(F^{(k)}) \cdot M^{-1}$.

(iii) The rational closure *$C(F)^{\mathrm{rc}}$ of $C(F)$ is the union of $C(F)$ with all rational cones that are adjacent to F, i.e.*

$$C(F)^{\mathrm{rc}} := C(F) \cup \bigcup_{F \prec F'} C(F') ,$$

where the union is taken over all rational boundary components F' with $F \prec F'$.

We can now formulate the conditions on the fan Σ in $\mathcal{P}'(F)$. The space $C(F)^{\mathrm{rc}}$ that we just defined will be the support of Σ which will guarantee that the fan Σ is big enough for our purposes.

Definition 1.42 *Let F be a rational boundary component of D_g. We call a fan Σ in $\mathcal{P}'(F)$* admissible *if it satisfies the following three conditions:*

(i) $|\Sigma| = C(F)^{\mathrm{rc}}$,

(ii) $\sigma \in \Sigma,\ M \in \bar{P}(F) \implies M \cdot \sigma \in \Sigma$,

(iii) $\Sigma / \bar{P}(F)$ is a finite set.

For an admissible fan Σ in $\mathcal{P}'(F)$ denote by T_Σ the toric variety defined by Σ. We can now construct the fiber bundle $\mathcal{X}_\Sigma(F)$.

Theorem 1.43 *Let F be a rational boundary component of D_g and Σ an admissible fan in $\mathcal{P}'(F)$. Furthermore let $\mathcal{X}_\Sigma(F) := \mathcal{X}(F) \times_T T_\Sigma$ be the associated* *fiber bundle with fiber T_Σ, where $\mathcal{X}(F)$ and T are defined as in Theorem 1.40. Let $X_\Sigma(F)$ be the interior of the closures of $X(F)$ in $\mathcal{X}_\Sigma(F)$. Then the operation of $P''(F)$ on $X(F)$ can be extended to a properly discontinuous operation on $X_\Sigma(F)$ in a unique way. The quotient space $Y_\Sigma(F) := X_\Sigma(F)/P''(F)$ is an analytic variety that contains $D_g/P(F)$ as open and dense subvariety. The boundary $Y_\Sigma(F) \setminus (D_g/P(F))$ is a purely 1–codimensional subvariety.*

Proof. This follows from [AMRT, Proposition III.6.2] □

In the sense of this theorem one can speak of $Y_\Sigma(F)$ as a partial compactification of $D_g/P(F)$. With the help of the map $p(F)$ from (12) we can "attach" $\partial Y_\Sigma(F) := Y_\Sigma(F) \setminus (D_g/P(F))$ as a boundary piece at F to the space D_g/Γ. In that sense we call $Y_\Sigma(F)$ a *partial compactification of D_g/Γ in the direction of F defined by Σ.*

We can construct such a partial compactification $Y_{\Sigma(F)}(F)$ for each rational boundary component F of D_g by choosing an admissible fan $\Sigma(F)$ for every F. We thereby obtain a collection

$$\widetilde{\Sigma} := \{\Sigma(F)\,;\ F \text{ a rational boundary component}\} \tag{15}$$

of fans. We now formulate the conditions on $\widetilde{\Sigma}$, that will allow us to "glue" together the partial compactifications $Y_{\Sigma(F)}(F)$ to obtain a compactification of D_g/Γ.

Definition 1.44 *A collection $\widetilde{\Sigma} := \{\Sigma(F)\,;\ F$ a rational boundary component$\}$ of fans $\Sigma(F)$ in $\mathcal{P}'(F)$ is called* admissible*, if it satisfies the following three conditions:*

(i) $\Sigma(F)$ is an admissible fan for every rational boundary component F.

(ii) If F, F' are two rational boundary components with $F = M \cdot F'$ for an $M \in \Gamma$, then $\Sigma(F) = M \cdot \Sigma(F') \cdot M^{-1}$.

(iii) For any pair $F \prec F'$ of adjacent rational boundary components the relation $\Sigma(F') = \Sigma(F) \cap \mathcal{P}'(F')$ holds.

Note that according to (iii) the collection $\widetilde{\Sigma}$ is already determined by the fans $\Sigma(F)$ corresponding to minimal rational boundary components F. We here call a rational boundary component F *minimal* if there exists no rational boundary component F' with $F' \prec F$.

The condition (ii) from Definition 1.44 guarantees that two Γ–equivalent rational boundary components lead to isomorphic partial compactifications:

Proposition 1.45 *Let F and F' be two rational boundary components such that $F = M \cdot F'$ for an $M \in \Gamma$. Furthermore let $\Sigma(F)$ and $\Sigma(F')$ be two admissible fans in $\mathcal{P}'(F)$ and $\mathcal{P}'(F')$ respectively which satisfy the condition $\Sigma(F) = M \cdot \Sigma(F') \cdot M^{-1}$. Then M induces the following isomorphisms:*

(i) There exists a natural isomorphism $\widetilde{M} : X_{\Sigma(F')}(F') \to X_{\Sigma(F)}(F)$, such that the diagram

$$\begin{CD} X_{\Sigma(F')}(F') @>{\widetilde{M}}>> X_{\Sigma(F)}(F) \\ @AAA @AAA \\ D_g/P'(F') @>{M}>> D_g/P'(F) \end{CD}$$

commutes.

(ii) There exists a natural isomorphism $\overline{M} : Y_{\Sigma(F')}(F') \to Y_{\Sigma(F)}(F)$, such that the diagram

$$\begin{CD} Y_{\Sigma(F')}(F') @>{\overline{M}}>> Y_{\Sigma(F)}(F) \\ @AAA @AAA \\ D_g/P(F') @>{M}>> D_g/P(F) \end{CD}$$

commutes.

Proof. The argument given in [HKW, Proposition 3.69] for the $g = 2$–case can easily be generalized. □

Let $F \prec F'$ be a pair of adjacent rational boundary components of D_g. According to [AMRT, Theorem III.4.3] we then have that $\mathcal{P}'(F') \subset \mathcal{P}'(F)$ and therefore also $P'(F') \subset P'(F)$. Hence there exists a natural quotient map

$$\pi_0(F', F) : X(F') \to X(F) \ . \tag{16}$$

We can use condition (iii) from Definition 1.44 to extend this map to an étale map from $X_{\Sigma(F')}(F')$ to $X_{\Sigma(F)}(F)$ (by an *étale map* we mean here a smooth map with discrete fibers):

Proposition 1.46 *Let $F \prec F'$ be a pair of adjacent rational boundary components. Furthermore let $\Sigma(F)$ and $\Sigma(F')$ be two admissible fans in $\mathcal{P}'(F)$ and $\mathcal{P}'(F')$ respectively which satisfy the condition $\Sigma(F') = \Sigma(F) \cap \mathcal{P}'(F')$. Then there exists an étale map*

$$\pi(F', F) : X_{\Sigma(F')}(F') \to X_{\Sigma(F)}(F) \ ,$$

that extends the map $\pi_0(F, F')$ in a natural way.

Proof. [AMRT, Lemma III.5.1] □

Lemma 1.47 *Let $\widetilde{\Sigma} := \{\Sigma(F)\}$ be an admissible collection of fans. Then for every pair $F \prec F'$ of adjacent rational boundary components and every $M \in \Gamma$ the following diagram commutes:*

$$\begin{array}{ccc} X_{\Sigma(F')}(F') & \xrightarrow{\pi(F',F)} & X_{\Sigma(F)}(F) \\ \downarrow{\scriptstyle \widetilde{M}} & & \downarrow{\scriptstyle \widetilde{M}} \\ X_{M\cdot\Sigma(F')\cdot M^{-1}}(M\cdot F') & \xrightarrow{\pi(M\cdot F',M\cdot F)} & X_{M\cdot\Sigma(F)\cdot M^{-1}}(M\cdot F) \end{array}$$

Proof. This is an immediate consequence of the two preceding propositions (cf. [HKW, Lemma 3.72]) □

We can now define the toroidal compactification of D_g/Γ as an identification space. For an admissible collection $\widetilde{\Sigma} := \{\Sigma(F)\}$ of fans let $X(\widetilde{\Sigma})$ be the disjoint union

$$X(\widetilde{\Sigma}) := \coprod_{F \text{ rat. boundary comp.}} X_{\Sigma(F)}(F)\ . \tag{17}$$

We now define an equivalence relation on $X(\widetilde{\Sigma})$:

Definition 1.48 *Let $\widetilde{\Sigma} := \{\Sigma(F)\}$ be an admissible collection of fans and let $X(\widetilde{\Sigma})$ be as in* (17)*. The equivalence relation $\sim$ on $X(\widetilde{\Sigma})$ is defined by the following two types of equivalences. For that, let $x \in X_{\Sigma(F)}(F)$ and $x' \in X_{\Sigma(F')}(F')$ for two rational boundary components F and F'. Then*

(i) $x \sim x'$, if there exists an $M \in \Gamma$ with $F = M \cdot F'$ such that $x = \widetilde{M} \cdot x'$, where $\widetilde{M}$ is defined as in Proposition 1.45.

(ii) $x \sim x'$, if $F \prec F'$ and $\pi(F', F)(x') = x$, where $\pi(F', F)$ is the étale mapping from Proposition 1.46.

With the help of this equivalence relation we can now define the toroidal compactification of D_g/Γ.

Definition 1.49 *Let $\widetilde{\Sigma} := \{\Sigma(F)\}$ be an admissible collection of fans. The toroidal compactification $(D_g/\Gamma)^*$ of D_g/Γ determined by $\widetilde{\Sigma}$ is the quotient space*

$$(D_g/\Gamma)^* := X(\widetilde{\Sigma})/\sim\ ,$$

where $X(\widetilde{\Sigma})$ is given as in (17) *and $\sim$ is the equivalence relation from Definition 1.48.*

Let F be a rational boundary component. Analogous to the $g = 2$–case (cf. [HKW, Remark 3.77 (i)]) the composition $X_{\Sigma(F)}(F) \hookrightarrow X(\widetilde{\Sigma}) \twoheadrightarrow X(\widetilde{\Sigma})/\sim$ yields an open map

$$p^*(F) : Y_{\Sigma(F)}(F) \to (D_g/\Gamma)^* \ , \tag{18}$$

which naturally extends the projection $p(F)$ from (12). The image of $p^*(F)$ is dense in $(D_g/\Gamma)^*$ for every rational boundary component F. Moreover, for every pair $F \prec F'$ of adjacent rational boundary components the image of $p^*(F')$ is contained in the image of $p^*(F)$ as a dense open subset. The set of images of $p^*(F)$ for all minimal rational boundary components F therefore form an open cover of $(D_g/\Gamma)^*$.

To obtain a stratification of $(D_g/\Gamma)^*$, we define:

Definition 1.50 *For a rational boundary component F of D_g we define the* open boundary component $\partial_F\,(D_g/\Gamma)^*$ *by*

$$\partial_F\,(D_g/\Gamma)^* := p^*(F)\left(Y_{\Sigma(F)}(F)\right) \setminus \bigcup_{F \prec F'} p^*(F')\left(Y_{\Sigma(F')}(F')\right) \ .$$

If F is a corank i–boundary component of D_g, we call $\partial_F\,(D_g/\Gamma)^*$ an open corank i–boundary component *of $\partial_F\,(D_g/\Gamma)^*$.*

In particular $\partial_{D_g}\,(D_g/\Gamma)^* = D_g/\Gamma$. We have a stratification

$$(D_g/\Gamma)^* = \coprod_F \partial_F\,(D_g/\Gamma)^* \ . \tag{19}$$

As a conclusion we summarize the results of the previous section in the following theorem:

Theorem 1.51 (Toroidal compactification) *Let Γ be a subgroup of $\mathrm{Sp}(2g, \mathbb{Z})$ of finite index and $\widetilde{\Sigma} = \{\Sigma(F)\}$ an admissible collection of fans. Then there exists a compact space $(D_g/\Gamma)^*$ which contains D_g/Γ as an open and dense subset with purely 1–codimensional boundary $(D_g/\Gamma)^* \setminus (D_g/\Gamma)$.*
The projections $p(F) : D_g/P(F) \to D_g/\Gamma$ can be extended to open maps $p^*(F) : Y_{\Sigma(F)}(F) \to (D_g/\Gamma)^*$ which are isomorphisms when restricted to a sufficiently small neighborhood of $Y_{\Sigma(F)}(F)$. Furthermore, $(D_g/\Gamma)^*$ can be represented as the union of all the images of the maps $p^*(F)$ for every boundary component F.

Note that in general $(D_g/\Gamma)^*$ is not projective. However, we can impose conditions on the admissible fans that will guarantee this property. To do this, consider the spaces

$$\Omega := \coprod_F C(F)\,, \quad \Omega_{\mathbb{Z}} := \coprod_F C(F) \cap P'(F)\ , \tag{20}$$

where the union is taken over all rational boundary components F of D_g. Using the operation of Γ on the cones $C(F)$ we can define the quotient spaces

$$\Xi := \Omega/\Gamma\,, \quad \Xi_{\mathbb{Z}} := \Omega_{\mathbb{Z}}/\Gamma\,. \tag{21}$$

We can now define the notion of a projective, admissible collection of fans.

Definition 1.52 *An admissible collection* $\widetilde{\Sigma} = \{\Sigma(F)\}$ *of fans is called* projective, *if there exists a piecewise linear function* $\varphi : \Xi \to \mathbb{R}$ *with the following three properties:*

(i) $\varphi(x) > 0$ *for all* $x \in \Xi \setminus \{0\}$*,*

(ii) φ *is linear on the image of each cone* $\sigma \in \Sigma(F)$ *in* Ξ*, and* $\Sigma(F)$ *is the largest fan in* $\mathcal{P}'(F)$ *with support* $C(F)^{\mathrm{rc}}$ *which has this property,*

(iii) φ *is integral on* $\Xi_{\mathbb{Z}}$*.*

These additional conditions on an admissible collection of fans ensure that the corresponding toroidal compactification is projective as the following theorem shows:

Theorem 1.53 *Let* $\widetilde{\Sigma} = \{\Sigma(F)\}$ *be a projective admissible collection of fans. Then the toroidal compactification* $(D_g/\Gamma)^*$ *defined by* $\widetilde{\Sigma}$ *is a projective variety.*

Proof. [AMRT, Theorem IV.2.1] □

1.5 Quotient singularities

The Siegel modular varieties $\mathcal{A}_\Gamma = \mathcal{H}_g/\Gamma$ introduced in Section 1.2 as well as their toroidal compactifications have in general singularities. These can occur at two different stages: during the construction when taking the quotient by Γ and during the compactification. However, we will see in this section that the singularities in both cases can be described locally as the quotient of $\mathbb{C}^n$ by some finite group, provided that the fans used in the compactification process are nonsingular. We will study these so–called quotient singularities and provide criteria which guarantee the extensibility of pluricanonical forms over them. For the general background on quotient singularities we refer the reader to the works of Cartan ([Car1]) and Prill ([Pri]).

We start by providing the general definition of a quotient singularity.

Definition 1.54 *A singularity which is isomorphic to a singularity of a quotient X/G of a complex manifold X by a properly discontinuous action of a finite group G is called a* quotient singularity *(or a V-*germ*).*

The following theorem allows us to reduce the study of quotient singularities to a special case.

Theorem 1.55 *Given a quotient singularity, there exist $n \in \mathbb{N}$ and a finite subgroup G of $\mathrm{GL}(n, \mathbb{C})$ such that the singularity is equivalent to the singularity of $\mathbb{C}^n/G$ at the origin.*

Proof. [Car1, p. 97] and [Pri, p. 380] □

To determine the type of a quotient singularity we have to study the elements of the finite group G. An important role is played by the following type of matrices.

Definition 1.56 *A matrix $g \in \mathrm{GL}(n, \mathbb{C})$ is called a* quasi–reflection *if the matrix $g - \mathbb{1}$ has rank 1.*

The importance of these quasi–reflections is made clear by the following lemma.

Lemma 1.57 *Let G be a subgroup of $\mathrm{GL}(n, \mathbb{C})$ containing only quasi–reflections. Then the quotient $\mathbb{C}^n/G$ is nonsingular.*

Proof. [Pri, p. 382] □

Let G be a finite subgroup of $\mathrm{GL}(n, \mathbb{C})$. Then the subgroup N of G generated by the quasi–reflections is normal in G. The factor group G/N is isomorphic to a subgroup K of $\mathrm{GL}(n, \mathbb{C})$ without quasi–reflections. Furthermore K can be chosen in such a way that the singularities at the origins of $\mathbb{C}^n/K$ and $\mathbb{C}^n/K$ are isomorphic. This gives the basic idea for the proof of the following proposition.

Proposition 1.58 *Given any finite subgroup G of $\mathrm{GL}(n, \mathbb{C})$, there exists a subgroup K of $\mathrm{GL}(n, \mathbb{C})$ without quasi–reflections such that the singularities at the origins of $\mathbb{C}^n/K$ and $\mathbb{C}^n/G$ are isomorphic.*

Proof. [Pri, Proposition 6] □

Thus every quotient singularity can be represented by a subgroup G of $\mathrm{GL}(n, \mathbb{C})$ without quasi–reflections. The singular locus of the quotient $\mathbb{C}^n/G$ can then be determined as follows.

Lemma 1.59 *Let G be a subgroup of* $\mathrm{GL}(n, \mathbb{C})$ *without quasi–reflections. Then the singular locus of* $\mathbb{C}^n/G$ *is given by*

$$\mathrm{Sing}(\mathbb{C}^n/G) = \{x \in \mathbb{C}^n;\ gx = x \text{ for one } g \in G\}/G\ .$$

Proof. [MS, Lemma 2.1] □

We are not only interested in the type of a quotient singularity, but also in the question whether pluricanonical forms can be extended over this singularity.

Before we can answer this question, we first have to define what we mean when we speak of the canonical bundle or a pluricanonical form on a normal, not necessarily nonsingular, variety. An easy way to do this is provided by the following definition.

Definition 1.60 *A* canonical divisor K_X *of a normal variety X is a Weil divisor on X that coincides with the canonical divisor on $X \setminus \mathrm{Sing}(X)$. The variety X is called* $\mathbb{Q}$–Gorenstein *if there is an integer m such that mK_X is Cartier.*

Another traditional way due to Zariski to define this is to say that the canonical bundle ω_X on a normal variety is given by $j_*(\Omega^n_{X^\circ})$, where $X^\circ := X \setminus \mathrm{Sing}(X)$ is the smooth locus of X and $j : X^\circ \to X$ its inclusion. The canonical divisor K_X is then the Weil divisor corresponding to ω_X. Alternatively, there is also a definition which explicitly gives the group of sections $\Gamma(U, \omega_X)$ for each open subset U which can be found in [Rei, (1.5) and (1.7)].

From now on we will assume that X is a quasi–projective variety which is $\mathbb{Q}$–Gorenstein. Hence there exists an integer such that the sheaf $\mathcal{O}_X(mK_X)$ is invertible. Let s be a local generator of $\mathcal{O}_X(mK_X)$ at a singular point $P \in X$ and let $f : Y \to X$ be a resolution. According to [Rei, (1.9)] s is a regular m–canonical form on X which can be considered as a rational differential form on Y, since the function fields $k(X)$ and $k(Y)$ coincide. At those points where f is an isomorphism s is again regular, but at the exceptional divisors s will in general have poles and thus fails to be regular. We define:

Definition 1.61 *Let X be a quasi–projective, normal variety. The variety X has* canonical singularities *if it satisfies the following two conditions:*

(i) X is $\mathbb{Q}$–Gorenstein, i.e. there is an integer m such that mK_X is Cartier.

(ii) For every resolution $f : Y \to X$ of X with exceptional prime divisors $\{E_i\}$ we have that

$$mK_Y = f^*(mK_X) + \sum a_i E_i$$

for some rational numbers a_i, called discrepancies, *satisfying $a_i \geq 0$.*

If in (ii) we even have that all discrepancies a_i satisfy $a_i > 0$, the variety X is said to have terminal singularities*; if $a_i > -1$ for all discrepancies, X is said to have* log–terminal singularities.

Condition (ii) guarantees that in the above setup where we have a regular m–canonical form s on X, the form s does not collect poles along the exceptional divisors of a resolution $f : Y \to X$ and thus defines a regular form on Y (cf. [Rei, (1.9)]). We summarize these observations in the following theorem:

Theorem 1.62 *Let X be a quasi–projective, normal variety with canonical singularities. Then every pluricanonical form on X can be extended to any resolution of X.*

To extend pluricanonical forms to resolutions of the Siegel modular varieties introduced in Section 1.2, we need to determine the discrepancies of the exceptional divisors. This will be done in detail in the following chapters. However, we can use the following general result to get a first estimate:

Proposition 1.63 *Let X be a smooth projective algebraic variety over $\mathbb{C}$ and G be a finite group acting on X. Then the quotient variety X/G has log–terminal singularities.*

Proof. [Bor, Proposition 7.9] □

Although the subgroups Γ of $\mathrm{Sp}(2g, \mathbb{Z})$ of finite index are infinite, their action can locally be given by finite subgroups.

Proposition 1.64 *Let Γ be a discrete subgroup of* $\mathrm{Sp}(2g, \mathbb{Z})$*. Then for every point $\tau \in \mathcal{H}_g$ the stabilizer* $\mathrm{Stab}_\Gamma(\tau)$ *of τ in Γ is finite.*

Proof. The finiteness follows from the fact that the action of Γ on $\mathcal{H}_g$ is properly discontinuous by Lemma 1.13. □

The above proposition allows us to apply Proposition 1.63 to conclude that Siegel modular varieties have log–terminal singularities, i.e. discrepancies greater than -1. To check whether the varieties even have canonical or terminal singularities, we have to know more about the elements of these finite stabilizers and their actions. We will study them in detail in the following chapters, but for now, we will just give the general theory needed for determining whether a given quotient singularity is canonical or not.

For that, recall that by Proposition 1.58 it suffices to consider the action of a finite subgroup Γ of $\mathrm{GL}(n, \mathbb{C})$ without quasi–reflections on $\mathbb{C}^n$. If $x \in \mathbb{C}^n$ is fixed by $\mathrm{id} \neq \gamma \in \Gamma$ with $\mathrm{ord}(\gamma) = m$, we can find for any primitive m–th root of unity ζ integers $a_1, \ldots, a_n$ with $0 \leq a_i < m$ such that the eigenvalues of the linearized action of γ on the tangent space of x are given by

$$\zeta^{a_1}, \ldots, \zeta^{a_n} \ . \tag{22}$$

In terms of these eigenvalues, the question whether a quotient singularity is canonical or not can be answered as follows.

Theorem 1.65 (Reid, Shepherd–Barron, and Tai) *Let Γ be a finite subgroup of $\mathrm{GL}(n, \mathbb{C})$ without quasi–reflections. Then the quotient X/Γ has canonical (resp. terminal) singularities if and only if for all $\mathrm{id} \neq \gamma \in \Gamma$, all primitive m–th roots of unity ζ, where m denotes the order of γ, and for each $x \in \mathrm{Fix}(\gamma)$ the condition*

$$\frac{1}{m}\Big(a_1 + \cdots + a_n\Big) \geq 1 \quad (\textit{resp.} \ > 1)$$

is satisfied, where $0 \leq a_i < m$ are determined by the eigenvalues of the action of γ on the tangent space of x as in (22).

Proof. This follows from [MS, Theorem 2.3 (ii)] and [Rei, Theorem 4.11]. □

1.6 Modular forms

In this section we will introduce modular forms. These are certain functions on the Siegel upper half space $\mathcal{H}_g$ which can be expanded into Fourier series due to the properties they possess. They play an important role in determining the Kodaira dimension of the Siegel modular varieties we introduced in the previous sections. Our main sources in this section are the books of Freitag [Fre] and Igusa [Igu3].

To keep the notation simple we first introduce the following operator. Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2g, \mathbb{R})$, $r \in \mathbb{Z}$ and f be a $\mathbb{C}$–valued function on $\mathcal{H}_g$. We then define the function $f \,|\, M$ on $\mathcal{H}_g$ via

$$f \,|\, M(\tau) := f \underset{r}{|} M(\tau) := f(M\tau) \det(C\tau + D)^{-r} \ .$$

It is easy to check that for any two matrices $M, N \in \mathrm{Sp}(2g, \mathbb{R})$ the identity $(f \,|\, M) \,|\, N = f \,|\, MN$ is satisfied.

We can now define the main object of this section. Note that $\mathcal{H}_g$ is an open subset of the space of symmetric complex $g \times g$ matrices which we can identify with $\mathbb{C}^{g(g+1)/2}$. As a consequence the notion of a holomorphic function on $\mathcal{H}_g$ is well–defined.

Definition 1.66 *Let $g \geq 2$ and Γ be a subgroup of* $\mathrm{Sp}(2g, \mathbb{Z})$ *of finite index. A holomorphic function $f : \mathcal{H}_g \to \mathbb{C}$ is called a* (degree g) modular form of weight $r \in \mathbb{Z}$ with respect to Γ, *if for all $M \in \Gamma$ the identity $f\,|_r\, M = f$ is satisfied.*

The modular forms of weight r with respect to a fixed group Γ form a $\mathbb{C}$–vector space which we denote by $[\Gamma, r]$. We have even more; let $f \in [\Gamma, r], g \in [\Gamma, s]$. It is then easy to verify that $f \cdot g \in [\Gamma, r+s]$ holds. This means that the space of modular forms with respect to Γ has the structure of a graded $\mathbb{C}$–algebra.

The set of all those symmetric matrices $S \in \mathrm{Sym}(g, \mathbb{R})$ for which the translation $\tau \mapsto \tau + S$ is contained in Γ defines a lattice L in the vector space $\mathrm{Sym}(g, \mathbb{R})$. We define its dual lattice in $\mathrm{Sym}(g, \mathbb{R})$ by

$$L_* := \{T \in \mathrm{Sym}(g, \mathbb{R});\ \mathrm{tr}(ST) \in \mathbb{Z} \text{ for all } S \in L\}\ .$$

Let $f \in [\Gamma, r]$ be a modular form. Then f is periodic in the following sense:

$$f(\tau + S) = f(\tau) \quad \text{for all } S \in L\ .$$

According to [Igu3, p. 198] we can thus expand f into a Fourier series as follows:

$$f(\tau) = \sum_{S \in L_*} a(S) e^{2\pi i\, \mathrm{tr}(S\tau)} \tag{23}$$

This representation can be simplified further as the *Koecher principle* shows.

Lemma 1.67 (Koecher principle) *Let $f \in [\Gamma, r]$ be a modular form. Then f has a Fourier expansion of the form*

$$f(\tau) = \sum_{S \in L_*, S \geq 0} a(S) e^{2\pi i\, \mathrm{tr}(S\tau)}\ .$$

Proof. According to [Fre, p. 129] we have that in the Fourier expansion given in (23) the implication

$$a(S) \neq 0 \Rightarrow S \geq 0$$

holds (cf. also [Fre, Chapter III, Hilfssatz 4.11]). This means that f has the Fourier expansion as claimed. □

We will now investigate the behavior of a modular form f at the boundary of $\mathcal{H}_g$. To do this, we first introduce the Siegel Φ–operator. Let $f : \mathcal{H}_g \to \mathbb{C}$ be a function for which the limit

$$\lim_{t \to +\infty} f \begin{pmatrix} \tau & 0 \\ 0 & it \end{pmatrix}$$

exists for all $\tau \in \mathcal{H}_{g-1}$. We thus obtain a function $f \,|\, \Phi$ on $\mathcal{H}_{g-1}$ defined by

$$f \,|\, \Phi(\tau) := \lim_{t \to +\infty} f \begin{pmatrix} \tau & 0 \\ 0 & it \end{pmatrix} .$$

If f is a modular form, we can express $f \,|\, \Phi$ again as a Fourier series as the following proposition shows.

Proposition 1.68 *Let $f \in [\Gamma, r]$ be a modular form with Fourier expansion*

$$f(\tau) = \sum_{S \in L_*, S \geq 0} a(S) e^{2\pi i \operatorname{tr}(S\tau)} .$$

Then the function $f \,|\, \Phi$ is well–defined and we have for its Fourier expansion

$$f \,|\, \Phi(\tau) = \sum_{\widetilde{S}} a \begin{pmatrix} \widetilde{S} & 0 \\ 0 & 0 \end{pmatrix} e^{2\pi i \operatorname{tr} S\tau} ,$$

where we sum over all $\widetilde{S} \in \mathrm{Sym}(g-1, \mathbb{R})$ for which $\begin{pmatrix} \widetilde{S} & 0 \\ 0 & 0 \end{pmatrix} \in L_*$.

Proof. [Fre, p. 129] □

The map that associates to every modular form f the function $f \,|\, \Phi$ is called *Siegel's Φ–operator*. It is a linear operator on $[\Gamma, r]$ which maps to a space of modular forms of degree $g-1$ with respect to a suitable subgroup of $\mathrm{Sp}(2g-2, \mathbb{R})$. We will use this operator to define a special subspace of $[\Gamma, r]$, the space of cusp forms. We need the following notion:

Definition 1.69 *A real matrix N is called* projectively rational *if there is a real number $t \neq 0$ such that tN is a rational matrix.*

We can now define the notion of a cusp form.

Definition 1.70 *A modular form $f \in [\Gamma, r]$ is called a* cusp form *if for all projectively rational matrices N in $\mathrm{Sp}(2g, \mathbb{R})$ the function $f \,|\, N$ is contained in the kernel of Siegel's Φ–operator, that is if the identity $(f \,|\, N) \,|\, \Phi \equiv 0$ holds.*

The cusp forms of weight r with respect to a fixed group Γ form a subspace of the $\mathbb{C}$–vector space $[\Gamma, r]$ which we denote by $[\Gamma, r]_0$.

Chapter 2

Siegel modular varieties

In this chapter we will motivate and state the main result of this thesis. Moreover, we will give a rough outline and the key ideas of its proof which will then be carried out in the following chapters.

2.1 Introduction

We have seen in the previous chapter that any subgroup Γ of $\mathrm{Sp}(2g, \mathbb{Z})$ of finite index defines an algebraic variety $\mathcal{A}_\Gamma := \mathcal{H}_g/\Gamma$ which is quasi–projective and of dimension $g(g+1)/2$. As stated in the preface there is the following conjecture regarding the Kodaira dimension of these Siegel modular varieties:

Conjecture 2.1 *There are only finitely many subgroups Γ of $\mathrm{Sp}(2g, \mathbb{Z})$ for any $g \geq 2$ such that the corresponding Siegel modular variety $\mathcal{A}_\Gamma$ is not of general type.*

Note that since $\mathrm{Sp}(2g, \mathbb{Z})$ is finitely generated it has only finitely many subgroups of a given index (cf. [Hal, Section 2]). This implies that the conjecture is equivalent to giving a bound on the index of Γ in $\mathrm{Sp}(2g, \mathbb{Z})$ such that for all subgroups Γ whose indices exceed this bound the corresponding Siegel modular varieties $\mathcal{A}_\Gamma$ are of general type.

For $\Gamma = \mathrm{Sp}(2g, \mathbb{Z})$, the full symplectic group, we have seen in the preface that the corresponding moduli space $\mathcal{A}_g$ is rational or unirational for small g and of general type for $g \geq 7$.

Theorem 2.2 *The moduli space of principally polarized abelian varieties $\mathcal{A}_g$ is of general type for $g \geq 7$.*

Proof. [Mum2] □

This result has an immediate consequence for the conjecture as the following corollary shows.

Corollary 2.3 *For any subgroup* Γ *of* $\mathrm{Sp}(2g, \mathbb{Z})$ *for* $g \geq 7$ *of finite index the corresponding moduli space* $\mathcal{A}_\Gamma$ *is of general type.*

Proof. Since Γ is a subgroup of $\mathrm{Sp}(2g, \mathbb{Z})$ we obtain a morphism from $\mathcal{A}_\Gamma$ to $\mathcal{A}_g$. Note that the fact that Γ has finite index implies that this morphism is finite. We can now use Proposition 1.23 together with Theorem 2.2 to conclude that $\mathcal{A}_\Gamma$ is of general type since $\mathcal{A}_g$ is. □

This implies that the conjecture can be shown by considering each $2 \leq g \leq 6$ separately. On the other hand, there is the finiteness theorem of Borisov in the $g = 2$–case which motivated this thesis.

Theorem 2.4 *There are only finitely many subgroups* Γ *of* $\mathrm{Sp}(4, \mathbb{Z})$ *of finite index such that* $\mathcal{A}_\Gamma$ *is not of general type.*

Proof. [Bor, Proposition 6.4] □

So the only open cases are $g = 3$, $g = 4$, $g = 5$ and $g = 6$. In this thesis we will focus on the $g = 3$–case, for which we will give a main result that proves the conjecture in this case up to some technical result. However, quite a few of our results are stated for arbitrary $g \geq 3$, e.g. the description of the singularities in the interior, and others can easily be generalized.

2.2 Pluricanonical sections

In order to show that for a given $\Gamma < \mathrm{Sp}(2g, \mathbb{Z})$ the corresponding moduli space $\mathcal{A}_\Gamma$ is of general type, we need to construct sufficiently many pluricanonical sections. A key role is played by Siegel modular forms with respect to Γ which we introduced in the previous chapter in Section 1.6. The importance becomes clear when we look at the canonical divisor of $\mathcal{A}_\Gamma$.

Recall that any element τ of the Siegel upper half space $\mathcal{H}_g$ (cf. Definition 1.7) can be considered as a symmetric matrix $\tau = (\tau_{ij})$. We can thus define

$$d\tau := d\tau_{11} \wedge d\tau_{12} \wedge \cdots \wedge d\tau_{gg}$$

as the usual $g(g+1)/2$–form on $\mathcal{H}_g$. If f is a modular form of weight $g + 1$ with respect to Γ, a short calculation shows that $f d\tau$ defines a differential form on $\mathcal{H}_g$

which is invariant under Γ. Hence, if Γ acts freely and the natural quotient map $\mathcal{H}_g \to \mathcal{A}_\Gamma$ is thus unramified, we obtain a canonical form on $\mathcal{A}_\Gamma$, i.e. a section in

$$K_{\mathcal{A}_\Gamma} = (g+1)L \ ,$$

where L is the line bundle of modular forms given by the automorphy factor $\det(C\tau + D)$. If Γ does not act freely the above statement is still true if one stays away from the branch locus of $\mathcal{H}_g \to \mathcal{A}_\Gamma$.

If one considers a toroidal compactification of $\mathcal{A}_\Gamma$ as introduced in Section 1.4, the relation with modular forms can be extended as follows. For now, we will assume that Γ acts freely and that we have a smooth compactification $\mathcal{A}_\Gamma^*$ with the following property: for every point in the boundary there exists a representative $x \in X_{\Sigma(F)}(F)$ for some boundary component F such that $X_{\Sigma(F)}(F)$ is smooth at x and that the action of $P''(F)$ on x is free (here we used the notation of Section 1.4). Then the canonical divisor on $\mathcal{A}_\Gamma^*$ is given by

$$K_{\mathcal{A}_\Gamma^*} = (g+1)L - D \ ,$$

where D is the boundary divisor of $\mathcal{A}_\Gamma^*$ and L is the extension of the line bundle of modular forms on $\mathcal{A}_\Gamma$ to the compactification $\mathcal{A}_\Gamma^*$. As before, if Γ or the compactification $\mathcal{A}_\Gamma^*$ does not satisfy these properties, the description of the canonical divisor still holds on a suitable open part of $\mathcal{A}_\Gamma^*$.

In that sense, to get a section of the canonical bundle, we need a modular form of weight $(g+1)$ with respect to Γ which vanishes at the boundary. These modular forms are just the cusp forms of weight $g+1$ introduced in Definition 1.70 as the following result of Freitag shows.

Theorem 2.5 *Let Γ be a subgroup of $\mathrm{Sp}(2g,\mathbb{Z})$ of finite index. If $\widetilde{\mathcal{A}}_\Gamma$ is a smooth projective model of the Siegel modular variety $\mathcal{A}_\Gamma$, then every cusp form f of weight $g+1$ with respect to Γ defines a differential form $f d\tau$ which extends to $\widetilde{\mathcal{A}}_\Gamma$. In particular, there is a natural isomorphism*

$$H^0\Big(\widetilde{\mathcal{A}}_\Gamma, \omega_{\widetilde{\mathcal{A}}_\Gamma}\Big) \cong [\Gamma, g+1]_0 \ ,$$

where $[\Gamma, g+1]_0$ denotes the space of cusp forms of weight $g+1$ with respect to Γ as introduced in Definition 1.70.

Proof. This is implied by [Fre, Satz III.2.6] and the remark following it. □

Remark 2.6 *Note carefully, that while it is still true that a weight $m(g+1)$ form defines an m–fold differential form on the open part of $\mathcal{A}_\Gamma$ where the quotient map $\mathcal{H}_g \to \mathcal{A}_\Gamma$ is unbranched, such a form does in general not extend to a smooth projective model of $\mathcal{A}_\Gamma$. For that it would need to have a higher order of vanishing at the boundary and at the points where Γ does not act freely.*

In general for arbitrary Γ it is very difficult to formulate these vanishing conditions explicitly and even more difficult to check them for a given cusp form. However, the situation becomes much better when one knows more about the geometry of the Siegel modular variety $\mathcal{A}_\Gamma$ and its compactification. There is a family of Siegel modular varieties, namely the moduli spaces $\mathcal{A}_g(n)$ of abelian varieties with a level n–structure, which has been studied extensively and whose geometry is understood rather well. Although they represent only a very small fraction of Siegel modular varieties, we will see in Section 2.5 how they can be used to formulate vanishing conditions for an arbitrary Siegel modular variety $\mathcal{A}_\Gamma$. Before we do this, we will first collect the results relevant to our situation in the following section.

2.3 The moduli spaces $\mathcal{A}_g(n)$

In this section we will collect some results on the moduli spaces $\mathcal{A}_g(n)$ of principally polarized g–dimensional abelian varieties with a level n–structure. Furthermore, we will present a toroidal compactification of $\mathcal{A}_g(n)$, the so–called Voronoi compactification $\mathcal{A}_g^{\mathrm{Vor}}(n)$.

Recall the definition of $\Gamma(n) = \Gamma_g(n)$ as given in Section 1.2 and the interpretation of the corresponding Siegel modular variety $\mathcal{A}_g(n)$ as stated in Theorem 1.16.

For the principal congruence subgroups there is an estimate for the dimension of the space of modular forms of weight k with respect to $\Gamma(n)$. For $g = 3$ there is even an exact formula for the dimension of the space of cusp forms which has been computed by Tsushima in [Tsu]. However, for our purposes it suffices to have the following estimate:

Proposition 2.7 *For $n \geq 3$ the dimension of the space of modular forms of weight $k(g+1)$ with respect to $\Gamma_g(n)$ grows as*

$$2^{(g-1)(g-2)/2}\,[\mathrm{Sp}(2g,\mathbb{Z}):\Gamma_g(n)]\prod_{j=1}^{g}\frac{(j-1)!}{(2j)!}(-1)^{j+1}B_{2j}\left[k(g+1)\right]^{g(g+1)/2}$$

as $k \to \infty$, where the B_{2j} are the Bernoulli numbers which can be defined by the identity

$$\frac{x}{e^x-1} = \sum_{\nu}^{\infty} B_\nu \frac{x^\nu}{\nu!}\ .$$

Proof. [Tai, Proposition 2.1] □

In particular, we obtain the following estimate for $g = 3$:

Corollary 2.8 *For $n \geq 3$ the dimension of the space of modular forms of degree 3 and weight $4k$ with respect to $\Gamma(n)$ is given by*

$$\dim[\Gamma(n), 4k] = \frac{1}{6!} \cdot \frac{1}{181440} [\mathrm{Sp}(6, \mathbb{Z}) : \Gamma(n)] (4k)^6 + O(k^5)$$

These modular forms, or more precisely the cusp forms with respect to $\Gamma(n)$, have been used in many cases to show that the moduli spaces $\mathcal{A}_g(n)$ are of general type for all but finitely many values of g and n. We summarize these results in the following theorem:

Theorem 2.9 *The moduli space $\mathcal{A}_g(n)$ is of general type for the following values of g and $n \geq n_0$:*

g	2	3	4	5	6	≥ 7
n_0	4	3	2	2	2	1

Proof. This theorem collects various results which can be found in [Tai], [Fre], [Mum2] and [Hul]. A similar collection with sketches of the proofs can be found in [HS, Theorem II.2.1]. □

All other cases are either known to be rational or unirational with the exception of $\mathcal{A}_6$: The spaces $\mathcal{A}_2(2)$ and $\mathcal{A}_2(3)$ are birational to the Segre cubic resp. the Burkhardt quartic in $\mathbb{P}^4$ and hence rational (cf. the papers of van der Geer [vdG1] resp. Todd [Tod] and Baker [Bak]). Rationality of $\mathcal{A}_3(2)$ was shown by van Geemen [vG] and Dolgachev and Ortland [DO]. For $n = 1$, we have that $\mathcal{A}_g(1)$ coincides with $\mathcal{A}_g$, the moduli space of principally polarized g–dimensional abelian varieties. Here Igusa [Igu1] showed that $\mathcal{A}_2$ is rational, whereas the rationality for $\mathcal{A}_3$ follows from the rationality of $\mathcal{M}_3$, the moduli space of curves of genus 3, which was shown by Katsylo [Kat]. The spaces $\mathcal{A}_4$ and $\mathcal{A}_5$ are unirational by the works of Clemens [Cle] for $g = 4$ and Donagi [Don], Mori and Mukai [MM] and Verra [Ver] for $g = 5$. The only open question is the one for the Kodaira dimension of $\mathcal{A}_6$.

It is well–known that for $n \geq 3$ the action of $\Gamma(n)$ on $\mathcal{H}_g$ is free and consequently the moduli space $\mathcal{A}_g(n)$ is smooth. However, the minimal projective compactification of $\mathcal{A}_g(n)$, the so–called Satake compactification $\mathcal{A}_g^{\mathrm{Sat}}(n)$ (cf. [Sat],[Bai]), has bad singularities at infinity. It is defined as the projective variety associated with the graded ring of modular forms with respect to $\Gamma(n)$ as introduced in Section 1.6. Set–theoretically $\mathcal{A}_g^{\mathrm{Sat}}(n)$ is the union of $\mathcal{A}_g(n)$ with some boundary components each of which is isomorphic to some moduli space $\mathcal{A}_d(n)$ of lower dimension (cf. [Nam, §5]):

$$\mathcal{A}_g^{\mathrm{Sat}}(n) = \mathcal{A}_g(n) \amalg \left(\coprod_{i_{g-1}} \mathcal{A}_{g-1}^{i_{g-1}}(n) \right) \amalg \ldots \amalg \left(\coprod_{i_0} \mathcal{A}_0^{i_0}(n) \right) \tag{1}$$

Using the method of toroidal compactification we can construct other compactifications which have milder singularities. For $n \geq 3$ there even exists a cone decomposition such that the corresponding toroidal compactification is smooth. However, we are at the same time interested in having a compactification which admits a simple geometric interpretation such that we can easily describe the action of $\mathrm{Sp}(2g, \mathbb{Z}/n\mathbb{Z})$ on the boundary and can formulate simple vanishing conditions. Therefore we will work with the so–called 2^{nd} Voronoi decomposition which possesses such a geometric interpretation. It will turn out that the corresponding compactification $\mathcal{A}_g^{\text{Vor}}(n)$ is in fact smooth for $g \leq 4$, but acquires singularities at the boundary for $g \geq 5$.

Recall the definitions and notations of Section 1.4. To obtain a toroidal compactification of $\mathcal{A}_g(n)$ we thus have to specify an admissible collection $\widetilde{\Sigma}$ of fans in the sense of Definition 1.44. To simplify matters we want this collection to be independent of the level n of $\mathcal{A}_g(n)$. It therefore suffices to give such a collection for the biggest group $\Gamma(n)$, i.e. $\Gamma(1) = \mathrm{Sp}(2g, \mathbb{Z})$. With respect to this group all the boundary components F are equivalent to one of the standard components $F^{(k)}$ given in (10) in Section 1.4, so we can restrict to specifying fans for these standard components. Moreover, according to the remark following Definition 1.44 it even suffices to specify just one fan Σ for the minimal standard component $F^{(0)}$. We thus have to define a decomposition of the open homogeneous cone $C(F^{(0)})$ in $\mathcal{P}(F^{(0)})$ associated to $F^{(0)}$, or more precisely of its rational closure $C(F^{(0)})^{\text{rc}}$ as defined in Definition 1.41 (iii). This latter space is isomorphic to the cone $\mathrm{Sym}_{\geq 0}(g, \mathbb{R})$ of semi–positive definite real symmetric $g \times g$ matrices.

There exist several decompositions of this cone in the literature. There are the 1^{st} *Voronoi* or *perfect cone* decomposition, the *central cone* decomposition and the 2^{nd} *Voronoi* decomposition. They are all described in [Nam, §8 and §9]. They lead to the *perfect cone* compactification $\mathcal{A}_g^{\text{Perf}}(n)$, the *Igusa* compactification $\mathcal{A}_g^{\text{Igu}}(n)$ and the 2^{nd} *Voronoi* compactification $\mathcal{A}_g^{\text{Igu}}(n)$ respectively. In general, all these compactifications have different properties and have thus certain advantages and disadvantages depending on ones view point. As mentioned before, we are interested in having a nice geometric interpretation and will therefore use the 2^{nd} Voronoi compactification, or just Voronoi compactification for short.

The 2^{nd} Voronoi decomposition is described in [Nam, §9] in terms of Delaunay and Voronoi cells for general g. However, for $g \leq 3$ all the three standard decompositions as given above coincide, so we can give a more explicit construction in this case.

For $g = 3$, we have the following 6–dimensional standard cone in $\mathrm{Sym}_{\geq 0}(3, \mathbb{R})$:

$$\sigma_3 := \mathbb{R}_{\geq 0}\,\alpha_1 + \mathbb{R}_{\geq 0}\,\alpha_2 + \mathbb{R}_{\geq 0}\,\alpha_3 + \mathbb{R}_{\geq 0}\,\beta_1 + \mathbb{R}_{\geq 0}\,\beta_2 + \mathbb{R}_{\geq 0}\,\beta_3\ ,$$

where

$$\alpha_1 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \alpha_2 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \alpha_3 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\beta_1 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}, \quad \beta_2 := \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}, \quad \beta_3 := \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We define an admissible fan Σ consisting of all the cones which are images of σ_3 or its faces by the action of $\mathrm{GL}(3,\mathbb{Z})$. This fan gives the 2$^{\text{nd}}$ Voronoi decomposition for $g = 3$. As we remarked at the beginning of the construction this fan for the minimal rational boundary component $F^{(0)}$ defines a whole admissible collection $\widetilde{\Sigma} = \{\Sigma(F)\}$ of fans for all boundary components F. We thus have the partial compactifications $Y_{\Sigma(F)}(F)$ in direction of each boundary component F and hence the Voronoi compactification $\mathcal{A}_3^{\mathrm{Vor}}(n)$ of $\mathcal{A}_3(n)$ for each level n by Theorem 1.51.

As already mentioned, the 2$^{\text{nd}}$ Voronoi decomposition of $\mathrm{Sym}_{\geq 0}(g,\mathbb{R})$ is more complicated for $g \geq 4$ (cf. [Nam, §9]). In particular, unlike in the $g = 3$–case where the whole decomposition is generated by σ_3 and its faces, there is no unique maximal cone in this decomposition up to the action of $\mathrm{GL}(g,\mathbb{Z})$ (cf. [Nam, p. 94]). However, it was shown by Alexeev ([Ale1]) that the decomposition is projective for all g. As a consequence we obtain by Theorem 1.53 that the Voronoi compactifications $\mathcal{A}_g^{\mathrm{Vor}}(n)$ are projective varieties.

For $g = 3$ the Voronoi compactification $\mathcal{A}_3^{\mathrm{Vor}}(n)$ coincides with the Igusa compactification for which Igusa showed that it is smooth provided $n \geq 3$ (cf. [Igu2, Theorem 2]). In the $g = 4$–case, the 2$^{\text{nd}}$ Voronoi decomposition is a basic refinement of the central cone decomposition (cf. [ER1],[ER2]) which implies that $\mathcal{A}_4^{\mathrm{Vor}}(n)$ can be obtained as a desingularization of the Igusa compactification $\mathcal{A}_4^{\mathrm{Igu}}(n)$. However, for $g \geq 5$, the Voronoi compactification $\mathcal{A}_g^{\mathrm{Vor}}(n)$ acquires singularities on the boundary.

Theorem 2.10 *The Voronoi compactification $\mathcal{A}_g^{\mathrm{Vor}}(n)$ of the moduli space $\mathcal{A}_g(n)$ is smooth for $n \geq 3$ and $g \leq 4$.*

As for any toroidal compactification of $\mathcal{A}_g(n)$ there is a projection

$$\pi : \mathcal{A}_g^{\mathrm{Vor}}(n) \to \mathcal{A}_g^{\mathrm{Sat}}(n) \stackrel{(1)}{=} \mathcal{A}_g(n) \amalg \left(\coprod_{i_{g-1}} \mathcal{A}_{g-1}^{i_{g-1}}(n) \right) \amalg \ldots \amalg \left(\coprod_{i_0} \mathcal{A}_0^{i_0}(n) \right) \qquad (2)$$

to the Satake compactification $\mathcal{A}_g^{\mathrm{Sat}}(n)$ which is sometimes also called the minimal compactification because of this property. Following the notation of [vdG3] we denote the preimage of the moduli spaces $\mathcal{A}_j^{i_j}(n)$ with $j \leq g-k$ under this projection by β_k, i.e.

$$\beta_k := \pi^{-1}\left(\coprod_{j\leq g-k}\coprod_{i_j}\mathcal{A}_j^{i_j}(n)\right) . \tag{3}$$

The preimage $\beta_k \subset \mathcal{A}_g^{\mathrm{Vor}}(n)$ can be interpreted as the locus of semi–abelian varieties with torus rank $\geq k$. In particular, we have that $\beta_0 = \mathcal{A}_g^{\mathrm{Vor}}(n)$ and $\beta_1 = \mathcal{A}_g^{\mathrm{Vor}}(n) \setminus \mathcal{A}_g(n)$. Furthermore, we define the space of rank $\leq k$–degenerations

$$\left(\mathcal{A}_g^{\mathrm{Vor}}(n)\right)^{(k)} = \mathcal{A}_g^{\mathrm{Vor}}(n) \setminus \beta_{k+1} . \tag{4}$$

We shall remark here that the space of rank 1–degenerations $\left(\mathcal{A}_g^{\mathrm{Vor}}(n)\right)^{(1)}$ is canonical in the sense that it does not depend on the toroidal compactification chosen (cf. [Mum2]).

The boundary of $\mathcal{A}_3^{\mathrm{Vor}}(n)$ is a divisor which can be expressed as follows:

$$D = \sum_j D_j ,$$

where the D_j are just the closures of the preimages of the top–dimensional components $\mathcal{A}_2^{i_2}(n) \subset \mathcal{A}_3^{\mathrm{Sat}}(n)$ under the map π.

Since for $n \geq 3$ the action of the group $\Gamma(n)$ is free and the spaces $\mathcal{A}_3^{\mathrm{Vor}}(n)$ are smooth, the canonical divisor of $\mathcal{A}_3^{\mathrm{Vor}}(n)$ is, according to our results from Section 2.2, given by

$$K_{\mathcal{A}_3^{\mathrm{Vor}}(n)} = 4L - D , \tag{5}$$

where L is the extension of the line bundle of modular forms to $\mathcal{A}_3^{\mathrm{Vor}}(n)$.

There is the following result of Hulek regarding the nef cone of $\mathcal{A}_3^{\mathrm{Vor}}(n)$:

Theorem 2.11 *A divisor $aL - bD$ on $\mathcal{A}_3^{\mathrm{Vor}}(n)$ is nef if and only if $b \geq 0$ and $a - 12\frac{b}{n} \geq 0$.*

Proof. [Hul, Theorem 0.2] □

As an immediate consequence we obtain that the canonical divisor $K_{\mathcal{A}_3^{\mathrm{Vor}}(n)}$ is nef for $n = 3$ and ample for $n \geq 4$ (cf. [Hul, Corollary 0.4]). Hence $\mathcal{A}_3^{\mathrm{Vor}}(n)$ is a minimal model for $n = 3$ and a canonical model for $n \geq 4$.

2.4 Compactification of $\mathcal{A}_\Gamma$

In this section we will see how the moduli spaces $\mathcal{A}_g(n)$ and its compactifications constructed in the previous section can be used to obtain compactifications for arbitrary Siegel modular varieties $\mathcal{A}_\Gamma$. Furthermore, we will explain how the vanishing conditions from Remark 2.6 on $\mathcal{A}_\Gamma$ can be reformulated on $\mathcal{A}_g^{\mathrm{Vor}}(n)$ instead.

For that, we first have to recall Theorem 1.18 from Chapter 1. There we have seen that for $g \geq 2$ every subgroup Γ of $\mathrm{Sp}(2g,\mathbb{Z})$ of finite index contains a principal congruence subgroup $\Gamma(n) = \Gamma_g(n)$ as a normal subgroup of finite index. We can thus realize every Siegel modular variety $\mathcal{A}_\Gamma$ as a quotient of $\mathcal{A}_g(n) = \mathcal{H}_g/\Gamma(n)$ by the action of the finite group $\Gamma/\Gamma(n)$ as follows:

$$\begin{array}{c} \mathcal{A}_g(n) = \mathcal{H}_g/\Gamma(n) \\ \big\downarrow \circlearrowright \Gamma/\Gamma(n) \\ \mathcal{A}_\Gamma = \mathcal{H}_g/\Gamma \end{array}$$

As a special case we obtain for $\Gamma = \mathrm{Sp}(2g,\mathbb{Z})$ the following diagram

$$\begin{array}{c} \mathcal{A}_g(n) = \mathcal{H}_g/\Gamma(n) \\ \big\downarrow \circlearrowright \mathrm{Sp}(2g,\mathbb{Z}/n\mathbb{Z}) \\ \mathcal{A}_g = \mathcal{H}_g/\,\mathrm{Sp}(2g,\mathbb{Z}) \end{array},$$

where we used that the factor group $\mathrm{Sp}(2g,\mathbb{Z})/\Gamma(n)$ is isomorphic to the symplectic group $G := \mathrm{Sp}(2g,\mathbb{Z}/n\mathbb{Z})$ defined over the finite ring $\mathbb{Z}/n\mathbb{Z}$ (cf. Definition 1.8 and Theorem 1.16). We can thus consider the factor group $\Gamma/\Gamma(n)$ as a subgroup H of G and consider the map from $\mathcal{A}_g(n)$ to $\mathcal{A}_\Gamma$ as a partial quotient map as the following diagram shows:

$$\begin{array}{ccc} & \mathcal{A}_g(n) & \\ \mathrm{G} \circlearrowright & \big\downarrow \quad \searrow \circlearrowright \mathrm{H} & \\ & & \mathcal{A}_\Gamma \\ & \mathcal{A}_g & \end{array}$$

Note that although we also have a morphism from $\mathcal{A}_\Gamma$ to $\mathcal{A}_g$, it can in general not be represented by the action of a finite group. For that we would need that H is normal in G, which is only the case if Γ is normal in $\mathrm{Sp}(2g,\mathbb{Z})$.

We will now consider the Voronoi compactification $\mathcal{A}_g^{\mathrm{Vor}}(n)$ of $\mathcal{A}_g(n)$ constructed in the previous section. According to [Igu2, p. 243] the action of the group $G = \mathrm{Sp}(2g,\mathbb{Z}/n\mathbb{Z})$ on $\mathcal{A}_g(n)$ can be extended to this compactification. It is easy to check that the quotient of $\mathcal{A}_g^{\mathrm{Vor}}(n)$ by this action defines a projective variety which coincides with the Voronoi compactification $\mathcal{A}_g^{\mathrm{Vor}}$ of $\mathcal{A}_g = \mathcal{A}_g(1)$. Furthermore, the partial quotient of $\mathcal{A}_g^{\mathrm{Vor}}(n)$ by the subgroup $H = \Gamma/\Gamma(n)$ corresponding to Γ, defines a projective variety that can be considered as a compactification of $\mathcal{A}_\Gamma$, which we denote by $\mathcal{A}_\Gamma^{\mathrm{Vor}}$. As a consequence, we can extend the description given in the above diagram to the compactifications as follows:

$$\begin{array}{ccc} & \mathcal{A}_g^{\mathrm{Vor}}(n) & \\ & \downarrow \quad \searrow & \circlearrowleft \mathrm{H} \\ \mathrm{G} \circlearrowright & & \mathcal{A}_\Gamma^{\mathrm{Vor}} \\ & \mathcal{A}_g^{\mathrm{Vor}} & \end{array} \qquad (6)$$

By abuse of notation we define the locus $\beta_k \subset \mathcal{A}_\Gamma^{\mathrm{Vor}}$ as the image of the corresponding locus $\beta_k \subset \mathcal{A}_g^{\mathrm{Vor}}(n)$ as defined in (3) and write

$$\left(\mathcal{A}_\Gamma^{\mathrm{Vor}}\right)^{(k)} = \mathcal{A}_\Gamma^{\mathrm{Vor}} \setminus \beta_{k+1} \qquad (7)$$

for the image of the space of rank $\leq k$ degenerations.

Our aim is to construct H–invariant forms on $\mathcal{A}_g^{\mathrm{Vor}}(n)$ which satisfy certain vanishing conditions which ensure that they give pluricanonical forms on a suitable resolution of $\mathcal{A}_\Gamma^{\mathrm{Vor}}$. Although these conditions clearly depend on H and thus on Γ, it is much easier to formulate and check them on $\mathcal{A}_g^{\mathrm{Vor}}(n)$ since there we have a much better understanding of the geometry.

We conclude this section by giving an estimate for the dimension of the space of modular forms with respect to Γ which is an immediate consequence of the corresponding result for the principal congruence subgroups $\Gamma(n)$.

Proposition 2.12 *Let Γ be a subgroup of $\mathrm{Sp}(2g,\mathbb{Z})$ containing a principal congruence subgroup $\Gamma(n)$ with $n \geq 3$. Then the dimension of the space of modular forms of weight $k(g+1)$ with respect to Γ is given as follows:*

$$\dim[\Gamma, k(g+1)] \sim \begin{cases} \dfrac{2}{[\Gamma : \Gamma(n)]} \dim[\Gamma(n), k(g+1)] & \text{if } -\mathbb{1} \in \Gamma\,, \\ \dfrac{1}{[\Gamma : \Gamma(n)]} \dim[\Gamma(n), k(g+1)] & \text{if } -\mathbb{1} \notin \Gamma\,, \end{cases}$$

as $k \to \infty$, where $[\Gamma(n), k(g+1)]$ is given as in Proposition 2.7.

Proof. We can calculate this dimension, as in [Tai, Proposition 2.1], by using the method of Hirzebruch in [Hir2]. Since $\Gamma(n)$ is a normal subgroup of Γ, we can consider the factor group $H = \Gamma/\Gamma(n)$ and have that the space $[\Gamma, k(g+1)]$ is just given by the H–invariant forms in $[\Gamma(n), k(g+1)]$, i.e.

$$[\Gamma, k(g+1)] = [\Gamma(n), k(g+1)]^H \ .$$

We then can conclude that

$$[\Gamma, k(g+1)] = \frac{1}{|H|} \sum_{\gamma \in H} \operatorname{tr}(\gamma^*|_{[\Gamma(n),k(g+1)]}) \ ,$$

where tr denotes the trace operator. Using the Atiyah–Bott fixed point theorem (cf. [Tai, Appendix to §2]) we have that

$$\operatorname{tr}(\gamma^*|_{[\Gamma(n),k(g+1)]}) = O(k^{\dim \operatorname{Fix}(\gamma)})$$

which tells us that we only need to consider $\gamma = \pm \mathbb{1}$, as otherwise $\dim \operatorname{Fix}(\gamma)$ is strictly less than $g(g+1)/2$ which means that we do not get a contribution to the leading term in this case. Since $-\mathbb{1}$ acts trivially, we get the result as claimed with a factor of 1 or 2 depending whether $-\mathbb{1}$ is in H or not. □

In particular, using Corollary 2.8 we obtain the following estimate for $g = 3$:

Corollary 2.13 *Let Γ be a subgroup of $\operatorname{Sp}(6, \mathbb{Z})$ containing a principal congruence subgroup $\Gamma(n)$ with $n \geq 3$. Then there is the following estimate for the dimension of the space of modular forms of weight $4k$ with respect to Γ:*

$$\dim [\Gamma, 4k] = \begin{cases} \frac{1}{6!} \cdot \frac{1}{90720} \, [\operatorname{Sp}(6,\mathbb{Z}) : \Gamma] \, (4k)^6 + O(k^5) & \text{if } -\mathbb{1} \in \Gamma \ , \\ \frac{1}{6!} \cdot \frac{1}{181440} \, [\operatorname{Sp}(6,\mathbb{Z}) : \Gamma] \, (4k)^6 + O(k^5) & \text{if } -\mathbb{1} \notin \Gamma \ . \end{cases}$$

2.5 Outline

In this section we will state the main result of this thesis and give a rough outline of its proof which will be carried out in the following chapters.

As stated in Section 2.1 it is conjectured that there are only finitely many subgroups Γ of $\operatorname{Sp}(2g, \mathbb{Z})$ of finite index such that the corresponding moduli space $\mathcal{A}_\Gamma$ is not of general type. As mentioned before we will focus in this thesis on the case where $g = 3$ which is mostly due to the fact that the geometry of the boundary becomes more complicated with higher g and that for $g \geq 5$ the spaces $\mathcal{A}_g^{\mathrm{Vor}}(n)$ are no longer nonsingular as we have seen in Section 2.3.

In the $g = 3$–case we will prove the conjecture up to some technical point. Namely, we will show that for all but finitely many subgroups Γ of $\operatorname{Sp}(6, \mathbb{Z})$ the space of pluricanonical sections on a resolution of the corresponding Siegel modular variety $\mathcal{A}_\Gamma^{\mathrm{Vor}}$ grows maximally away from the locus β_3 as defined in Section 2.4. We say here that the space of pluricanonical sections on a smooth projective variety X *grows maximally* if $\dim H^0(X, \mathcal{O}_X(mK_X))$ grows as $m^{\dim X}$ as m tends to infinity.

To make this statement precise, we first have to introduce some notations. For each group Γ we take the Voronoi compactification $\mathcal{A}_\Gamma^{\mathrm{Vor}}$ of $\mathcal{A}_\Gamma$ as defined in Section 2.4 and consider a desingularization $\widetilde{\mathcal{A}}_\Gamma^{\mathrm{Vor}}$. The locus β_3 which lies in codimension 3 in $\mathcal{A}_\Gamma^{\mathrm{Vor}}$ might contain singularities which will then be resolved by this desingularization. Thus the preimage of β_3 in $\widetilde{\mathcal{A}}_\Gamma^{\mathrm{Vor}}$ which we denote by $\widetilde{\beta}_3$ is in general 1–codimensional. As in (7) we denote the subvariety of $\widetilde{\mathcal{A}}_\Gamma^{\mathrm{Vor}}$ which is given as the complement of this locus by $\left(\widetilde{\mathcal{A}}_\Gamma^{\mathrm{Vor}}\right)^{(2)}$, i.e.

$$\left(\widetilde{\mathcal{A}}_\Gamma^{\mathrm{Vor}}\right)^{(2)} = \widetilde{\mathcal{A}}_\Gamma^{\mathrm{Vor}} \setminus \widetilde{\beta}_3 \, . \tag{8}$$

We can now give the following precise formulation of the main result:

Theorem 2.14 *There are only finitely many subgroups* Γ *of* $\mathrm{Sp}(6,\mathbb{Z})$ *of finite index such that the space of pluricanonical sections on* $\left(\widetilde{\mathcal{A}}_\Gamma^{\mathrm{Vor}}\right)^{(2)}$ *does not grow maximally.*

To obtain the conjecture from this result in the $g = 3$–case one has to show that there are at most finitely many subgroups for which the pluricanonical sections on $\left(\widetilde{\mathcal{A}}_\Gamma^{\mathrm{Vor}}\right)^{(2)}$ can not be extended over $\widetilde{\beta}_3$, i.e. over the singularities in $\mathcal{A}_\Gamma^{\mathrm{Vor}}$ lying in the boundary in codimension ≥ 3. The author hopes to do so in a forthcoming paper.

We will conclude this chapter by giving a rough outline of the proof of the main result. As we have seen in Section 2.4 every subgroup Γ of $\mathrm{Sp}(6,\mathbb{Z})$ of finite index contains a principal subgroup $\Gamma(n)$ of some level n. To obtain sufficiently many pluricanonical sections on $\left(\widetilde{\mathcal{A}}_\Gamma^{\mathrm{Vor}}\right)^{(2)}$, we will start with pluricanonical forms on $\mathcal{A}_3^{\mathrm{Vor}}(n)$ rather than on the Voronoi compactification of $\mathcal{A}_\Gamma$ itself. We will only consider forms on $\mathcal{A}_3^{\mathrm{Vor}}(n)$ which are invariant under the action of the factor group $\Gamma/\Gamma(n)$ and impose certain vanishing conditions on them. These conditions will guarantee two things: First, that these forms can be extended over the branch locus of the map from $\mathcal{A}_3^{\mathrm{Vor}}(n) \to \mathcal{A}_\Gamma^{\mathrm{Vor}}$ and therefore define pluricanonical sections on $\mathcal{A}_\Gamma^{\mathrm{Vor}}$. Second, that they vanish of sufficiently high order at the singularities of $\mathcal{A}_\Gamma^{\mathrm{Vor}}$ outside β_3 and can thus be extended to $\left(\widetilde{\mathcal{A}}_\Gamma^{\mathrm{Vor}}\right)^{(2)}$.

Once, we have formulated the required vanishing conditions for each subgroup Γ, we will calculate the obstructions imposed by them on the spaces of pluricanonical forms on $\mathcal{A}_3^{\mathrm{Vor}}(n)$. We will then be able to conclude that there are only finitely many Γ for which there do not exist sufficiently many pluricanonical forms satisfying these conditions - which proves the main result.

Both for formulating the vanishing conditions and for concluding that they are satisfied for all but finitely many groups Γ, we need to study the spaces $\mathcal{A}_3^{\mathrm{Vor}}(n)$ and $\mathcal{A}_\Gamma^{\mathrm{Vor}}$, their geometry and their singularities. This can be done by considering the action of the finite group $\mathrm{Sp}(2g, \mathbb{Z}/n\mathbb{Z})$ on $\mathcal{A}_3^{\mathrm{Vor}}(n)$, or more precisely the action of its subgroup $\Gamma/\Gamma(n)$. To do this we need to have a good understanding of the action on the boundary of $\mathcal{A}_3^{\mathrm{Vor}}(n)$, in particular at the intersection of several boundary divisors. For that, we will provide a characterization of the geometry of the boundary in terms of so–called primitive vectors in $(\mathbb{Z}/n\mathbb{Z})^6$ in Chapter 3.

In Chapter 4 we will study the ramification divisor of the map $\mathcal{A}_3^{\mathrm{Vor}}(n) \to \mathcal{A}_3^{\mathrm{Vor}}$ which is just the union of the boundary divisors of $\mathcal{A}_3^{\mathrm{Vor}}(n)$. The singularities at the interior, i.e. in $\mathcal{A}_\Gamma$ itself, and at the boundary of $\mathcal{A}_\Gamma^{\mathrm{Vor}}$ are described in Chapters 5 and 6 respectively. In these three chapters we will also relate the number of elements in $\Gamma/\Gamma(n)$ which fix certain components of $\mathcal{A}_3^{\mathrm{Vor}}(n)$ pointwise – and thus cause ramification or singularities – to the index of Γ in $\mathrm{Sp}(2g, \mathbb{Z})$. This will be used in Chapter 7, where we put all the results together and calculate the obstructions, to conclude that subgroups Γ of sufficiently large index do not pose too many obstructions. This will give us a bound on the index of Γ in $\mathrm{Sp}(2g, \mathbb{Z})$ which is equivalent to excluding finitely many subgroups and thus gives us the main result.

Chapter 3

Geometry of the boundary

In this chapter we want to study the intersection of divisors contained in the boundary of the Voronoi compactification of $\mathcal{A}_3(n)$, the moduli space of principally–polarized abelian threefolds with level n structure. Most of it can be done using toric geometry and can be found in [Tsu]. However, we want to have a rather explicit description of the combinatorial data involved. For that we will introduce a finite module over $\mathbb{Z}/n\mathbb{Z}$ and characterize both the geometry of the Satake and the Voronoi compactification in these terms. Most of our work can be done for arbitrary dimension. In fact we will give a description of the Satake compactification of $\mathcal{A}_g(n)$ for arbitrary g and it will only be in the last section when we talk about the Voronoi compactification that we restrict to the case $g = 3$.

3.1 The group $\mathrm{Sp}(2g, \mathbb{Z}/n\mathbb{Z})$ and the set of primitive vectors

In this section we will recall some basic facts about the group $\mathrm{Sp}(2g, \mathbb{Z}/n\mathbb{Z})$ which acts on both the Satake and the Voronoi compactification of $\mathcal{A}_g(n)$. Moreover, we will introduce the set of primitive vectors and give an interpretation of $\mathrm{Sp}(2g, \mathbb{Z}/n\mathbb{Z})$ in terms of these vectors. Primitive vectors will play a key role in giving a nice description of the geometry of the compactification of $\mathcal{A}_g(n)$.

Recall from Definition 1.8 that $\mathrm{Sp}(2g, \mathbb{Z}/n\mathbb{Z})$ is defined as the group that contains all matrices $M \in \mathrm{GL}(2g, \mathbb{Z}/n\mathbb{Z})$ which satisfy the symplectic relations given by

$$M^T J M = J \;, \qquad \text{where } J = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} \;.$$

It is the image of the map on $\mathrm{Sp}(2g,\mathbb{Z})$ given by reduction modulo n and thus can also be described as the quotient of $\mathrm{Sp}(2g,\mathbb{Z})$ by its normal subgroup $\Gamma(n)$, the principal congruence subgroup of level n defined in Section 1.2.

Remark 3.1 *If $n = k \cdot l$ with $\gcd(k,l) = 1$, we can write $\mathrm{Sp}(2g,\mathbb{Z}/n\mathbb{Z})$ as a cartesian product, namely*

$$\mathrm{Sp}(2g,\mathbb{Z}/n\mathbb{Z}) = \mathrm{Sp}(2g,\mathbb{Z}/k\mathbb{Z}) \times \mathrm{Sp}(2g,\mathbb{Z}/l\mathbb{Z})\,.$$

This allows us to work mostly with the case where $n = p^t$ is a power of a prime p.

Let $V_k(n)$ be the $\mathbb{Z}$–module $(\mathbb{Z}/n\mathbb{Z})^k$. The columns of matrices in $\mathrm{Sp}(2g,\mathbb{Z}/n\mathbb{Z})$ can then be considered as vectors in $V_{2g}(n)$. Moreover, as we will see shortly the columns satisfy a special condition.

Definition 3.2 *We call a vector $v = (v^1,\dots,v^k) \in V_k(n)$* primitive *if the coordinates satisfy $\gcd(v^1,\dots,v^k,n) = 1$ or equivalently if v has order exactly n.*

Here again we can make a small observation which allows us to assume that $n = p^t$ in most cases.

Remark 3.3 *If $n = r \cdot s$ with $\gcd(r,s) = 1$, we have that the $\mathbb{Z}$–module $V_k(n)$ is isomorphic to $V_k(r) \times V_k(s)$ and a vector $(u,w) \in V_k(r) \times V_k(s)$ is primitive iff $u \in V_k(r)$ and $w \in V_k(s)$ are both primitive.*

In the following sections we will sometimes need some combinatorics. The next result tells us how many primitive vectors there are.

Lemma 3.4 *The number of primitive vectors in $V_k(n) = (\mathbb{Z}/n\mathbb{Z})^k$ is given by*

$$\mu(k,n) := \sharp\,\{v \in V_k(n)\,;\ v \text{ is primitive}\} = n^k \prod_{p|n} (1 - p^{-k})\ .$$

Proof. The statement is easy to check for $n = p^t$ a prime power. For composite n the claim then follows from Remark 3.3. □

Among the column vectors there holds a relation which can be understood by looking at the skew form on $V_{2g}(n)$ given by

$$\begin{aligned}\Big\langle (v^1,\dots,v^{2g}),(w^1,\dots,w^{2g})\Big\rangle &:= (v^1,\dots,v^{2g}) \cdot J \cdot \begin{pmatrix} w^1 \\ \vdots \\ w^{2g} \end{pmatrix} \\ &= v^1 w^{g+1} + \cdots + v^g w^{2g} - v^{g+1} w^1 - \cdots - v^{2g} w^g\ .\end{aligned}$$

We can now rephrase the symplectic relations with the new terminology we just introduced.

Proposition 3.5 *A matrix $M \in \mathrm{Mat}(2g, \mathbb{Z}/n\mathbb{Z})$ is symplectic iff its columns $v_1, \ldots, v_{2g} \in V_{2g}(n)$ are primitive vectors which satisfy the following conditions for all $1 \leq i \leq j \leq 2g$:*

$$\langle v_i, v_j \rangle = \begin{cases} 1 & \text{if } j = i + g \\ 0 & \text{otherwise} \end{cases}$$

Proof. It suffices to notice that the conditions on the column vectors are equivalent to the symplectic relations satisfied by the matrix. The claim that the column vectors are primitive follows from the well–known fact that symplectic matrices have determinant 1. □

We can use the skew form to introduce another notion which will play an important role in the following section.

Definition 3.6 *We call a submodule W of $V_{2g}(n)$* isotropic *if for all $v, w \in W$ the relation $\langle v, w \rangle = 0$ holds.*

Remark 3.7 *There is also the notion of an isotropic subspace $U \subset \mathbb{R}^{2g}$ as given in Definition 1.33. If U is defined over $\mathbb{Q}$, its restriction to $\mathbb{Z}^{2g}$ can be considered as an isotropic submodule $\overline{U}$ of $\mathbb{Z}^{2g}$. Note carefully that if we reduce $\overline{U}$ modulo n, we obtain an isotropic submodule of $V_{2g}(n) = (\mathbb{Z}/n\mathbb{Z})^{2g}$. However, in general not every isotropic submodule of $V_{2g}(n)$ can be obtained in this way - this is due to the fact that we have zero divisors in $\mathbb{Z}/n\mathbb{Z}$ unless $n = p$ is a prime.*

The group $\mathrm{Sp}(2g, \mathbb{Z}/n\mathbb{Z})$ acts on $V_{2g}(n)$ by left multiplication. It is easy to check that this action preserves the property of being primitive and thus can be restricted to the set of primitive vectors.

Lemma 3.8 *The action of $\mathrm{Sp}(2g, \mathbb{Z}/n\mathbb{Z})$ on the set of primitive vectors in $V_{2g}(n)$ is transitive.*

Proof. It suffices to show that every primitive vector in $V_{2g}(n)$ can appear in the first column of a matrix in $\mathrm{Sp}(2g, \mathbb{Z}/n\mathbb{Z})$. So let $v_1 \in V_{2g}(n)$ be an arbitrary primitive vector. Since v_1 is primitive, we can find a primitive vector v_{g+1} such that $< v_1, v_{g+1} >= 1$. The set of vectors in $V_{2g}(n)$ orthogonal to the module generated by v_1 and v_{g+1} is isomorphic to $(\mathbb{Z}/n\mathbb{Z})^{2g-2}$. Now the claim follows by induction on g. □

Remark 3.9 *In fact we have shown a much stronger result. The proof shows that given any g primitive vectors $v_1, \ldots, v_g \in V_{2g}(n)$ which generate an isotropic submodule of $V_{2g}(n)$ of dimension g, we can find a symplectic matrix which has exactly the vectors $v_1, \ldots, v_g$ as its first g columns.*

Following the proof we can easily compute the order of $\mathrm{Sp}(2g, \mathbb{Z}/n\mathbb{Z})$ which is known to be

$$\gamma_g(n) := [\mathrm{Sp}(2g, \mathbb{Z}) : \Gamma(n)] = n^{g(2g+1)} \prod_{p|n} \prod_{1 \leq l \leq g} (1 - p^{-2l}) \,.$$

3.2 The geometry of the Satake compactification of $\mathcal{A}_g(n)$

In this section we study the Satake compactification of $\mathcal{A}_g(n)$ and give a description in terms of the objects introduced in the previous section. We will make frequent use of results given in [Nam](cf. also [HKW, Chapter 3]).

From now on we assume that $n \geq 3$. Recall the stratification of the Satake compactification $\mathcal{A}_g^{\mathrm{Sat}}(n)$ given in Section 2.3:

$$\mathcal{A}_g^{\mathrm{Sat}}(n) = \mathcal{A}_g(n) \amalg \left(\coprod_{i_{g-1}} \mathcal{A}_{g-1}^{i_{g-1}}(n) \right) \amalg \ldots \amalg \left(\coprod_{i_0} \mathcal{A}_0^{i_0}(n) \right)$$

As remarked there this means that $\mathcal{A}_g^{\mathrm{Sat}}(n)$ is set-theoretically the union of $\mathcal{A}_g(n)$ with some boundary components each of which is isomorphic to some moduli space $\mathcal{A}_d(n)$ of lower dimension.

The group $G := \mathrm{Sp}(2g, \mathbb{Z}/n\mathbb{Z})$ acts both on the components of $\mathcal{A}_g^{\mathrm{Sat}}(n)$ and on $V_{2g}(n)$ by left multiplication. The action of G on $\left\{ \mathcal{A}_d^{i_d}(n) \right\}$ is transitive for each $d = 0, \ldots, g-1$ (cf. [Nam, Remark 4.16]). We have seen in the previous section that the action on the primitive vectors of $V_{2g}(n)$ is also transitive. In fact, we will establish a 1–to–1 G–equivariant correspondence between the top–dimensional components $\mathcal{A}_{g-1}^{i_{g-1}}(n)$ and primitive $\pm$vectors $\pm v_{i_{g-1}} \in V_{2g}(n)$, where from now on for a vector $v \in V_{2g}(n)$ we identify $+v$ and $-v$ and just write $\pm v$. Instead of proving this directly we will show a more general result first of which this will be an easy consequence. We want to describe the components of higher codimension and for that we need the notion of isotropic submodules $W \subset V_{2g}(n)$ which we introduced in the previous section and the concept of non–degenerate alternating multilinear forms $f : V_{2g}(n) \times \cdots \times V_{2g}(n) \to \mathbb{Z}/n\mathbb{Z}$, where non-degeneracy means that $1 \in f(V_{2g}(n), \ldots, V_{2g}(n))$.

Proposition 3.10 *Let* $0 \leq d \leq g-1$.

(i) The components $\mathcal{A}_d^{i_d}(n)$ *are in 1–to–1 G-equivariant correspondence with pairs* $(W_d^{i_d}, \pm f_d^{i_d})$, *where*

- $W_d^{i_d} \cong (\mathbb{Z}/n\mathbb{Z})^{g-d}$ *is an isotropic submodule of* $V_{2g}(n)$
- $f_d^{i_d}$ *is a non–degenerate alternating* $(g-d)$*–linear form on* $W_d^{i_d}$.

(ii) The number of components $\mathcal{A}_d^{i_d}(n)$ *is given by*

$$\sharp\left\{i_d\,;\,\mathcal{A}_d^{i_d}(n)\right\} = \frac{1}{2}\mu(1,n)\prod_{k=d}^{g-1}\frac{\mu(2(k+1),n)}{\mu(g-k,n)} = \frac{1}{2}\mu(1,n)\prod_{k=d}^{g-1}\frac{2\mu_{k+1}(n)}{\mu(g-k,n)}\ ,$$

where $\mu(k,n)$ *is given as in Lemma 3.4 and* $\mu_k(n) := \frac{1}{2}\mu(2k,n)$ *is the number of maximal dimensional components in* $\mathcal{A}_k(n)$.

Proof.

(i) According to [Nam, Chapter 4] the components $\mathcal{A}_d^{i_d}(n)$ are in 1–to–1 G-equivariant correspondence with isotropic subspaces $\widetilde{W}_d^{i_d}$ in $\mathbb{Q}^{2g}$, or more precisely with equivalence classes $[\widetilde{W}_d^{i_d}]$ of such spaces under the action of $\Gamma(n)$. Since the action of G on the components $\mathcal{A}_d^{i_d}(n)$ is transitive, so is the action on the equivalence classes $[\widetilde{W}_d^{i_d}]$ and it suffices to define the correspondence for one equivalence class which we will call *standard*. Namely, consider the equivalence class which contains the isotropic subspace $\widetilde{W}_d^0 \subset \mathbb{Q}^{2g}$ given by

$$\widetilde{W}_d^0 := (\underbrace{0,\ldots,0}_{d\text{ times}},\underbrace{*,\ldots,*}_{g-d\text{ times}},0,\ldots,0) \subset \mathbb{Q}^{2g}\ . \tag{1}$$

We associate $\widetilde{W}_d^0$ with the *standard pair* $(W_d^0, \pm f_d^0)$ given by

$$\begin{aligned} W_d^0 &:= (\underbrace{0,\ldots,0}_{d\text{ times}},\underbrace{*,\ldots,*}_{g-d\text{ times}},0,\ldots,0) \subset V_{2g}(n)\ ,\\ f_d^0(e_{d+1},\ldots,e_g) &:= 1 \pmod n\ , \end{aligned} \tag{2}$$

where $e_i \in V_{2g}(n)$ is the i-th vector of the canonical basis of $V_{2g}(n)$. The group G acts on the set of pairs $(W_d^{i_d}, \pm f_d^{i_d})$. By using Remark 3.9 we can not only conclude that all $W_d^{i_d}$ are equivalent under the action of G, but also that on a given $W_d^{i_d}$ all alternating forms $f_d^{i_d}$ are equivalent under the action of the stabilizer of $W_d^{i_d}$ in G. This implies that G acts transitively on the set of pairs $(W_d^{i_d}, \pm f_d^{i_d})$. To establish our 1–to–1 G-equivariant correspondence it suffices now to check that the stabilizers of the standard elements $[\widetilde{W}_d^0]$ and $(W_d^0, \pm f_d^0)$ coincide. It can be shown that the stabilizer of $[\widetilde{W}_d^0]$ in G can be described as the quotient of the stabilizer of $\widetilde{W}_d^0$ in $\mathrm{Sp}(2g,\mathbb{Z})$ by the stabilizer of $\widetilde{W}_d^0$ in $\Gamma(n)$ or in other words it is the image of the stabilizer of $\widetilde{W}_d^0$ in $\mathrm{Sp}(2g,\mathbb{Z})$ under the map to G given by reduction modulo n. A straightforward calculation shows that (cf. [Nam, Proposition 4.8])

$$\mathrm{Stab}_{\mathrm{Sp}(2g,\mathbb{Z})}(\widetilde{W}_d^0) = \left\{ \left(\begin{array}{cc:cc} A & 0 & B & * \\ * & U^T & * & * \\ \hdashline C & 0 & D & * \\ 0 & 0 & 0 & U^{-1} \end{array} \right) \subset \mathrm{Sp}(2g,\mathbb{Z}) \; ; \right.$$
$$\left. \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2d,\mathbb{Z}) \; , \; U \in \mathrm{GL}(g-d,\mathbb{Z}) \right\} \; .$$

We now need to calculate the image of this stabilizer under the map given by reduction modulo n. Recall that the image of $\mathrm{Sp}(2d,\mathbb{Z})$ under this map is $\mathrm{Sp}(2d,\mathbb{Z}/n\mathbb{Z})$ and that the image of $\mathrm{GL}(g-d,\mathbb{Z})$ is given by $\{U \in \mathrm{GL}(g-d,\mathbb{Z}/n\mathbb{Z}) \; ; \; \det U = \pm 1\}$. This and some further calculations show that

$$\mathrm{Stab}_G([\widetilde{W}_d^0]) = \left\{ \left(\begin{array}{cc:cc} A & 0 & B & * \\ * & U^T & * & * \\ \hdashline C & 0 & D & * \\ 0 & 0 & 0 & U^{-1} \end{array} \right) \subset G \; ; \right.$$
$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2d,\mathbb{Z}/n\mathbb{Z}) \; ,$$
$$\left. U \in \mathrm{GL}(g-d,\mathbb{Z}/n\mathbb{Z}) \; , \; \det U = \pm 1 \right\} \; .$$

Without the condition that $\det U = \pm 1$ this group is exactly the stabilizer of the isotropic submodule $W_d^0 \subset V_{2g}(n)$ in G. The fact that we are considering pairs and thus need to stabilize the alternating form f_d^0 up to a sign as well gives us then the condition $\det U = \pm 1$, so that the stabilizers of the standard elements coincide as claimed.

(ii) We proceed by induction on $g-d$. For $d = g-1$ the components $\mathcal{A}_{g-1}^{i_{g-1}}(n)$ correspond to isotropic submodules $W_{g-1}^{i_{g-1}} \cong \mathbb{Z}/n\mathbb{Z}$ equipped with a non–degenerate linear form (up to ± 1). These in turn correspond to primitive $\pm$vectors in $V_{2g}(n)$. The number of these is given by $\mu(2g,n)$ as seen in Lemma 3.4. Hence the formula is true for $d = g-1$.
We now consider components $\mathcal{A}_d^{i_d}(n)$ corresponding to pairs $(W_d^{i_d}, \pm f_d^{i_d})$. We will first determine how many isotropic submodules $W_d^{i_d} \cong (\mathbb{Z}/n\mathbb{Z})^{g-d}$ we have in $V_{2g}(n)$. For that consider any isotropic submodule W_{d+1} isomorphic to $(\mathbb{Z}/n\mathbb{Z})^{g-d-1}$. By the induction hypothesis there are

$$\prod_{k=d+1}^{g-1} \frac{\mu(2(k+1),n)}{\mu(g-k,n)}$$

such submodules. Taking a primitive vector w_0 in $V_{2g}(n)$ which is orthogonal to W_{d+1} with respect to the skew form $\langle\cdot,\cdot\rangle$ and adding it to W_{d+1} we get an isotropic submodule $W_d^{i_d} \cong (\mathbb{Z}/n\mathbb{Z})^{g-d}$ and every such submodule can be obtained in this way. The submodule of vectors w_0 orthogonal to a given W_{d+1} is isomorphic to $(\mathbb{Z}/n\mathbb{Z})^{2(d+1)}$. This module contains $\mu(2(d+1),n)$ primitive vectors w_0 but $\mu(1,n)$ of them give the same submodule when added to W_{d+1}. So we have just seen that each W_{d+1} is contained in $\mu(2(d+1),n)/\mu(1,n)$ different isotropic submodules $W_d^{i_d}$. Conversely, every isotropic submodule $W_d^{i_d}$ contains $\mu(g-d,n)/\mu(1,n)$ different submodules $W_{d+1} \cong (\mathbb{Z}/n\mathbb{Z})^{g-d-1}$, so we obtain that there are

$$\frac{\mu(2(d+1),n)}{\mu(1,n)} \cdot \frac{\mu(1,n)}{\mu(g-d,n)} \cdot \prod_{k=d+1}^{g-1} \frac{2\mu_{k+1}(n)}{\mu(g-k,n)} = \prod_{k=d}^{g-1} \frac{2\mu_{k+1}(n)}{\mu(g-k,n)}$$

isotropic submodules $W_d^{i_d} \cong (\mathbb{Z}/n\mathbb{Z})^{g-d}$ in $V_{2g}(n)$. Since there are $\frac{1}{2}\mu(1,n)$ up to a sign different non–degenerate alternating forms on each of them, the claim follows.

□

In the case of top–dimensional components we can rephrase the proposition and obtain the following corollary.

Corollary 3.11 *The components $\mathcal{A}_{g-1}^{i_{g-1}}(n)$ are in 1–to–1 G-equivariant correspondence with primitive ±vectors $\pm v_{i_{g-1}} \in V_{2g}(n)$.*

Proof. It suffices to notice that primitive ±vectors are in 1–to–1 correspondence with isotropic submodules $W_{g-1}^{i_{g-1}} \cong \mathbb{Z}/n\mathbb{Z}$ equipped with a non–degenerate linear form (up to ±1) and then use Proposition 3.10. □

Between the various boundary components of $\mathcal{A}_g^{\mathrm{Sat}}(n)$ we have several adjacency relations. These are described in [Nam] in terms of chains of isotropic subspaces of $\mathbb{Q}^{2g}$ modulo the action of $\Gamma(n)$ (cf. also [HKW, Chapter 3]). We will now express these relations using isotropic subspaces of $V_{2g}(n)$.

Proposition 3.12 *Let $0 \le r < s \le g-1$.*

(i) The component $\mathcal{A}_r^{i_r}(n)$ is contained in $\overline{\mathcal{A}_s^{i_s}}(n)$ iff $W_s^{i_s} \subset W_r^{i_r}$.

(ii) The number of components $\mathcal{A}_r^{i_r}(n)$ that are contained in a given component $\overline{\mathcal{A}_s^{i_s}}(n)$ is given by

$$\sharp\left\{i_r\,;\,\mathcal{A}_r^{i_r}(n) \subset \overline{\mathcal{A}_s^{i_s}}(n)\right\} = \frac{1}{2}\mu(1,n) \prod_{k=r}^{s-1} \frac{2\mu_{k+1}(n)}{\mu(s-k,n)}\ .$$

(iii) *The number of components* $\overline{\mathcal{A}_s^{i_s}}(n)$ *that contain a given component* $\mathcal{A}_r^{i_r}(n)$ *is given by*

$$\sharp\left\{i_s\,;\,\mathcal{A}_1^{i_r}(n)\subset\overline{\mathcal{A}_s^{i_s}}(n)\right\}=\frac{1}{2}\mu(1,n)\prod_{k=0}^{(g-s)-1}\frac{\mu((g-r)-k,n)}{\mu((g-s)-k,n)}\ .$$

Proof.

(i) We know already from the description in [Nam] that $\mathcal{A}_r^{i_r}(n)$ is contained in $\overline{\mathcal{A}_s^{i_s}}(n)$ iff the subspace $\widetilde{W}_s^{i_s}$ in $\mathbb{Q}^{2g}$ modulo the action of $\Gamma(n)$ corresponding to $\mathcal{A}_s^{i_s}(n)$ is contained in the one corresponding to $\mathcal{A}_r^{i_r}(n)$, say $\widetilde{W}_r^{i_r}$. So take any such inclusion $\widetilde{W}_s^{i_s}\subset\widetilde{W}_r^{i_r}$. By [Nam, Theorem 4.14 iii) and Remark 4.16] we can assume w.l.o.g. that $\widetilde{W}_r^{i_r}$ and $\widetilde{W}_s^{i_s}$ are standard, i.e. $\widetilde{W}_r^{i_r}=\widetilde{W}_r^0$ and $\widetilde{W}_s^{i_s}=\widetilde{W}_s^0$ as given in (1). These correspond to the standard pairs $(W_r^0,\pm f_r^0)$ and $(W_s^0,\pm f_s^0)$ (cf. (2)) and we have $W_s^0\subset W_r^0$ as claimed.

For the converse take any two pairs $(W_s^{i_s},\pm f_s^{i_s})$ and $(W_r^{i_r},\pm f_r^{i_r})$ such that $W_s^{i_s}\subset W_r^{i_r}$. We can assume w.l.o.g. that $(W_r^{i_r},\pm f_r^{i_r})=(W_r^0,\pm f_r^0)$ is the standard pair and additionally that $W_s^{i_s}=W_s^0$ is standard, too. Note that the intersection of the stabilizers of $(W_r^0,\pm f_r^0)$ and of W_s^0 contains matrices of the form

$$\left(\begin{array}{cc:cc}\mathbb{1}&0&0&0\\0&U^T&0&0\\\hdashline 0&0&\mathbb{1}&0\\0&0&0&U\end{array}\right),$$

where

$$U=\begin{pmatrix}U_1&0\\0&U_2\end{pmatrix}\in\mathrm{GL}(g-r,\mathbb{Z}/n\mathbb{Z})\ ,\det U=\pm1\ ,U_2\in\mathrm{GL}(g-s,\mathbb{Z}/n\mathbb{Z})\ .$$

We can hence choose $U_2\in\mathrm{GL}(g-s,\mathbb{Z}/n\mathbb{Z})$ such that $f_s^{i_s}$ is mapped to f_s^0 and by choosing $U_1\in\mathrm{GL}(s-r,\mathbb{Z}/n\mathbb{Z})$ accordingly, we can guarantee that $\det U=\pm1$ so that $(W_r^0,\pm f_r^0)$ is left invariant. This shows that we can furthermore assume that $f_s^{i_s}=f_s^0$, so both pairs are standard. These correspond to the standard isotropic subspaces $[\widetilde{W}_s^0]\subset[\widetilde{W}_r^0]$ which completes the proof.

(ii) Fix the component $\mathcal{A}_s^{i_s}(n)$. It corresponds to a pair $(W_s^{i_s},\pm f_s^{i_s})$ with $W_s^{i_s}\cong(\mathbb{Z}/n\mathbb{Z})^{g-s}$ an isotropic submodule of $V_{2g}(n)$. Any component $\mathcal{A}_r^{i_r}(n)$ that is contained in $\overline{\mathcal{A}_s^{i_s}}(n)$ corresponds to a pair $(W_r^{i_r},\pm f_r^{i_r})$ with $W_r^{i_r}\cong(\mathbb{Z}/n\mathbb{Z})^{g-r}$ satisfying $W_s^{i_s}\subset W_r^{i_r}$ by (i). So we need to know how many isotropic submodules $W_r^{i_r}$ there are which contain the given submodule $W_s^{i_s}$. Consider the subset of all vectors in $V_{2g}(n)$ orthogonal to $W_s^{i_s}$. It is a submodule isomorphic to $(\mathbb{Z}/n\mathbb{Z})^{2s}=V_{2s}(n)$. Choosing $W_r^{i_r}$ containing

$W_s^{i_s}$ is equivalent to taking an $(s-r)$–dimensional isotropic submodule of $V_{2s}(n)$. By Proposition 3.10 (ii) there are $\prod_{k=r}^{s-1} \frac{2\mu_{k+1}(n)}{\mu(s-k,n)}$ such submodules. Together with the choice of $\pm f_r^{i_r}$ this proves the claim.

(iii) We fix the component $\mathcal{A}_r^{i_r}(n)$ and consider all components $\mathcal{A}_s^{i_s}(n)$ such that $\mathcal{A}_r^{i_r}(n)$ is contained in their closure. By (i) this is equivalent to looking at a fixed isotropic submodule $W_r^{i_r} \cong (\mathbb{Z}/n\mathbb{Z})^{g-r}$ of $V_{2g}(n)$ and considering all isotropic submodules $W_s^{i_s} \cong (\mathbb{Z}/n\mathbb{Z})^{g-s}$ contained in it. Since any submodule of an isotropic submodule is isotropic, the number of $W_s^{i_s}$ is just the number of $(g-s)$–dimensional submodules in $(\mathbb{Z}/n\mathbb{Z})^{g-r}$. It can be easily computed by induction and is given by

$$\prod_{k=0}^{(g-s)-1} \frac{\mu((g-r)-k,n)}{\mu((g-s)-k,n)}.$$

We can choose $\pm f_s^{i_s}$ freely and thus obtain the desired formula.

□

For the top–dimensional components $\mathcal{A}_{g-1}^{i_{g-1}}(n)$ we can again rephrase this result in terms of primitive ±vectors.

Corollary 3.13 *Let* $0 \leq d < g-1$. *The component* $\mathcal{A}_d^{i_d}(n)$ *is contained in* $\overline{\mathcal{A}_{g-1}^{i_{g-1}}(n)}$ *iff* $\pm v_{i_{g-1}} \in W_d^{i_d}$.

Proof. Recall that we can identify pairs $(W_{g-1}^{i_{g-1}}, \pm f_{g-1}^{i_{g-1}})$ with primitive ±vectors $\pm v_{i_{g-1}}$. If $\mathcal{A}_d^{i_d}(n)$ is contained in $\overline{\mathcal{A}_{g-1}^{i_{g-1}}(n)}$ then $\pm v_{i_{g-1}} \in W_{g-1}^{i_{g-1}} \subset W_d^{i_d}$ by Proposition 3.12. For the converse note that since $W_d^{i_d}$ is a submodule, it contains $W_{g-1}^{i_{g-1}}$ if it contains $\pm v_{i_{g-1}}$. □

3.3 The geometry of the Voronoi compactification of $\mathcal{A}_3(n)$

The Satake compactification of $\mathcal{A}_g(n)$ as described in the previous section is highly singular along its boundary. Moreover, the boundary has codimension g. We will now specialize to the case $g=3$ and consider the Voronoi compactification $\mathcal{A}_3^{\text{Vor}}(n)$ given by the 2nd Voronoi decomposition as described in Section 2.3. As in the previous section we assume $n \geq 3$ in which case $\mathcal{A}_3^{\text{Vor}}(n)$ is nonsingular (cf. Theorem 2.10). Recall further from Section 2.3 that we have a projection

$\pi : \mathcal{A}_3^{\mathrm{Vor}}(n) \to \mathcal{A}_3^{\mathrm{Sat}}(n)$ to the Satake compactification and that the boundary is a divisor D which has several components

$$D = \sum_j D_j \ ,$$

where the D_j are just the closures of the preimages of the top–dimensional components $\mathcal{A}_2^{i_2}(n) \subset \mathcal{A}_3^{\mathrm{Sat}}(n)$ under the projection π.

We have seen that the 2^{nd} Voronoi decomposition is a rational polyhedral decomposition of the set of non–negative integral quadratic forms of degree 3, or more precisely of its convex hull in the space of real quadratic forms. The divisors D_j correspond to the 1–dimensional cones in this decomposition and their intersection behavior can be understood completely from this toric picture (cf. [Tsu, Remark 4.5]). Recall that there is a 6–dimensional cone σ_3 which is spanned by six 1–dimensional rays $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3$ which together with its faces generates the whole decomposition under the action of $\mathrm{GL}(3, \mathbb{Z})$. We will mostly work over this principal cone and follow the notation of Tsushima introduced in [Tsu, Section 3].

We start by looking at intersections of two boundary divisors. The group G acts on them and this action is transitive as the following lemma shows.

Lemma 3.14 *All non–trivial intersections $D_{j_1} \cap D_{j_2}$ of two boundary divisors in $\mathcal{A}_g^{\mathrm{Vor}}(n)$ are equivalent under the action of G.*

Proof. W.l.o.g. we can assume that the intersection corresponds to a 2–dimensional face of the standard cone. We will show that every such face is equivalent to $\alpha_1 * \alpha_2$, the cone spanned by α_1 and α_2, under the action of $\mathrm{GL}(3, \mathbb{Z})$. Using $u_{ij} \in \mathrm{GL}(3, \mathbb{Z})$ (cf. [Tsu, p. 949] for a definition of u_{ij}) we can show that every face of the form $\alpha_k * \alpha_l$ is equivalent to $\alpha_1 * \alpha_2$. The same is true for $\alpha_1 * \beta_3$ by u_I and hence for all $\alpha_k * \beta_l$ with $k \neq l$. Faces of the form $\alpha_k * \beta_k$ are equivalent to $\alpha_3 * \beta_3$ and hence to $\alpha_2 * \alpha_3$ by u_I. It remains to consider $\beta_k * \beta_l$ which is equivalent to $\beta_2 * \beta_3$ which can be transformed to $\alpha_1 * \beta_1$ by u_V which completes the proof. □

We will now give a criterion for determining whether two given divisors intersect over a given component of $\mathcal{A}_3^{\mathrm{Sat}}(n)$ in terms of the terminology we introduced in the previous sections.

Proposition 3.15 *Two divisors D_{j_1} and D_{j_2} intersect in $\mathcal{A}_3^{\mathrm{Vor}}(n)$ over the component $\mathcal{A}_1^{i_1}(n)$ iff $v_{j_1}, v_{j_2} \in W_1^{i_1}$ and $f_1^{i_1}(v_{j_1}, v_{j_2}) = \pm 1$.*

Proof. Take two divisors D_{j_1}, D_{j_2} intersecting over $\mathcal{A}_1^{i_1}(n)$. We can assume w.l.o.g. that they correspond to α_2 and α_3 in the standard cone. Hence $\mathcal{A}_1^{i_1}(n)$

has to correspond to the equivalence class $[\widetilde{W}_1^0]$ in $\mathbb{Q}^6$ modulo the action of $\Gamma(n)$ given by the standard isotropic subspace $\widetilde{W}_1^0$ as in (1). This corresponds to the standard pair $(W_1^0, \pm f_1^0)$ as given in (2). Since α_2 and α_3 correspond to the $\pm$vectors e_2 and e_3 of the canonical basis of $V_6(n)$ the condition is satisfied. Conversely take any two $\pm$vectors v_{j_1}, v_{j_2} and any pair $(W_1^{i_1}, \pm f_1^{i_1})$ satisfying the conditions. W.l.o.g. the pair $(W_1^{i_1}, \pm f_1^{i_1}) = (W_1^0, \pm f_1^0)$ can be assumed to be standard. Since by the hypothesis $f_1^0(v_{j_1}, v_{j_2}) = \pm 1$, we can transform v_{j_1} and v_{j_2} within the stabilizer of $(W_1^0, \pm f_1^0)$ in G to e_2 and e_3 respectively. So they correspond to $\alpha_2 * \alpha_3$ which intersect over $\mathcal{A}_1^0(n)$ which completes the proof. □

We now turn our attention to the intersection of three divisors. As before G acts on them, however, in this case this action is no longer transitive as the following result shows.

Lemma 3.16 *There are two orbits of non–trivial intersections $D_{j_1} \cap D_{j_2} \cap D_{j_3}$ of three boundary divisors in $\mathcal{A}_3^{\mathrm{Vor}}(n)$ under the action of G, namely one containing the intersection corresponding to $\alpha_1 * \alpha_2 * \alpha_3$ and one containing $\beta_1 * \beta_2 * \beta_3$. Following Tsushima we call intersections of the first kind of* local type *and the latter of* global type.

Proof. We can again work over the standard cone and have to show that all of its 3–dimensional faces are either equivalent to $\alpha_1 * \alpha_2 * \alpha_3$ or $\beta_1 * \beta_2 * \beta_3$. Since the argument is similar to the one given in Lemma 3.14 we omit it here. To see that $\alpha_1 * \alpha_2 * \alpha_3$ and $\beta_1 * \beta_2 * \beta_3$ are inequivalent, note that all matrices in the interior of $\alpha_1 * \alpha_2 * \alpha_3$ are rank 3 matrices whereas all matrices in the interior of $\beta_1 * \beta_2 * \beta_3$ have rank 2, so they cannot be equivalent in $GL(3, \mathbb{Z})$. □

Intersections of local type and of global type can both be characterized in terms of primitive $\pm$vectors.

Proposition 3.17 *(i) Three divisors $D_{j_1}, D_{j_2}, D_{j_3}$ intersect in $\mathcal{A}_3^{\mathrm{Vor}}(n)$ over the component $\mathcal{A}_1^{i_1}(n)$ (of global type) iff $v_{j_1}, v_{j_2}, v_{j_3} \in W_1^{i_1}$, the set of combinations $\{\pm v_{j_1} \pm v_{j_2} \pm v_{j_3}\}$ contains 0 and $f_1^{i_1}(v_{j_1}, v_{j_2}) = \pm 1$.*

(ii) Three divisors $D_{j_1}, D_{j_2}, D_{j_3}$ intersect in $\mathcal{A}_3^{\mathrm{Vor}}(n)$ over the component $\mathcal{A}_0^{i_0}(n)$ (of local type) iff $v_{j_1}, v_{j_2}, v_{j_3} \in W_0^{i_0}$ and $f_0^{i_0}(v_{j_1}, v_{j_2}, v_{j_3}) = \pm 1$.

Proof.

(i) We can assume w.l.o.g. that $D_{j_1}, D_{j_2}, D_{j_3}$ correspond to $\alpha_2, \alpha_3, \beta_1$ in the standard cone. This implies that $\mathcal{A}_1^{i_1}(n)$ has to correspond to the standard pair $(W_1^0, \pm f_1^0)$. The cones $\alpha_2, \alpha_3, \beta_1$ correspond to the primitive $\pm$vectors e_2, e_3 and $(e_2 - e_3)$ respectively. For these the given conditions are easy to

verify.
For the converse we can assume w.l.o.g. that the pair $(W_1^{i_1}, \pm f_1^{i_1})$ is standard and that $v_{j_1} = e_2$ and $v_{j_2} = e_3$. Hence $\pm v_{j_3} \in \{e_2+e_3, e_2-e_3\}$. In the latter case the intersection then corresponds to $\alpha_2 * \alpha_3 * \beta_1$ which is of global type over $\mathcal{A}_1^0(n)$. In the first case the intersection is given by $\alpha_2 * \alpha_3 * \gamma_1$, where γ_1 is defined as in [Tsu, p. 949]. This is transformed to $\alpha_1 * \alpha_3 * \beta_2$ by u_{II}^{-1} which is of global type, too.

(ii) The argument is essentially the same as in Proposition 3.15, so we omit it here.

□

In the case of intersection of four divisors we again have two orbits under the action of G.

Lemma 3.18 *Under the action of G there are two orbits of non–trivial intersections $D_{j_1} \cap \ldots \cap D_{j_4}$ of four boundary divisors in $\mathcal{A}_3^{\mathrm{Vor}}(n)$. One containing the intersection corresponding to $\alpha_1 * \alpha_2 * \alpha_3 * \beta_3$ and one containing $\alpha_1 * \alpha_3 * \beta_1 * \beta_3$. We call them of* type I *and of* type II *respectively.*

Proof. The same argument as in Lemma 3.14 shows that every intersection of four divisors can be transformed to one of these two types. Each of these 4–dimensional cones has four 3–dimensional faces; for $\alpha_1 * \alpha_3 * \beta_1 * \beta_3$ these are all of local type whereas $\alpha_1 * \alpha_2 * \alpha_3 * \beta_3$ contains exactly one 3–dimensional face of global type. This shows that the two types are inequivalent. □

Proposition 3.19 *(i) Four divisors $D_{j_1}, \ldots, D_{j_4}$ intersect in $\mathcal{A}_3^{\mathrm{Vor}}(n)$ over the component $\mathcal{A}_0^{i_0}(n)$ of type I iff $v_{j_1}, \ldots, v_{j_4} \in W_0^{i_0}$ and there is a permutation σ of $\{j_1, \ldots, j_4\}$ such that the set $\{\pm v_{\sigma(j_1)} \pm v_{\sigma(j_2)} \pm v_{\sigma(j_3)}\}$ contains 0 and $f_0^{i_0}(v_{\sigma(j_1)}, v_{\sigma(j_2)}, v_{\sigma(j_4)}) = \pm 1$.*

(ii) Four divisors $D_{j_1}, \ldots, D_{j_4}$ intersect in $\mathcal{A}_3^{\mathrm{Vor}}(n)$ over the component $\mathcal{A}_0^{i_0}(n)$ of type II iff $v_{j_1}, \ldots, v_{j_4} \in W_0^{i_0}$, the set $\{\pm v_{j_1} \pm v_{j_2} \pm v_{j_3} \pm v_{j_4}\}$ contains 0 and $f_0^{i_0}(v_{j_1}, v_{j_2}, v_{j_3}) = \pm 1$.

Proof.

(i) We can again assume w.l.o.g. that $D_{j_1}, \ldots, D_{j_4}$ correspond to the rays $\alpha_1, \alpha_2, \alpha_3, \beta_3$ over the standard cone, i.e. $\mathcal{A}_0^{i_0}(n)$ is then given by the standard pair $(W_0^0, \pm f_0^0)$. The cones $\alpha_1, \alpha_2, \alpha_3, \beta_3$ are represented by the primitive $\pm$vectors e_1, e_2, e_3 and $(e_1 - e_2)$ respectively. Then $f_0^0(e_1, e_2, e_3) = \pm 1$ and $-e_1 + e_2 + (e_1 - e_2) = 0$.
For the other direction we can assume that $(W_0^{i_0}, \pm f_0^{i_0}) = (W_0^0, \pm f_0^0)$ is

standard and that $v_{j_1} = e_1, v_{j_2} = e_2$ and $v_{j_4} = e_3$. Since $\{\pm v_{j_1} \pm v_{j_2} \pm v_{j_3}\}$ has to contain 0, we can conclude that $\pm v_{j_3} \in \{e_1 + e_2, e_1 - e_2\}$. These two possibilities correspond to the cones γ_3 and β_3 respectively. By u_{III}^{-1} the cone $\alpha_1 * \alpha_2 * \alpha_3 * \gamma_3$ is equivalent to $\alpha_1 * \alpha_2 * \alpha_3 * \beta_2$ which is of type I. The same is true for $\alpha_1 * \alpha_2 * \alpha_3 * \beta_3$.

(ii) We can consider w.l.o.g. $\alpha_1 * \alpha_3 * \beta_1 * \beta_3$ in the standard cone. Then $\mathcal{A}_0^{i_0}(n)$ corresponds to $(W_0^0, \pm f_0^0)$. The primitive $\pm$vectors representing $\alpha_1, \alpha_3, \beta_1$ and β_3 are given by $e_1, e_3, (e_2 - e_3)$ and $(e_1 - e_2)$ respectively. These satisfy the conditions of the Proposition.
Conversely, since $f_0^{i_0}(v_{j_1}, v_{j_2}, v_{j_3}) = \pm 1$ we can assume that $(W_0^{i_0}, \pm f_0^{i_0}) = (W_0^0, \pm f_0^0)$ is standard and that $v_{j_1} = e_1, v_{j_2} = e_2$ and $v_{j_3} = e_3$. Then since $\{\pm v_{j_1} \pm v_{j_2} \pm v_{j_3} \pm v_{j_4}\}$ contains 0, the vector $\pm v_{j_4}$ has to lie in $\{e_1 + e_2 + e_3, -e_1 + e_2 + e_3, e_1 - e_2 + e_3, e_1 + e_2 - e_3\}$. These four cases correspond to the cones $\varepsilon, \delta_1, \delta_2$ and δ_3 resp. as given in [Tsu, p. 949]. By u_{IV}^{-1} the first case $\alpha_1 * \alpha_2 * \alpha_3 * \varepsilon$ is equivalent to $\alpha_1 * \alpha_3 * \beta_1 * \beta_3$ which is of type II. The other three cases can all be transformed to $\alpha_1 * \alpha_2 * \alpha_3 * \delta_1$ by u_{i1} which is mapped to $\alpha_1 * \alpha_3 * \beta_1 * \beta_3$ by u_I^{-1}.

□

For the intersection of five divisors there is again only one orbit under the action of G.

Lemma 3.20 *All non–trivial intersections of $D_{j_1} \cap \ldots \cap D_{j_5}$ of five boundary divisors in $\mathcal{A}_3^{\mathrm{Vor}}(n)$ are equivalent under the action of G.*

Proof. We can again look at the standard cone. Using u_{ij} the only thing we need to show is that $\alpha_1 * \alpha_2 * \alpha_3 * \beta_2 * \beta_3$ and $\alpha_1 * \alpha_3 * \beta_1 * \beta_2 * \beta_3$ are equivalent. This is done by u_V. □

Proposition 3.21 *Five divisors $D_{j_1}, \ldots, D_{j_5}$ intersect in $\mathcal{A}_3^{\mathrm{Vor}}(n)$ over the component $\mathcal{A}_0^{i_0}(n)$ iff $v_{j_1}, \ldots, v_{j_5} \in W_0^{i_0}$ and there is a permutation σ of $\{j_1, \ldots, j_5\}$ such that $\{v_{\sigma(j_1)}, \ldots, v_{\sigma(j_4)}\}$ satisfy the condition of Proposition 3.19 (ii) and both $\{\pm v_{\sigma(j_1)} \pm v_{\sigma(j_2)} \pm v_{\sigma(j_5)}\}$ and $\{\pm v_{\sigma(j_3)} \pm v_{\sigma(j_4)} \pm v_{\sigma(j_5)}\}$ contain* 0.

Proof. It suffices to consider the case where $\mathcal{A}_0^{i_0}(n)$ is given by the standard pair $(W_0^0, \pm f_0^0)$. Furthermore we can assume that $D_{j_1}, \ldots, D_{j_5}$ correspond to $\alpha_1, \alpha_2, \alpha_3, \beta_2, \beta_3$. These are given by the primitive $\pm$vectors $e_1, e_2, e_3, (e_1 - e_3)$ and $(e_1 - e_2)$ respectively. Then $\{e_2, e_3, e_1 - e_3, e_1 - e_2\}$ satisfy the condition of Proposition 3.19 (ii), since they correspond to the cone $\alpha_2 * \alpha_3 * \beta_2 * \beta_3$ which is of type II. The identities $e_1 - e_2 - (e_1 - e_2) = 0$ and $e_1 - e_3 - (e_1 - e_3) = 0$ show that the last two conditions are also satisfied.

For the converse, we can assume as usual that $(W_0^{i_0}, \pm f_0^{i_0}) = (W_0^0, \pm f_0^0)$ is standard and that $v_{j_1} = e_1, v_{j_2} = e_2$ and $v_{j_3} = e_3$. The condition of Proposition 3.19 (ii) then implies that $\pm v_{j_4} \in \{e_1+e_2+e_3, -e_1+e_2+e_3, e_1-e_2+e_3, e_1+e_2-e_3\}$. If $\pm v_{j_4} = e_1 + e_2 + e_3$ the last two conditions give us that the vector $\pm v_{j_5}$ has to lie in $\{e_1 + e_2, e_1 + e_3, e_2 + e_3\}$. These three cases all correspond to $\alpha_1 * \alpha_2 * \alpha_3 * \varepsilon * \gamma_i$ which can be transformed to $\alpha_1 * \alpha_2 * \alpha_3 * \varepsilon * \gamma_3$ which is equivalent to $\alpha_1 * \alpha_2 * \alpha_3 * \beta_1 * \beta_3$ under the action of u_{IV}^{-1}.
If $\pm v_{j_4} = -e_1+e_2+e_3$ then we can conclude that $\pm v_{j_5} \in \{e_1-e_2, e_1-e_3, e_2+e_3\}$. So we have either $\alpha_1 * \alpha_2 * \alpha_3 * \gamma_1 * \beta_i$ with $i \neq 1$ or $\alpha_1 * \alpha_2 * \alpha_3 * \gamma_1 * \gamma_3$. In the first case we can get $\alpha_1 * \alpha_2 * \alpha_3 * \gamma_1 * \beta_3$ and hence $\alpha_1 * \alpha_2 * \alpha_3 * \beta_2 * \beta_3$ by u_{II}^{-1}. The latter case can be transformed to $\alpha_1 * \alpha_2 * \alpha_3 * \gamma_2 * \gamma_3$ which is equivalent to $\alpha_1 * \alpha_2 * \alpha_3 * \beta_1 * \beta_2$ by u_{III}^{-1}.
The other two cases for $\pm v_{j_4}$ can be done in the same way for symmetry reasons.
□

We now deal with the last case, the intersection of six boundary divisors in $\mathcal{A}_3(n)$.

Lemma 3.22 *All non–trivial intersections of $D_{j_1} \cap \ldots \cap D_{j_6}$ of six boundary divisors in $\mathcal{A}_3^{\mathrm{Vor}}(n)$ are equivalent under the action of G.*

Proof. Since all 6–dimensional cones are equivalent to the standard cone the claim follows. □

Proposition 3.23 *Six divisors $D_{j_1}, \ldots, D_{j_6}$ intersect in $\mathcal{A}_3^{\mathrm{Vor}}(n)$ over the component $\mathcal{A}_0^{i_0}(n)$ iff $v_{j_1}, \ldots, v_{j_6} \in W_0^{i_0}$ and there is a permutation σ of $\{j_1, \ldots, j_6\}$ such that $\{v_{\sigma(j_1)}, \ldots, v_{\sigma(j_4)}\}$ satisfy the condition of Proposition 3.19 (ii) and both $\{v_{\sigma(j_1)}, \ldots, v_{\sigma(j_4)}, v_{\sigma(j_5)}\}$ and $\{v_{\sigma(j_1)}, \ldots, v_{\sigma(j_4)}, v_{\sigma(j_6)}\}$ satisfy Proposition 3.21.*

Proof. We can just consider $\alpha_1 * \alpha_2 * \alpha_3 * \beta_1 * \beta_2 * \beta_3$ and $(W_0^0, \pm f_0^0)$. Then $v_{j_1} = e_1, v_{j_2} = e_2, v_{j_3} = e_3, v_{j_4} = e_2 - e_3, v_{j_5} = e_1 - e_3$ and $v_{j_6} = e_1 - e_2$. Now the vectors $\{e_1, e_2, e_2 - e_3, e_1 - e_3\}$ satisfy the condition of Proposition 3.19 (ii), because it corresponds to the type II cone $\alpha_1 * \alpha_2 * \beta_1 * \beta_2$. Likewise the sets $\{e_1, e_2, e_2 - e_3, e_1 - e_3, e_3\}$ and $\{e_1, e_2, e_2 - e_3, e_1 - e_3, e_1 - e_2\}$ satisfy Proposition 3.21, since they represent the cones $\alpha_1 * \alpha_2 * \alpha_3 * \beta_1 * \beta_2$ and $\alpha_1 * \alpha_2 * \beta_1 * \beta_2 * \beta_3$ respectively.

For the other direction we can assume that $(W_0^{i_0}, \pm f_0^{i_0}) = (W_0^0, \pm f_0^0)$ is standard and that $v_{j_1} = e_1, v_{j_2} = e_2$ and $v_{j_3} = e_3$. We can then conclude that $\pm v_{j_4} \in \{e_1 + e_2 + e_3, -e_1 + e_2 + e_3, e_1 - e_2 + e_3, e_1 + e_2 - e_3\}$ by applying the condition

of Proposition 3.19 (ii).
If $\pm v_{j_4} = e_1 + e_2 + e_3$ the last two conditions imply that the last two vectors satisfy $\pm v_{j_5}, \pm v_{j_6} \in \{e_1 + e_2, e_1 + e_3, e_2 + e_3\}$. This gives us three cases, namely $\alpha_1 * \alpha_2 * \alpha_3 * \varepsilon * \gamma_k * \gamma_l$ with $k \neq l$. These are all equivalent and $\alpha_1 * \alpha_2 * \alpha_3 * \varepsilon * \gamma_2 * \gamma_3$ can be transformed to $\alpha_1 * \alpha_2 * \alpha_3 * \beta_1 * \beta_2 * \beta_3$ by u_{IV}^{-1}.
If $\pm v_{j_4} = -e_1 + e_2 + e_3$, it follows that $\pm v_{j_5}, \pm v_{j_6} \in \{e_1 - e_2, e_1 - e_3, e_2 + e_3\}$. Hence we get the following three 6–dimensional cones: $\alpha_1 * \alpha_2 * \alpha_3 * \delta_1 * \beta_2 * \beta_3$ and $\alpha_1 * \alpha_2 * \alpha_3 * \delta_1 * \beta_k * \gamma_1$, $k = 2, 3$. The last two are clearly equivalent and can be transformed to $\alpha_1 * \alpha_2 * \alpha_3 * \beta_1 * \beta_2 * \beta_3$ by u_{II}^{-1} for $k = 3$. The first is equivalent to $\alpha_1 * \alpha_2 * \alpha_3 * \beta_1 * \beta_2 * \beta_3$ by applying u_I^{-1}.
The last two cases for $\pm v_{j_4}$ follow out of symmetry. □

We will finish this section by giving some combinatorics about the intersections.

Lemma 3.24 *The number of different non–trivial intersections of d divisors in $\mathcal{A}_3^{\mathrm{Vor}}(n)$ is given by*

(i) $\mu_3(n)$, *if* $d = 1$,

(ii) $\frac{1}{2}n\mu_2(n)\mu_3(n)$, *if* $d = 2$,

(iii) *(a)* *(of global type)* $\frac{1}{3}n\mu_2(n)\mu_3(n)$,

(b) *(of local type)* $\frac{1}{6}n^3\mu_1(n)\mu_2(n)\mu_3(n)$, *if* $d = 3$,

(iv) *(a)* *(of type I)* $\frac{1}{3}n^3\mu_1(n)\mu_2(n)\mu_3(n)$,

(b) *(of type II)* $\frac{1}{6}n^3\mu_1(n)\mu_2(n)\mu_3(n)$, *if* $d = 4$,

(v) $\frac{1}{2}n^3\mu_1(n)\mu_2(n)\mu_3(n)$, *if* $d = 5$,

(vi) $\frac{1}{6}n^3\mu_1(n)\mu_2(n)\mu_3(n)$, *if* $d = 6$.

Proof. cf. [Tsu, Lemma 7.1] □

In principal such a description of the geometry of $\mathcal{A}_g^{\mathrm{Vor}}(n)$ in terms of primitive vectors can be given for arbitrary g. However, for $g \geq 4$ the 2^{nd} Voronoi decomposition does not have a unique maximal cone up to the action of $\mathrm{GL}(g, \mathbb{Z})$ as was the case for $g = 3$. In fact, there appears another maximal cone besides the principal one which turns out not to be regular. This gives us a much richer geometry which is more difficult to describe but can be done at least for the principal cone.

Chapter 4

Ramification mean

In this chapter we will study elements in $\mathrm{Sp}(2g, \mathbb{Z}/n\mathbb{Z})$ which fix divisors in the boundary of $\mathcal{A}_g^{\mathrm{Vor}}(n)$, the Voronoi compactification of the moduli space of abelian threefolds with level n structure as introduced in Section 2.3. More precisely, we will show that subgroups of $\mathrm{Sp}(2g, \mathbb{Z}/n\mathbb{Z})$ which contain many elements that fix boundary divisors pointwise, have small index in $\mathrm{Sp}(2g, \mathbb{Z}/n\mathbb{Z})$. We will use this later in Chapter 7 to conclude that subgroups of sufficiently large index do not have too many *bad* elements, i.e. elements that either give us ramification divisors or non–canonical singularities when we consider the quotient of $\mathcal{A}_g^{\mathrm{Vor}}(n)$ by them.

4.1 Definitions

In this section we want to describe the elements in $\mathrm{Sp}(2g, \mathbb{Z}/n\mathbb{Z})$ which fix boundary divisors, the so–called transvections. Furthermore we will introduce the notion of the ramification mean for subgroups of $\mathrm{Sp}(2g, \mathbb{Z})$ which will allow us to replace the statement that a subgroup contains many *bad* elements with a precise formulation, the main result of this chapter.

Recall the notion of a primitive vector given in Definition 3.2. Each primitive vector in $(\mathbb{Z}/n\mathbb{Z})^{2g}$ defines a subgroup of $\mathrm{Sp}(2g, \mathbb{Z}/n\mathbb{Z})$ as follows:

Definition 4.1 *For any primitive vector $v \in (\mathbb{Z}/n\mathbb{Z})^{2g}$ we define $\mathrm{Ram}_G(v)$ to be the subgroup of $G := \mathrm{Sp}(2g, \mathbb{Z}/n\mathbb{Z})$ consisting of* transvections*, which are operators of the form*

$$r_{v,\alpha} : w \mapsto w + \alpha \langle w, v \rangle v , \quad \alpha \in \mathbb{Z}/n\mathbb{Z} . \tag{1}$$

Note that $\mathrm{Ram}_G(v) \cong \mathbb{Z}/n\mathbb{Z}$. Rather than working with the full symplectic group G, we will consider subgroups H of G and thus in general not have all transvections in H. We thus define for any subgroup $H < G$

$$\mathrm{Ram}_H(v) := H \cap \mathrm{Ram}_G(v) \ . \tag{2}$$

Recall from Sections 3.2 and 3.3 that each primitive vector v_j corresponds to a boundary divisors D_j of the Voronoi compactification $\mathcal{A}_3^{\mathrm{Vor}}(n)$. We will see later in Chapter 7 that $\mathrm{Ram}_H(v_j)$ is exactly the subgroup of H that fixes the corresponding divisor D_j pointwise. Because of this relation we will sometimes also write $\mathrm{Ram}_H(D_j)$ instead of $\mathrm{Ram}_H(v_j)$.

$\mathrm{Ram}_H(v)$ as a subgroup of the cyclic group $\mathrm{Ram}_G(v)$ is again cyclic and in fact uniquely determined by its order. We normalize the order and define the ramification of v with respect to H as

$$\mathrm{ram}_H(v) := \frac{1}{n}\Big|\,\mathrm{Ram}_H(v)\Big| \ .$$

Note that with this definition $0 < \mathrm{ram}_H(v) \leq 1$, or more precisely, $\mathrm{ram}_H(v) = k/n$ for some $k \in \{1, \ldots, n\}$.

While $\mathrm{ram}_H(v_j)$ describes the behavior of H at a single boundary divisor D_j, we will be more interested in the action of H on all boundary divisors at once and thus consider a certain mean.

Definition 4.2 *For any subgroup H of G we define the ramification mean of H to be*

$$\frac{1}{\#v} \sum_v \mathrm{ram}_H(v) \ ,$$

where the sum is taken over all primitive vectors $v \in (\mathbb{Z}/n\mathbb{Z})^{2g}$.

If we take any subgroup $\Gamma < \mathrm{Sp}(2g, \mathbb{Z})$ of finite index, it will contain a principal congruence subgroup $\Gamma(n)$ by Theorem 1.18 provided that $g \geq 2$. We can then consider the factor group $H := \Gamma/\Gamma(n)$. Although H is not uniquely determined by Γ alone but also depends on n, the ramification mean of H is in fact independent of n because of our normalization. We thus can define the ramification mean of $\Gamma < \mathrm{Sp}(2g, \mathbb{Z})$ to be the ramification mean of H.

We can now state the main result of this chapter using the terminology we just introduced.

Theorem 4.3 *For every $\varepsilon > 0$ there are only finitely many subgroups $\Gamma < \mathrm{Sp}(2g, \mathbb{Z})$ of finite index with ramification mean at least ε.*

The rest of this chapter will be dedicated to the proof of this theorem. We will reduce the complexity of this problem in several steps.

4.2 A first reduction

Recall that by Theorem 1.18 any subgroup Γ of $\mathrm{Sp}(2g,\mathbb{Z})$ of finite index is in fact a congruence subgroup and contains therefore a principal congruence subgroup $\Gamma(n)$ of finite index. However, the level n is not uniquely determined by Γ. In fact, if Γ contains $\Gamma(n)$ for some level n then Γ also contains any $\Gamma(m)$ whose level m is a multiple of n. Nevertheless, we can associate to any such group Γ an unique level n_Γ by taking the smallest integer n with the property that $\Gamma(n)$ is contained in Γ, i.e. by defining for any subgroup Γ of $\mathrm{Sp}(2g,\mathbb{Z})$

$$n_\Gamma := \min\{n;\ \Gamma(n) < \Gamma\}\ . \tag{3}$$

Throughout this chapter, whenever we consider a principal congruence subgroup $\Gamma(n)$ contained in a given group Γ, we will assume that the level n is minimal, i.e. $n = n_\Gamma$ unless stated explicitly otherwise. When looking at ramification we will replace Γ with the factor group $H = \Gamma/\Gamma(n)$ which can be considered as a subgroup of $\mathrm{Sp}(2g,\mathbb{Z}/n\mathbb{Z})$. Note that if $n = p$ is a prime the situation becomes much simpler. Not only do the elements of the group $H < \mathrm{Sp}(2g,\mathbb{Z}/n\mathbb{Z})$ now have coefficients in the field $\mathbb{Z}/p\mathbb{Z}$, but also, as a consequence, the ramification group $\mathrm{Ram}_H(v)$ for any primitive vector $v \in (\mathbb{Z}/p\mathbb{Z})^{2g}$ is either trivial or equal to the full group $\mathrm{Ram}_G(v) \cong \mathbb{Z}/p\mathbb{Z}$. We therefore wish to reduce the problem to the case where Γ contains a principal congruence subgroup $\Gamma(p)$ of prime level p.

We will later see in Section 4.4 how the result for arbitrary n follows from the corresponding result for when $n = p^t$ is a prime power. In this section we will show that in turn the $n = p^t$–case can be reduced to the case where $n = p$ is a prime. This will be done by showing that any subgroup Γ which contains $\Gamma(p^t)$ and has ramification mean $\varepsilon > 0$ contains in fact $\Gamma(p)$, provided p is sufficiently large in relation to ε^{-1}. For smaller p we will at least be able to give an upper bound for t. This reduces the problem effectively to the $n = p$–case except for finitely many cases.

We will start with the reduction from prime powers $n = p^t$ to primes $n = p$ for sufficiently big p.

Proposition 4.4 *Let $\Gamma \lneq \mathrm{Sp}(2g,\mathbb{Z})$ s.t. $n_\Gamma = p^t$ for some t and some prime p. Let $\varepsilon > 0$ denote the ramification mean of Γ. If $p > \max(3\varepsilon^{-1}, 2)$ then $t = 1$.*

Proof. Let $\Gamma \lneq \mathrm{Sp}(2g,\mathbb{Z})$ be arbitrary with $n_\Gamma = p^t$. i.e. $\Gamma(p^t) < \Gamma$, but $\Gamma(m) \not< \Gamma$ for all $m < p^t$, in particular $\Gamma(p^{t-1}) \not< \Gamma$. Assume further that Γ has ramification mean $\varepsilon > 0$ satisfying $p > \max(3\varepsilon^{-1}, 2)$. As usual we denote by H the quotient $\Gamma/\Gamma(p^t)$. Assume that $t > 1$. Consider the quotient

$$\Gamma(p^{t-1})/\Gamma(p^t) < \Gamma(1)/\Gamma(p^t) \cong \mathrm{Sp}(2g,\mathbb{Z}/p^t\mathbb{Z})$$

which allows us to think of this quotient as a subgroup of $\mathrm{Sp}(2g, \mathbb{Z}/p^t\mathbb{Z})$. More precisely, under this identification $\Gamma(p^{t-1})/\Gamma(p^t)$ is given as the kernel of the map from $\mathrm{Sp}(2g, \mathbb{Z}/p^t\mathbb{Z})$ to $\mathrm{Sp}(2g, \mathbb{Z}/p^{t-1}\mathbb{Z})$ given by reduction modulo p^{t-1}. A short calculation shows that this kernel is abelian and of order $p^{g(2g+1)}$, hence

$$\Gamma(p^{t-1})/\Gamma(p^t) \cong \ker\Big(\mathrm{Sp}(2g, \mathbb{Z}/p^t\mathbb{Z}) \to \mathrm{Sp}(2g, \mathbb{Z}/p^{t-1}\mathbb{Z})\Big) \cong (\mathbb{Z}/p\mathbb{Z})^{g(2g+1)} . \quad (4)$$

Now consider a primitive vector $v = (v_1, \ldots, v_{2g}) \in (\mathbb{Z}/p^t\mathbb{Z})^{2g}$. For each v we obtain a transvection $r_{v,p^{t-1}}$ as defined in (1) which can be thought of as an element in $\Gamma(p^{t-1})/\Gamma(p^t) \cong (\mathbb{Z}/p\mathbb{Z})^{g(2g+1)}$. This defines a map from the set of primitive vectors in $(\mathbb{Z}/p^t\mathbb{Z})^{2g}$ to $(\mathbb{Z}/p\mathbb{Z})^{g(2g+1)}$ which can be written in an appropriate basis as

$$(v_1, \ldots, v_{2g}) \mapsto (v_j \cdot v_k)_{\substack{j=1,\ldots,2g \\ k=j,\ldots,2g}} \pmod p . \quad (5)$$

Since $\Gamma(p^{t-1}) \not< \Gamma$ the quotient $\Gamma \cap \Gamma(p^{t-1})/\Gamma(p^t)$ defines a proper subspace of $(\mathbb{Z}/p\mathbb{Z})^{g(2g+1)}$, i.e. it is contained in some subspace isomorphic to $(\mathbb{Z}/p\mathbb{Z})^{g(2g+1)-1}$, which is given by a linear relation in the $g(2g+1)$ coordinates. Using (5) this can be considered as a quadratic relation in the v_i. Note that we have $\mathrm{ram}_H(v) = 1/p^t$ for all primitive $v = (v_1, \ldots, v_{2g})$ not satisfying this quadratic relation. We will now estimate the number of primitive vectors satisfying the relation and thus obtain an upper bound for the ramification mean of Γ.

By completing the square we can diagonalize the quadratic relation for $p > 2$. In fact, all we need is to write it as

$$\widetilde{v}_1^2 \equiv P(v_2, \ldots, v_{2g}) \pmod p ,$$

where P is some quadratic polynomial in $v_2, \ldots, v_{2g}$ and $\widetilde{v}_1$ is obtained from v_1 by adding a linear combination in $v_2, \ldots, v_{2g}$ (w.l.o.g. we can assume that the coefficient in front of v_1^2 in the relation is nonzero). This equation in $\mathbb{Z}/p\mathbb{Z}$ has at most $2p^{2g-1}$ solutions $(v_1, \ldots, v_{2g}) \in (\mathbb{Z}/p\mathbb{Z})^{2g}$. Since every coordinate $v_i \pmod p$ gives rise to p^{t-1} different $v_i \in \mathbb{Z}/p^t\mathbb{Z}$, we have at most $2p^{2gt-1}$ primitive vectors $(v_1, \ldots, v_{2g}) \in (\mathbb{Z}/p^t\mathbb{Z})^{2g}$ satisfying this quadratic relation. We obtain for the ramification mean of Γ that

$$\varepsilon \cdot (\#v) = \sum_v \mathrm{ram}_H(v) \le 2p^{2gt-1} \cdot 1 + \Big((\#v) - 2p^{2gt-1}\Big) \cdot \frac{1}{p^t} .$$

where $(\#v)$ denotes the number of primitive vectors. If $\varepsilon > 3/p$ this can be rewritten as

$$(\#v) \le 2p^{2gt-1} \frac{p^t - 1}{3p^{t-1} - 1} ,$$

which is strictly less than $(4/5)p^{2gt}$ provided $p > 2$. On the other hand, the number of primitive vectors in $(\mathbb{Z}/p^t\mathbb{Z})^{2g}$ as calculated in Lemma 3.4 is given by $p^{2gt}(1 - p^{-2g})$ which gives the desired contradiction. □

The previous result allows us for sufficiently large p to consider only the case where $\Gamma(p) < \Gamma$, i.e. the $t = 1$ case. We will now establish a bound on t for all primes p.

Proposition 4.5 *Let $\Gamma \lneq \mathrm{Sp}(2g, \mathbb{Z})$ s.t. $n_\Gamma = p^t$ for some prime p and some t. Let $\varepsilon > 0$ denote the ramification mean of Γ.*

(i) If $p > 2$ and $\varepsilon > \frac{7}{3} p^{\frac{1}{4}(1-t)}$, then $t = 1$.

(ii) If $p = 2$ and $\varepsilon > \frac{26}{3} \cdot 2^{\frac{1}{4}(1-t))}$, then $t < 4$.

Proof. We will show (i) and (ii) simultaneously. The proof follows the ideas of the proof of Proposition 4.4.

Fix a prime $p \geq 2$. Let $\Gamma(p^t) < \Gamma \lneq \mathrm{Sp}(2g, \mathbb{Z})$ be arbitrary with ramification mean $\varepsilon > \frac{7}{3} p^{\frac{1}{4}(1-t)}$ ($\varepsilon > \frac{26}{3} \cdot 2^{\frac{1}{4}(1-t)}$ if $p = 2$) and $n_\Gamma = p^t$, i.e. $\Gamma(p^t)$ is the principal congruence subgroup with the smallest level contained in Γ.

We assume that $t \geq 2$ ($t \geq 4$ if $p = 2$) and will show that this leads to a contradiction. Let $s := \lceil \frac{t}{2} \rceil$. Then $s < t$ and consequently $\Gamma(p^s) \not< \Gamma$. As before the quotient $\Gamma(p^s)/\Gamma(p^t)$ can be identified with the kernel of the map from $\mathrm{Sp}(2g, \mathbb{Z}/p^t\mathbb{Z})$ to $\mathrm{Sp}(2g, \mathbb{Z}/p^s\mathbb{Z})$ given by reduction modulo p^s. It is easy to check that $\Gamma(p^t)$ contains the commutator subgroup of $\Gamma(p^s)$ provided $s \geq \lceil \frac{t}{2} \rceil$ which implies that this kernel is abelian. By calculations similar to those in (4) we can conclude that

$$\Gamma(p^s)/\Gamma(p^t) \cong \ker\Big(\mathrm{Sp}(2g, \mathbb{Z}/p^t\mathbb{Z}) \to \mathrm{Sp}(2g, \mathbb{Z}/p^s\mathbb{Z})\Big) \cong (\mathbb{Z}/p^{t-s}\mathbb{Z})^{g(2g+1)} . \quad (6)$$

We can proceed as in the proof of Proposition 4.4 where we have to pay attention to the fact that $(\mathbb{Z}/p^{t-s}\mathbb{Z})^{g(2g+1)}$ is now in general no longer a vector space but a free module over $\mathbb{Z}/p^{t-s}\mathbb{Z}$. We can define a map from the set of primitive vectors in $(\mathbb{Z}/p^t\mathbb{Z})^{2g}$ to $(\mathbb{Z}/p^{t-s}\mathbb{Z})^{g(2g+1)}$ by sending $v = (v_1, \ldots, v_{2g}) \in (\mathbb{Z}/p^t\mathbb{Z})^{2g}$ to the transvection $r_{v,p^s} \in \Gamma(p^s)/\Gamma(p^t) \cong (\mathbb{Z}/p^{t-s}\mathbb{Z})^{g(2g+1)}$. As before we can find a suitable basis such that this map is given by

$$(v_1, \ldots, v_{2g}) \mapsto (v_j \cdot v_k)_{\substack{j=1,\ldots,2g \\ k=j,\ldots,2g}} \pmod{p^{t-s}} . \quad (7)$$

Since $\Gamma(p^s) \not< \Gamma$ the quotient $\Gamma \cap \Gamma(p^s)/\Gamma(p^t)$ defines a proper submodule of $(\mathbb{Z}/p^{t-s}\mathbb{Z})^{g(2g+1)}$. So it must be contained in a submodule given by one linear relation in the $g(2g+1)$ coordinates. Note that there is at least one coefficient in this linear relation which is not divisible by p. In fact, if all the coefficients were divisible by p, all the elements of the group $\Gamma(p^{t-1})/\Gamma(p^t) < \Gamma(p^s)/\Gamma(p^t) \cong (\mathbb{Z}/p^{t-s}\mathbb{Z})^{g(2g+1)}$ would satisfy this relation, which would imply that $\Gamma(p^{t-1}) < \Gamma$ which contradicts the minimality of t. In terms of (7) this relation can be rewritten as a quadratic relation in the v_i. Since all the primitive vectors $v \in$

$(\mathbb{Z}/p^t\mathbb{Z})^{2g}$ not satisfying this relation have $\mathrm{ram}_H(v) = 1/p^t$, we can get a bound for the ramification mean of Γ by estimating the number of solutions of this relation. Since this number depends on the question whether 2 is a zero divisor in $\mathbb{Z}/p^{t-s}\mathbb{Z}$ or not, we have to distinguish the following two cases:

If $p > 2$ we can complete the square to obtain a relation of the following type

$$\widetilde{v}_1^2 \equiv P(v_2, \ldots, v_{2g}) \pmod{p^{t-s}}\ , \tag{8}$$

where P is some quadratic polynomial in $v_2, \ldots, v_{2g}$ and $\widetilde{v}_1$ is obtained from v_1 by adding a suitable linear combination in the other coordinates (we might have to change the order of the v_i to ensure that the coefficient of v_1^2 is a unit).

Given $a \in \mathbb{Z}/p^{t-s}\mathbb{Z}$ the number of solutions to the equation $x^2 \equiv a \pmod{p^{t-s}}$ is at most 2 if $a \neq 0$ and $p^{\lfloor \frac{t-s}{2} \rfloor}$ if $a = 0$. In any case there are at most $2p^{\lfloor \frac{t-s}{2} \rfloor}$ solutions. So we have at most $2p^{\lfloor \frac{t-s}{2} \rfloor} \cdot p^{(2g-1)(t-s)}$ solutions for the relation (8) which gives us at most $2p^{\lfloor \frac{t-s}{2} \rfloor} \cdot p^{(2g-1)t+s}$ primitive vectors $v = (v_1, \ldots, v_{2g}) \in (\mathbb{Z}/p^t\mathbb{Z})^{2g}$ satisfying this quadratic relation.

Using that $\lfloor \frac{t-s}{2} \rfloor + s \leq \frac{3}{4}t + \frac{1}{4}$ we can further estimate the number of solutions to be no more than $2p^{(2g-\frac{1}{4})t+\frac{1}{4}}$. We can now conclude that we have for the ramification mean

$$\varepsilon \cdot (\#v) = \sum_v \mathrm{ram}_H(v) \leq 2p^{(2g-\frac{1}{4})t+\frac{1}{4}} \cdot 1 + \left((\#v) - 2p^{(2g-\frac{1}{4})t+\frac{1}{4}}\right) \cdot \frac{1}{p^t}$$

where $(\#v)$ denotes the number of primitive vectors. This is equivalent to

$$(\#v) \leq 2p^{(2g-\frac{1}{4})t+\frac{1}{4}} \frac{p^t - 1}{\varepsilon p^t - 1}$$

provided $\varepsilon p^t - 1 > 0$. Since we have $\varepsilon > \frac{7}{3}p^{\frac{1}{4}(1-t)}$ this is indeed the case, furthermore, a short calculation gives us that

$$(\#v) < 2p^{(2g-\frac{1}{4})t+\frac{1}{4}} \cdot \frac{63}{128} p^{\frac{1}{4}(t-1)} = \frac{63}{64} p^{2gt}\ .$$

Comparing this with the number of primitive vectors in $(\mathbb{Z}/p^t\mathbb{Z})^{2g}$, which is given by $p^{2gt} - p^{2gt-2g}$ as we calculated in Lemma 3.4, this gives the desired contradiction for $p > 2$.

If $p = 2$ we can again assume w.l.o.g. that the coefficient of v_1^2 is 1. However to complete the square, we need that the coefficients of $v_1 v_i\ , i = 2, \ldots, 2g$ are divisible by 2. To ensure this, we multiply the relation by 2. Since by assumption $t \geq 4$ and thus $t - s \geq 2$, this still is a nontrivial quadratic relation, only the number of primitive vectors satisfying the relation has possibly increased. We now complete the square and obtain

$$2\widetilde{v}_1^2 \equiv P(v_2, \ldots, v_{2g}) \pmod{2^{t-s}}\ , \tag{9}$$

where P is again some quadratic polynomial and $\widetilde{v}_1$ is the sum of v_1 with a linear combination in $v_2, \ldots, v_{2g}$.

As above we can show that the number of solutions to the quadratic equation $2x^2 \equiv a \pmod{2^{t-s}}$ is at most $4 \cdot 2^{\lfloor \frac{t-s}{2} \rfloor}$. Consequently there are at most $4 \cdot 2^{\lfloor \frac{t-s}{2} \rfloor} \cdot 2^{(2g-1)t+s}$ primitive vectors $v = (v_1, \ldots, v_{2g}) \in (\mathbb{Z}/2^t\mathbb{Z})^{2g}$ satisfying the quadratic relation in this case.

We can now proceed analogously to the $p > 2$–case to estimate the ramification mean and thus get an upper bound for the number of primitive vectors which then gives a contradiction as desired. □

We will need a bound on t which is valid for all $\varepsilon > 0$, not only for sufficiently large ε. Thus we rephrase the statement of this Proposition to obtain an ε-dependent bound on t for all $\varepsilon > 0$.

Corollary 4.6 *Let $\Gamma \lneq \mathrm{Sp}(2g, \mathbb{Z})$ be a subgroup with $n_\Gamma = p^t$ for some prime $p \geq 2$ and some integer t and let $\varepsilon > 0$ denote its ramification mean.*

(i) If $p > 2$ then $t \leq 1 - 4\log_p(\frac{3}{7}\varepsilon)$.

(ii) If $p = 2$ then $t \leq \max(1 - 4\log_2(\frac{3}{26}\varepsilon), 3)$.

Proof.

(i) Assume that $t > 1 - 4\log_p(\frac{3}{7}\varepsilon)$. Note that since $0 < \varepsilon \leq 1$, we have that $\log_p(\frac{3}{7}\varepsilon) < 0$, so in particular $t > 1$. On the other hand our assumption is equivalent to $\varepsilon > \frac{7}{3}p^{\frac{1}{4}(1-t)}$ which implies $t = 1$ by the above proposition.

(ii) The argument here is completely analogous to the case (i).

□

Another Corollary recovers the result of Proposition 4.4, although with a somewhat weaker bound.

Corollary 4.7 *Let $\Gamma \lneq \mathrm{Sp}(2g, \mathbb{Z})$ s.t. $n_\Gamma = p^t$ for some t and some prime p and let $\varepsilon > 0$ denote the ramification mean of Γ. If $p > \max(\frac{7}{3}\varepsilon^{-4}, 2)$ then $t = 1$.*

4.3 The n = p–case

Recall that $\mathrm{Sp}(2g, \mathbb{Z})$ is finitely generated (cf. [Fre, Anhang V]). Hence it has only finitely many subgroups of a given index which means that the finiteness statement of Theorem 4.3 is equivalent to giving for every $\varepsilon > 0$ a bound on the index of subgroups $\Gamma < \mathrm{Sp}(2g, \mathbb{Z})$ with ramification mean at least ε. In this section we will prove this version of the theorem in the case where $n = p$ is a prime. Together with the reduction steps of the previous section the corresponding statement for prime powers $n = p^t$ will then be an easy consequence.

Lemma 4.8 *Let $\Gamma < \mathrm{Sp}(2g, \mathbb{Z})$ be a subgroup containing $\Gamma(p)$ for some prime p. Let $\varepsilon > 0$ denote the ramification mean of Γ. If $p \geq \max(3, 8 \cdot \left(\frac{8}{7}\right)^{g-1} \varepsilon^{-1})$ then the index of Γ in $\mathrm{Sp}(2g, \mathbb{Z})$ is at most $\frac{4^{g-1} \cdot 8}{3} \cdot \left(\frac{8}{7}\right)^{\frac{g(g-1)}{2}} \varepsilon^{-g}$.*

To prove this lemma we will proceed by induction on g. Since the proof is rather involved, we will split it up into several parts. In the case where $g = 1$, the symplectic group $\mathrm{Sp}(2, \mathbb{Z})$ coincides with the special linear group $\mathrm{SL}(2, \mathbb{Z})$. The statement we are trying to prove is then given by the following proposition:

Proposition 4.9 *Let $\Gamma < \mathrm{SL}(2, \mathbb{Z})$ contain $\Gamma(p)$ for some prime p. If $\varepsilon > 0$ denotes the ramification mean of Γ and $p \geq \max(3, 8\varepsilon^{-1})$ then Γ has index at most $\frac{8}{3}\varepsilon^{-1}$ in $\mathrm{SL}(2, \mathbb{Z})$.*

Proof. We identify the quotient $\mathrm{SL}(2, \mathbb{Z})/\Gamma(p)$ with $\mathrm{SL}(2, \mathbb{Z}/p\mathbb{Z})$ and thus can consider $H := \Gamma/\Gamma(p)$ as a subgroup of the finite group $\mathrm{SL}(2, \mathbb{Z}/p\mathbb{Z})$. Note that the index of H in $\mathrm{SL}(2, \mathbb{Z}/p\mathbb{Z})$ is just the same as the index of Γ in $\mathrm{SL}(2, \mathbb{Z})$, the one we want to bound.

Let $p \geq \max(3, 8\varepsilon^{-1})$. Consider the space $V := (\mathbb{Z}/p\mathbb{Z})^2$. For any subset $W \subset V$ we define the ramification mean of W with respect to H as

$$\mathrm{rammean}_H(W) := \frac{1}{(\#W^*)} \sum_{w \in W^*} \mathrm{ram}_H(w) , \tag{10}$$

where $W^* := W \setminus \{0\}$.

Consider all subgroups $W_1 \subset V$ with $W_1 \cong \mathbb{Z}/p\mathbb{Z}$. Since the action of $\mathrm{SL}(2, \mathbb{Z}/p\mathbb{Z})$ on V^* is transitive, all primitive vectors $v \in V^*$ appear in the same number of subgroups. This implies that

$$\begin{aligned}
\varepsilon &= \frac{1}{\#v} \sum_v \mathrm{ram}_H(v) \\
&= \frac{1}{\#\{W_1 \subset V\,;\ W_1 \cong \mathbb{Z}/p\mathbb{Z}\}} \sum_{\substack{W_1 \subset V\,, \\ W_1 \cong \mathbb{Z}/p\mathbb{Z}}} \frac{1}{(\#W_1^*)} \sum_{w \in W_1^*} \mathrm{ram}_H(w) \\
&= \frac{1}{\#\{W_1 \subset V\,;\ W_1 \cong \mathbb{Z}/p\mathbb{Z}\}} \sum_{\substack{W_1 \subset V\,, \\ W_1 \cong \mathbb{Z}/p\mathbb{Z}}} \mathrm{rammean}_H(W_1) .
\end{aligned}$$

Hence there must be one such W_1 with $\mathrm{rammean}_H(W_1) \geq \varepsilon$. W.l.o.g. we can assume that $\mathrm{rammean}_H(V_1) \geq \varepsilon$, where $V_1 = (*, 0) \subset V$. Indeed, by the transitivity of the action of $\mathrm{SL}(2, \mathbb{Z}/p\mathbb{Z})$ on V^*, we can conclude that all subgroups W_1 are conjugate with respect to this action. Since replacing H by a conjugate in $\mathrm{SL}(2, \mathbb{Z}/p\mathbb{Z})$ does neither change the ramification mean nor does it change the index, we can replace H by a suitable conjugate to obtain the desired property for V_1.

Note that the condition on p given by $p \geq 8\varepsilon^{-1}$ can be rewritten as $\varepsilon \geq 8/p$, so $\mathrm{rammean}_H(V_1) \geq 8/p$. Since p is a prime we have for any primitive vector $v \in V_1$ that either $\mathrm{ram}_H(v) = 1/p$ or $\mathrm{ram}_H(v) = 1$. Thus there must be at least one primitive vector $v \in V_1$ with $\mathrm{ram}_H(v) = 1$. The corresponding transvection $r_{v,1} \in \mathrm{SL}(2, \mathbb{Z}/p\mathbb{Z}) \cap H$ generates the group

$$G_{V_1} = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \; ; \; b \in \mathbb{Z}/p\mathbb{Z} \right\} < H \; .$$

Now consider two primitive elements $v = (v_1, v_2), \widetilde{v} = (\widetilde{v_1}, \widetilde{v_2}) \in V^*$ with $v_2, \widetilde{v_2} \neq 0$ and the transvections

$$r_{v,1} = \begin{pmatrix} 1 + v_1 v_2 & -v_1^2 \\ v_2^2 & 1 - v_1 v_2 \end{pmatrix} \quad \text{and} \quad r_{\widetilde{v},1} = \begin{pmatrix} 1 + \widetilde{v_1}\widetilde{v_2} & -\widetilde{v_1}^2 \\ \widetilde{v_2}^2 & 1 - \widetilde{v_1}\widetilde{v_2} \end{pmatrix} \; .$$

Consider the cosets of $r_{v,1}$ and $r_{\widetilde{v},1}$ with respect to G_{V_1}:

$$\begin{pmatrix} 1 + v_1 v_2 & -v_1^2 \\ v_2^2 & 1 - v_1 v_2 \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 + v_1 v_2 & b(1 + v_1 v_2) - v_1^2 \\ v_2^2 & b v_2^2 + (1 - v_1 v_2) \end{pmatrix}$$

and likewise for $r_{\widetilde{v},1}$. Note that v and $\widetilde{v}$ can only lie in the same coset if $v_2^2 = \widetilde{v_2}^2$. Once we have chosen $v_2 \neq 0$ ($\widetilde{v_2} \neq 0$) the other coordinate v_1 (resp. $\widetilde{v_1}$) is uniquely determined. This shows that at most two primitive vectors $v = (v_1, v_2) \in V^*$ with $v_2 \neq 0$ can lie in the same coset.

We will now estimate how many primitive vectors $v \in V^*$ we have with $r_{v,1} \in H$, i.e. which satisfy $\mathrm{ram}_H(v) = 1$. Since $\mathrm{ram}_H(v) \in \{1/p, 1\}$ this is equivalent to asking how many $v \in V^*$ we are guaranteed to have with $\mathrm{ram}_H(v) > 1/p$. We are thus in the situation of Proposition $B.1$ from the appendix which tells us that this number which we denote by γ is at least

$$\gamma \geq \frac{\varepsilon - (1/p)}{1 - (1/p)}(p^2 - 1) = (p\varepsilon - 1)(p + 1) = \varepsilon p^2 + (\varepsilon - 1)p - 1 \; . \tag{11}$$

Using that $\varepsilon \geq 8/p$ we can further estimate that

$$\gamma \geq \frac{3}{4}\varepsilon p^2 + \underbrace{\frac{1}{4}\varepsilon p^2}_{\geq 2p} + (\varepsilon - 1)p - 1 \geq \frac{3}{4}\varepsilon p^2 + (p - 1) \; . \tag{12}$$

Since there are only $p-1$ primitive vectors $v=(v_1,v_2)\in V^*$ with $v_2=0$, we can conclude that we have at least $(3/4)\varepsilon p^2$ primitive vectors $v=(v_1,v_2)\in V^*$ with $v_2\neq 0$ such that $r_{v,1}\in H$. By our above considerations these give at least $(3/8)\varepsilon p^2$ elements in $(3/8)\varepsilon p^2$ different cosets with respect to G_{V_1}. Hence H has at least $(3/8)\varepsilon p^3$ elements. For the index of H in $\mathrm{SL}(2,\mathbb{Z}/p\mathbb{Z})$ this gives us

$$[\mathrm{SL}(2,\mathbb{Z}/p\mathbb{Z}):H]\leq\frac{p(p^2-1)}{(3/8)\varepsilon p^3}<\frac{8}{3}\varepsilon^{-1}$$

as claimed. □

Proof of Lemma 4.8. With the case $g=1$ settled by Proposition 4.9, we will now do the induction step from $g-1$ to g. This part requires a number of technical results which we will put into several individual claims to be shown at the end of this proof.

We will again write H for the quotient $\Gamma/\Gamma(p)$ which we consider as a subgroup of $G:=\mathrm{Sp}(2g,\mathbb{Z}/p\mathbb{Z})$. We need to bound the index of H in G. Consider the space $V=(\mathbb{Z}/p\mathbb{Z})^{2g}$. As in (10) we define for any subset $W\subset V$ the ramification mean of W with respect to H as follows:

$$\mathrm{rammean}_H(W):=\frac{1}{(\#W^*)}\sum_{w\in W^*}\mathrm{ram}_H(w)$$

Using the transitivity of the action of G on V^* we can conclude as in the case $g=1$ that among all subgroups $W_{2g-1}\subset V$ which are isomorphic to $(\mathbb{Z}/p\mathbb{Z})^{2g-1}$ there must be at least one with ramification mean greater than or equal to ε. W.l.o.g. we can assume that this is the case for the subgroup $V_{2g-1}=(*,\ldots,*,0)\subset V$ (by replacing H with a suitable conjugate if necessary), that is

$$\mathrm{rammean}_H(V_{2g-1})\geq\varepsilon\ . \tag{13}$$

The stabilizer of V_{2g-1} in G can be computed to be given by

$$S_{2g-1}:=\left\{\left(\begin{array}{cc:cc}A&0&B&m_3\\ m_1^T&u&m_2^T&m_4\\ \hdashline C&0&D&m_5\\ 0&0&0&u^{-1}\end{array}\right);\ \begin{pmatrix}A&B\\C&D\end{pmatrix}\in\mathrm{Sp}(2g-2,\mathbb{Z}/p\mathbb{Z}),\right.$$
$$\left.\begin{array}{l}u\in(\mathbb{Z}/p\mathbb{Z})^*,m_1,m_2,m_3,m_5\in(\mathbb{Z}/p\mathbb{Z})^{g-1},m_4\in\mathbb{Z}/p\mathbb{Z},\\ A\cdot m_2-B\cdot m_1=u\cdot m_3\,,\\ C\cdot m_2-D\cdot m_1=u\cdot m_5\end{array}\right\}. \tag{14}$$

We will need various results about S_{2g-1} which can be found in the appendix (cf. Section A).

Consider the set of subgroups $W_g \subset V_{2g-1}$ which are isomorphic to $(\mathbb{Z}/p\mathbb{Z})^g$. The group S_{2g-1} acts on this set, however, this action is not transitive. We want to conclude that we have w.l.o.g. $\mathrm{rammean}_H(V_g) \geq \varepsilon/2$, where

$$V_g := (\underbrace{*, \ldots, *}_{g \text{ times}}, \underbrace{0, \ldots, 0}_{g \text{ times}}) \subset V \ .$$

To achieve this, we restrict ourselves to subgroups $W_g \subset V_{2g-1}$ which are in the orbit of V_g under the action of S_{2g-1}.

Claim 1: There exists a subgroup $W_g \in \mathrm{orb}_{S_{2g-1}}(V_g)$ with $\mathrm{rammean}_H(W_g) \geq \varepsilon/2$.

As mentioned earlier, we postpone the proof of this technical result and continue with the proof of the lemma. By using conjugation we can again assume w.l.o.g. that we have $\mathrm{rammean}_H(V_g) \geq \varepsilon/2$. Note that since we are conjugating with $S_{2g-1} = \mathrm{Stab}_G(V_{2g-1})$ this leaves V_{2g-1} invariant.

Primitive vectors $v \in V_g^*$ correspond to transvections $r_{v,1}$ which lie in

$$G_{V_g} := \left\{ \begin{pmatrix} \mathbb{1} & B \\ 0 & \mathbb{1} \end{pmatrix} \ ; \ B \in \mathrm{Sym}(g, \mathbb{Z}/p\mathbb{Z}) \right\} < \mathrm{Sp}(2g, \mathbb{Z}/p\mathbb{Z}) \ .$$

Note that transvections $r_{v,1}$ given by a primitive vectors $v \in V_g^*$ with $\mathrm{ram}_H(v) = 1$ are not only elements of G_{V_g} but also of H. The subgroup generated by them is in fact all of G_{V_g} provided the ramification mean of V_g is sufficiently big as the following claim shows.

Claim 2: If $\mathrm{rammean}_H(V_g) \geq \varepsilon/2$ then $G_{V_g} < H$.

Our next goal is to show that most of the elements in S_{2g-1} are in fact in H, i.e. to get a bound on the index of $H \cap S_{2g-1}$ in S_{2g-1}. We have just found some elements in $H \cap S_{2g-1}$, namely all the elements of G_{V_g}. They will play an important role in getting an estimate for the index.

Recall the description of S_{2g-1} given in (14). We define a surjective group homomorphism $\Psi : S_{2g-1} \to \mathrm{Sp}(2g-2, \mathbb{Z}/p\mathbb{Z})$ by

$$\left(\begin{array}{cc:cc} A & 0 & B & m_3 \\ m_1^T & u & m_2^T & m_4 \\ \hdashline C & 0 & D & m_5 \\ 0 & 0 & 0 & u^{-1} \end{array} \right) \mapsto \begin{pmatrix} A & B \\ C & D \end{pmatrix} \ .$$

We will estimate the index of $\mathrm{Im}(\Psi|_{H \cap S_{2g-1}})$ in $\mathrm{Im}(\Psi) = \mathrm{Sp}(2g-2, \mathbb{Z}/p\mathbb{Z})$ and the index of $\ker(\Psi|_{H \cap S_{2g-1}})$ in $\ker(\Psi)$ and thus get a bound for the index of $H \cap S_{2g-1}$ in S_{2g-1}. These calculations require some tedious counting arguments and matrix computations. We therefore state the results here and show them together with the other claims at the end of this proof.

Claim 3: $\left[\mathrm{Sp}(2g-2,\mathbb{Z}/p\mathbb{Z}) : \mathrm{Im}(\Psi|_{H\cap S_{2g-1}})\right] \le \frac{4^{g-2}\cdot 8}{3}\cdot\left(\frac{8}{7}\right)^{\frac{g(g-1)}{2}}\varepsilon^{1-g}$

Claim 4: $\ker(\Psi) < H$

With these results we can estimate the index of $S_{2g-1}\cap H$ in S_{2g-1} as follows:

$$\begin{aligned}
[S_{2g-1} : S_{2g-1}\cap H] &= \frac{|\mathrm{Im}(\Psi)|\,/\,|\ker(\Psi)|}{\left|\mathrm{Im}(\Psi|_{H\cap S_{2g-1}})\right|\,/\,\left|\ker(\Psi)|_{H\cap S_{2g-1}}\right|} \\
&\overset{\text{Claim 4}}{=} \frac{|\mathrm{Im}(\Psi)|}{\left|\mathrm{Im}(\Psi|_{H\cap S_{2g-1}})\right|} = \left[\mathrm{Sp}(2g-2,\mathbb{Z}/p\mathbb{Z}) : \mathrm{Im}(\Psi|_{H\cap S_{2g-1}})\right] \\
&\overset{\text{Claim 3}}{\le} \frac{4^{g-2}\cdot 8}{3}\cdot\left(\frac{8}{7}\right)^{\frac{g(g-1)}{2}}\varepsilon^{1-g}\ . \qquad (15)
\end{aligned}$$

As our final step in this proof we will now estimate the index of H in $\mathrm{Sp}(2g,\mathbb{Z}/p\mathbb{Z})$. Consider primitive vectors $v = (v_1,\dots,v_{2g}) \in V^*$ with $v_{2g}\neq 0$ and the corresponding transvections $r_{v,1}$. We want to know how many of these can lie in the same coset of S_{2g-1} in G.

Let e_g denote the g-th vector of the canonical basis of $V = (\mathbb{Z}/p\mathbb{Z})^{2g}$. Note that the action of S_{2g-1} given by left multiplication leaves the subspace generated by e_g invariant whereas $r_{v,1}$ maps e_g to

$$e_g + v_{2g}\cdot(v_1,\dots,v_{2g}) = (v_{2g}\cdot v_1,\dots,v_{2g}\cdot v_{g-1}, 1+v_{2g}\cdot v_g, v_{2g}\cdot v_{g+1},\dots,v_{2g}^2)\ .$$

So there are at most $2p$ primitive vectors v with $v_{2g}\neq 0$ that lie in the same coset of S_{2g-1}.

Using that $\mathrm{rammean}(V)=\varepsilon$ a short calculation as in (11) and (12) using Proposition B.1 shows that we have at least $(3/4)\varepsilon p^{2g}+p^{2g-1}$ primitive vectors $v\in V^*$ with $\mathrm{ram}_H(v)=1$ and hence $r_{v,1}\in H$. At least $(3/4)\varepsilon p^{2g}$ of them have $v_{2g}\neq 0$, so they lie in at least $(3/8)\varepsilon p^{2g-1}$ different cosets of S_{2g-1}. This implies that

$$[\mathrm{Sp}(2g,\mathbb{Z}/p\mathbb{Z}) : H] = \frac{|\mathrm{Sp}(2g,\mathbb{Z}/p\mathbb{Z})|}{|H|} \le \frac{|\mathrm{Sp}(2g,\mathbb{Z}/p\mathbb{Z})|}{|H\cap S_{2g-1}|\cdot(3/8)\varepsilon p^{2g-1}}\ .$$

Using Proposition A.1 we obtain

$$\begin{aligned}
[\mathrm{Sp}(2g,\mathbb{Z}/p\mathbb{Z}) : H] &\le \frac{|\mathrm{Sp}(2g,\mathbb{Z}/p\mathbb{Z})|}{|H\cap S_{2g-1}|\cdot(3/8)\varepsilon p^{2g-1}} \\
&\overset{A.1}{=} \frac{((p^{2g}-1)/(p-1))\cdot|S_{2g-1}|}{|H\cap S_{2g-1}|\cdot(3/8)\varepsilon p^{2g-1}} \\
&\le \frac{4}{\varepsilon}\cdot[S_{2g-1} : H\cap S_{2g-1}] \overset{(15)}{\le} \frac{4^{g-1}\cdot 8}{3}\cdot\left(\frac{8}{7}\right)^{\frac{g(g-1)}{2}}\varepsilon^{-g}
\end{aligned}$$

which completes the proof of the lemma.

We will now prove the technical results we used in this proof.

Claim 1: There exists a subgroup $W_g \in \mathrm{orb}_{S_{2g-1}}(V_g)$ with $\mathrm{rammean}_H(W_g) \geq \varepsilon/2$.

Proof of Claim 1: Assume not. Then we have for all W_g lying in $\mathrm{orb}_{S_{2g-1}}(V_g)$ that $\mathrm{rammean}_H(W_g) < \varepsilon/2$. This implies

$$\begin{aligned}\left|\mathrm{orb}_{S_{2g-1}}(V_g)\right| \cdot (\varepsilon/2) &> \sum_{W_g \in \mathrm{orb}_{S_{2g-1}}(V_g)} \mathrm{rammean}_H(W_g) \\ &= \frac{1}{p^g - 1} \sum_{W_g \in \mathrm{orb}_{S_{2g-1}}(V_g)} \sum_{w \in W_g^*} \mathrm{ram}_H(w) \ .\end{aligned}$$

Since every primitive vector $v \in V_{2g-1}^*$ appears at least $\prod_{i=1}^{g-2}(p^i+1)$ times in this double sum by Corollary A.5, we can conclude that

$$\left|\mathrm{orb}_{S_{2g-1}}(V_g)\right| \cdot (\varepsilon/2) > \frac{\prod_{i=1}^{g-2}(p^i+1)}{p^g-1} \sum_{v \in V_{2g-1}^*} \mathrm{ram}_H(v) \ .$$

Using that $\left|\mathrm{orb}_{S_{2g-1}}(V_g)\right| = \prod_{i=1}^{g-1}(p^i+1)$ as shown in Corollary A.3, we get that this implies that

$$\begin{aligned}(p^{g-1}+1) \cdot (\varepsilon/2) > \frac{1}{p^g-1} \sum_{v \in V_{2g-1}^*} \mathrm{ram}_H(v) &= \frac{p^{2g-1}-1}{p^g-1} \underbrace{\mathrm{rammean}_H(V_{2g-1})}_{\geq \varepsilon \text{ by (13)}} \\ \geq \frac{p^{2g-1}-1}{p^g-1} \cdot \varepsilon &\end{aligned}$$

which leads to a contradiction and thus proves the claim.

Claim 2: If $\mathrm{rammean}_H(V_g) \geq \varepsilon/2$ then $G_{V_g} < H$.

Proof of Claim 2: We will proceed by induction on g. The case $g = 1$ for the group G_{V_1} has been shown already in the proof of Proposition 4.9.

For the induction step let $g \geq 2$. Consider the subspace

$$V_{g-1}^g = (\underbrace{*, \ldots, *}_{g-1 \text{ times}}, \underbrace{0, \ldots, 0}_{g+1 \text{ times}}) \subset V \ .$$

Using that $\mathrm{rammean}_H(V_g) \geq \varepsilon/2$ we want to conclude that w.l.o.g. we have that $\mathrm{rammean}_H(V_{g-1}^g) \geq \varepsilon/2$. For that note that the stabilizer of V_g in G acts transitively on the set of all $v \in V_g^*$ which means that all such v appear in the same number of subgroups W_{g-1}^g isomorphic to $(\mathbb{Z}/p\mathbb{Z})^{g-1}$ of V_g. As before this implies that there must be at least one such subgroup W_{g-1}^g with

$\mathrm{rammean}_H(W^g_{g-1}) \geq \varepsilon/2$. Since the action of $\mathrm{Stab}_G(V_g)$ on V_g is transitive, all subgroups $W^g_{g-1} \cong (\mathbb{Z}/p\mathbb{Z})^{g-1}$ are conjugate in $\mathrm{Stab}_G(V_g)$, so we can assume w.l.o.g. that $\mathrm{rammean}_H(V^g_{g-1}) \geq \varepsilon/2$ by replacing H with a suitable conjugate. Note however, that we can not assume this outside the proof of this claim, since this conjugation might not leave V_{2g-1} invariant.

If we take a primitive vector $v \in V^g_{g-1}$ it defines a transvection $r_{v,1}$ which lies in

$$G_{V^g_{g-1}} := \left\{ \left(\begin{array}{c|c} \mathbb{1} & \begin{matrix} B & 0 \\ 0 & 0 \end{matrix} \\ \hline 0 & \mathbb{1} \end{array} \right) ; \ B \in \mathrm{Sym}(g-1, \mathbb{Z}/p\mathbb{Z}) \right\} .$$

We will now show that $G_{V^g_{g-1}}$ is contained in H. For that we use that $V^g_{g-1} \cong V_{g-1} \subset (\mathbb{Z}/p\mathbb{Z})^{2(g-1)}$ and that $G_{V^g_{g-1}} \cong G_{V_{g-1}} < \mathrm{Sp}(2(g-1), \mathbb{Z}/p\mathbb{Z})$. Under this isomorphism $H \cap G_{V^g_{g-1}}$ is identified with some group $\widetilde{H} < \mathrm{Sp}(2(g-1), \mathbb{Z}/p\mathbb{Z})$ and we have that $\mathrm{rammean}_H(V^g_{g-1}) = \mathrm{rammean}_{\widetilde{H}}(V_{g-1})$. Therefore we can conclude by the induction hypothesis that $G_{V_{g-1}} < \widetilde{H}$ which implies that $G_{V^g_{g-1}} < H$.

Define $\lambda : G_{V_g} \to (\mathbb{Z}/p\mathbb{Z})^g$ as the projection of $\begin{pmatrix} \mathbb{1} & B \\ 0 & \mathbb{1} \end{pmatrix}$ to the last row of B. We have just seen that $\ker(\lambda) = G_{V^g_{g-1}} < H$. If we can show that $\mathrm{Im}(\lambda|_{H \cap G_{V_g}}) = \mathrm{Im}(\lambda) = (\mathbb{Z}/p\mathbb{Z})^g$, we can conclude that $H \cap G_{V_g} = G_{V_g}$ which implies $G_{V_g} < H$.

We will do this by a counting argument using that $\mathrm{rammean}_H(V_g) \geq \varepsilon/2$. If γ denotes the number of primitive vectors $v \in V_g^*$ with $\mathrm{ram}_H(v) = 1$ (or equivalently with $\mathrm{ram}_H(v) > 1/p$), then we have by Proposition B.1 that

$$\gamma \geq \frac{(\varepsilon/2) - (1/p)}{1 - (1/p)} (p^g - 1) = ((\varepsilon/2)p - 1) \frac{p^g - 1}{p - 1} > ((\varepsilon/2)p - 1)\, p^{g-1} .$$

Using that $\varepsilon \geq (8/p) \cdot (\frac{8}{7})^{g-1} \geq (8/p)$ we can deduce that $\gamma > 3p^{g-1}$, which means that we have at least that many primitive vectors in $v \in V_g^*$ with $\mathrm{ram}_H(v) = 1$. In particular we are guaranteed to have $2p^{g-1} + 1$ such primitive vectors $v = (v_1, \ldots, v_g, 0, \ldots, 0)$ which additionally satisfy $v_g \neq 0$. Each of these v defines a transvection $r_{v,1} \in H \cap G_{V_g}$ and if we apply λ to it, we obtain

$$(-v_g) \cdot (v_1, \ldots, v_g) = (-v_1 v_g, -v_2 v_g, \ldots, -v_g^2) \in (\mathbb{Z}/p\mathbb{Z})^g .$$

Observe that there are at most two primitive vectors $v = (v_1, \ldots, v_g, 0, \ldots, 0)$ with $v_g \neq 0$ having the same image in $(\mathbb{Z}/p\mathbb{Z})^g$. So we have at least $p^{g-1} + 1/2$ different elements in $\mathrm{Im}(\lambda|_{H \cap G_{V_g}}) \subset (\mathbb{Z}/p\mathbb{Z})^g$. Since they generate a subgroup of $(\mathbb{Z}/p\mathbb{Z})^g$ they must generate all of $(\mathbb{Z}/p\mathbb{Z})^g$, which gives us $\mathrm{Im}(\lambda|_{H \cap G_{V_g}}) = (\mathbb{Z}/p\mathbb{Z})^g$ as desired.

Claim 3: $\Big[\mathrm{Sp}(2g-2, \mathbb{Z}/p\mathbb{Z}) : \mathrm{Im}(\Psi|_{H \cap S_{2g-1}})\Big] \leq \frac{4^{g-2} \cdot 8}{3} \cdot \left(\frac{8}{7}\right)^{\frac{g(g-1)}{2}} \varepsilon^{1-g}$

Proof of Claim 3: We will eventually use the induction hypothesis of Lemma 4.8 on the subgroup $\Psi(S_{2g-1} \cap H)$ of $\mathrm{Sp}(2g-2, \mathbb{Z}/p\mathbb{Z})$ (or more precisely on the subgroup of $\mathrm{Sp}(2g-2, \mathbb{Z})$ defined by this coset in $\mathrm{Sp}(2g-2, \mathbb{Z})/\Gamma_{g-1}(p)$). To do this, we need to get an estimate on the ramification mean of this subgroup.

For that, define a map $\psi : V_{2g-1} \longrightarrow (\mathbb{Z}/p\mathbb{Z})^{2g-2}$ via

$$(v_1, \ldots, v_{2g-1}, 0) \mapsto (v_1, \ldots, v_{g-1}, v_{g+1}, \ldots, v_{2g-1}) \ .$$

Note that for any primitive vector $v \in V^*_{2g-1}$ the image $\psi(v) \in (\mathbb{Z}/p\mathbb{Z})^{2g-2}$ is again primitive except in the $p-1$ cases where v_g is the only nonzero coordinate. We write $\widetilde{V}_{2g-1}$ for the subset of V_{2g-1} where $\psi(v)$ is primitive. If we consider the corresponding transvections $r_{v,1} \in S_{2g-1}$ and $r_{\psi(v),1} \in \mathrm{Sp}(2g-2, \mathbb{Z}/p\mathbb{Z})$, we obtain the following connection between the maps Ψ and ψ:

$$\Psi(r_{v,1}) = r_{\psi(v),1}$$

This implies that for all $v \in \widetilde{V}_{2g-1}$ we have that

$$\mathrm{ram}_H(v) = \mathrm{ram}_{S_{2g-1} \cap H}(v) = \mathrm{ram}_{\Psi(S_{2g-1} \cap H)}(\psi(v)) \ .$$

Since ψ is surjective and p–to–1 we get that the ramification mean of $\widetilde{V}_{2g-1}$ with respect to $H < \mathrm{Sp}(2g, \mathbb{Z}/p\mathbb{Z})$ is just the desired ramification mean of the subgroup $\Psi(S_{2g-1} \cap H)$ of $\mathrm{Sp}(2g-2, \mathbb{Z}/p\mathbb{Z})$, namely

$$\mathrm{rammean}_H(\widetilde{V}_{2g-1}) = \frac{1}{\#w} \sum_{w \in (\mathbb{Z}/p\mathbb{Z})^{2g-2} \setminus \{0\}} \mathrm{ram}_{\Psi(S_{2g-1} \cap H)}(w) \ . \tag{16}$$

Using that $\mathrm{rammean}_H(V_{2g-1}) \geq \varepsilon$ by (13) it is straightforward to show that for its subset $\widetilde{V}_{2g-1}$ we have $\mathrm{rammean}_H(\widetilde{V}_{2g-1}) \geq (7/8)\varepsilon$ (We could certainly find a better bound, but this bound will suffice for the proof of this claim).

We thus have that the subgroup $\Psi(S_{2g-1} \cap H)$ of $\mathrm{Sp}(2g-2, \mathbb{Z}/p\mathbb{Z})$ has ramification mean at least $(7/8)\varepsilon$ by (16). It now follows from the induction hypothesis (of Lemma 4.8) on $\Psi(S_{2g-1} \cap H)$ that

$$\begin{aligned} [\mathrm{Sp}(2g-2, \mathbb{Z}/p\mathbb{Z}) : \Psi(S_{2g-1} \cap H)] &\leq \frac{4^{g-2} \cdot 8}{3} \cdot \left(\frac{8}{7}\right)^{\frac{(g-1)(g-2)}{2}} \left(\frac{7}{8}\,\varepsilon\right)^{1-g} \\ &= \frac{4^{g-2} \cdot 8}{3} \cdot \left(\frac{8}{7}\right)^{\frac{g(g-1)}{2}} \varepsilon^{1-g} \ , \end{aligned}$$

which proves the claim.

Claim 4: $\ker(\Psi) < H$

Proof of Claim 4: The kernel of Ψ can be calculated explicitly to be

$$\ker(\Psi) = \left\{ \left(\begin{array}{cc:cc} \mathbb{1} & 0 & 0 & m_2 \\ m_1^T & 1 & m_2^T & m_4 \\ \hdashline 0 & 0 & \mathbb{1} & -m_1 \\ 0 & 0 & 0 & 1 \end{array} \right) ;\ m_1, m_2 \in (\mathbb{Z}/p\mathbb{Z})^{g-1}, m_4 \in \mathbb{Z}/p\mathbb{Z} \right\}$$

which is isomorphic to $(\mathbb{Z}/p\mathbb{Z})^{2g-1}$. We already know that some of the elements of $\ker(\Psi)$ are also in H, namely the elements of

$$G_{V_g} \cap \ker(\Psi) = \left\{ \left(\begin{array}{cc:cc} \mathbb{1} & 0 & 0 & b_1 \\ 0 & 1 & b_1^T & b_2 \\ \hdashline 0 & 0 & \mathbb{1} & 0 \\ 0 & 0 & 0 & 1 \end{array} \right) ; b_1 \in (\mathbb{Z}/p\mathbb{Z})^{g-1}, b_2 \in \mathbb{Z}/p\mathbb{Z} \right\} < H \cap \ker(\Psi)\ ,$$

since $G_{V_g} < H$ by Claim 2. Note that this group is naturally isomorphic to $(\mathbb{Z}/p\mathbb{Z})^g$ and that under this isomorphism the multiplication of matrices corresponds to the addition of elements in $(\mathbb{Z}/p\mathbb{Z})^g$.

We will use the elements of $G_{V_g} \cap \ker(\Psi)$ to construct more elements in $H \cap \ker(\Psi)$. For that let

$$h := \mathbb{1} + \left(\begin{array}{c:cc} 0 & 0 & 0 \\ & 0 & \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} \\ \hdashline 0 & & 0 \end{array} \right) \in G_{V_g} \cap \ker(\Psi) < H\ .$$

Take any primitive vector $v = (v_1, \ldots, v_{2g-1}, 0) \in V_{2g-1}^*$ and consider the corresponding transvection $r_{v,1} \in \mathrm{Sp}(2g, \mathbb{Z}/p\mathbb{Z})$. A short calculation shows that

$$r_{v,1} \cdot h \cdot r_{v,1}^{-1} = h \ + \ v_{2g-1} \left(\begin{array}{cccc:cccc} & 0 & & & & 0 & & \begin{matrix} v_1 \\ \vdots \\ v_{g-1} \end{matrix} \\ -v_{g+1} & \cdots & -v_{2g-1} & 0 & v_1 & \cdots & v_{g-1} & 2v_g \\ \hdashline & 0 & & & & 0 & & \begin{matrix} v_{g+1} \\ \vdots \\ v_{2g-1} \\ 0 \end{matrix} \end{array} \right) .$$

Observe that if we multiply an element of $G_{V_g} \cap \ker(\Psi)$ with this matrix, the product is essentially obtained by adding the coefficients in the upper right corners of the matrices. It is thus easy to find a suitable matrix in $G_{V_g} \cap \ker(\Psi)$ such that its product with $r_{v,1} \cdot h \cdot r_{v,1}^{-1}$ is given by

$$\mathbb{1} \; + \; v_{2g-1} \left(\begin{array}{c:c} \begin{matrix} & 0 & \\ -v_{g+1} & \cdots \; - v_{2g-1} & 0 \end{matrix} & 0 \\ \hdashline 0 & 0 \quad \begin{matrix} v_{g+1} \\ \vdots \\ v_{2g-1} \\ 0 \end{matrix} \end{array} \right) . \tag{17}$$

If $r_{v,1} \in H$ then this matrix is in $H \cap \ker(\Psi)$. Note that if we only consider primitive vectors $v = (v_1, \ldots, v_{2g-1}, 0) \in V^*_{2g-1}$ with $v_{2g-1} \neq 0$, then at most $2p^g$ of them give the same matrix.

Our next step will be to estimate how many different primitive vectors v with $r_{v,1} \in H$ and $v_{2g-1} \neq 0$, and as a consequence how many different matrices in $H \cap \ker(\Psi)$, we are guaranteed to have. We will then conclude that these matrices already generate all of $\ker(\Psi)$ which will prove the claim.

Using that $\mathrm{rammean}_H(V_{2g-1}) \geq \varepsilon$ by (13), we can apply Proposition B.1 to conclude that we have at least

$$\frac{\varepsilon - (1/p)}{1 - (1/p)} \, (p^{2g-1} - 1) = (\varepsilon p - 1) \, \frac{p^{2g-1} - 1}{p - 1} > (\varepsilon p - 1) \, p^{2g-2} \; .$$

vectors $v \in V^*_{2g-1}$ which satisfy $\mathrm{ram}_H(v) > 1/p$. Since $\mathrm{ram}_H(v) \in \{1/p, 1\}$ these in fact have $\mathrm{ram}_H(v) = 1$ which implies that $r_{v,1} \in H$. We can further estimate this number to be no less than $7p^{2g-2}$ by using that $\varepsilon \geq (8/p)$. If we now discount the $p^{2g-2} - 1$ primitive vectors $v \in V^*_{2g-1}$ with $v_{2g-1} = 0$, we still are guaranteed to have at least $6p^{2g-2}$ vectors v with $r_{v,1} \in H$ which additionally satisfy $v_{2g-1} \neq 0$. These gives us at least $3p^{g-2}$ different matrices in $H \cap \ker(\Psi)$ of the form given in (17). Together with the matrices in $G_{V_g} \cap \ker(\Psi)$ we thus have at least $3p^{2g-2}$ elements in $H \cap \ker(\Psi)$. Comparing this with $\left|\ker(\Psi)\right| = \left|(\mathbb{Z}/p\mathbb{Z})^{2g-1}\right| = p^{2g-1}$, we get that in fact $\ker(\Psi) < H$. □

4.4 Putting it all together

In this section we will assemble the results of the previous sections to obtain a proof of Theorem 4.3.

We start with the corresponding statement for subgroups $\Gamma < \mathrm{Sp}(2g, \mathbb{Z})$ containing principal congruence subgroups $\Gamma(p^t)$ which follows directly from the $n = p$–case and our considerations in Section 4.2.

Lemma 4.10 *For every $\varepsilon > 0$ there are only finitely many subgroups $\Gamma <$ $\mathrm{Sp}(2g, \mathbb{Z})$ for which there is a prime p and an integer t with $\Gamma(p^t) < \Gamma$ and which have ramification mean at least ε.*

Proof. Since $\mathrm{Sp}(2g, \mathbb{Z})$ is finitely generated it suffices to show that there is a number only depending on ε which bounds the index of every subgroup Γ of $\mathrm{Sp}(2g, \mathbb{Z})$ with the given properties.

Let $\Gamma < \mathrm{Sp}(2g, \mathbb{Z})$ be a subgroup containing $\Gamma(p^t)$ for some prime power p^t and having ramification mean at least ε. If p is sufficiently big (namely $p > \max(3\varepsilon^{-1}, 2)$), we can conclude by Proposition 4.4 that in fact $t = 1$, so $\Gamma(p) < \Gamma$. If p even satisfies $p \geq \max(3, 8 \cdot \left(\frac{8}{7}\right)^{g-1} \varepsilon^{-1})$ we get an upper bound for the index by Lemma 4.8.

This leaves us with those cases where p is not sufficiently big. However, by Corollary 4.6 we can for each p give a bound on t. So in fact, we only have to deal with a finite number of cases for p^t. But for every prime power p^t there are only finitely many groups $\Gamma(p^t) \subset \Gamma < \mathrm{Sp}(2g, \mathbb{Z})$ since every such Γ corresponds to exactly one subgroup of the finite group $\mathrm{Sp}(2g, \mathbb{Z}/p^t\mathbb{Z})$. This means that we have only finitely many subgroups Γ of $\mathrm{Sp}(2g, \mathbb{Z})$ containing some group $\Gamma(p^t)$ which do not satisfy the hypothesis of Lemma 4.8 and we are done. □

To obtain the general result we will use the fact that every integer n can be decomposed into distinct prime powers. This gives us a decomposition of the ring $\mathbb{Z}/n\mathbb{Z}$ which induces a decomposition of the group $\mathrm{Sp}(2g, \mathbb{Z}/n\mathbb{Z})$. We then can use the lemma we have just shown on each factor and finish the proof of the main result of this chapter.

Theorem 4.3 *For every $\varepsilon > 0$ there are only finitely many subgroups $\Gamma <$ $\mathrm{Sp}(2g, \mathbb{Z})$ of finite index with ramification mean at least ε.*

Proof. Let Γ be a subgroup of $\mathrm{Sp}(2g, \mathbb{Z}/n\mathbb{Z})$ of finite index. Let $n = n_\Gamma$, i.e. the minimal level n such that the principal congruence subgroup $\Gamma(n)$ is contained in Γ. Consider the factor group $H := \Gamma/\Gamma(n)$ which can be identified with a subgroup of $\mathrm{Sp}(2g, \mathbb{Z}/n\mathbb{Z})$. If we decompose n into primes, say

$$n = p_1^{t_1} \cdot \ldots p_k^{t_k}\ ,\ (p_i, p_j) = 1 \text{ for } i \neq j\ ,$$

we also obtain a factorization of $\mathrm{Sp}(2g, \mathbb{Z}/n\mathbb{Z})$ as follows:

$$\mathrm{Sp}(2g, \mathbb{Z}/n\mathbb{Z}) \cong \mathrm{Sp}(2g, \mathbb{Z}/p_1^{t_1}\mathbb{Z}) \times \cdots \times \mathrm{Sp}(2g, \mathbb{Z}/p_k^{t_k}\mathbb{Z})$$

We will now describe what happens under this factorization with the subgroup H. For that we define

$$H_i := H \cap \Big(\{\mathbb{1}\} \times \cdots \times \{\mathbb{1}\} \times \mathrm{Sp}(2g, \mathbb{Z}/p_i^{t_i}\mathbb{Z}) \times \{\mathbb{1}\} \times \cdots \times \{\mathbb{1}\}\Big)\ .$$

We abuse notation and write H_i for the projection to the i-th coordinate of H_i also. Note that all H_i are proper subgroups of $\mathrm{Sp}(2g, \mathbb{Z}/p_i^{t_i}\mathbb{Z})$ by the minimality of n. The group H certainly contains the cartesian product of the H_i, but is in general not generated by it:

$$H > H_1 \times \cdots \times H_k$$

However, we will see shortly that the corresponding description of the ramification groups of H and H_i in fact gives the desired equality. Recall that a vector $v \in (\mathbb{Z}/n\mathbb{Z})^{2g}$ (given as $v = (v_1, \ldots, v_k) \in (\mathbb{Z}/p_1^{t_1}\mathbb{Z})^{2g} \times \cdots \times (\mathbb{Z}/p_k^{t_k}\mathbb{Z})^{2g}$) is primitive if and only if $v_1, \ldots, v_k$ are primitive. Now it is easy to check that

$$\mathrm{Ram}_{\mathrm{Sp}(2g,\mathbb{Z}/n\mathbb{Z})}(v) \cong \mathrm{Ram}_{\mathrm{Sp}(2g,\mathbb{Z}/p_1^{t_1}\mathbb{Z})}(v_1) \times \cdots \times \mathrm{Ram}_{\mathrm{Sp}(2g,\mathbb{Z}/p_k^{t_k}\mathbb{Z})}(v_k) \ . \tag{18}$$

Recall that $\mathrm{Ram}_{\mathrm{Sp}(2g,\mathbb{Z}/p_i^{t_i}\mathbb{Z})}(v_i) \cong \mathbb{Z}/p_i^{t_i}\mathbb{Z}$, so the orders of $\mathrm{Ram}_{\mathrm{Sp}(2g,\mathbb{Z}/p_i^{t_i}\mathbb{Z})}(v_k)$ are all coprime to each other. Therefore the group $\mathrm{Ram}_H(v) < \mathrm{Ram}_{\mathrm{Sp}(2g,\mathbb{Z}/n\mathbb{Z})}(v)$ under the identification given in (18) can be expressed as a cartesian product of subgroups of $\mathrm{Ram}_{\mathrm{Sp}(2g,\mathbb{Z}/p_i^{t_i}\mathbb{Z})}(v_i)$. Hence

$$\mathrm{Ram}_H(v) \cong \mathrm{Ram}_{H_1}(v_1) \times \cdots \times \mathrm{Ram}_{H_k}(v_k) \ .$$

We can now compare the ramification means of H and of H_i and obtain

$$\begin{aligned} \varepsilon &\le \frac{1}{\#v}\sum_v \mathrm{ram}_H(v) = \frac{1}{\#v}\sum_{v_1}\cdots\sum_{v_k} \mathrm{ram}_{H_1}(v_1)\cdot\dots\cdot \mathrm{ram}_{H_k}(v_k) \\ &= \left(\frac{1}{\#v_1}\sum_{v_1}\mathrm{ram}_{H_1}(v_1)\right)\cdot\dots\cdot\left(\frac{1}{\#v_k}\sum_{v_k}\mathrm{ram}_{H_k}(v_k)\right) =: \varepsilon_1\cdot\dots\cdot\varepsilon_k \ . \end{aligned}$$

So the ramification mean of H is equal to the product of the ramification means of the H_i. Since $0 < \varepsilon_i \le 1$ this implies that

$$\varepsilon_i \ge \varepsilon \quad \text{for all } i = 1, \ldots, k \ .$$

Given $1 \ge \varepsilon > 0$ there are by Lemma 4.10 only finitely many subgroups Γ of $\mathrm{Sp}(2g, \mathbb{Z})$ containing $\Gamma(p^t)$ for some prime p and some integer t having ramification mean at least ε. Let $\mathcal{B}$ be the set containing these (or equivalently their images $H = \Gamma/\Gamma(p^t)$). By what we have just seen all H_i must be contained in this set (recall that H_i is a proper subgroup of $\mathrm{Sp}(2g, \mathbb{Z}/p_i^{t_i}\mathbb{Z})$). Since the number of H_i is thus finite, so is the number of combinations $H_1 \times \cdots \times H_k$ with $H_i \neq H_j$ for $i \neq j$. This gives us a bound on n and thus a bound for the index of any subgroup Γ with ramification mean at least ε. □

Remark 4.11 *We could give an explicit bound on the index of subgroups* $\Gamma < \mathrm{Sp}(2g, \mathbb{Z})$ *with ramification mean at least* $\varepsilon > 0$ *using the bounds given in the propositions of the previous sections. However, this bound would be far away from being optimal and be in fact much too large to be of any practical use, so we omit it here.*

We will need this result in Chapter 7 to conclude that subgroups Γ of $\mathrm{Sp}(2g, \mathbb{Z})$ of sufficiently large index have relatively small ramifications means and thus do not pose too many obstructions to extending pluricanonical forms. Moreover this theorem will allow us to control the ramification occurring at boundary components of higher codimension, which will be done in Chapter 6.

Chapter 5

Singularities in the interior

In this chapter we will study the singularities in the interior of $\mathcal{A}_\Gamma$. They occur at the images of those points in $\mathcal{H}_g$ that have non–trivial stabilizers in Γ. We will start in the first section by considering the stabilizers with respect to the full symplectic group $\mathrm{Sp}(2g, \mathbb{Z})$. This will give us an explicit description of the non–canonical singularities in the interior of the moduli space of abelian varieties $\mathcal{A}_g$. We will use this knowledge in the following section to derive a similar description for the space $\mathcal{A}_\Gamma$. Finally, we will relate the number of elements in Γ (or more precisely their equivalence classes in $\Gamma(n)$) which lead to non–canonical singularities in $\mathcal{A}_\Gamma$ to the index of Γ in $\mathrm{Sp}(2g, \mathbb{Z})$. This will play an important role when we calculate the obstructions to extending pluricanonical forms over these singularities in Chapter 7.

For the general background on canonical and non–canonical singularities we refer the reader to Section 1.5 where we introduced the notion of quotient singularities, in particular with regard to Siegel modular varieties. Throughout this chapter we will make frequent use of the results provided in that section.

5.1 Singularities in $\mathcal{A}_g$

In this section we will study the singularities in $\mathcal{A}_g$, or more precisely, determine the locus of non–canonical singularities lying in the interior of $\mathcal{A}_g$. For $g = 2$ this question has been answered by Borisov (cf. [Bor, Section 4]). On the other hand, for $g \geq 5$ there is a paper by Tai in which he shows that $\mathcal{A}_g$ has only canonical singularities (cf. [Tai, Section 4]). A careful analysis of his proof will allow us to derive the non–canonical singularities in the cases $g = 3$ and $g = 4$.

Since we will follow Tai's proof, we will also use his notation in this section to avoid confusion. We first recall some general results from [Tai, Section 3] which will be used in the following discussion.

Let Γ be a finite group acting linearly on $\mathbb{C}^n$ and denote the quotient of this action by X. For any element $\gamma \in \Gamma$, we can also consider $X_\gamma := \mathbb{C}^n / \langle\gamma\rangle$, the quotient of $\mathbb{C}^n$ by the subgroup of Γ generated by γ. Let $\widetilde{X}$ and $\widetilde{X}_\gamma$ denote nonsingular models of X and X_γ respectively. When we are concerned with the extension of pluricanonical forms to $\widetilde{X}$, the following proposition allows us to restrict ourselves to the study of cyclic subgroups.

Proposition 5.1 *Let η be a Γ–invariant pluricanonical form on $\mathbb{C}^n$. The form η extends to $\widetilde{X}$ if and only if it extends to $\widetilde{X}_\gamma$ for every $\gamma \in \Gamma$.*

Proof. [Tai, Proposition 3.1] □

Remark 5.2 *Note that this result does not imply that X has canonical singularities if and only if all X_γ have canonical singularities. While the if part of this statement is certainly true, the only if part fails since one then would have to extend forms which are only invariant with respect to γ and not necessarily with respect to Γ.*

Let $\gamma \in \Gamma$ and $x \in \mathbb{C}^n$ such that x is fixed by γ. Suppose that the action of γ on the tangent space of x is given by multiplication in each coordinate by $e^{2\pi i S_j}$ with $S_j \in \mathbb{Q}, 0 \leq S_j < 1$. We then define the Reid–Tai sum $\{\gamma, x\}$ of x with respect to γ by

$$\{\gamma, x\} := \sum_{j=1}^{N} S_j \; . \tag{1}$$

With the help of this sum, the Reid–Shepherd–Barron–Tai criterion (cf. Theorem 1.65) can be reformulated as follows to answer the question of extensibility of pluricanonical forms:

Theorem 5.3 (Reid–Shepherd–Barron–Tai criterion) *Let η be a pluricanonical form on $\mathbb{C}^n$ which is invariant under the action of a finite group Γ acting linearly on $\mathbb{C}^n$. The form η extends to a nonsingular model of $\mathbb{C}^n/\Gamma$ if for every $\mathrm{id} \neq \gamma \in \Gamma$ and every $x \in \mathrm{Fix}(\gamma)$*

$$\{\gamma, x\} \geq 1 \; .$$

Proof. [Tai, Theorem 3.3] □

In our situation, the case of the moduli space of g–dimensional abelian varieties $\mathcal{A}_g$, we have the group $\mathrm{Sp}(2g, \mathbb{Z})$ acting on the Siegel upper half space $\mathcal{H}_g$ of genus

g. The stabilizer of any point $Z \in \mathcal{H}_g$ is a finite group (cf. Proposition 1.64) which can be linearized locally. By Proposition 5.1 it suffices to consider cyclic subgroups of these stabilizers or equivalently their generators. So let $Z \in \mathcal{H}_g$ and $\gamma \in \mathrm{Sp}(2g, \mathbb{Z})$ such that γ fixes Z. Note that γ has finite order $m = \mathrm{ord}(\gamma)$ and thus can be diagonalized. Using the symplectic relation, we can even conclude that γ is conjugate to

$$\begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda^{-1} \end{pmatrix}, \ \Lambda = \mathrm{diag}(\zeta^{t_1}, \ldots, \zeta^{t_g}), \zeta = e^{2\pi i/m} \tag{2}$$

for some $t_j \in \mathbb{Z}$. According to [Tai, Lemma 4.1], we can then choose a local coordinate system (x_{ij}) around Z such that the action of γ is given by

$$(x_{ij}) \mapsto (\zeta^{t_i+t_j} x_{ij}) \ .$$

If $m \neq 1, 2, 3, 4, 6$ Tai shows that in this case $\{\gamma, x\} \geq 1$ (This is basically due to the fact that all primitive m-th roots of unity appear as eigenvalues, since the characteristic polynomial of γ has rational coefficients (cf. [Tai, Lemma 4.4])). Moreover if γ has order 1,2 or 4 easy combinatorics show that then also $\{\gamma, x\} \geq 1$.

This leaves the case where $\mathrm{ord}(\gamma) = 6$. Here Tai shows that $\{\gamma, x\} \geq 1$ is again satisfied for $g \geq 3$ unless all eigenvalues of γ have either order 6 or order 1 (cf. [Tai, Proof of Lemma 4.5]). If $g \geq 5$ a short computation shows that this case also gives $\{\gamma, x\} \geq 1$ which implies that $\mathcal{A}_g$ then only has canonical singularities by Theorem 5.3. However, for both $g = 3$ and $g = 4$ we have exactly one case each which gives $\{\gamma, x\} < 1$, namely if γ has up to permutation the eigenvalues $(1, 1, \varrho, 1, 1, \varrho^5)$ (resp. $(1, 1, 1, \varrho, 1, 1, 1, \varrho^5)$), where ϱ is a primitive 6th root of unity.

Note that these calculations show that $\mathrm{Sp}(2g, \mathbb{Z})$ has no quasi–reflections for $g \geq 3$ in the sense of Definition 1.56. Indeed, such an element has exactly one eigenvalue different from 1 which implies that its Reid–Tai sum must be strictly less than 1. But we have just seen that the only elements with this property have two primitive 6th roots of unity.

In the case of a group without quasi–reflections the Reid–Shepherd–Barron–Tai criterion is in fact a characterization for canonical singularities (cf. Theorem 1.65). Thus the elements we have found for $g = 3$ and $g = 4$ having Reid–Tai sum strictly less than 1 give indeed non–canonical singularities.

We summarize our discussion in the following theorem:

Theorem 5.4 *Let ϱ be a primitive 6th root of unity.*

(i) A point in $\mathcal{A}_3$ is a non–canonical singularity if and only if it is the image of a point in $\mathcal{H}_3$ whose stabilizer in $\mathrm{Sp}(6, \mathbb{Z})$ contains a matrix $\gamma \in \mathrm{Sp}(6, \mathbb{Z})$ which has up to permutation the eigenvalues $(1, 1, \varrho, 1, 1, \varrho^5)$.

(ii) *A point in* $\mathcal{A}_4$ *is a non–canonical singularity if and only if it is the image of a point in* $\mathcal{H}_4$ *whose stabilizer in* $\mathrm{Sp}(8,\mathbb{Z})$ *contains a matrix* $\gamma \in \mathrm{Sp}(8,\mathbb{Z})$ *which has up to permutation the eigenvalues* $(1,1,1,\varrho,1,1,1,\varrho^5)$.

(iii) *For* $g \geq 5$ *the moduli space* $\mathcal{A}_g$ *has only canonical singularities in the open part.*

In the remainder of this section we will look for $g = 3$ and $g = 4$ at automorphisms of abelian varieties to determine the locus in $\mathcal{A}_g$ where non–canonical singularities can occur. For abelian threefolds the automorphisms have been studied by Birkenhake, Gonzáles–Aguilera and Lange (cf. [BGAL]) and also by Schmidt in his thesis(cf. [Sch]).

Let $A = \mathbb{C}^g/\Lambda$ be an abelian variety of dimension g and let α be an automorphism of A, that is a biholomorphic map $\alpha : A \to A$ respecting the group law on A. Any automorphism α has an analytic and a rational representation, denoted by $\varrho_{\mathrm{a}}(\alpha)$ and $\varrho_{\mathrm{r}}(\alpha)$ respectively (cf. [BGAL, p. 3] for an explicit construction). The latter, $\varrho_{\mathrm{r}}(\alpha)$, can be considered as an element of $\mathrm{Sp}(2g,\mathbb{Z})$ fixing the point in $\mathcal{H}_g$ corresponding to the abelian variety A. In the light of our results from Theorem 5.4 we are interested in those abelian varieties A that admit an automorphism $\alpha \in \mathrm{Aut}(A)$ whose rational representation $\varrho_{\mathrm{r}}(\alpha)$ has exactly the eigenvalues given for the matrix γ from the theorem.

For every $\alpha \in \mathrm{Aut}(A)$ we can consider $\mathrm{Fix}(\alpha)$, the subgroup of A containing the fixed points of α given by

$$\mathrm{Fix}(\alpha) := \ker(\mathrm{id}_A - \alpha)\ . \tag{3}$$

The number of elements in $\mathrm{Fix}(\alpha)$ is related to the eigenvalues of $\varrho_{\mathrm{r}}(\alpha)$ by the following proposition:

Proposition 5.5 *The set of fixed points* $\mathrm{Fix}(\alpha)$ *on an abelian variety* A *is finite with respect to an automorphism* α *if and only if all eigenvalues of a rational representation* $\varrho_{\mathrm{r}}(\alpha)$ *are primitive* d_α*–th roots of unity, where* d_α *is the order of* α.

Proof. [BGAL, Prop. 1.4] □

This immediately tells us that we need to investigate the case where we have infinitely many fixed points. For this we can use the following decomposition theorem due to Roan:

Theorem 5.6 *Let* A *be an abelian variety and let* α *be an automorphism of* A *of finite order. Let* $1 \leq d_1 < d_2 < \cdots < d_r$ *be the orders of the eigenvalues of* $\varrho_{\mathrm{r}}(\alpha)$*, a rational representation of* α*. Then there are* α*–stable abelian subvarieties* $A_1, \ldots, A_r$ *of* A *such that* *(a)* $\alpha_i := \alpha|_{A_i}$ *is of order* d_i*,*

(b) the set of fixed points $\mathrm{Fix}(\alpha_i) \subset A_i$ *under the action of* α_i *on* A_i *is finite for* $i > 1$*,*

(c) the addition map

$$\mu : A_1 \times \cdots \times A_r \to A$$

is an isogeny.

Proof. [BGAL, Theorem 2.1] □

Using this theorem we can conclude that we have to consider the product of an elliptic curve E with an abelian variety A_{g-1} of dimension $g-1$. Under the isogeny

$$\mu : E \times A_{g-1} \to A$$

the automorphism α has to have order 6 on the elliptic curve E and must be trivial on A_{g-1}. Since 6 is not a prime power, we can conclude by the fixed point formula for complex tori that $\mathrm{Fix}(\alpha|_E)$ is trivial on E (cf. [BGAL, Lemma 1.2 and Corollary 1.7]). This implies by [Sch, Satz 3.4] that the kernel of μ is trivial, which means that μ is in fact an isomorphism (cf. also [BGAL, Proposition 5.2]).

We have to look at automorphisms of elliptic curves of order 6. Up to isomorphism there is only one elliptic curve which admits such an automorphism, the curve E_0 having j–invariant 0. The automorphism is given by multiplication by $-\varrho$ where $\varrho = e^{2\pi i/3}$ (cf. [BGAL, Example 1.9]).

Its rational representation is given by the matrix

$$\beta_6 := \left(\begin{array}{cc|cc} \mathbb{1} & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ \hline 0 & 0 & \mathbb{1} & 0 \\ 0 & -1 & 0 & 0 \end{array}\right) \in \mathrm{Sp}(2g, \mathbb{Z}) \ . \tag{4}$$

As an element of $\mathrm{Sp}(2g, \mathbb{Z})$ it operates on the Siegel upper half space $\mathcal{H}_g$ and we can consider its set of fixed points which is given by

$$\mathrm{Fix}(\beta_6) = \left\{ \begin{pmatrix} Z & 0 \\ 0 & \varrho \end{pmatrix} ;\ Z \in \mathcal{H}_{g-1} \right\} \ .$$

Note that each point in $\mathrm{Fix}(\beta_6)$ corresponds to a product $E_0 \times A_{g-1}$ as desired. All abelian varieties isomorphic to such a product can be obtained by looking at the orbit of $\mathrm{Fix}(\beta_6)$ in $\mathrm{Sp}(2g, \mathbb{Z})$, that is

$$\bigcup_{M \in \mathrm{Sp}(2g,\mathbb{Z})} M \cdot \mathrm{Fix}(\beta_6) \ .$$

For $M \in \mathrm{Sp}(2g,\mathbb{Z})$ we have the identity

$$\mathrm{Fix}(M\beta_6 M^{-1}) = M \cdot \mathrm{Fix}(\beta_6)\ ,$$

which implies that the respective automorphism is given by taking the conjugate of β_6 with respect to M.

The following theorem summarizes our results:

Theorem 5.7 *For $g = 3$ and $g = 4$ the locus of non–canonical singularities in the interior of $\mathcal{A}_g$ is exactly the image of*

$$\bigcup_{M\in \mathrm{Sp}(2g,\mathbb{Z})} \mathrm{Fix}(M\beta_6 M^{-1}) \subset \mathcal{H}_g\ ,$$

where $\beta_6 \in \mathrm{Sp}(2g,\mathbb{Z})$ is the matrix defined in (4)*.*

Our discussion allows us also to give a geometric interpretation in terms of abelian varieties. Though it will not be needed in the rest of this thesis, it might be of independent interest, so we put it in the following remark:

Remark 5.8 *Each point in $\mathrm{Fix}(\beta_6)$ corresponds to a product of an elliptic curve E_0 with j–invariant 0 with an abelian variety A_{g-1} of dimension $g-1$. Considered as an automorphism in $\mathrm{Aut}(E_0 \times A_{g-1})$, β_6 acts trivially on A_{g-1} and by multiplication by $-\varrho$ on E_0, where $\varrho = e^{2\pi i/3}$.*

5.2 Singularities in $\mathcal{A}_\Gamma$

In this section we will describe for any subgroup Γ of $\mathrm{Sp}(2g,\mathbb{Z})$ the non–canonical singularities in the interior of the corresponding moduli space $\mathcal{A}_\Gamma$ using the results on $\mathcal{A}_g$ from the previous section.

Let Γ be a subgroup of $\mathrm{Sp}(2g,\mathbb{Z})$ of finite index for any $g \geq 3$. To describe the singularities in the corresponding moduli space $\mathcal{A}_\Gamma := \mathcal{H}_g/\Gamma$ we have to look at fixed points in $\mathcal{H}_g$ under the action of Γ. The stabilizer of any point $Z \in \mathcal{H}_g$ in Γ is contained in the stabilizer of Z in $\mathrm{Sp}(2g,\mathbb{Z})$ since Γ is a subgroup of $\mathrm{Sp}(2g,\mathbb{Z})$. Hence we can use our results from the previous section on the Reid–Tai sums for elements in $\mathrm{Sp}(2g,\mathbb{Z})$ to conclude that $\mathcal{A}_\Gamma$ can only have a non–canonical singularity at the image of a point $Z \in \mathcal{H}_g$ if $\mathcal{A}_g$ has one at its image. This tells us immediately that $\mathcal{A}_\Gamma$ has only canonical singularities for $g \geq 5$. For $g = 3$ and $g = 4$ we obtain that the image of a point $Z \in \mathcal{H}_g$ in $\mathcal{A}_\Gamma$ is a non–canonical singularity if and only if the stabilizer of Z in Γ contains a conjugate of β_6.

Remark 5.9 *Note carefully that the mere fact that $\mathcal{A}_g$ has a canonical singularity at the image of a point $Z \in \mathcal{H}_g$ is in general not enough to conclude that $\mathcal{A}_\Gamma$ has a canonical singularity at the image of that point. This is due to the fact that the Reid–Shepherd–Barron–Tai criterion as stated in Theorem 5.3 only works in one direction, namely that we have a canonical singularity if the Reid–Tai sum is no less than 1. The converse is only true if you add the assumption that there are no quasi–reflections (cf. Theorem 1.65).*
Since in our case this assumption is true for $g \geq 3$, we could have used this converse here. However, in the preceding argument we used the explicit knowledge of the Reid–Tai sums calculated in the previous section instead.

The following theorem follows from our discussion:

Theorem 5.10 *Let Γ be a subgroup of $\mathrm{Sp}(2g,\mathbb{Z})$ of finite index and $\mathcal{A}_\Gamma := \mathcal{H}_g/\Gamma$ be the corresponding moduli space.*

(i) For $g = 3$ and $g = 4$ a point in the interior of $\mathcal{A}_\Gamma$ is a non–canonical singularity if and only if it lies in the image of

$$\bigcup_{M\beta_6M^{-1}\in\Gamma} \mathrm{Fix}(M\beta_6M^{-1}) \subset \mathcal{H}_g \ ,$$

where the union is taken over all $M \in \mathrm{Sp}(2g,\mathbb{Z})$ such that $M\beta_6M^{-1} \in \Gamma$.

(ii) For $g \geq 5$ the moduli space $\mathcal{A}_\Gamma$ has only canonical singularities in the open part.

Let Γ be a subgroup of $\mathrm{Sp}(2g,\mathbb{Z})$ of finite index. By Theorem 1.18 it contains a principal congruence subgroup $\Gamma(n)$ of some level n and we can describe the moduli space $\mathcal{A}_\Gamma$ as the quotient of $\mathcal{A}_g(n) := \mathcal{H}_g/\Gamma(n)$ by the action of the finite group $H := \Gamma/\Gamma(n)$. Using the quotient map $p_{g,n} : \mathrm{Sp}(2g,\mathbb{Z}) \to \mathrm{Sp}(2g,\mathbb{Z}/n\mathbb{Z}) \cong \mathrm{Sp}(2g,\mathbb{Z})/\Gamma(n)$ we can consider β_6 and its conjugates as elements of $\mathrm{Sp}(2g,\mathbb{Z}/n\mathbb{Z})$ which will be useful in the following section. Moreover, instead of $p_{g,n}(\beta_6)$ we can also work with the involution given by its third power

$$\varphi_0 := (p_{g,n}(\beta_6))^3 = \mathrm{diag}(\underbrace{1,\ldots,1}_{g-1\text{ times}},-1,\underbrace{1,\ldots,1}_{g-1\text{ times}},-1) \in \mathrm{Sp}(2g,\mathbb{Z}/n\mathbb{Z}) \ . \tag{5}$$

Its set of fixed points in $\mathcal{A}_g(n)$ is given as the image of

$$\mathrm{Fix}(\beta_6^3) = \left\{ \begin{pmatrix} Z & 0 \\ 0 & \tau \end{pmatrix} ; \ Z \in \mathcal{H}_{g-1}, \tau \in \mathcal{H}_1 \right\} \subset \mathcal{H}_g \ .$$

under the natural quotient map $\pi_{g,n} : \mathcal{H}_g \to \mathcal{A}_g(n)$. Each point in this set corresponds to the product of an abelian variety A_{g-1} with an arbitrary elliptic curve E. We denote the image in $\mathcal{A}_g(n)$ by

$$X_0 := \pi_{g,n}(\mathrm{Fix}(\beta_6^3)) \ . \tag{6}$$

Let $\{\varphi_\alpha\}$ be the set containing all $\mathrm{Sp}(2g, \mathbb{Z}/n\mathbb{Z})$–conjugates of φ_0 and $\{X_\alpha\}$ be the corresponding sets of fixed points in $\mathcal{A}_g(n)$. Note that if a certain conjugate of β_6 is in Γ, the corresponding involution φ_α is in H, since it is just the image of the 3rd power of this conjugate. This implies the following corollary:

Corollary 5.11 *Let Γ be a subgroup of $\mathrm{Sp}(2g, \mathbb{Z})$ for $g = 3$ or $g = 4$ such that $\Gamma(n) < \Gamma$ for some n. If $\mathcal{A}_\Gamma$ has a non–canonical singularity at the image of a point $Z \in \mathcal{A}_g(n)$, then there is an index α such that $Z \in X_\alpha$ and $\varphi_\alpha \in H$, where $H := \Gamma/\Gamma(n)$.*

Note carefully that the converse is no longer true, since we replaced β_6 with the involution φ_0. However, this description will be sufficient for the further discussion; in fact, it will simplify the proofs of the following section.

In the last part of this section we will establish a correspondence between the φ_α and certain pairs of complementary submodules of $V = (\mathbb{Z}/n\mathbb{Z})^{2g}$.

For that, consider the submodules of V given by

$$W_0^1 := (\underbrace{*, \ldots, *}_{g-1 \text{ times}}, 0, \underbrace{*, \ldots, *}_{g-1 \text{ times}}, 0) \quad \text{and} \quad W_0^2 := (\underbrace{0, \ldots, 0}_{g-1 \text{ times}}, *, \underbrace{0, \ldots, 0}_{g-1 \text{ times}}, *) \,. \tag{7}$$

As an element of $G := \mathrm{Sp}(2g, \mathbb{Z}/n\mathbb{Z})$ the involution φ_0 acts on V by left multiplication and thus on W_0^1 and W_0^2. There we have that

$$\varphi_0|_{W_0^1} \equiv \mathrm{id}|_{W_0^1} \quad \text{and} \quad \varphi_0|_{W_0^2} \equiv -\mathrm{id}|_{W_0^2} \,. \tag{8}$$

We can extend this to a G–equivariant correspondence as follows.

Proposition 5.12 *(i) The involutions $\{\varphi_\alpha\}$ are in one–to–one G–equivariant correspondence with pairs (W_α^1, W_α^2) of submodules $W_\alpha^1, W_\alpha^2 \subset V$ with the following properties:*

(a) $W_\alpha^1 \cong (\mathbb{Z}/n\mathbb{Z})^{2g-2}$, $W_\alpha^2 \cong (\mathbb{Z}/n\mathbb{Z})^2$,

(b) W_α^1 is orthogonal to W_α^2 with respect to the standard skew form on V,

(c) $\langle W_\alpha^1, W_\alpha^2 \rangle = V$.

(ii) In (i) condition (c) can be replaced by the following equivalent condition:

(c') The restriction of the standard skew form $\langle\ ,\ \rangle$ on V to W_α^2 satisfies the following property: For each primitive vector $w_2 \in W_\alpha^2$ there exists a vector $w_2' \in W_\alpha^2$ such that $\langle w_2, w_2' \rangle = 1$.

(iii) Every involution φ_α together with its corresponding pair (W_α^1, W_α^2) satisfies the relations

$$\varphi_\alpha|_{W_\alpha^1} \equiv \mathrm{id}|_{W_\alpha^1} \quad \textit{and} \quad \varphi_\alpha|_{W_\alpha^2} \equiv -\mathrm{id}|_{W_\alpha^2} \,.$$

Proof.

(ii) We start by showing the equivalence of conditions (c) and (c').

Let (c) be satisfied and let w_2 be any primitive vector in W_α^2. Then there exists a primitive vector $\widetilde{w}_2$ in the full module V with $\langle w_2, \widetilde{w}_2\rangle = 1$ (a suitable multiple of one of the vectors $e_1, \dots, e_g$ of the canonical basis of V will satisfy this condition). By (c) this vector $\widetilde{w}_2$ can be written as

$$\widetilde{w}_2 = w_1' + w_2' \ ,$$

where $w_1' \in W_\alpha^1$ and $w_2' \in W_\alpha^2$.

By condition (b) we have that $\langle w_2, w_1'\rangle = 0$, so

$$1 = \langle w_2, \widetilde{w}_2\rangle = \underbrace{\langle w_2, w_1'\rangle}_{=0} + \langle w_2, w_2'\rangle = \langle w_2, w_2'\rangle \ ,$$

which shows condition (c').

The converse can be shown by a similar argument using the fact that W_α^1 is orthogonal to W_α^2 as given in (b).

(i) A straightforward calculation shows that the stabilizers in G of φ_0 and of the standard pair (W_0^1, W_0^2) coincide and are given by:

$$\left\{ \left(\begin{array}{cc:cc} A & 0 & B & 0 \\ 0 & a & 0 & b \\ \hdashline C & 0 & D & 0 \\ 0 & c & 0 & d \end{array} \right); \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2g-2, \mathbb{Z}/n\mathbb{Z}), \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}/n\mathbb{Z}) \right\}$$

Now it suffices to check that every pair (W_α^1, W_α^2) with the given properties is conjugate to the standard one.

(iii) We have seen this in (8) for the standard involution and the standard pair. The claim thus follows from the G–equivariance of the correspondence.

□

Remark 5.13 *Note that by part (ii) of the above proposition the involutions $\{\varphi_\alpha\}$ are determined by W_α^2 alone. Indeed, for any W_α^2 with the given properties, W_α^1 is uniquely determined as the orthogonal complement of W_α^2.*

With the knowledge of the stabilizer of the standard involution, we can calculate the number of conjugates in G.

Corollary 5.14 *The number of conjugates* $\{\varphi_\alpha\}$ *of the standard involution* φ_0 *in* $G = \mathrm{Sp}(2g, \mathbb{Z}/n\mathbb{Z})$ *(or equivalently the number of* X_α*) is given by*

$$\#\alpha := \#\varphi_\alpha = \frac{|\mathrm{Sp}(2g, \mathbb{Z}/n\mathbb{Z})|}{|\mathrm{Sp}(2g-2, \mathbb{Z}/n\mathbb{Z})| \cdot |\mathrm{SL}(2, \mathbb{Z}/n\mathbb{Z})|} = n^{4(g-1)} \prod_{i=1}^{k} \frac{1 - p_i^{-2g}}{1 - p_i^{-2}} \ ,$$

where $n = p_1^{t_1} \cdot \ldots p_k^{t_k}$ *with* $(p_i, p_j) = 1$ *for* $i \neq j$*.*

Proof. We have just seen in the proof of Proposition 5.12 (i) that the stabilizer of φ_0 in G is isomorphic to

$$\mathrm{Stab}_G(\varphi_0) \cong \mathrm{Sp}(2g-2, \mathbb{Z}/n\mathbb{Z}) \times \mathrm{SL}(2, \mathbb{Z}/n\mathbb{Z}) \ .$$

Since the action of G on the set of conjugates φ_α is by definition transitive, this implies the result. □

5.3 Ramification

We have seen in the previous section that the question if the moduli space $\mathcal{A}_\Gamma$ has a non–canonical singularity at the image of a point in $X_\alpha \subset \mathcal{A}_g(n)$ depends on whether the corresponding involution φ_α is in $H = \Gamma/\Gamma(n)$ or not. We will give a measure for the number of involutions in H by defining a ramification mean similar to the one introduced in Chapter 4 for boundary divisors. In fact, the results from that chapter will be used to relate the ramification mean for the involutions to the index of Γ in $\mathrm{Sp}(2g, \mathbb{Z})$.

Let Γ be a subgroup of $\mathrm{Sp}(2g, \mathbb{Z})$ containing a principal congruence subgroup $\Gamma(n)$ for some level n. As usual we denote the group $\mathrm{Sp}(2g, \mathbb{Z}/n\mathbb{Z})$ by G and its subgroup $\Gamma/\Gamma(n)$ by H.

Definition 5.15 *For any subgroup* H *of* G *we define* $\mathrm{ram}_H(X_\alpha)$ *to be equal to* 1 *if the corresponding involution* φ_α *is in* H *and* 0 *otherwise.*

Rather than looking at an individual X_α we will usually consider all conjugates at once and look at the following mean:

$$\frac{1}{\#\alpha} \sum_\alpha \mathrm{ram}_H(X_\alpha) \ , \tag{9}$$

where $\#\alpha$ denotes the number of X_α as calculated in Corollary 5.14. Note that this mean can be associated to any subgroup Γ of $\mathrm{Sp}(2g, \mathbb{Z})$ of finite index; in particular it does not depend on the level n of the principal congruence subgroup $\Gamma(n)$ contained in Γ.

We will now formulate and prove the main result of this section.

Theorem 5.16 *For every $\varepsilon > 0$ there are only finitely many subgroups $\Gamma < \mathrm{Sp}(2g, \mathbb{Z})$ with the following properties:*

(i) Γ has finite index in $\mathrm{Sp}(2g, \mathbb{Z})$, which means that it contains a principal congruence subgroup $\Gamma(n)$ for some level n.

(ii) $\sum_\alpha \mathrm{ram}_H(X_\alpha) \geq \varepsilon \cdot \#\alpha$, where H denotes the factor group $\Gamma/\Gamma(n)$.

Proof. We will first reduce the problem to the case where $n = p^t$ is a prime power. This can be done by an argument analogous to the one used in the proof of Theorem 4.3 in the previous section with $\mathrm{ram}_H(v)$ replaced by $\mathrm{ram}_H(X_\alpha)$.

Let Γ be a subgroup of $\mathrm{Sp}(2g, \mathbb{Z})$ containing $\Gamma(n)$ for some n. We will assume that n is minimal with this property, i.e. $n = n_\Gamma$ in the sense of Section 4.2. Decomposing n into primes, say

$$n = p_1^{t_1} \cdot \ldots p_k^{t_k} \; , \; (p_i, p_j) = 1 \text{ for } i \neq j \; ,$$

we obtain a factorization of $\mathrm{Sp}(2g, \mathbb{Z}/n\mathbb{Z})$ as follows:

$$\mathrm{Sp}(2g, \mathbb{Z}/n\mathbb{Z}) \cong \mathrm{Sp}(2g, \mathbb{Z}/p_1^{t_1}\mathbb{Z}) \times \cdots \times \mathrm{Sp}(2g, \mathbb{Z}/p_k^{t_k}\mathbb{Z})$$

Note that under this isomorphism the involution $\varphi_\alpha \in \mathrm{Sp}(2g, \mathbb{Z}/n\mathbb{Z})$ is identified with $(\varphi_\alpha^1, \ldots, \varphi_\alpha^k)$ where each $\varphi_\alpha^i \in \mathrm{Sp}(2g, \mathbb{Z}/p_i^{t_i}\mathbb{Z})$ is a conjugate of the corresponding standard involution $\varphi_0^i \in \mathrm{Sp}(2g, \mathbb{Z}/p_i^{t_i}\mathbb{Z})$ defined in (5). If we denote the fixed set corresponding to φ_α^i by X_α^i, we obtain that

$$\mathrm{ram}_H(X_\alpha) = \prod_{i=1}^k \mathrm{ram}_{H_i}(X_\alpha^i) \; , \tag{10}$$

where H_i is defined as in the proof of Theorem 4.3. Indeed, φ_α is in H if and only if all its components φ_α^i are contained in the H_i.

An easy calculation using Corollary 5.14 shows that the number of components X_α is related to $\#\alpha_i$, the number of components X_α^i, as follows:

$$\#\alpha = \# \{X_\alpha\} = \prod_{i=1}^k \#\alpha_i$$

As in the proof of Theorem 4.3 this together with the description of $\mathrm{ram}_H(X_\alpha)$ in (10) implies that the ramification mean of H is equal to the product of the ramification means ε_i of the H_i, i.e.

$$\varepsilon \leq \frac{1}{\#\alpha} \sum_\alpha \mathrm{ram}_H(X_\alpha) = \varepsilon_1 \cdot \cdots \cdot \varepsilon_k \; .$$

Since $0 < \varepsilon_i \leq 1$ we can conclude that $\varepsilon_i \geq \varepsilon$ for all $i = 1, \ldots, k$. We can then use the corresponding finiteness result for the H_i which we will show next. Since the argument is no different from the one at the end of the proof of Theorem 4.3 we omit it here.

We thus have reduced the proof of the theorem to the case where $n = p^t$ is a power of a prime. In this case we will make use of the corresponding result for the ramification mean of boundary divisors given in Lemma 4.10. Namely, we will show that the condition

$$\sum_\alpha \mathrm{ram}_H(X_\alpha) \geq \varepsilon \cdot \#\alpha$$

implies that

$$\sum_\alpha \mathrm{ram}_H(v_\alpha) > r(\varepsilon) \cdot (\#v_\alpha) \ ,$$

where r is a function in ε which is independent of n and H (cf. Definition 4.2 for the definition of $\mathrm{ram}_H(v_\alpha)$). We then can conclude by Lemma 4.10 that Γ belongs to some finite set of subgroups of $\mathrm{Sp}(2g, \mathbb{Z})$.

So let $\Gamma < \mathrm{Sp}(2g, \mathbb{Z})$ such that $\Gamma(p^t) < \Gamma$ for some prime p and some integer t. For any set I of indices we define the ramification mean of I with respect to $H = \Gamma/\Gamma(p^t)$ to be

$$\mathrm{rammean}_H(I) := \frac{1}{|I|} \sum_{\alpha \in I} \mathrm{ram}_H(X_\alpha) \ . \tag{11}$$

Consider for any primitive vector $v \in V = (\mathbb{Z}/p^t\mathbb{Z})^{2g}$ the set I_v of indices α such that v is an eigenvector of the involution φ_α for the eigenvalue -1, i.e.

$$I_v := \{\alpha \, ; \ \varphi_\alpha \cdot v = -v\} \ . \tag{12}$$

Note that all sets I_v contain the same number of indices α. Indeed, take any set I_v and a primitive vector $w \in V$. Since the action of $\mathrm{Sp}(2g, \mathbb{Z}/p^t\mathbb{Z})$ on the set of primitive vectors is transitive there exists a matrix $M \in \mathrm{Sp}(2g, \mathbb{Z}/p^t\mathbb{Z})$ such that $w = M \cdot v$. It is easy to check that given an involution φ_α with $\alpha \in I_v$, i.e. $\varphi_\alpha \cdot v = -v$, the involution $M\varphi_\alpha M^{-1}$ has $w = M \cdot v$ as an eigenvector and its index is thus in I_w. This shows $|I_v| \leq |I_w|$. Since v was arbitrary we have in fact equality.

On the other hand, since all the matrices φ_α are conjugate, each φ_α has the same number of eigenvectors, this means that each α belongs to the same number of sets I_v. These two observations imply that

$$\begin{aligned} \varepsilon \leq \frac{1}{\#\alpha} \sum_\alpha \mathrm{ram}_H(X_\alpha) &= \frac{1}{\#v} \sum_v \frac{1}{\#I_v} \sum_{\alpha \in I_v} \mathrm{ram}_H(X_\alpha) \\ &= \frac{1}{\#v} \sum_v \mathrm{rammean}_H(I_v) \ . \end{aligned} \tag{13}$$

It now follows that there are at least $(\varepsilon/2) \cdot (\#v)$ primitive vectors $v \in V$ such that the ramification mean of I_v is bigger than $(\varepsilon/2)$. Indeed, assume this were not the case. Then we would have

$$\sum_v \operatorname{rammean}_H(I_v) < (\varepsilon/2) \cdot (\#v) \cdot 1 + ((\#v) - (\varepsilon/2) \cdot (\#v)) \cdot (\varepsilon/2)$$
$$= \varepsilon \cdot (\#v) - (\varepsilon^2/4) \cdot (\#v) < \varepsilon \cdot (\#v)$$

which contradicts (13).

We will now estimate $\operatorname{ram}_H(v)$ (cf. Definition 4.2) for any primitive vector v with the property that $\operatorname{rammean}_H(I_v) > (\varepsilon/2)$. It will turn out that we have

$$\operatorname{ram}_H(v) > \quad (\varepsilon/2)^{(2g-1)/2g-2)}$$

if $p > 2$ and

$$\operatorname{ram}_H(v) > \frac{1}{8}(\varepsilon/2)^{(2g-1)/2g-2)}$$

for $p = 2$. Since we are guaranteed to have at least $(\varepsilon/2) \cdot (\#v)$ such primitive vectors, we can then conclude that

$$\sum_v \operatorname{ram}_H(v) > \quad (\varepsilon/2)^{(2g-1)/2g-2)} \cdot (\varepsilon/2) \cdot (\#v) = \quad (\varepsilon/2)^{(4g-3)/(2g-2)} \cdot (\#v)$$

if $p > 2$ and

$$\sum_v \operatorname{ram}_H(v) > \frac{1}{8}(\varepsilon/2)^{(2g-1)/2g-2)} \cdot (\varepsilon/2) \cdot (\#v) = \frac{1}{8}(\varepsilon/2)^{(4g-3)/(2g-2)} \cdot (\#v)$$

for $p = 2$. We can then apply Lemma 4.10 to finish the proof once we have shown the estimate for $\operatorname{ram}_H(v)$ which will be established by the following lemma. □

Lemma 5.17 *Let* Γ *be a subgroup of* $\operatorname{Sp}(2g, \mathbb{Z})$ *such that* $\Gamma(p^t)$ *is contained in* Γ *for some prime power* p^t*. Let* $\varepsilon > 0$ *and* $v \in (\mathbb{Z}/p^t\mathbb{Z})^{2g}$ *be a primitive vector with* $\operatorname{rammean}_H(I_v) > \varepsilon$*, where* $H = \Gamma/\Gamma(p^t)$ *and* $\operatorname{rammean}_H(I_v)$ *is defined as in* (11)*. Then its ramification satisfies*

$$\operatorname{ram}_H(v) > \begin{cases} \varepsilon^{(2g-1)/(2g-2)} & \text{if } p > 2 \\ \frac{1}{8}\varepsilon^{(2g-1)/(2g-2)} & \text{if } p = 2 \end{cases}.$$

Proof. Since $\operatorname{Sp}(2g, \mathbb{Z}/p^t\mathbb{Z})$ acts transitively on the set of primitive vectors, we can assume w.l.o.g. that $v = e_g$, the g–th vector of the canonical basis of $V = (\mathbb{Z}/p^t\mathbb{Z})^{2g}$.

We will now determine explicitly all involutions φ_α with $\alpha \in I_v$. Recall that by Proposition 5.12 (i) the involutions φ_α are in one–to–one correspondence with certain pairs (W^1_α, W^2_α) of submodules of V. Moreover, by Remark 5.13 every involution φ_α is uniquely determined by the choice of W^2_α alone.

Let $\alpha \in I_v$ and (W^1_α, W^2_α) be the corresponding pair of submodules. We will now choose a basis $\{b_1, \ldots, b_{2g}\}$ of V such that

$$\langle b_1, \ldots, b_{g-1}, b_{g+1}, \ldots, b_{2g-1}\rangle = W^1_\alpha, \quad \langle b_g, b_{2g}\rangle = W^2_\alpha \, . \tag{14}$$

Recall that by definition of I_v as given in (12) we have that $\varphi_\alpha \cdot v = -v$. Since $\varphi_\alpha|_{W^1_\alpha} \equiv \mathrm{id}|_{W^1_\alpha}$ and $\varphi_\alpha|_{W^2_\alpha} \equiv -\mathrm{id}|_{W^2_\alpha}$ as we have seen in Proposition 5.12 (iii), we thus can conclude that $v \in W^2_\alpha$. Furthermore, since v is primitive, we can choose the basis in (14) in such a way that

$$b_g = v \, .$$

It then follows from Proposition 5.12 (ii) that the other basis vector b_{2g} is given by

$$b_{2g} = (\beta_1, \ldots, \beta_{g-1}, 0, \beta_{g+1}, \ldots, \beta_{2g-1}, 1)$$

for some $\beta_i \in \mathbb{Z}/p^t\mathbb{Z}$.

Since W^1_α is uniquely determined by W^2_α as its orthogonal complement with respect to the standard skew form on V, we can choose the following basis $\{b_1, \ldots, b_{g-1}, b_{g+1}, \ldots, b_{2g-1}\}$ for W^1_α:

$$b_i = \begin{cases} e_i - \beta_{g+i}\, e_g & \text{for } i = 1, \ldots, g-1 \, , \\ e_i + \beta_{i-g}\, e_g & \text{for } i = g+1, \ldots, 2g-1 \, . \end{cases}$$

Using that φ_α maps b_g and b_{2g} to $-b_g$ and $-b_{2g}$ respectively and fixes all other b_i, a simple calculation tells us that φ_α is represented with respect to the canonical basis $\{e_1, \ldots, e_{2g}\}$ of V by the following matrix:

$$\left(\begin{array}{cc:cc} \mathbb{1} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & 0 & \begin{matrix} -2\beta_1 \\ \vdots \\ -2\beta_{g-1} \end{matrix} \\ -2\beta_{g+1} \cdots -2\beta_{2g-1} & -1 & 2\beta_1 \ldots 2\beta_{g-1} & 0 \\ \hdashline & & & \begin{matrix} -2\beta_{g+1} \\ \vdots \\ -2\beta_{2g-1} \end{matrix} \\ 0 & & \mathbb{1} & \\ & & 0 \ \ldots \ 0 & -1 \end{array}\right)$$

We denote this matrix by $\varphi_{(\beta_1, \ldots, \beta_{g-1}, \beta_{g+1}, \ldots, \beta_{2g-1})}$, or for short by $\varphi_{(\beta_1, \ldots, \beta_{2g-1})}$. Our discussion shows that these are exactly the involutions φ_α with $\alpha \in I_v$.

Since $\mathrm{rammean}_H(I_v) > \varepsilon > 0$ we have at least one of these involutions in H. We may assume w.l.o.g. that this involution is $\varphi_0 = \varphi_{(0,\dots,0)}$. Indeed, we can replace H with a suitable conjugate in $\mathrm{Stab}_G(v)$ while respecting our previous assumption on v.

For two involutions $\varphi_{(\beta_1^1,\dots,\beta_{2g-1}^1)}, \varphi_{(\beta_1^2,\dots,\beta_{2g-1}^2)}$ we compute

$$\left(\varphi_{(\beta_1^1,\dots,\beta_{2g-1}^1)} \cdot \varphi_0 \cdot \varphi_{(\beta_1^2,\dots,\beta_{2g-1}^2)}\right)^2 = \left(\begin{array}{c:c} \mathbb{1} & \begin{matrix} 0 & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ 0\dots0 & \eta \end{matrix} \\ \hdashline 0 & \mathbb{1} \end{array}\right) ,$$

where

$$\eta := 8\left(\beta_1^1\beta_{g+1}^2 + \cdots + \beta_{g-1}^1\beta_{2g-1}^2 - (\beta_{g+1}^1\beta_1^2 + \cdots + \beta_{2g-1}^1\beta_{g-1}^2)\right) . \tag{15}$$

Note that if both involutions $\varphi_{(\beta_1^1,\dots,\beta_{2g-1}^1)}$ and $\varphi_{(\beta_1^2,\dots,\beta_{2g-1}^2)}$ are contained in H, then this gives us an element of $\mathrm{Ram}_H(v)$. To get an estimate for the order of the subgroup generated by this element, and thus an estimate for $\mathrm{ram}_H(v)$, we will need to consider

$$\gcd\left(\beta_1^1\beta_{g+1}^2 + \cdots + \beta_{g-1}^1\beta_{2g-1}^2 - (\beta_{g+1}^1\beta_1^2 + \cdots + \beta_{2g-1}^1\beta_{g-1}^2), p^t\right) . \tag{16}$$

The rest of this proof will thus be dedicated to showing that the fact that $\mathrm{rammean}_H(I_v) > \varepsilon$ implies, that we have so many different $\varphi_{(\beta_1,\dots,\beta_{2g-1})} \in H$, such that we can find two of them for which the greatest common divisor as given in (16) is sufficiently small, so we can conclude that $\mathrm{ram}_H(v)$ is as big as claimed. This will require some combinatorics and some number theoretic computations for which we will mostly refer to Section B of the appendix.

We have $\mathrm{rammean}_H(I_v) > \varepsilon$ which means that we have more than $\varepsilon(p^t)^{2g-2}$ involutions $\varphi_{(\beta_1,\dots,\beta_{2g-1})}$ in H. By Proposition B.2 there are at most $\varepsilon(p^t)^{2g-2}$ different $(\beta_1^1,\dots,\beta_{g-1}^1,\beta_{g+1}^1,\dots,\beta_{2g-1}^1) \in (\mathbb{Z}/p^t\mathbb{Z})^{2g-2}$ with

$$\gcd(\beta_1^1,\dots,\beta_{g-1}^1,\beta_{g+1}^1,\dots,\beta_{2g-1}^1,p^t) \geq \varepsilon^{-1/(2g-2)} .$$

This implies that we have at least one involution $\varphi_{(\beta_1^1,\dots,\beta_{2g-1}^1)} \in H$ where the β_i^1 satisfy

$$\gcd(\beta_1^1,\dots,\beta_{g-1}^1,\beta_{g+1}^1,\dots,\beta_{2g-1}^1,p^t) < \varepsilon^{-1/(2g-2)} , \tag{17}$$

say this greatest common divisor is p^s for some $0 \leq s \leq t$.

We need to find $(\beta_1^2,\dots,\beta_{g-1}^2,\beta_{g+1}^2,\dots,\beta_{2g-1}^2) \in (\mathbb{Z}/p^t\mathbb{Z})^{2g-2}$ such that the greatest common divisor given in (16) is sufficiently small. Note that this quantity can be rewritten as

$$p^s \cdot \gcd\left(\widetilde{\beta}_1^1\beta_{g+1}^2 + \cdots + \widetilde{\beta}_{g-1}^1\beta_{2g-1}^2 - (\widetilde{\beta}_{g+1}^1\beta_1^2 + \cdots + \widetilde{\beta}_{2g-1}^1\beta_{g-1}^2), p^{t-s}\right) ,$$

where $\widetilde{\beta}_i^1 := \beta_i^1/p^s$. We now have that

$$\gcd(\widetilde{\beta}_1^1, \dots, \widetilde{\beta}_{g-1}^1, \widetilde{\beta}_{g+1}^1, \dots, \widetilde{\beta}_{2g-1}^1, p^{t-s}) = 1$$

and can thus use Proposition B.3 to conclude that for any $r > 0$ there are at most

$$((p^{t-s})^{2g-2}/r) \cdot (p^s)^{2g-2} = (p^t)^{2g-2}/r$$

different $(\beta_1^2, \dots, \beta_{g-1}^2, \beta_{g+1}^2, \dots, \beta_{2g-1}^2) \in (\mathbb{Z}/p^t\mathbb{Z})^{2g-2}$ with

$$\gcd\left(\widetilde{\beta}_1^1\beta_{g+1}^2 + \cdots + \widetilde{\beta}_{g-1}^1\beta_{2g-1}^2 - (\widetilde{\beta}_{g+1}^1\beta_1^2 + \cdots + \widetilde{\beta}_{2g-1}^1\beta_{g-1}^2), p^{t-s}\right) \geq r \ .$$

So by setting $r = 1/\varepsilon$ and comparing this with the number of involutions in H, we can conclude that there is at least one tuple $(\beta_1^2, \dots, \beta_{g-1}^2, \beta_{g+1}^2, \dots, \beta_{2g-1}^2) \in (\mathbb{Z}/p^t\mathbb{Z})^{2g-2}$ such that the corresponding involution $\varphi_{(\beta_1^2,\dots,\beta_{2g-1}^2)}$ is in H and which satisfies

$$\gcd\left(\widetilde{\beta}_1^1\beta_{g+1}^2 + \cdots + \widetilde{\beta}_{g-1}^1\beta_{2g-1}^2 - (\widetilde{\beta}_{g+1}^1\beta_1^2 + \cdots + \widetilde{\beta}_{2g-1}^1\beta_{g-1}^2), p^{t-s}\right) < 1/\varepsilon \ . \quad (18)$$

Putting the results of (17) and (18) together we obtain that

$$\gcd\left(\beta_1^1\beta_{g+1}^2 + \cdots + \beta_{g-1}^1\beta_{2g-1}^2 - (\beta_{g+1}^1\beta_1^2 + \cdots + \beta_{2g-1}^1\beta_{g-1}^2), p^t\right) < \varepsilon^{-(2g-1)/(2g-2)}.$$

For $p > 2$ we then have for η as defined in (15) that

$$\gcd(\eta, p^t) < \varepsilon^{-(2g-1)/(2g-2)} \ ,$$

which implies that the subgroup of $\mathrm{Ram}_G(v)$ generated by

$$\left(\varphi_{(\beta_1^1,\dots,\beta_{2g-1}^1)} \cdot \varphi_0 \cdot \varphi_{(\beta_1^2,\dots,\beta_{2g-1}^2)}\right)^2 \in H$$

has order at least $\varepsilon^{(2g-1)/(2g-2)} \cdot p^t$, i.e. $\mathrm{ram}_H(v) \geq \varepsilon^{(2g-1)/(2g-2)}$ as claimed.

In the case $p = 2$ we have to take the factor of 8 in η into account and thus obtain $\mathrm{ram}_H(v) \geq (1/8)\varepsilon^{(2g-1)/(2g-2)}$. □

Remark 5.18 *Note that the proof of Theorem 5.16 could be used to give an explicit bound on the index of* Γ *in* $\mathrm{Sp}(2g, \mathbb{Z})$*. But as in the case of the ramification mean for boundary divisors this bound is far from being optimal, so we just give the finiteness statement.*

This main result will be used in Chapter 7 to conclude that subgroups Γ of sufficiently large index in $\mathrm{Sp}(2g, \mathbb{Z})$ do not pose too many obstructions to extending pluricanonical forms over the singularities in the interior of $\mathcal{A}_\Gamma$.

Chapter 6

Singularities in the boundary

In this chapter we will study elements in $\mathrm{Sp}(2g, \mathbb{Z}/n\mathbb{Z})$ which fix boundary components of $\mathcal{A}_g^{\mathrm{Vor}}(n)$ pointwise. Since for our main result as stated in Theorem 2.14 we can ignore boundary components lying in β_3, the locus of semi–abelian varieties with torus rank ≥ 3, we will restrict our study to components of the space of rank ≤ 2–degenerations $(\mathcal{A}_g^{\mathrm{Vor}}(n))^{(2)}$ (cf. Section 2.3 for the definitions of β_3 and $(\mathcal{A}_g^{\mathrm{Vor}}(n))^{(2)}$).

In Chapter 4 we have already considered the case of boundary divisors and have shown there that subgroups of $\mathrm{Sp}(2g, \mathbb{Z}/n\mathbb{Z})$ which fix many boundary divisors pointwise have small index in $\mathrm{Sp}(2g, \mathbb{Z}/n\mathbb{Z})$ (cf. Theorem 4.3). In fact, we will use this theorem here to get similar results for the components of higher codimension in $(\mathcal{A}_g^{\mathrm{Vor}}(n))^{(2)}$, namely the intersections $D_{i_1} \cap D_{i_2}$ of two boundary divisors and the intersections $D_{i_1} \cap D_{i_2} \cap D_{i_3}$ of three boundary divisors of global type. These results will play an important role in Chapter 7 when we consider the obstructions coming from singularities in the boundary of $\mathcal{A}_3^{\mathrm{Vor}}(n)$. Since we will specialize there to the case $g = 3$, we will give the results in this chapter only for this case, although they can easily be generalized to arbitrary g.

6.1 Intersections of two boundary divisors

In this section we will consider intersections $D_{i_1} \cap D_{i_2}$ of two boundary divisors as described in Proposition 3.15.

Recall the correspondence between primitive $\pm$vectors and boundary divisors established in Corollary 3.11. Let D_1 and D_2 denote the divisors corresponding to the primitive vectors $\pm e_1$ and $\pm e_2$, where $\{e_k\}_{k=1,\dots,6}$ denotes the canonical basis of $(\mathbb{Z}/n\mathbb{Z})^6$. We now define the subgroup $\mathrm{Ram}_G(D_1 \cap D_2)$ of $G = \mathrm{Sp}(6, \mathbb{Z}/n\mathbb{Z})$

by

$$\begin{aligned}\mathrm{Ram}_G(D_1\cap D_2) &:= \mathrm{Ram}_G(D_1)\oplus \mathrm{Ram}_G(D_2)\\ &= \left\{ \left(\begin{array}{c|c} \mathbb{1} & \begin{matrix} \begin{smallmatrix} b_1 & 0 \\ 0 & b_2 \end{smallmatrix} & 0 \\ 0 & 0 \end{matrix} \\ \hline 0 & \mathbb{1} \end{array}\right) ;\ b_1,b_2\in \mathbb{Z}/n\mathbb{Z} \right\},\end{aligned}$$

where $\mathrm{Ram}_G(D_1) := \mathrm{Ram}_G(e_1)$ and $\mathrm{Ram}_G(D_2) := \mathrm{Ram}_G(e_2)$ are defined as in Definition 4.1. We will see later in Chapter 7 that this is the group fixing the general point on $D_1\cap D_2$ if we add the transposition switching D_1 and D_2.

Definition 6.1 *Let D_{i_1}, D_{i_2} be boundary divisors and $M \in G = \mathrm{Sp}(6,\mathbb{Z}/n\mathbb{Z})$ such that $D_{i_1}\cap D_{i_2} = M\cdot(D_1\cap D_2)$. Then the* ramification group of $D_{i_1}\cap D_{i_2}$ with respect to G *is defined by*

$$\mathrm{Ram}_G(D_{i_1}\cap D_{i_2}) := \mathrm{Ram}_G(M\cdot(D_1\cap D_2)) := M\cdot \mathrm{Ram}_G(D_1\cap D_2)\cdot M^{-1}\ .$$

Remark 6.2 *Note that this defines the ramification group for any non–trivial intersection $D_{i_1}\cap D_{i_2}$ of two boundary divisors, since by Lemma 3.14 all such intersections are equivalent to the standard intersection $D_1\cap D_2$, i.e. there always exists a matrix $M\in G$ such that $D_{i_1}\cap D_{i_2} = M\cdot(D_1\cap D_2)$.*

To simplify notation we define

$$\mathcal{I}_2 := \Big\{\{i_1,i_2\};\ D_{i_1}, D_{i_2} \text{ intersect non-trivially}\Big\} \tag{1}$$

and write $D_I := D_{i_1}\cap D_{i_2}$ for the intersection of two divisors D_{i_1}, D_{i_2} with $I=\{i_1,i_2\}\in\mathcal{I}_2$ and $\mathrm{Ram}_G(D_I)$ for their ramification group.

Since we will usually work with subgroups of G, we extend this notion to any subgroup H by setting

$$\mathrm{Ram}_H(D_I) := H\cap \mathrm{Ram}_G(D_I) \tag{2}$$

for any $I\in\mathcal{I}_2$. Furthermore we define $\mathrm{ram}_H(D_I)$ to be the maximum order of the elements in $\mathrm{Ram}_H(D_I)$ divided by n, i.e.

$$\mathrm{ram}_H(D_I) := \frac{1}{n}\max_{M\in \mathrm{Ram}_H(D_I)} \mathrm{ord}(M)\ . \tag{3}$$

Note that since $\mathrm{Ram}_G(D_I)\cong(\mathbb{Z}/n\mathbb{Z})^2$, we have that every element of $\mathrm{Ram}_H(D_I)$ has order dividing n. Hence $\mathrm{ram}_H(D_I) = (k/n)$ for some $k\in\{1,\dots,n\}$. In particular $\mathrm{ram}_G(D_I)=1$ for all $I\in\mathcal{I}_2$.

We will be not so much interested in the values of $\mathrm{ram}_H(D_I)$ for individual $I \in \mathcal{I}_2$, but more in the mean for all I, and thus consider for each subgroup H of G

$$\frac{1}{\#\mathcal{I}_2} \sum_{I \in \mathcal{I}_2} \mathrm{ram}_H(D_I) \ . \tag{4}$$

Note that this mean can be associated to any subgroup Γ of $\mathrm{Sp}(6,\mathbb{Z})$ of finite index. Indeed, every such subgroup contains a principal congruence subgroup $\Gamma(n)$ of some level n by Theorem 1.18 and we can consider the mean as defined in (4) for the factor group $H = \Gamma/\Gamma(n)$. Note also that this mean is independent of the level n and thus does only depend on Γ.

We are now ready to formulate the main result of this section.

Theorem 6.3 *For every* $\varepsilon > 0$ *there are only finitely many subgroups* $\Gamma < \mathrm{Sp}(6,\mathbb{Z})$ *with the following properties:*

(i) Γ *has finite index in* $\mathrm{Sp}(6,\mathbb{Z})$, *which means that it contains a principal congruence subgroup* $\Gamma(n)$ *for some level* n.

(ii) $\frac{1}{\#\mathcal{I}_2} \sum_{I \in \mathcal{I}_2} \mathrm{ram}_H(D_I) > \varepsilon$, *where* H *denotes the factor group* $\Gamma/\Gamma(n)$.

Since we want to use the corresponding result for the ramification mean of boundary divisors given in Theorem 4.3, we need to do some reduction steps before we can give the proof of this theorem. For that we will need the following definition which, as we will see, can be used to relate the ramification of an intersection $D_{i_1} \cap D_{i_2}$ of two divisors to the ramifications of the individual divisors D_{i_1} and D_{i_2} in the sense of Definition 4.2.

Definition 6.4 *For any* $I = \{i_1, i_2\} \in \mathcal{I}_2$ *and* $k \in \{1,2\}$ *we define the group* $\mathrm{Ram}_H(D_I \subset D_{i_k})$ *to be the image of*

$$\mathrm{Ram}_H(D_I) \subset \mathrm{Ram}_G(D_I) \cong \mathrm{Ram}_G(D_{i_1}) \oplus \mathrm{Ram}_G(D_{i_2})$$

under the projection to $\mathrm{Ram}_G(D_{i_k})$. *We write* $\mathrm{ram}_H(D_I \subset D_{i_k})$ *for the order of this subgroup of* $\mathrm{Ram}_G(D_{i_k})$ *divided by* n, *i.e.*

$$\mathrm{ram}_H(D_I \subset D_{i_k}) := \frac{1}{n} \Big| \mathrm{Ram}_H(D_I \subset D_{i_k}) \Big| \ .$$

We can interpret the quantity $\mathrm{ram}_H(D_I \subset D_{i_k})$ as follows:

Remark 6.5 *If* $D_{i_1} = D_1$, *the standard divisor corresponding to the primitive* $\pm$*vector* e_1, *we have that* $\mathrm{ram}_H(D_I \subset D_{i_1})$ *is the inverse of the minimum* $\gcd(b,n)$

in the set of all $b \in \mathbb{Z}/n\mathbb{Z}$ *such that*

$$\left(\begin{array}{c:cc} \mathbb{1} & \begin{matrix} b & 0 \\ 0 & 0 \end{matrix} \\ \hdashline 0 & \mathbb{1} \end{array}\right) \cdot M \in H \tag{5}$$

for some $M \in \mathrm{Ram}_G(D_{i_2})$. *To see this, it suffices to note that the order of the first matrix in the above product is given by* $(n/\gcd(b,n))$.

We will now investigate the relation between $\mathrm{ram}_H(D_I)$ and $\mathrm{ram}_H(D_I \subset D_{i_k})$. Clearly, we always have that

$$\mathrm{ram}_H(D_I) \geq \mathrm{ram}_H(D_I \subset D_{i_k})$$

for $k \in \{1,2\}$. For general n this is all we can say. However, if $n = p^t$ is a prime power, we can say more.

Proposition 6.6 *Let* $n = p^t$ *for some prime* p *and some integer* t. *Then for any subgroup* H *of* $G = \mathrm{Sp}(6, \mathbb{Z}/p^t\mathbb{Z})$ *the following equality holds*

$$\mathrm{ram}_H(D_I) = \max_{k \in \{1,2\}} \{\mathrm{ram}_H(D_I \subset D_{i_k})\} \tag{6}$$

for all $I = \{i_1, i_2\} \in \mathcal{I}_2$.

Proof. Say $\mathrm{ram}_H(D_I) = 1/p^s$ for some $0 \leq s \leq t$. This means that there is an element of order p^{t-s} in $\mathrm{Ram}_H(D_I)$. Since $n = p^t$ is a prime power we can conclude that under the embedding

$$\mathrm{Ram}_H(D_I) \subset \mathrm{Ram}_G(D_I) \cong \mathrm{Ram}_G(D_{i_1}) \oplus \mathrm{Ram}_G(D_{i_2})$$

this element must have the same order p^{t-s} in one of the two components, i.e. $\mathrm{ram}_H(D_I \subset D_{i_k}) = 1/p^s$ for $k = 1$ or $k = 2$.

□

With the help of this proposition we will now prove Theorem 6.3 in the case where $n = p^t$ is a prime power.

Proposition 6.7 *For every* $\varepsilon > 0$ *there are only finitely many subgroups* Γ *of* $\mathrm{Sp}(6, \mathbb{Z})$ *containing* $\Gamma(p^t)$ *for some prime* p *and some integer* t *which satisfy*

$$\sum_{I \in \mathcal{I}_2} \mathrm{ram}_H(D_I) \geq \varepsilon \cdot \# \mathcal{I}_2$$

for the factor group $H = \Gamma/\Gamma(p^t)$.

Proof. Let Γ be any subgroup of $\mathrm{Sp}(6,\mathbb{Z})$ with the above properties. Since for each prime power p^t there are only finitely many groups Γ containing $\Gamma(p^t)$, we might as well assume that $p^t > 2$ and that $p^t > 16/\varepsilon$.

For any boundary divisor D_{i_1} we can consider the set of all divisors D_{i_2} intersecting D_{i_1} non–trivially. We define

$$\mathcal{I}_2(D_{i_1}) := \{i_2;\ D_{i_2} \text{ intersects } D_{i_1} \text{ non-trivially}\}\ . \tag{7}$$

The ramification mean of this divisor D_{i_1} with respect to $\mathcal{I}_2(D_{i_1})$ is then given by

$$\mathrm{rammean}_H(D_{i_1},\mathcal{I}_2(D_{i_1})) := \frac{1}{\#\mathcal{I}_2(D_{i_1})} \sum_{i_2\in\mathcal{I}_2(D_{i_1})} \mathrm{ram}_H(D_{i_1}\cap D_{i_2} \subset D_{i_1})\ . \tag{8}$$

As an immediate consequence of Proposition 6.6 we have that

$$\mathrm{ram}_H(D_I) < \mathrm{ram}_H(D_I \subset D_{i_1}) + \mathrm{ram}_H(D_I \subset D_{i_2}) \tag{9}$$

for all $I = \{i_1, i_2\} \in \mathcal{I}_2$. We can use this to obtain

$$\begin{aligned}
\varepsilon\cdot\#\mathcal{I}_2 &\leq \sum_{I\in\mathcal{I}_2} \mathrm{ram}_H(D_I)\\
&\overset{(9)}{<} \sum_{I=\{i_1,i_2\}\in\mathcal{I}_2} \Big(\mathrm{ram}_H(D_I\subset D_{i_1}) + \mathrm{ram}_H(D_I\subset D_{i_2})\Big)\\
&= \sum_{i_1}\sum_{i_2} \mathrm{ram}_H(D_{i_1}\cap D_{i_2}\subset D_{i_2})\\
&= \sum_{i_1} (\#\mathcal{I}_2(D_{i_1}))\cdot \mathrm{rammean}_H(D_{i_1},\mathcal{I}_2(D_{i_1}))\ .
\end{aligned}$$

Note that since all boundary divisors D_{i_1} are equivalent under the action of $\mathrm{Sp}(6,\mathbb{Z}/p^t\mathbb{Z})$, the number $\#\mathcal{I}_2(D_{i_1})$ of divisors intersecting a given divisor D_{i_1} is the same for all divisors D_{i_1} and does only depend on p^t. We can thus divide the above equation by $\#\mathcal{I}_2(D_{i_1})$ and get that

$$\sum_{i_1} \mathrm{rammean}_H(D_{i_1},\mathcal{I}_2(D_{i_1})) > (\varepsilon/2)\cdot(\#D_{i_1})\ . \tag{10}$$

We are interested in divisors D_{i_1} with sufficiently big ramification mean with respect to $\mathcal{I}_2(D_{i_1})$. We can apply Proposition B.1 to conclude that at least

$$\frac{(\varepsilon/2)-(\varepsilon/4)}{1-(\varepsilon/4)}\cdot(\#D_{i_1}) > (\varepsilon/4)\cdot(\#D_{i_1})$$

of the divisors D_{i_1} have $\mathrm{rammean}_H(D_{i_1},\mathcal{I}_2(D_{i_1})) > (\varepsilon/4)$.

We will show in the next lemma that all these divisors have $\mathrm{ram}_H(D_{i_1}) > \frac{1}{3}(\varepsilon/8)^7$ in the sense of Chapter 4, so

$$\sum_{i_1} \mathrm{ram}_H(D_{i_1}) \geq \frac{1}{3}(\varepsilon/8)^7 \cdot (\varepsilon/4) \cdot (\#D_{i_1}) = \frac{2}{3}(\varepsilon/8)^8 \cdot (\#D_{i_1}) \,,$$

i.e. H and thus Γ has ramification mean at least $\frac{2}{3}(\varepsilon/8)^8$ (cf. Definition 4.2). The claim then follows from Lemma 4.10. □

Lemma 6.8 *Let* Γ *be a subgroup of* $\mathrm{Sp}(6,\mathbb{Z})$ *such that* $\Gamma(p^t)$ *is contained in* Γ *for some prime power* p^t*. Let* $\varepsilon > 0$ *and* D_i *be a boundary divisor with* $\mathrm{rammean}_H(D_i, \mathcal{I}_2(D_i)) > \varepsilon$*, where* $H = \Gamma/\Gamma(p^t)$ *and* $\mathrm{rammean}_H(D_i, \mathcal{I}_2(D_i))$ *is defined as in* (8)*. If* $p^t > 4/\varepsilon$*, then the ramification of* D_i *satisfies* $\mathrm{ram}_H(D_i) > \frac{1}{3}(\varepsilon/2)^7$*.*

Proof. W.l.o.g. we can assume that the divisor D_i is the standard divisor, i.e. $D_i = D_0$ where D_0 is the divisor corresponding to the primitive ±vector e_3, the third vector of the canonical basis of $(\mathbb{Z}/p^t\mathbb{Z})^6$.

We will start by estimating how many divisors D_{i_2} which intersect D_0 non–trivially and satisfy $\mathrm{ram}_H(D_0 \cap D_{i_2} \subset D_0) > \varepsilon/2$ we are guaranteed to have. Since $\mathrm{rammean}_H(D_0, \mathcal{I}_2(D_0)) > \varepsilon$ we can apply Proposition B.1 to conclude that we have at least

$$\frac{\varepsilon - (\varepsilon/2)}{1 - (\varepsilon/2)} \cdot (\#\mathcal{I}_2(D_0)) > (\varepsilon/2) \cdot (\#\mathcal{I}_2(D_0)) \tag{11}$$

divisors D_{i_2} with these properties where $\#\mathcal{I}_2(D_0)$ denotes the total number of all divisors intersecting D_0 non–trivially as in (7).

Recall that we have a map $\pi : \mathcal{A}_3^{\mathrm{Vor}}(p^t) \to \mathcal{A}_3^{\mathrm{Sat}}(p^t)$ from the Voronoi compactification to the Satake compactification and that there is a stratification of $\mathcal{A}_3^{\mathrm{Sat}}(p^t)$ into several components as described in Section 2.3. The boundary divisors D_i are the closures of the preimages of the top–dimensional components $\mathcal{A}_2^{j_2}(p^t)$ of this stratification. We denote the top–dimensional component corresponding to the standard divisor D_0 by $\mathcal{A}_2^0(p^t)$. Non–trivial intersections of divisors D_{i_2} with D_0 then occur over the top–dimensional cusps of $\overline{\mathcal{A}_2^0}(p^t)$, i.e. those components $\mathcal{A}_1^{j_1}(p^t)$ of the stratification which are contained in $\overline{\mathcal{A}_2^0}(p^t)$. We know from Proposition 3.12 (ii) that we have $\mu_2(p^t) = \frac{1}{2}p^{4t}(1-p^{-4})$ such cusps $\mathcal{A}_1^{j_1}(p^t)$.

To determine the number of divisors D_{i_2} intersecting D_0 over a given cusp $\mathcal{A}_1^{j_1}(p^t)$ we can either use the geometric interpretation of $D_0 \to \overline{\mathcal{A}_2^0}(p^t)$ given in [Hul, Chapter 3] or use the correspondence between the components $\mathcal{A}_1^{j_1}(p^t)$ of the stratification and pairs $(W_1^{j_1}, \pm f_1^{j_1})$ established in Proposition 3.10 (i), where $W_1^{j_1}$ is a 2–dimensional isotropic submodule of $(\mathbb{Z}/p^t\mathbb{Z})^6$ and $f_1^{j_1}$ is a non–degenerate alternating bilinear form on $W_1^{j_1}$. We will use the correspondence here, since we will need it later in the proof anyway.

The standard component $\mathcal{A}_1^0(p^t)$ corresponds to the pair $(W_1^0, \pm f_1^0)$ given by

$$W_1^0 := (0,*,*,0,0,0) \subset (\mathbb{Z}/p^t\mathbb{Z})^6, \quad f_1^0(e_2,e_3) = 1 \ (\text{mod } p^t) .$$

Using that D_0 corresponds to the primitive $\pm$vector e_3 it now follows from Proposition 3.15 that the divisors D_{i_2} intersecting D_0 over $\mathcal{A}_1^0(p^t)$ are exactly those corresponding to the primitive vectors

$$\pm(0,1,a,0,0,0) \in (\mathbb{Z}/p^t\mathbb{Z})^6, \ a \in \mathbb{Z}/p^t\mathbb{Z} . \tag{12}$$

Since all cusps $\mathcal{A}_1^{j_1}(p^t)$ of $\overline{\mathcal{A}_2^0}(p^t)$ are equivalent, we can conclude that over each cusp there are exactly p^t divisors D_{i_2} intersecting D_0.

This means that we have

$$\#\mathcal{I}_2(D_0) = \mu_2(p^t) \cdot p^t = \frac{1}{2}p^{5t}(1-p^{-4})$$

different divisors D_{i_2} intersecting D_0. Comparing this with the number of such divisors with $\mathrm{ram}_H(D_0 \cap D_{i_2} \subset D_0) > \varepsilon/2$ computed in (11), we can conclude that there must be at least one cusp $\mathcal{A}_1^{j_1}(p^t)$ over which there are at least $(\varepsilon/2)\cdot p^t$ intersections with D_0 with this property. W.l.o.g. we can assume that this is the standard cusp $\mathcal{A}_1^0(p^t)$. Note that the fact that $p^t > 4/\varepsilon$ implies that we have at least two such intersections over $\mathcal{A}_1^0(p^t)$.

We can thus use the description of the divisors D_{i_2} given in (12). By Remark 6.5 we have for each divisor D_{i_2} corresponding to $\pm(0,1,a,0,0,0)$ for some $a \in \mathbb{Z}/p^t\mathbb{Z}$ with the property that $\mathrm{ram}_H(D_0 \cap D_{i_2} \subset D_0) > \varepsilon/2$, that there are $b, c \in \mathbb{Z}/p^t\mathbb{Z}$ with $\gcd(b,p^t) < 2/\varepsilon$ such that

$$\left(\begin{array}{c:ccc} & 0 & 0 & 0 \\ \mathbb{1} & 0 & b & ab \\ & 0 & ab & a^2b+c \\ \hdashline 0 & & \mathbb{1} & \end{array}\right) \in H .$$

Moreover, by taking appropriate powers of this matrix if necessary, we can find one such $b_0 = b \in \mathbb{Z}/p^t\mathbb{Z}$ with $\gcd(b_0,p^t) < 2/\varepsilon$ which works for all such a, i.e. all such divisors D_{i_2}. We will now choose a_1 and a_2 (resp. two divisors D_{i_2}) that give us matrices of the above form in H for some $c_1, c_2 \in \mathbb{Z}/p^t\mathbb{Z}$ and additionally satisfy $\gcd(a_2 - a_1, p^t) < 2/\varepsilon$. The existence of such a_1, a_2 follows from an easy number theoretic argument or from Proposition B.2 and the fact that we have at least $(\varepsilon/2)\cdot p^t$ divisors D_{i_2} with the property that $\mathrm{ram}_H(D_0 \cap D_{i_2} \subset D_0) > \varepsilon/2$.

If we multiply one of these matrices with the inverse of the other, we obtain

$$\varphi_0 := \left(\begin{array}{c:ccc} \mathbb{1} & \begin{matrix}0\\0\\0\end{matrix} & \begin{matrix}0\\0\\(a_2-a_1)b_0\end{matrix} & \begin{matrix}0\\(a_2-a_1)b_0\\(a_2^2-a_1^2)b_0+(c_2-c_1)\end{matrix} \\ \hdashline 0 & & \mathbb{1} & \end{array}\right) \in H\ .$$

Our goal is to find a different matrix in H such that when we multiply φ_0 with it, we obtain an element of $\mathrm{Ram}_H(D_0)$. For that we have to consider a different cusp $\mathcal{A}_1^{j_1}(p^t)$. Note that for a divisor D_{i_2} to intersect D_0 over some cusp it has to correspond to a primitive vector of the form $(d,e,f,g,h,0)$ with (d,e,g,h) primitive in $(\mathbb{Z}/p^t\mathbb{Z}^4)$. This is a consequence of Proposition 3.15. These are $\#\mathcal{I}_2(D_0) = \frac{1}{2}p^{5t}(1-p^{-4})$ primitive vectors and we know from our calculation in (11) that at least $(\varepsilon/2)\cdot\#\mathcal{I}_2(D_0)$ of the corresponding divisors D_{i_2} satisfy $\mathrm{ram}_H(D_0 \cap D_{i_2} \subset D_0) > \varepsilon/2$. A short calculation using Proposition B.2 now shows that at least one of them additionally satisfies $\gcd(h,p^t) < (2/\varepsilon)\cdot(1-p^{-4})^{-1}$. By Remark 6.5 this implies that H contains a matrix of the form

$$\varphi_1 := b_0 \left(\begin{array}{ccc:ccc} dg & dh & 0 & -d^2 & -de & -df \\ eg & eh & 0 & -de & -e^2 & -ef \\ fg & fh & 0 & -df & -ef & -f^2 \\ \hdashline g^2 & gh & 0 & -dg & -eg & -fg \\ gh & h^2 & 0 & -dh & -eh & -fh \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) + \left(\begin{array}{c:ccc} \mathbb{1} & \begin{matrix}0\\0\\0\end{matrix} & \begin{matrix}0\\0\\0\end{matrix} & \begin{matrix}0\\0\\\widetilde{c}\end{matrix} \\ \hdashline 0 & & \mathbb{1} & \end{array}\right)$$

for some $\widetilde{c} \in \mathbb{Z}/p^t\mathbb{Z}$ and with $\gcd(h,p^t) < (2/\varepsilon)\cdot(1-p^{-4})^{-1}$. Here we used again that the b from Remark 6.5 can be chosen in such a way that it coincides with b_0 by taking appropriate powers if necessary.

A calculation then gives that

$$\varphi_1\,\varphi_0\,\varphi_1^{-1}\varphi_0^{-1}\varphi_1\,\varphi_0^{-1}\varphi_1^{-1}\varphi_0 = \left(\begin{array}{c:ccc} \mathbb{1} & \begin{matrix}0\\0\\0\end{matrix} & \begin{matrix}0\\0\\0\end{matrix} & \begin{matrix}0\\0\\2\,h^2b_0^3(a_2-a_1)^2\end{matrix} \\ \hdashline 0 & & \mathbb{1} & \end{array}\right) \in H\ , \qquad (13)$$

which is in fact an element of $\mathrm{Ram}_H(D_0)$.

By our assumptions on the greatest common divisors of b_0,a_2-a_1 and h with p^t, we can conclude that

$$\gcd(2\,h^2b_0^3(a_2-a_1)^2, p^t) < 2\cdot(2/\varepsilon)^7(1-p^{-4})^{-2} < 3\cdot(2/\varepsilon)^7\ .$$

Hence the matrix in (13) has order greater than $\frac{1}{3}(\varepsilon/2)^7\cdot p^t$ which means that $\mathrm{ram}_H(D_0) > \frac{1}{3}(\varepsilon/2)^7$ as claimed. □

Theorem 6.3 now follows from a reduction argument and the above proposition.

Theorem 6.3 *For every $\varepsilon > 0$ there are only finitely many subgroups $\Gamma < \mathrm{Sp}(6, \mathbb{Z})$ with the following properties:*

(i) Γ *has finite index in* $\mathrm{Sp}(6, \mathbb{Z})$*, which means that it contains a principal congruence subgroup* $\Gamma(n)$ *for some level* n*.*

(ii) $\dfrac{1}{\#\mathcal{I}_2} \sum_{I \in \mathcal{I}_2} \mathrm{ram}_H(D_I) > \varepsilon$*, where* H *denotes the factor group* $\Gamma/\Gamma(n)$*.*

Proof. The proof can be reduced to the case $n = p^t$ which has been taken care of in Proposition 6.7. The reduction argument is completely analogous to the one given in the proof of Theorem 4.3 with $\mathrm{Ram}_H(D) = \mathrm{Ram}_H(v)$ replaced by $\mathrm{Ram}_H(D_{i_1} \cap D_{i_2})$, so we omit it here. □

We will use this result in Chapter 7 to conclude that subgroups Γ of $\mathrm{Sp}(6, \mathbb{Z})$ of sufficiently big index do not pose too many obstructions to extending pluricanonical forms over the singularities at points lying on the intersection of two boundary divisors.

6.2 Intersections of three boundary divisors of global type

We will now turn our attention in this section to intersections $D_{i_1} \cap D_{i_2} \cap D_{i_3}$ of three boundary divisors of global type.

As in the previous section we use the correspondence between primitive ±vectors and boundary divisors to define ramification groups for the intersections of three boundary divisors of global type. As before, let D_1 and D_2 denote the divisors corresponding to the vectors $\pm e_1$ and $\pm e_2$ respectively. Furthermore, we denote the divisor corresponding to $\pm(e_1 + e_2)$ by D_{12}. The subgroup $\mathrm{Ram}_G(D_1 \cap D_2 \cap D_{12})$ of $G = \mathrm{Sp}(6, \mathbb{Z}/n\mathbb{Z})$ is then defined by

$$\begin{aligned}\mathrm{Ram}_G(D_1 \cap D_2 \cap D_{12}) &:= \mathrm{Ram}_G(D_1) \oplus \mathrm{Ram}_G(D_2) \oplus \mathrm{Ram}_G(D_{12}) \\ &= \left\{ \left(\begin{array}{c:c} \mathbb{1} & \begin{matrix} b_1 & b_3 & 0 \\ b_3 & b_2 & \\ & 0 & 0 \end{matrix} \\ \hdashline 0 & \mathbb{1} \end{array} \right) ;\ b_1, b_2, b_3 \in \mathbb{Z}/n\mathbb{Z} \right\}.\end{aligned}$$

It will turn out that this is in fact the stabilizer of the general point on the intersection $D_1 \cap D_2 \cap D_{12}$ in G if we add the group permuting these three divisors which is isomorphic to S_3.

Definition 6.9 *Let* $D_{i_1}, D_{i_2}, D_{i_3}$ *be boundary divisors and let* $M \in G = \mathrm{Sp}(6, \mathbb{Z}/n\mathbb{Z})$ *such that* $D_{i_1} \cap D_{i_2} \cap D_{i_3} = M \cdot (D_1 \cap D_2 \cap D_{12})$. *Then the* ramification group of $D_{i_1} \cap D_{i_2} \cap D_{i_3}$ with respect to G *is defined by*

$$\begin{aligned}\mathrm{Ram}_G(D_{i_1} \cap D_{i_2} \cap D_{i_3}) &:= \mathrm{Ram}_G(M \cdot (D_1 \cap D_2 \cap D_{12})) \\ &:= M \cdot \mathrm{Ram}_G(D_1 \cap D_2 \cap D_{12}) \cdot M^{-1} \ .\end{aligned}$$

Unlike in the case of the intersection of two boundary divisors, not all intersections of three boundary divisors are equivalent. In fact, there are two disjoint orbits as we have seen in Lemma 3.16, containing intersections of global and of local type respectively. However, since we are in this section only interested in intersections of global type, we can still proceed as in the previous section. We thus have that all intersections of global type are equivalent to the standard intersection $D_1 \cap D_2 \cap D_{12}$. Hence the above definition defines the ramification groups for all intersections of global type.

In analogy to the set $\mathcal{I}_2$, we define

$$\mathcal{I}_3^{\mathrm{glob}} := \Big\{\{i_1, i_2, i_3\};\ D_{i_1}, D_{i_2}, D_{i_3} \text{ intersect of global type}\Big\} \qquad (14)$$

and write $D_I := D_{i_1} \cap D_{i_2} \cap D_{i_3}$ for the intersection given by $I = \{i_1, i_2, i_3\} \in \mathcal{I}_3^{\mathrm{glob}}$ and $\mathrm{Ram}_G(D_I)$ for its ramification group.

For any subgroup H of G and any $I \in \mathcal{I}_3^{\mathrm{glob}}$ we set

$$\mathrm{Ram}_H(D_I) := H \cap \mathrm{Ram}_G(D_I)$$

and write $\mathrm{ram}_H(D_I)$ for the maximum order of the elements in $\mathrm{Ram}_H(D_I)$ divided by n, i.e.

$$\mathrm{ram}_H(D_I) := \frac{1}{n} \max_{M \in \mathrm{Ram}_H(D_I)} \mathrm{ord}(M) \ .$$

Our aim in this section is to describe the singularities occurring at the image of the general point of each intersection D_I when we take the quotient by $\mathrm{Ram}_H(D_I)$. For each D_I and each general point $P \in D_I$ there is a natural choice of coordinates $(x_1, \ldots, x_6)$ in a neighborhood of P with the following two properties:

(i) D_I is locally given by $\{x_1 = x_2 = x_3 = 0\}$,

(ii) $\mathrm{Ram}_H(D_I) \subset \mathrm{Ram}_G(D_I) = \mathrm{Ram}_G(D_{i_1}) \oplus \mathrm{Ram}_G(D_{i_2}) \oplus \mathrm{Ram}_G(D_{i_3}) \cong (\mathbb{Z}/n\mathbb{Z})^3$ acts on this neighborhood by

$$(\xi_1, \xi_2, \xi_3) \cdot (x_1, \ldots, x_6) = (e^{2\pi i \xi_1/n}\, x_1, e^{2\pi i \xi_2/n}\, x_2, e^{2\pi i \xi_3/n}\, x_3, x_4, x_5, x_6)$$

for each $(\xi_1, \xi_2, \xi_3) \in \mathrm{Ram}_H(D_I) \subset \mathrm{Ram}_G(D_I) \cong (\mathbb{Z}/n\mathbb{Z})^3$.

One way to describe the singularity at the image of the point P is to determine the $\mathrm{Ram}_H(D_I)$–invariant monomials $x_1^{k_1} \cdot \ldots \cdot x_6^{k_6}$. While it is clear that with this choice of coordinates the monomials x_4, x_5 and x_6 are invariant, the invariance of monomials involving powers of x_1, x_2 or x_3 depends on the group $\mathrm{Ram}_H(D_I)$ and thus on H. We will therefore only consider monomials in the first three coordinates. Instead of calculating all these monomials we will look at their orders and define

$$\delta(H, D_I) := \delta(H, P) := \frac{1}{n} \min_{(k_1,k_2,k_3)\neq 0} (k_1 + k_2 + k_3) \,, \qquad (15)$$

where the minimum is taken over all non–trivial $\mathrm{Ram}_H(D_I)$–invariant monomials $x_1^{k_1} x_2^{k_2} x_3^{k_3}$. Note carefully that $\delta(H, P)$ does not depend on the point P or the coordinates chosen, but only on $\mathrm{Ram}_H(D_I)$. It thus makes sense to denote the δ for the general point of D_I by $\delta(H, D_I)$.

Although $\delta(H, D_I)$ does not describe the invariant ring precisely, it nevertheless gives a good measure on how big this ring is. If $\delta(H, D_I)$ is small, there tend to be quite a few invariant monomials and the resulting singularity at the image of P is usually well–behaved. On the other hand if $\delta(H, D_I)$ is big, we have only a couple of invariant monomials and tend to get *bad* singularities.

As in the previous section, where we considered the ramification mean, we will now consider the mean of all $\delta(H, D_I)$ over all intersections D_I and show that if the index of H in G is sufficiently big, this mean can be bounded from above and consequently there are not too many *bad* singularities. More precisely, we will show the following theorem:

Theorem 6.10 *For every $\varepsilon > 0$ there are only finitely many subgroups $\Gamma < \mathrm{Sp}(6, \mathbb{Z})$ with the following properties:*

(i) *Γ has finite index in $\mathrm{Sp}(6, \mathbb{Z})$, which means that it contains a principal congruence subgroup $\Gamma(n)$ for some level n.*

(ii) $\dfrac{1}{\#\mathcal{I}_3^{\mathrm{glob}}} \displaystyle\sum_{I \in \mathcal{I}_3^{\mathrm{glob}}} \delta(H, D_I) \geq \varepsilon$*, where H denotes the factor group $\Gamma/\Gamma(n)$.*

Before we start proving this theorem, we relate the situation to the $g = 2$–case. Locally, each intersection $D_I = D_{i_1} \cap D_{i_2} \cap D_{i_3}$ for $I \in \mathcal{I}_3^{\mathrm{glob}}$ is isomorphic to the product of a point $P_{\alpha\beta\gamma}$ with $\mathbb{C}^3$. This point $P_{\alpha\beta\gamma}$ is one of the *deepest points* in $\mathcal{A}_2^{\mathrm{Vor}}(n)$, i.e. one of the points at the intersection of three boundary divisors D_α, D_β, D_γ in $\mathcal{A}_2^{\mathrm{Vor}}(n)$. To see this, one has to look at the toroidal compactification. Recall that both D_I and $P_{\alpha\beta\gamma}$ are given as the quotient of a toric variety $T_{\Sigma_{D_I}}$ resp. $T_{\Sigma_{\alpha\beta\gamma}}$ by some finite group. We know from Chapter 3 that these intersections both lie over components of the Satake compactification which correspond to 2–dimensional isotropic submodules of $(\mathbb{Z}/n\mathbb{Z})^6$ resp. $(\mathbb{Z}/n\mathbb{Z})^4$ equipped with some alternating bilinear form (cf. Proposition 3.10 (i)). In the sense of toroidal compactifications this means that in both cases the toric varieties $T_{\Sigma_{D_I}}$ resp. $T_{\Sigma_{\alpha\beta\gamma}}$ are constructed by giving a decomposition of a three–dimensional cone. It is now easy to check by looking at the standard components that these two cones are essentially the same; the one corresponding to $T_{\Sigma_{D_I}}$ is just the embedding of the other one into a 6–dimensional space. Hence the same is true for the decompositions and we have that $T_{\Sigma_{D_I}} \cong T_{\Sigma_{\alpha\beta\gamma}} \times \mathbb{C}^3$. Since the action of the finite group is the same on the three torus coordinates, this isomorphism extends to the quotients and we get the desired isomorphism as claimed. We summarize our discussion in the following proposition:

Proposition 6.11 *For each $I \in \mathcal{I}_3^{\mathrm{glob}}$ the intersection $D_I = D_{i_1} \cap D_{i_2} \cap D_{i_3}$ is locally isomorphic to a product of a deepest point $P_{\alpha\beta\gamma}$ in $\mathcal{A}_2^{\mathrm{Vor}}(n)$ with $\mathbb{C}^3$.*

If we consider a general point on D_I this proposition allows us to make use of the results of Borisov for the $g = 2$–case (cf. [Bor, Section 3]). We start by proving Theorem 6.10 in the case where $n = p^t$ is a prime power. This is essentially an easy consequence of the corresponding result for points $P_{\alpha\beta\gamma}$.

Proposition 6.12 *For every $\varepsilon > 0$ there are only finitely many subgroups Γ of $\mathrm{Sp}(6, \mathbb{Z})$ containing $\Gamma(p^t)$ for some prime p and some integer t which satisfy*

$$\sum_{I \in \mathcal{I}_3^{\mathrm{glob}}} \delta(H, D_I) \geq \varepsilon \cdot \#\mathcal{I}_3^{\mathrm{glob}}$$

for the factor group $H = \Gamma/\Gamma(p^t)$.

Proof. Recall the correspondence between boundary divisors D_i in $\mathcal{A}_3^{\mathrm{Vor}}(p^t)$ and primitive $\pm$vectors v_i in $(\mathbb{Z}/p^t\mathbb{Z})^6$ described in Corollary 3.11. Given any intersection $D_I = D_{i_1} \cap D_{i_2} \cap D_{i_3}$ of three boundary divisors which is of global type, there is by Proposition 3.17 (i) a unique 2–dimensional isotropic submodule $W_1^{j_1}$ in $(\mathbb{Z}/p^t\mathbb{Z})^6$ containing the three corresponding primitive $\pm$vectors $v_{i_1}, v_{i_2}, v_{i_3}$. For simplicity, we will say in this case that the intersection is contained in $W_1^{j_1}$. Conversely, every such submodule contains an intersection of global type in that sense and the number of these intersections contained is the same for every such submodule by Lemma 3.16 (this number is given by $(1/12)p^{4t}(1 - p^{-2})(1 - p^{-1})$ as can be easily calculated by combining the formulas in Lemma 3.24 (iii) (a) and Lemma 3.10 (ii)). We write $\mathcal{I}_3^{\mathrm{glob}}(W_1^{j_1})$ for the subset of $\mathcal{I}_3^{\mathrm{glob}}$ of the intersections contained in $W_1^{j_1}$, i.e.

$$\mathcal{I}_3^{\mathrm{glob}}(W_1^{j_1}) := \left\{ \{i_1, i_2, i_3\} \in \mathcal{I}_3^{\mathrm{glob}};\ v_{i_1}, v_{i_2}, v_{i_3} \in W_1^{j_1} \right\} .$$

By our above observations the set $\mathcal{W}_1$ of all 2–dimensional isotropic submodules defines a partition $\{\mathcal{I}_3^{\mathrm{glob}}(W_1^{j_1});\ W_1^{j_1} \in \mathcal{W}_1\}$ of $\mathcal{I}_3^{\mathrm{glob}}$. Hence

$$\sum_{W_1^{j_1} \in \mathcal{W}_1} \sum_{I \in \mathcal{I}_3^{\mathrm{glob}}(W_1^{j_1})} \delta(H, D_I) = \sum_{I \in \mathcal{I}_3^{\mathrm{glob}}} \delta(H, D_I) \geq \varepsilon \cdot \#\mathcal{I}_3^{\mathrm{glob}} . \qquad (16)$$

Note that the cardinality $\#\mathcal{I}_3^{\text{glob}}(W_1^{j_1}) = (1/12)p^{4t}(1-p^{-2})(1-p^{-1})$ is the same for each $W_1^{j_1} \in \mathcal{W}_1$. We thus have

$$\#\mathcal{I}_3 = \#\mathcal{W}_1 \cdot \#\mathcal{I}_3^{\text{glob}}(W_1^{j_1}) ,$$

which implies that the inequality in (16) can be rewritten as follows:

$$\sum_{W_1^{j_1} \in \mathcal{W}_1} \frac{1}{\#\mathcal{I}_3^{\text{glob}}(W_1^{j_1})} \sum_{I \in \mathcal{I}_3^{\text{glob}}(W_1^{j_1})} \delta(H, D_I) \geq \varepsilon \cdot \#\mathcal{W}_1 . \tag{17}$$

Since $1/p^t \leq \delta(H, D_I) \leq 1$, we can conclude from this inequality by Proposition B.1 that there are at least $(\varepsilon/2) \cdot \#\mathcal{W}_1$ submodules $W_1^{j_1} \in \mathcal{W}_1$ which satisfy

$$\sum_{I \in \mathcal{I}_3^{\text{glob}}(W_1^{j_1})} \delta(H, D_I) \geq (\varepsilon/2) \cdot \#\mathcal{I}_3^{\text{glob}}(W_1^{j_1}) . \tag{18}$$

Recall from (1) that $\mathcal{I}_2$ contains all those sets $\{i_1, i_2\}$ such that the intersection $D_{i_1} \cap D_{i_2}$ of the corresponding divisors is non–trivial. In analogy to the definition of the subset $\mathcal{I}_3^{\text{glob}}(W_1^{j_1})$ of $\mathcal{I}_3^{\text{glob}}$, we define the following subset of $\mathcal{I}_2$:

$$\mathcal{I}_2(W_1^{j_1}) := \Big\{ \{i_1, i_2\} \in \mathcal{I}_2; v_{i_1}, v_{i_2} \in W_1^{j_1} \Big\}$$

We will now show that each of the submodules $W_1^{j_1}$ satisfying (18) satisfies also

$$\sum_{I \in \mathcal{I}_2(W_1^{j_1})} \text{ram}_H(D_I) \geq 2^{-52}\varepsilon^{16} \cdot \#\mathcal{I}_2(W_1^{j_1}) \tag{19}$$

and then use Proposition 6.7.

Since all $W_1^{j_1}$ are equivalent under the action of $\text{Sp}(6, \mathbb{Z}/p^t\mathbb{Z})$ it suffices to consider the submodule W_1^0 given by

$$W_1^0 := (0, *, *, 0, 0, 0) \subset (\mathbb{Z}/p^t\mathbb{Z})^6 .$$

This is naturally isomorphic to the 2–dimensional isotropic submodule V_2 of $(\mathbb{Z}/p^t\mathbb{Z})^4$ given by

$$V_2 := (*, *, 0, 0) \subset (\mathbb{Z}/p^t\mathbb{Z})^4 .$$

For every $I = \{i_1, i_2, i_3\} \in \mathcal{I}_3^{\text{glob}}(W_1^0)$ the primitive vectors $v_{i_1}, v_{i_2}, v_{i_3} \in W_1^0$ are mapped under this isomorphism to primitive vectors $v_\alpha, v_\beta, v_\gamma \in V_2$. Moreover, these primitive vectors correspond by [Bor, Proposition 2.5] to three boundary divisors $D_\alpha, D_\beta, D_\gamma$ in $\mathcal{A}_2^{\text{Vor}}(p^t)$ which intersect in a point $P_{\alpha\beta\gamma}$. This point $P_{\alpha\beta\gamma}$ is exactly that point which corresponds to the intersection $D_I \subset \mathcal{A}_3^{\text{Vor}}(p^t)$ in the sense of Proposition 6.11.

In the beginning of the proof of [Bor, Proposition 3.21] Borisov defines a number $\delta(H, P_{\alpha\beta\gamma})$ which motivated the definition of δ in (15). This δ only depends on the action of the ramification group of $P_{\alpha\beta\gamma}$ in a neighborhood of $P_{\alpha\beta\gamma}$. Note that by Definition 4.1 the isomorphism $W_1^0 \cong V_2$ induces an isomorphism $\mathrm{Ram}_G(D_I) \cong \mathrm{Ram}_{\widetilde{G}}(P_{\alpha\beta\gamma})$ which respects the actions on D_I resp. $P_{\alpha\beta\gamma}$, where $\mathrm{Ram}_{\widetilde{G}}(P_{\alpha\beta\gamma})$ denotes the ramification group of $P_{\alpha\beta\gamma}$ in $\widetilde{G} := \mathrm{Sp}(4, \mathbb{Z}/p^t\mathbb{Z})$ in the sense of [Bor, Definition 3.19]. Under this isomorphism $\mathrm{Ram}_H(D_I) < \mathrm{Ram}_G(D_I)$ is identified with a subgroup K of $\mathrm{Ram}_{\widetilde{G}}(P_{\alpha\beta\gamma})$ which can be considered as a ramification group $\mathrm{Ram}_{\widetilde{H}}(P_{\alpha\beta\gamma})$ of $P_{\alpha\beta\gamma}$ with respect to a suitable subgroup $\widetilde{H}$ of $\widetilde{G}$ (one natural choice for $\widetilde{H}$ is the subgroup K itself). Note that the groups $\mathrm{Ram}_H(D_I)$ and $K = \mathrm{Ram}_{\widetilde{H}}(P_{\alpha\beta\gamma})$ do not depend on I as long as $I \in \mathcal{I}_3^{\mathrm{glob}}(W_1^0)$ (although the action of these groups does depend on D_I resp. $P_{\alpha\beta\gamma}$). Therefore we can choose the same group $\widetilde{H}$ for all $I \in \mathcal{I}_3^{\mathrm{glob}}(W_1^0)$ and have that

$$\delta_2(\widetilde{H}, P_{\alpha\beta\gamma}) = \delta(H, D_I) \ , \tag{20}$$

where we wrote δ_2 for the δ defined in Borisov's paper to have a clear distinction. This implies that

$$\begin{aligned}\sum_{v_\alpha, v_\beta, v_\gamma \in V_2} \delta_2(\widetilde{H}, P_{\alpha\beta\gamma}) &\overset{(20)}{=} \sum_{I \in \mathcal{I}_3^{\mathrm{glob}}(W_1^0)} \delta(H, D_I) \\ &\overset{(18)}{\geq} (\varepsilon/2) \cdot \#\mathcal{I}_3^{\mathrm{glob}}(W_1^0) = (\varepsilon/2) \cdot \#(v_\alpha, v_\beta, v_\gamma \in V_2) \ .\end{aligned}$$

It now follows from the proof of [Bor, Proposition 3.21] that

$$\sum_{v_\alpha, v_\beta \in V_2} \mathrm{ram}_{\widetilde{H}}(l_{\alpha\beta}) \geq \varepsilon_1 \cdot \#(v_\alpha, v_\beta \in V_2) \ , \tag{21}$$

where $\mathrm{ram}_{\widetilde{H}}(l_{\alpha\beta})$ is the analogue of $\mathrm{ram}_H(D_{i_1} \cap D_{i_2})$ for the $g = 2$–case in Borisov's notation and $\varepsilon_1 \geq 2^{-52}\varepsilon^{16}$ (there is an obvious misprint in Borisov's paper).

As before we can now use the isomorphism $V_2 \cong W_1^0$ to conclude that

$$\mathrm{ram}_{\widetilde{H}}(l_{\alpha\beta}) = \mathrm{ram}_H(D_{i_1} \cap D_{i_2}) \tag{22}$$

for each $I = \{i_1, i_2\} \in \mathcal{I}_2(W_1^{j_1})$. Hence

$$\begin{aligned}\sum_{I \in \mathcal{I}_2(W_1^0)} \mathrm{ram}_H(D_I) &\overset{(22)}{=} \sum_{v_\alpha, v_\beta \in V_2} \mathrm{ram}_{\widetilde{H}}(l_{\alpha\beta}) \\ &\overset{(21)}{\geq} \varepsilon_1 \cdot \#(v_\alpha, v_\beta \in V_2) \geq 2^{-52}\varepsilon^{16} \cdot \#\mathcal{I}_2(W_1^0)\end{aligned}$$

as claimed in (19).

By our considerations at the beginning of this proof we are guaranteed to have at least $(\varepsilon/2) \cdot \#\mathcal{W}_1$ submodules $W_1^{j_1} \in \mathcal{W}_1$ with this property. Since the sets $\left\{\mathcal{I}_2(W_1^{j_1}\right\}$ define a partition of $\mathcal{I}_2$ we can take the sum and obtain that

$$\sum_{I \in \mathcal{I}_2} \mathrm{ram}_H(D_I) = \sum_{W_1^{j_1}} \sum_{I \in \mathcal{I}_2(W_1^{j_1})} \mathrm{ram}_H(D_I) \overset{(19)}{\geq} 2^{-53}\varepsilon^{17} \cdot \#\mathcal{I}_2 \tag{23}$$

where we used that $\#\mathcal{I}_2(W_1^{j_1})$ is the same for each isotropic submodule $W_1^{j_1}$ which implies that we have the identity

$$\#\mathcal{I}_2 = \#\mathcal{W}_1 \cdot \#\mathcal{I}_2(W_1^{j_1}) \; .$$

The result now follows from Proposition 6.7. □

The proof of Theorem 6.10 now follows by the usual reduction arguments.

Theorem 6.10 *For every $\varepsilon > 0$ there are only finitely many subgroups $\Gamma < \mathrm{Sp}(6,\mathbb{Z})$ with the following properties:*

(i) *Γ has finite index in $\mathrm{Sp}(6,\mathbb{Z})$, which means that it contains a principal congruence subgroup $\Gamma(n)$ for some level n.*

(ii) $\dfrac{1}{\#\mathcal{I}_3^{\mathrm{glob}}} \sum\limits_{I\in\mathcal{I}_3^{\mathrm{glob}}} \delta(H, D_I) \geq \varepsilon$, *where H denotes the factor group $\Gamma/\Gamma(n)$.*

Proof. This proof is a slight variation of the one given for Theorem 4.3. We will reduce the claim to the case where $n = p^t$ is a prime power which has been taken care of in Proposition 6.12. We decompose n into distinct prime powers, say

$$n = p_1^{t_1} \cdot \ldots p_k^{t_k} \; , \; (p_i, p_j) = 1 \text{ for } i \neq j \; ,$$

This gives us the following factorization of $\mathrm{Sp}(6,\mathbb{Z}/n\mathbb{Z})$:

$$\mathrm{Sp}(6,\mathbb{Z}/n\mathbb{Z}) \cong \mathrm{Sp}(6,\mathbb{Z}/p_1^{t_1}\mathbb{Z}) \times \cdots \times \mathrm{Sp}(6,\mathbb{Z}/p_k^{t_k}\mathbb{Z})$$

Exactly as in the proof of Theorem 4.3, we obtain a description of $\mathrm{Ram}_H(D_j) = \mathrm{Ram}_H(v_j)$ as

$$\mathrm{Ram}_H(v_j) \cong \mathrm{Ram}_{H_1}(v_{j_1}) \times \cdots \times \mathrm{Ram}_{H_k}(v_{j_k}) \; ,$$

where the H_i are certain projections of H and each v_{j_i} is a primitive vector in $(\mathbb{Z}/p_i^{t_i}\mathbb{Z})^6$. This induces a decomposition on

$$\begin{aligned} \mathrm{Ram}_H(D_I) &= \mathrm{Ram}_H(D_{i_1}) \oplus \mathrm{Ram}_H(D_{i_2}) \oplus \mathrm{Ram}_H(D_{i_3}) \\ &\cong \mathrm{Ram}_{H_1}(D_I^{j_1}) \times \cdots \times \mathrm{Ram}_{H_k}(D_I^{j_k}) \end{aligned}$$

for each $I = \{i_1, i_2, i_3\} \in \mathcal{I}_3^{\mathrm{glob}}$, which in particular allows us to regard $\mathrm{Ram}_H(D_I)$ as a subgroup of $(\mathbb{Z}/p_1^{t_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/p_k^{t_k}\mathbb{Z})^3$. Under this decomposition the action of $\mathrm{Ram}_H(D_I)$ in a neighborhood of a general point $P \in D_I$ is given by

$$\begin{aligned} &\Big((\xi_1^1, \ldots, \xi_1^k), (\xi_2^1, \ldots, \xi_2^k), (\xi_3^1, \ldots, \xi_3^k)\Big) \cdot (x_1, \ldots, x_6) \\ &= (e^{2\pi i(\xi_1^1/p_1^{t_1} + \ldots + \xi_1^k/p_k^{t_k})}\, x_1, e^{2\pi i(\xi_2^1/p_1^{t_1} + \ldots + \xi_2^k/p_k^{t_k})}\, x_2, e^{2\pi i(\xi_3^1/p_1^{t_1} + \ldots + \xi_3^k/p_k^{t_k})}\, x_3, x_4, x_5, x_6) \end{aligned}$$

for each $\Big((\xi_1^1,\ldots,\xi_1^k),(\xi_2^1,\ldots,\xi_2^k),(\xi_3^1,\ldots,\xi_3^k)\Big)\in \mathrm{Ram}_H(D_I)$. Note that a monomial is H–invariant with respect to this action if and only if it is H_i–invariant since all the orders of $\mathrm{Ram}_{H_i}(D_I)$ are coprime.

We can now compare $\delta(H,D_I)$ with $\delta(H_i,D_I^{j_i})$ and claim that

$$\delta(H,D_I)\le\delta(H_i,D_I^{j_i})$$

for each $i=1,\ldots,k$. Indeed, say $x_1^{l_1}x_2^{l_2}x_3^{l_3}$ is a non–zero H_i–invariant monomial with

$$\delta(H_i,D_I^{j_i})=\frac{1}{p_i^{t_i}}(l_1+l_2+l_3)\ .$$

If we consider the $(n/p_i^{t_i})$–th power of this monomial we obtain a monomial which is not only still H_i–invariant, but also H_j–invariant for all $j\neq i$ since $x_1^{l_1\cdot(n/p_i^{t_i})}$ is a power of $x_1^{(p_j^{t_j})}$ and likewise for x_2 and x_3. By our observation it is thus also H–invariant and we can estimate

$$\delta(H,D_I)\le\frac{1}{n}\cdot\left(\frac{n}{p_i^{t_i}}\cdot(l_1+l_2+l_3)\right)=\frac{1}{p_i^{t_i}}\cdot(l_1+l_2+l_3)=\delta(H_i,D_I^{j_i})$$

as claimed.

Hence we have as a necessary condition that $\delta(H_i,D_I^{j_i})\ge\varepsilon$ for all $i=1,\ldots,k$ in order to have $\delta(H,D_I)\ge\varepsilon$. Since for each $\varepsilon>0$ there are only finitely many H_i with this property by Proposition 6.12 we can conclude by the same arguments as in the proof of Theorem 4.3 that the same is true for H and we are done. □

In Chapter 7 we will use this result to control the obstructions to extending pluricanonical forms over the point lying at the intersection of three boundary divisors of global type.

Chapter 7

Putting it all together

In this chapter we will assemble all the parts from the previous chapters to finally prove the main result of this thesis:

Theorem 7.14 *There are only finitely many subgroups* Γ *of* $\mathrm{Sp}(6,\mathbb{Z})$ *of finite index such that the space of pluricanonical sections on* $\left(\widetilde{\mathcal{A}}_\Gamma^{\mathrm{Vor}}\right)^{(2)}$ *does not grow maximally.*

Throughout this chapter we will restrict to the case $g = 3$ and Γ will thus be a subgroup of $\mathrm{Sp}(6,\mathbb{Z})$ of finite index. By Theorem 1.18 it contains a principal congruence subgroup $\Gamma(n)$ of some level n. Note that with respect to the main result as stated above we can always assume that n is sufficiently big, since for each integer n there are only finitely many subgroups Γ which contain the principal congruence subgroup $\Gamma(n)$.

We will give conditions which ensure that the space of pluricanonical sections on the moduli space defined by Γ (or more precisely on the open part $\left(\widetilde{\mathcal{A}}_\Gamma^{\mathrm{Vor}}\right)^{(2)}$ away from $\widetilde{\beta}_3$ of a smooth projective model) grows maximally and will eventually show that these conditions are violated by only a finite number of subgroups Γ.

In the first two sections we will reduce the problem in several steps to some calculation of obstructions on the Voronoi compactification $\mathcal{A}_3^{\mathrm{Vor}}(n)$ of the moduli space given by $\Gamma(n)$, the moduli space of principally polarized abelian threefolds with a level n–structure. These obstructions will be calculated in Section 7.3 and will be used to complete the proof of the main result in the last section.

7.1 A first reduction

Recall from Chapter 2 that for each subgroup Γ of $\mathrm{Sp}(6,\mathbb{Z})$ of finite index the corresponding moduli space $\mathcal{A}_\Gamma := \mathcal{H}_3/\Gamma$ can be described as the quotient of the moduli space $\mathcal{A}_3(n) := \mathcal{H}_3/\Gamma(n)$ by the action of the finite group $H := \Gamma/\Gamma(n)$ which we can consider as a subgroup of $G := \mathrm{Sp}(6,\mathbb{Z}/n\mathbb{Z})$. We have seen in Section 2.4 that the Voronoi compactification $\mathcal{A}_3^{\mathrm{Vor}}(n)$ of $\mathcal{A}_3(n)$ induces a compactification of $\mathcal{A}_\Gamma$ via the action of H which we denoted by $\mathcal{A}_\Gamma^{\mathrm{Vor}}$.

Unlike $\mathcal{A}_3^{\mathrm{Vor}}(n)$, this variety is in general singular and thus needs to be desingularized to obtain a smooth projective model. Let $\widetilde{\pi}_\Gamma : \widetilde{\mathcal{A}}_\Gamma^{\mathrm{Vor}} \to \mathcal{A}_\Gamma^{\mathrm{Vor}}$ be such a desingularization of $\mathcal{A}_\Gamma^{\mathrm{Vor}}$. We write $-1+\delta$ for its minimum discrepancy in the sense of Definition 1.61. Note that as the quotient of a smooth projective variety $\mathcal{A}_3^{\mathrm{Vor}}(n)$ by a finite group H the variety $\mathcal{A}_\Gamma^{\mathrm{Vor}}$ has at most log–terminal singularities which implies that δ is a positive rational number (cf. Proposition 1.63).

With respect to the main theorem we are interested in the space of pluricanonical sections on an open subvariety of this desingularization. For that, recall from Section 2.3 the definitions of $\beta_3 \subset \mathcal{A}_3^{\mathrm{Vor}}(n)$, the locus of semi–abelian varieties with torus rank ≥ 3, and the space of rank ≤ 2–degenerations $\left(\mathcal{A}_3^{\mathrm{Vor}}(n)\right)^{(2)} = \mathcal{A}_3^{\mathrm{Vor}}(n) \setminus \beta_3$. By abuse of notation we denoted the image of β_3 under the quotient map $p : \mathcal{A}_3^{\mathrm{Vor}}(n) \to \mathcal{A}_\Gamma^{\mathrm{Vor}}$ by the same letter and wrote $\left(\mathcal{A}_\Gamma^{\mathrm{Vor}}\right)^{(2)}$ for its complement in $\mathcal{A}_\Gamma^{\mathrm{Vor}}$. This description can be extended to $\widetilde{\mathcal{A}}_\Gamma^{\mathrm{Vor}}$, if we define $\widetilde{\beta}_3$ as the preimage of β_3 under the desingularization $\widetilde{\pi}_\Gamma : \widetilde{\mathcal{A}}_\Gamma^{\mathrm{Vor}} \to \mathcal{A}_\Gamma^{\mathrm{Vor}}$ and set $\left(\widetilde{\mathcal{A}}_\Gamma^{\mathrm{Vor}}\right)^{(2)} := \widetilde{\mathcal{A}}_\Gamma^{\mathrm{Vor}} \setminus \widetilde{\beta}_3$ (cf. also Section 2.5).

For brevity we will from now on write $X(n)$ for $\mathcal{A}_3^{\mathrm{Vor}}(n)$ and X for $\mathcal{A}_\Gamma^{\mathrm{Vor}}$ whenever the group Γ resp. $\Gamma(n)$ is clear. Likewise, we denote the desingularization $\widetilde{\pi}_\Gamma$ of X by $\widetilde{\pi} : \widetilde{X} \to X$. The open subvarieties we just defined are then represented by $X(n)^{(2)}$, $X^{(2)}$ and $\widetilde{X}^{(2)}$ respectively.

We denote the canonical sheaves on $X^{(2)}$ and $\widetilde{X}^{(2)}$ by $\omega_{X^{(2)}}$ resp. $\omega_{\widetilde{X}^{(2)}}$ (cf. Definition 1.60 for the notion of a canonical sheaf on a normal variety). To obtain pluricanonical sections on $\widetilde{X}^{(2)}$, we want to have global sections in the pluricanonical sheaf $\omega_{X^{(2)}}^{\otimes m}$ which vanish of order at least $m(1-\delta)$ at each non–canonical singularity of $X^{(2)}$, i.e. which lie in $\mathfrak{m}_{X^{(2)},x}^{m(1-\delta)}\left(\omega_{X^{(2)}}^{\otimes m}\right)_x$ for every non–canonical singularity $x \in X^{(2)}$. In other words, we are interested in sections in

$$H^0(X^{(2)}, \omega_{X^{(2)}}^{\otimes m} \otimes \mathcal{J}_{X_{\mathrm{nc}}^{(2)}}^{m(1-\delta)}) ,$$

where $X_{\mathrm{nc}}^{(2)}$ denotes the locus of non–canonical singularities in $X^{(2)}$ and $\mathcal{J}_{X_{\mathrm{nc}}^{(2)}}$ its ideal sheaf. Here and from now on, we will assume that m is sufficiently big and sufficiently divisible whenever necessary. We can do this, since for the main theorem we only need to consider the space of m–canonical sections as m tends to infinity. At this point we also refer the reader to the preface in which we introduced various symbols and notations which will be used subsequently to describe the growth of certain dimensions with respect to m.

We start by pulling the problem from $\widetilde{X}^{(2)}$ to $X^{(2)}$.

Proposition 7.1 *We have*

$$\dim H^0(\widetilde{X}^{(2)}, \omega_{\widetilde{X}^{(2)}}^{\otimes m}) \geq \dim H^0(X^{(2)}, \omega_{X^{(2)}}^{\otimes m} \otimes \mathcal{J}_{X_{\mathrm{nc}}^{(2)}}^{m(1-\delta)}) \ .$$

Proof. It suffices to note that the pullback $\widetilde{\pi}^* s$ of a section $s \in H^0(X^{(2)}, \omega_{X^{(2)}}^{\otimes m} \otimes \mathcal{J}_{X_{\mathrm{nc}}^{(2)}}^{m(1-\delta)})$ vanishes outside $\widetilde{\pi}^{-1}\beta_3$ of order at least $m(1-\delta)$ at exceptional divisors with negative discrepancies. Hence $\widetilde{\pi}^* s$ is in fact a global section in the pluricanonical sheaf $\omega_{\widetilde{X}^{(2)}}^{\otimes m}$ if we restrict it to the dense open subset $\widetilde{X}^{(2)} \subset \widetilde{X}$. This defines an injective linear map from $H^0(X^{(2)}, \omega_{X^{(2)}}^{\otimes m} \otimes \mathcal{J}_{X_{\mathrm{nc}}^{(2)}}^{m(1-\delta)})$ to $H^0(\widetilde{X}^{(2)}, \omega_{\widetilde{X}^{(2)}}^{\otimes m})$ which proves the claim. □

The global sections in the pluricanonical sheaf $\omega_{X^{(2)}}^{\otimes m}$ have to vanish of order at least $m(1-\delta)$ at each non–canonical singularity of $X_{\mathrm{nc}}^{(2)}$, in particular at those non–canonical singularities which lie at the image of the intersection of three boundary divisors in $X(n)$ under the quotient map $p : X(n) \to X$. As we already remarked in Section 6.2 these singularities are closely related to the ones occurring at the deepest points in the $g = 2$–case. To estimate their obstructions we will thus be able to reduce the calculations to the corresponding result for $g = 2$ which has been shown by Borisov in [Bor, Proposition 5.6].

To carry out this reduction, we first have to introduce some notations. We define $X^{(2^\circ)}$ as the open subset of $X^{(2)}$ obtained by taking out the images of the intersections of three boundary divisors of global type, i.e.

$$X^{(2^\circ)} := X^{(2)} \setminus \{p(D_I)\, ;\ I \in \mathcal{I}_3^{\mathrm{glob}}\} \ , \tag{1}$$

where D_I and $\mathcal{I}_3^{\mathrm{glob}}$ are given as in Section 6.2. Then $X_{\mathrm{nc}}^{(2^\circ)} := X_{\mathrm{nc}}^{(2)} \cap X^{(2^\circ)}$ can be interpreted as the locus of non–canonical singularities in $X^{(2)}$ which do not lie at the image of the intersection of three boundary divisors in $X(n)$. Note that although $X_{\mathrm{nc}}^{(2^\circ)}$ is an open subset of $X^{(2)}$, it is closed in $X^{(2^\circ)}$, so it makes sense to talk about its ideal sheaf $\mathcal{J}_{X_{\mathrm{nc}}^{(2^\circ)}}$ in $X^{(2^\circ)}$. Via the natural inclusion $\iota : X^{(2^\circ)} \to X^{(2)}$, we can push this sheaf to $X^{(2)}$. We can then interpret

$$\dim H^0\Big(X^{(2)}, \omega_{X^{(2)}}^{\otimes m} \otimes \iota_* \mathcal{J}_{X_{\mathrm{nc}}^{(2^\circ)}}^{m(1-\delta)}\Big)$$

as the space of global sections in the pluricanonical sheaf $\omega_{X^{(2)}}^{\otimes m}$ which vanish of order at least $m(1-\delta)$ at each non–canonical singularity in $X_{\mathrm{nc}}^{(2^\circ)}$, i.e. which lie in $\mathfrak{m}_{X^{(2)},x}^{m(1-\delta)}\left(\omega_{X^{(2)}}^{\otimes m}\right)_x$ for every $x \in X_{\mathrm{nc}}^{(2^\circ)}$.

Proposition 7.2 *Given $\varepsilon > 0$, for all but finitely many subgroups Γ of $\mathrm{Sp}(6,\mathbb{Z})$ of finite index, the obstructions coming from singularities in $X^{(2)}$ which lie at the image of the intersection of three boundary divisors in $X(n)$ grow no faster than $\varepsilon\, m^6\, |G:H|$, i.e.*

$$\dim H^0\Big(X^{(2)}, \omega_{X^{(2)}}^{\otimes m} \otimes \iota_* \mathcal{J}_{X_{\mathrm{nc}}^{(2^\circ)}}^{m(1-\delta)}\Big) - \dim H^0\Big(X^{(2)}, \omega_{X^{(2)}}^{\otimes m} \otimes \mathcal{J}_{X_{\mathrm{nc}}^{(2)}}^{m(1-\delta)}\Big) \preceq \varepsilon\, m^6\, |G:H|$$

as $m \to \infty$.

Proof. We start by relating the obstructions to the ones in the $g = 2$–case, so we can make use of Borisov's results. The claim then will follow from our calculations in Section 6.2, in particular from Theorem 6.10.

Recall that we have a finite number of intersections D_I, $I \in \mathcal{I}_3^{\mathrm{glob}}$, of three boundary divisors in $X(n)$. We want to consider the obstructions coming from the singularities at the image of these intersections under the map $p : X(n) \to X$ separately for each D_I. For that, we make the following rather weak estimate:

$$\begin{aligned}
&\dim H^0\Big(X^{(2)}, \omega_{X^{(2)}}^{\otimes m} \otimes \iota_* \mathcal{J}_{X_{\mathrm{nc}}^{(2^\circ)}}^{m(1-\delta)}\Big) - \dim H^0\Big(X^{(2)}, \omega_{X^{(2)}}^{\otimes m} \otimes \mathcal{J}_{X_{\mathrm{nc}}^{(2)}}^{m(1-\delta)}\Big)\\
\le \sum_{p(D_I)} &\Big[\dim H^0\Big(X^{(2)}, \omega_{X^{(2)}}^{\otimes m} \otimes \iota_* \mathcal{J}_{X_{\mathrm{nc}}^{(2^\circ)}}^{m(1-\delta)}\Big)\\
&\qquad - \dim H^0\Big(X^{(2)}, \omega_{X^{(2)}}^{\otimes m} \otimes \iota_* \mathcal{J}_{X_{\mathrm{nc}}^{(2^\circ)}}^{m(1-\delta)} \otimes \mathcal{J}_{p(D_I)}^{m(1-\delta)}\Big)\Big], \qquad (2)
\end{aligned}$$

where the sum is taken in such a way that we have exactly one image $p(D_I)$ for each orbit of the D_I, $I \in \mathcal{I}_3^{\mathrm{glob}}$, in H.

For each $p(D_I)$ we have the exact sequence

$$\begin{aligned}
0 \to \omega_{X^{(2)}}^{\otimes m} \otimes \iota_* \mathcal{J}_{X_{\mathrm{nc}}^{(2^\circ)}}^{m(1-\delta)} \otimes \mathcal{J}_{p(D_I)}^{m(1-\delta)} &\hookrightarrow \omega_{X^{(2)}}^{\otimes m} \otimes \iota_* \mathcal{J}_{X_{\mathrm{nc}}^{(2^\circ)}}^{m(1-\delta)}\\
&\twoheadrightarrow \Big(\omega_{X^{(2)}}^{\otimes m} \otimes \iota_* \mathcal{J}_{X_{\mathrm{nc}}^{(2^\circ)}}^{m(1-\delta)}\Big) \Big/ \Big(\omega_{X^{(2)}}^{\otimes m} \otimes \iota_* \mathcal{J}_{X_{\mathrm{nc}}^{(2^\circ)}}^{m(1-\delta)} \otimes \mathcal{J}_{p(D_I)}^{m(1-\delta)}\Big) \to 0\,.
\end{aligned}$$

As a consequence we can estimate each summand in (2) as follows:

$$\begin{aligned}
&\dim H^0\Big(X^{(2)}, \omega_{X^{(2)}}^{\otimes m} \otimes \iota_* \mathcal{J}_{X_{\mathrm{nc}}^{(2^\circ)}}^{m(1-\delta)}\Big) - \dim H^0\Big(X^{(2)}, \omega_{X^{(2)}}^{\otimes m} \otimes \iota_* \mathcal{J}_{X_{\mathrm{nc}}^{(2^\circ)}}^{m(1-\delta)} \otimes \mathcal{J}_{p(D_I)}^{m(1-\delta)}\Big)\\
&\le \dim H^0\Big(X^{(2)}, \Big(\omega_{X^{(2)}}^{\otimes m} \otimes \iota_* \mathcal{J}_{X_{\mathrm{nc}}^{(2^\circ)}}^{m(1-\delta)}\Big) \Big/ \Big(\omega_{X^{(2)}}^{\otimes m} \otimes \iota_* \mathcal{J}_{X_{\mathrm{nc}}^{(2^\circ)}}^{m(1-\delta)} \otimes \mathcal{J}_{p(D_I)}^{m(1-\delta)}\Big)\Big)\,. \qquad (3)
\end{aligned}$$

Note that the support of this quotient sheaf lies on $p(D_I)$ which means that we can consider this problem in a neighborhood of $p(D_I)$.

Recall from Proposition 6.11 that on $X(n) = \mathcal{A}_3^{\mathrm{Vor}}(n)$ the intersection of three boundary divisors D_I is locally isomorphic to a product of a deepest point $P_{\alpha\beta\gamma}$ in $\mathcal{A}_2^{\mathrm{Vor}}(n)$ with $\mathbb{C}^3$. This description can be extended to $p(D_I) \subset X = \mathcal{A}_3^{\mathrm{Vor}}(n)/H$ as follows: $p(D_I)$ is locally given as the quotient of D_I by some subgroup of H, essentially by $\mathrm{Ram}_H(D_I)$. As in the proof of Proposition 6.12 we can find a subgroup $\widetilde{H}$ of $\widetilde{G} = \mathrm{Sp}(4, \mathbb{Z}/n\mathbb{Z})$ such that the induced action on $P_{\alpha\beta\gamma}$ is given by $\mathrm{Ram}_{\widetilde{H}}(P_{\alpha\beta\gamma}) \cong \mathrm{Ram}_H(D_I)$. If we denote the quotient map on $\mathcal{A}_2^{\mathrm{Vor}}(n)$ given by $\widetilde{H}$ by $\widetilde{p}$, we have the following diagram:

$$\begin{array}{ccccccc} X(n) \supset & D_I & \cong & P_{\alpha\beta\gamma} \times \mathbb{C}^3 & \hookleftarrow & P_{\alpha\beta\gamma} & \subset \mathcal{A}_2^{\mathrm{Vor}}(n) \\ & \Big\downarrow p & & & & \Big\downarrow \widetilde{p} & \\ X \supset & p(D_I) & \cong & \widetilde{p}(P_{\alpha\beta\gamma}) \times \mathbb{C}^3 & \hookleftarrow & \widetilde{p}(P_{\alpha\beta\gamma}) & \subset \mathcal{A}_2^{\mathrm{Vor}}(n)/\widetilde{H} \end{array}$$

Thus $p(D_I)$ is locally isomorphic to the product of the image of $P_{\alpha\beta\gamma}$ with $\mathbb{C}^3$. Moreover, the singularities on $p(D_I) \cap X^{(2)}$ are all just copies of the singularity at $\widetilde{p}(P_{\alpha\beta\gamma})$ and their obstructions can be estimated using Borisov's calculation for this singularity given in [Bor, Proposition 5.6]. This gives us that

$$\dim H^0\Big(X^{(2)}, \omega_{X^{(2)}}^{\otimes m} \otimes \iota_* \mathcal{J}_{X_{\mathrm{nc}}^{(2^\circ)}}^{m(1-\delta)}\Big) - \dim H^0\Big(X^{(2)}, \omega_{X^{(2)}}^{\otimes m} \otimes \iota_* \mathcal{J}_{X_{\mathrm{nc}}^{(2^\circ)}}^{m(1-\delta)} \otimes \mathcal{J}_{p(D_I)}^{m(1-\delta)}\Big)$$
$$\leq \mathrm{mult}_{\widetilde{p}(P_{\alpha\beta\gamma})} \frac{m^6}{6} \tag{4}$$

for all sufficiently big m, where $\widetilde{p}(P_{\alpha\beta\gamma})$ is the point corresponding to the singularities of $p(D_I) \cap X^{(2)}$ as above and $\mathrm{mult}_{\widetilde{p}(P_{\alpha\beta\gamma})}$ is the multiplicity of the local ring of $\mathcal{A}_2^{\mathrm{Vor}}(n)/\widetilde{H}$ at $\widetilde{p}(P_{\alpha\beta\gamma})$.

To be able to use Theorem 6.10 to finish the proof, we have to relate this multiplicity to $\delta(H, D_I)$ as defined in (15) in Section 6.2. For that, consider the singularity at the image of $P_{\alpha\beta\gamma}$ in the quotient of a neighborhood of $P_{\alpha\beta\gamma}$ by the group $\mathrm{Ram}_{\widetilde{H}}(P_{\alpha\beta\gamma})$. If $\mathrm{mult}_{\widetilde{H}}(P_{\alpha\beta\gamma})$ denotes the multiplicity of this singular point, this multiplicity is by [Bor, Lemma 5.7] related to $\mathrm{mult}_{\widetilde{p}(P_{\alpha\beta\gamma})}$ as follows:

$$\mathrm{mult}_{\widetilde{p}(P_{\alpha\beta\gamma})} \leq 6^3 \, \mathrm{mult}_{\widetilde{H}}(P_{\alpha\beta\gamma}) \tag{5}$$

Recall the definition of $\delta(H, D_I)$ as given in (15) in Section 6.2. This rough measure for the ring of $\mathrm{Ram}_H(D_I)$–invariant monomials coincides by (20) in the same section with $\delta_2(\widetilde{H}, P_{\alpha\beta\gamma})$ as defined by Borisov, which measures the ring of $\mathrm{Ram}_{\widetilde{H}}(P_{\alpha\beta\gamma})$–invariant monomials. By [Bor, Proposition 7.14] we can compare the multiplicities we considered above to these deltas and obtain

$$\mathrm{mult}_{\widetilde{H}}(P_{\alpha\beta\gamma}) \leq \frac{n^3 \cdot \delta_2(\widetilde{H}, P_{\alpha\beta\gamma})}{\big|\mathrm{Ram}_{\widetilde{H}}(P_{\alpha\beta\gamma})\big|} = \frac{n^3 \cdot \delta(H, D_I)}{|\mathrm{Ram}_H(D_I)|} \, . \tag{6}$$

Combining (2), (4), (5) and (6) we thus have for all sufficiently big m

$$\begin{aligned}&\dim H^0\Big(X^{(2)}, \omega_{X^{(2)}}^{\otimes m} \otimes \iota_* \mathcal{J}_{X_{\mathrm{nc}}^{(2^\circ)}}^{m(1-\delta)}\Big) - \dim H^0\Big(X^{(2)}, \omega_{X^{(2)}}^{\otimes m} \otimes \mathcal{J}_{X_{\mathrm{nc}}^{(2)}}^{m(1-\delta)}\Big) \\ &\overset{(2),(4)}{\leq} \frac{m^6}{6} \sum_{p(D_I)} \mathrm{mult}_{\widetilde{p}(P_{\alpha\beta\gamma})} \overset{(5),(6)}{\leq} 6^2 m^6 \sum_{p(D_I)} \frac{n^3}{|\mathrm{Ram}_H(D_I)|} \cdot \delta(H, D_I) \ . \end{aligned} \tag{7}$$

The last part of this proof will be dedicated to relating the sum over $p(D_I) \subset X$ in (7) to the sum over all $D_I \subset X(n)$, $I \in \mathcal{I}_3^{\mathrm{glob}}$, as given in Theorem 6.10.

Recall from Lemma 3.16 that the action of G on the intersections of global type D_I in $X(n)$ is transitive. While this is no longer true for the subgroup H, we nevertheless know how many D_I are in each orbit of this action. This number is given by

$$\frac{|H|}{\mathrm{Stab}_H(D_I)}$$

for each D_I, where $\mathrm{Stab}_H(D_I)$ denotes the stabilizer of D_I in H (not pointwise, but in the sense of an invariant subset).

Since $\mathrm{Stab}_H(D_I)$, $\mathrm{Ram}_H(D_I)$ and $\delta(H, D_I)$ are invariant within each orbit, we can rewrite the sum in (7) in terms of intersections D_I in $X(n)$ as follows:

$$\sum_{p(D_I)} \frac{n^3}{|\mathrm{Ram}_H(D_I)|} \cdot \delta(H, D_I) = \sum_{I \in \mathcal{I}_3^{\mathrm{glob}}} \frac{\mathrm{Stab}_H(D_I)}{|H|} \cdot \frac{n^3}{|\mathrm{Ram}_H(D_I)|} \cdot \delta(H, D_I) \ . \tag{8}$$

Note that

$$\frac{|\mathrm{Stab}_H(D_I)|}{|\mathrm{Ram}_H(D_I)|} \leq \frac{|\mathrm{Stab}_G(D_I)|}{|\mathrm{Ram}_G(D_I)|} \ . \tag{9}$$

Indeed, $\mathrm{Ram}_H(D_I)$ is just the intersection of $\mathrm{Ram}_G(D_I)$ with $\mathrm{Stab}_H(D_I)$ and we then have

$$\begin{aligned}\mathrm{Stab}_H(D_I)/\,\mathrm{Ram}_H(D_I) = &\quad \mathrm{Stab}_H(D_I)/\Big(\mathrm{Ram}_G(D_I) \cap \mathrm{Stab}_H(D_I)\Big) \\ \cong &\ \Big(\mathrm{Stab}_H(D_I) \cdot \mathrm{Ram}_G(D_I)\Big)/\,\mathrm{Ram}_G(D_I) \\ < &\quad \mathrm{Stab}_G(D_I)/\,\mathrm{Ram}_G(D_I)\end{aligned}$$

which shows (9).

This implies that for all sufficiently big m

$$\begin{aligned}&\dim H^0\Big(X^{(2)}, \omega_{X^{(2)}}^{\otimes m} \otimes \iota_* \mathcal{J}_{X_{\mathrm{nc}}^{(2^\circ)}}^{m(1-\delta)}\Big) - \dim H^0\Big(X^{(2)}, \omega_{X^{(2)}}^{\otimes m} \otimes \mathcal{J}_{X_{\mathrm{nc}}^{(2)}}^{m(1-\delta)}\Big) \\ &\overset{(7),(8)}{\leq} 6^2 m^6 \sum_{I \in \mathcal{I}_3^{\mathrm{glob}}} \frac{\mathrm{Stab}_H(D_I)}{|H|} \cdot \frac{n^3}{|\mathrm{Ram}_H(D_I)|} \cdot \delta(H, D_I) \\ &\overset{(9)}{\leq} 6^2 m^6 \sum_{I \in \mathcal{I}_3^{\mathrm{glob}}} \frac{\mathrm{Stab}_G(D_I)}{|H|} \cdot \delta(H, D_I) \ , \end{aligned} \tag{10}$$

where we also used that $|\mathrm{Ram}_G(D_I)| = n^3$.

Using that the action of G on the intersections D_I of global type is transitive, we obtain the identity

$$\#\mathcal{I}_3^{\mathrm{glob}} = \frac{|G|}{|\mathrm{Stab}_G(D_I)|}$$

which allows us to rewrite (10) as follows:

$$\begin{aligned} &\dim H^0\Big(X^{(2)}, \omega_{X^{(2)}}^{\otimes m} \otimes \iota_* \mathcal{J}_{X_{\mathrm{nc}}^{(2^\circ)}}^{m(1-\delta)}\Big) - \dim H^0\Big(X^{(2)}, \omega_{X^{(2)}}^{\otimes m} \otimes \mathcal{J}_{X_{\mathrm{nc}}^{(2)}}^{m(1-\delta)}\Big) \\ \leq & 6^2 m^6 \, [G:H] \cdot \Bigg(\frac{1}{\#\mathcal{I}_3^{\mathrm{glob}}} \sum_{I \in \mathcal{I}_3^{\mathrm{glob}}} \delta(H, D_I) \Bigg) . \end{aligned} \tag{11}$$

The claim now follows from Theorem 6.10. $\square$

Recall that global sections in $\omega_{X^{(2)}}^{\otimes m} \otimes \iota_* \mathcal{J}_{X_{\mathrm{nc}}^{(2^\circ)}}^{m(1-\delta)}$ are coming from Γ–invariant modular forms of weight $4m$ which satisfy certain vanishing conditions at the boundary and at $X_{\mathrm{nc}}^{(2^\circ)}$ (cf. Theorem 2.5 and Remark 2.6). Rather than looking at the space of all these forms, we will look at modular forms which can be expressed as the product of a weight $3m$ form with a weight m form. More precisely, we will show that there is a special weight $3m$ modular form f_{3m} which satisfies all the necessary vanishing conditions and multiply it with an arbitrary modular form f_m. The resulting form $f_{3m} \cdot f_m$ has weight $4m$ and gives a global section in $\omega_{X^{(2)}}^{\otimes m} \otimes \iota_* \mathcal{J}_{X_{\mathrm{nc}}^{(2^\circ)}}^{m(1-\delta)}$ as desired. Since f_m was arbitrary we will obtain sufficiently many different sections which can be lifted to $\widetilde{X}^{(2)}$ to conclude that the space of m–canonical sections on $\widetilde{X}^{(2)}$ grows maximally, i.e. as m^6, as we will show in the next proposition.

However, we first have to impose an extra condition on the group Γ which ensures the extensibility of these forms over the singularities we took care of in Proposition 7.2.

Remark 7.3 *Given* $\varepsilon = 1/(6! \cdot 362880)$ *there are by the above proposition only finitely many subgroups* Γ *of* $\mathrm{Sp}(6, \mathbb{Z})$ *of finite index not satisfying*

$$\begin{aligned} &\dim H^0\Big(X^{(2)}, \omega_{X^{(2)}}^{\otimes m} \otimes \iota_* \mathcal{J}_{X_{\mathrm{nc}}^{(2^\circ)}}^{m(1-\delta)}\Big) - \dim H^0\Big(X^{(2)}, \omega_{X^{(2)}}^{\otimes m} \otimes \mathcal{J}_{X_{\mathrm{nc}}^{(2)}}^{m(1-\delta)}\Big) \\ \preceq & \frac{1}{6!} \cdot \frac{1}{362880} \, m^6 \, |G:H| \end{aligned} \tag{$*$}$$

as $m \to \infty$.
We will say that Γ *satisfies* $(*)$ *if* Γ *does not belong to this finite exceptional set.*

We can now formulate the proposition.

Proposition 7.4 *Let Γ be a subgroup of* $\mathrm{Sp}(6,\mathbb{Z})$ *satisfying* $(*)$. *If there is a non–trivial global section in* $H^0(X^{(2)}, \mathcal{O}_{X^{(2)}}(m(K_{X^{(2)}} - L)) \otimes \iota_* \mathcal{J}^{m(1-\delta)}_{X^{(2^\circ)}_{\mathrm{nc}}})$ *for all sufficiently big m, then the space of pluricanonical sections on $\widetilde{X}^{(2)}$ grows maximally.*

Proof. Note that any global section in $H^0(X^{(2)}, \mathcal{O}_{X^{(2)}}(mL))$ multiplied with the global section in $H^0(X^{(2)}, \mathcal{O}_{X^{(2)}}(m(K_{X^{(2)}} - L)) \otimes \iota_* \mathcal{J}^{m(1-\delta)}_{X^{(2^\circ)}_{\mathrm{nc}}})$ gives a section in $H^0(X^{(2)}, \mathcal{O}_{X^{(2)}}(mK_{X^{(2)}}) \otimes \iota_* \mathcal{J}^{m(1-\delta)}_{X^{(2^\circ)}_{\mathrm{nc}}})$. Hence

$$\dim H^0(X^{(2)}, \mathcal{O}_{X^{(2)}}(mK_{X^{(2)}}) \otimes \iota_* \mathcal{J}^{m(1-\delta)}_{X^{(2^\circ)}_{\mathrm{nc}}}) \geq \dim H^0(X^{(2)}, \mathcal{O}_{X^{(2)}}(mL)) \,, \qquad (12)$$

provided that $H^0(X^{(2)}, \mathcal{O}_{X^{(2)}}(m(K_{X^{(2)}} - L)) \otimes \iota_* \mathcal{J}^{m(1-\delta)}_{X^{(2^\circ)}_{\mathrm{nc}}})$ is non–trivial as guaranteed by our hypothesis.

The dimension of the space on the right hand side of this inequality is just the dimension of the space of modular forms of weight m with respect to Γ which we calculated in Corollary 2.13 to be

$$\dim H^0(X^{(2)}, \mathcal{O}_{X^{(2)}}(mL)) = \dim[\Gamma, m] = \frac{1}{6!} \cdot \frac{1}{181440} \cdot [\mathrm{Sp}(6,\mathbb{Z}) : \Gamma]\, m^6 \qquad (13)$$

if $-\mathbb{1} \notin \Gamma$ and

$$\dim H^0(X^{(2)}, \mathcal{O}_{X^{(2)}}(mL)) = \dim[\Gamma, m] = \frac{1}{6!} \cdot \frac{1}{90720} \cdot [\mathrm{Sp}(6,\mathbb{Z}) : \Gamma]\, m^6 \qquad (14)$$

otherwise. By Proposition 7.1 and Proposition 7.2 we thus have

$$\begin{aligned}
\dim H^0(\widetilde{X}^{(2)}, \omega^{\otimes m}_{\widetilde{X}^{(2)}}) \geq\ & \dim H^0(X^{(2)}, \omega^{\otimes m}_{X^{(2)}} \otimes \mathcal{J}^{m(1-\delta)}_{X^{(2)}_{\mathrm{nc}}}) \\
\overset{(*)}{\geq}\ & H^0(X^{(2)}, \omega^{\otimes m}_{X^{(2)}} \otimes \iota_* \mathcal{J}^{m(1-\delta)}_{X^{(2^\circ)}_{\mathrm{nc}}}) - \frac{1}{6!} \cdot \frac{1}{362880}\, m^6\, |G:H| \\
\overset{(12)}{\geq}\ & H^0(X^{(2)}, \mathcal{O}_{X^{(2)}}(mL)) - \frac{1}{6!} \cdot \frac{1}{362880}\, m^6\, |G:H| \\
\overset{(13),(14)}{\geq}\ & \frac{1}{6!} \cdot \frac{1}{362880}\, m^6 \cdot |G:H| \ .
\end{aligned}$$

This means that the space of m–canonical sections on $\widetilde{X}^{(2)}$ grows as fast as m^6 as claimed. □

As a first step we have thus reduced the problem on the desingularization $\widetilde{X}$ to a problem on X itself.

7.2 Reduction to $\mathcal{A}_3^{\mathrm{Vor}}(n)$

We know much more about the space $X(n)$ than we know about X. Not only do we have a better understanding of the geometry at the boundary but also and most importantly we know the G–invariant part of the Chow ring of $X(n)$. Therefore we want to do our calculations for the obstructions on $X(n)$ rather than on X itself.

Recall that X is obtained from $X(n)$ by taking the quotient by $H < \mathrm{Sp}(6, \mathbb{Z}/n\mathbb{Z})$, so we have a natural quotient map $p : X(n) \to X$. Note that the involution $-\mathbb{1} \in \mathrm{Sp}(6, \mathbb{Z}/n\mathbb{Z})$ acts trivially, which means that the map p is effectively given by $H/\{\pm\mathbb{1}\}$. For simplicity we will from now on assume w.l.o.g. that $-\mathbb{1} \in H$ and consider H as a subgroup of $\mathrm{PSp}(6, \mathbb{Z}/n\mathbb{Z})$ whenever we consider the quotient map p.

We need to impose conditions on an H–invariant pluricanonical form on $X(n)$ such that it descends to a global section in $H^0(X^{(2)}, \mathcal{O}_{X^{(2)}}(m(K_{X^{(2)}} - L)) \otimes \iota_* \mathcal{J}^{m(1-\delta)}_{X^{(2^\circ)}_{\mathrm{nc}}})$ under the quotient map p. For that, we have to take into account that p is branched. Thus we will not only have to ensure that the pluricanonical form on $X(n)$ vanishes of sufficiently high order along the preimage of $X^{(2^\circ)}_{\mathrm{nc}}$ but also along the branch divisors. We will address each of these problems individually in the following sections and then assemble all the parts.

7.2.1 Branch divisors

Consider the morphism $\mu : \mathcal{A}_3^{\mathrm{Vor}}(n) \to \mathcal{A}_3^{\mathrm{Vor}}$ which is given by taking the quotient by the group $\mathrm{PSp}(6, \mathbb{Z}/n\mathbb{Z})$. It is branched of order n along the boundary and its ramification divisor is thus given by $(n-1)\sum_\alpha D_\alpha$. Note that unlike in the $g = 2$–case where we also have ramification divisors in the interior of $\mathcal{A}_2^{\mathrm{Vor}}(n)$ (cf. [Bor, Proposition 5.12]), although there is ramification in the interior for $g \geq 3$, it occurs only in higher codimension and thus does not contribute to the ramification divisor (cf. [Tai, p. 439]).

Recall that the morphism $p : X(n) = \mathcal{A}_3^{\mathrm{Vor}}(n) \to X$ is given by the action of the subgroup $H = \Gamma/\Gamma(n)$ of $\mathrm{PSp}(6, \mathbb{Z}/n\mathbb{Z})$ and can thus be considered as a partial quotient map when compared to μ. Hence its ramification divisor is contained in the ramification divisor of μ. The subgroup of $\mathrm{PSp}(6, \mathbb{Z}/n\mathbb{Z})$ fixing all points of the divisor D_α has order n and is just the group $\mathrm{Ram}_G(D_\alpha)$ we introduced in Chapter 4. The corresponding subgroup of H is then given by $\mathrm{Ram}_H(D_\alpha)$ and has order $n\,\mathrm{ram}_H(D_\alpha)$. We summarize our discussion on the branch divisors in the following proposition:

Proposition 7.5 *The ramification divisor of the map $p : X(n) \to X$ equals $\sum_\alpha (n\,\mathrm{ram}_H(D_\alpha) - 1)\, D_\alpha$.*

7.2.2 Singularities

We will now determine the vanishing conditions at the preimage of $X_{\mathrm{nc}}^{(2^\circ)}$, the locus of non–canonical singularities in $X^{(2)}$ which do not lie at the image of the intersection of three boundary divisors. As we have seen in Proposition 7.4 we need a global section in $H^0(X^{(2)}, \mathcal{O}_{X^{(2)}}(m(K_{X^{(2)}} - L)) \otimes \iota_* \mathcal{J}_{X_{\mathrm{nc}}^{(2^\circ)}}^{m(1-\delta)})$, so in particular we need the section to vanish of order at least $m(1-\delta)$ along $X_{\mathrm{nc}}^{(2^\circ)}$. For that, we have to find out what order of vanishing is required on the preimage of $X_{\mathrm{nc}}^{(2^\circ)}$ in $X(n)$ to ensure that we get the desired order on $X_{\mathrm{nc}}^{(2^\circ)}$ after pushing down to X. This is essentially a question on the local rings of the corresponding points in $X(n)$ and X and their maximal ideals. The answer can be found in the appendix of Borisov's paper. But first, we need to recall some definitions from group theory.

Recall that a group G is called *solvable* if it has a normal series whose factor groups are all abelian, i.e. if there is a sequence of normal subgroups

$$\{\mathrm{id}\} = G_0 \lhd G_1 \lhd \ldots \lhd G_s = G$$

such that G_i/G_{i-1} is abelian for all $i = 1, \ldots, s$. If G is also finite, all the factor groups G_i/G_{i-1} are finite and we can denote the *exponent* of G_i/G_{i-1} by k_i, i.e. k_i is the smallest positive integer n such that $g^n = \mathrm{id}$ for all $g \in G$. We can then consider the product $k := k_1 \cdot \cdots \cdot k_s$ and define $k(G)$ to be the smallest integer k that can be obtained in this way if we consider all possible normal series of G. Clearly $k(G)$ is bounded by the order of G.

Proposition 7.6 *Let X be a projective algebraic variety with an action of a finite solvable group G and $Y = X/G$ be the corresponding quotient. Then there is a constant N such that for all $x \in X$ which satisfy $gx = x$ for all $g \in G$, the inclusion*

$$\mathfrak{m}_{X,x}^{k(G)l+N} \cap \mathcal{O}_{X,x}^G \subset \mathfrak{n}_{Y,y}^l$$

holds for all $l \geq 0$, where $(\mathcal{O}_{X,x}, \mathfrak{m}_{X,x})$ is the local ring of x in X and $(\mathcal{O}_{Y,y}, \mathfrak{n}_{Y,y}) = (\mathcal{O}_{X,x}^G, \mathfrak{m}_{X,x}^G)$ is the local ring of the image $y \in Y$ of x under the quotient morphism.

Proof. [Bor, Proposition 7.10] □

Remark 7.7 *This proposition can be interpreted for $l >> 0$ as follows. The constant N is then dominated by l and we obtain that a function which vanishes* *at a point $x \in X$ of order $k(G)l$ vanishes roughly of order l at its image in Y, or more precisely: for all sufficiently big $l >> 0$ and all $x \in X$ satisfying the hypothesis of the proposition we have the inclusion $\mathfrak{m}_{X,x}^{(k(G)+1)l} \cap \mathcal{O}_{X,x}^G \subset \mathfrak{n}_{Y,y}^l$.*

To formulate the necessary vanishing conditions on the preimage of $X_{\mathrm{nc}}^{(2^\circ)}$ in $X(n)$ we thus need to know the stabilizers of these points in $H = \Gamma/\Gamma(n)$. We start by looking at the ones corresponding to non–canonical singularities in the interior.

7.2.2.1 Singularities in the interior

Recall that by Corollary 5.11 the preimage of the locus of these non–canonical singularities is contained in the union of those X_α for which the corresponding involution φ_α is an element of H. While the stabilizer of a general point of such an X_α just contains this involution, we will in general have larger stabilizers at special points. These special points correspond to abelian varieties with extra automorphisms. Recall that each point in X_α corresponds to a product of an elliptic curve E with an abelian surface A. We can thus determine the stabilizers by looking at the automorphisms of special elliptic curves and special abelian surfaces.

It is well–known that the order of the automorphism group of an elliptic curve is at most 6 (realized by E_0). For an abelian surface Borisov showed in the proof of Proposition 4.3 in [Bor] that the stabilizer of any point in $\mathcal{H}_2$ is a solvable group of order at most 72. Since the direct product of solvable groups is again solvable, we obtain a solvable group of order at most 432. Additionally, there might be automorphisms permuting components of the product $E \times A$, which can be the case if the abelian surface A itself is a product of two elliptic curves. This would then be a subgroup of the permutation group S_3. However, those permutations which fix E and only permute components in A are already included in our calculation of the order of the stabilizer of a point in $\mathcal{H}_2$. The other permutations thus only contribute a factor of at most 3. Overall, we get that the stabilizer of any point in X_α is a group of order at most $3 \cdot 432 = 1296$. The fact that its normal subgroup of index at most 3 containing the trivial permutations is solvable implies that the stabilizer itself is solvable. This finishes the proof of the following proposition:

Proposition 7.8 *Let $x \in X_\alpha$ for some index α. Then the order of the stabilizer in H of the point x is a solvable group of order at most* 1296.

Remark 7.9 *This order is realized by the automorphism group of the product of three copies of the elliptic curve E_0. There is an automorphism of order* 6 *on each component of this product and an action of S_3 permuting the factors which gives the order of* $6^4 = 1296$.

7.2.2.2 Singularities in the boundary

We will now determine the vanishing conditions for the preimages of the non–canonical singularities in the boundary. We want to use Proposition 7.6 and thus need to determine the stabilizers of these points. Unlike the stabilizers of points in the interior, the stabilizers in the boundary depend on the level n. However, we will see that the only contribution of n comes from the ramification groups we already studied in Chapter 4.

Recall that any divisor D_α can be considered as the closure of the preimage of a top–dimensional component $\mathcal{A}_2^\alpha(n)$ of the Satake compactification. If we just consider the preimage D_α° of a component $\mathcal{A}_2^\alpha(n)$, we obtain a fibration $D_\alpha^\circ \to \mathcal{A}_{g-1}^\alpha(n) = \mathcal{A}_2(n)$ which is the universal family of abelian surfaces with level–n structure for $n \geq 3$ (cf. [Hul, Lemma 2.1]). This fibration can be extended to a map $D_\alpha \to \mathcal{A}_2^{\mathrm{Vor}}(n)$ which still has a geometric interpretation as given in [Hul, Proposition 3.1]. When we now look at the action of $H < G = \mathrm{PSp}(6, \mathbb{Z}/n\mathbb{Z})$ on D_α, we can interpret it in terms of this fibration.

From the description in Chapter 3 we know that the divisors D_α are in one–to–one correspondence with primitive $\pm$vectors which are in turn in one–to–one correspondence with isotropic lines. Since all divisors D_α are equivalent under the action of G, it suffices to look at the standard divisor resp. the standard line. Its stabilizer is generated by the image in G of certain elements $g_1, g_2, g_3, g_4 \in \mathrm{PSp}(6, \mathbb{Z})$ given on page 260 of [Hul]. They operate on the fibration $D_\alpha \to \mathcal{A}_2^{\mathrm{Vor}}(n)$ by a combination of modular transformations of the base, additions of points of order n in the fibers, and the involution $x \mapsto -x$ of the fibers. Note that the elements of type g_4 form the group $\mathrm{Ram}_G(D_\alpha)$ as described in Chapter 4. It is easy to check that they fix the divisor D_α pointwise and are in fact the stabilizer of the general point on D_α. Hence if we look at the stabilizer of a general point on D_α in the subgroup H, we obtain that it is given by $\mathrm{Ram}_H(D_\alpha)$. However, at certain special points the stabilizer might be larger.

Proposition 7.10 *Let $x \in D_\alpha^\circ$ for some index α. Then $\mathrm{Stab}_H(x)/\mathrm{Ram}_H(D_\alpha)$ is a solvable group of order at most 144.*

Proof. We have seen in the above discussion that the stabilizer of the general point is given by $\mathrm{Ram}_H(D_\alpha)$, the group generated by elements of type g_4 in H. It thus remains to look at the actions of the elements g_1, g_2 and g_3. The operation of g_1 on the base of the fibration $D_\alpha^\circ \to \mathcal{A}_2(n)$ is the one coming from modular transformations on $\mathcal{H}_2$. As we already remarked in the previous section this gives us a solvable group of order at most 72. While g_2 is just an involution, it is easy to check that g_3 operates on D_α° without fix points. This gives the bound of 144 for the order. The solvability follows from the solvability of its normal subgroup of order at most 72. □

We will now determine the stabilizers of the points lying on the intersection $D_{i_1} \cap D_{i_2}$ of two boundary divisors. Since G acts on the set of boundary divisors, every element in the stabilizer in G has to either leave D_{i_1} and D_{i_2} invariant or has to switch them. We will restrict to the subgroup of index 2 which leaves both D_{i_1} and D_{i_2} invariant. Certainly we have the groups $\operatorname{Ram}_G(D_{i_1})$ and $\operatorname{Ram}_G(D_{i_2})$ in this stabilizer since they fix all points of D_{i_1} resp. D_{i_2} as we have seen in the above discussion. They generate the group $\operatorname{Ram}_G(D_{i_1} \cap D_{i_2})$ we introduced in Section 6.1. It will follow from our discussion that this is the stabilizer of the general point on $D_{i_1} \cap D_{i_2}$. If we consider the subgroup H of G, we thus get that the stabilizer in H of the general point is given by $\operatorname{Ram}_H(D_{i_1} \cap D_{i_2})$.

Proposition 7.11 *Let $x \in D_{i_1} \cap D_{i_2}$ be a point lying on the intersection of two boundary divisors which does not lie on the intersection of three or more divisors. Then $\operatorname{Stab}_H(x)/\operatorname{Ram}_H(D_{i_1} \cap D_{i_2})$ is a solvable group of order at most 24.*

Proof. As we already remarked $\operatorname{Stab}_H(x)$ contains a subgroup of index 2 which leaves both D_{i_1} and D_{i_2} invariant. We can thus consider the map $D_{i_1} \to \mathcal{A}_2^{\text{Vor}}(n)$ we used above when we determined the stabilizer of a point on D_{i_1}. We are now over a point P in $\mathcal{A}_2^{\text{Vor}}(n) \setminus \mathcal{A}_2$ of type II (cf. [Hul, p. 266]), which means that P lies on the open part of a boundary divisor of $\mathcal{A}_2^{\text{Vor}}(n)$. It follows from [Bor, Proposition 5.13] that the contribution coming from this point P is a group of order at most $6\,|\operatorname{Ram}_H(D_{i_2})|$.

In terms of the elements $g_1, g_2, g_3, g_4 \in \operatorname{PSp}(6, \mathbb{Z})$ given on page 260 of [Hul], we still have to consider g_2 and g_4. While g_2 is just an involution, the elements of type g_4 generate $\operatorname{Ram}_H(D_{i_1})$. Regarding the order of $\operatorname{Stab}_H(x)/\operatorname{Ram}_H(D_{i_1} \cap D_{i_2})$ we have a contribution of a factor of 2 from g_2, another factor of 2 from our assumption on the invariance of D_{i_1} and D_{i_2} and a factor of 6 from Borisov's result. This gives the order of 24 as claimed. The solvability follows from the fact that the smallest non–solvable group is A_5 having order 60 or can be seen directly by a short calculation. □

7.2.3 Conclusions

With the results of the previous sections we can now formulate conditions on a pluricanonical form on $X(n)$ which guarantees that it can be pushed down to a global section in $H^0\Big(X^{(2)}, \mathcal{O}_{X^{(2)}}(m(K_{X^{(2)}} - L)) \otimes \iota_* \mathcal{J}^{m(1-\delta)}_{X^{(2^\circ)}_{\mathrm{nc}}}\Big)$ under the quotient map p.

Proposition 7.12 *If Γ satisfies $(*)$ and if there is a non–trivial global section in*

$$H^0\Big(X(n), \mathcal{O}_{X(n)}\Big(m(K-L) - m\sum_\alpha n\,\mathrm{ram}_H(D_\alpha)144D_\alpha\Big) \otimes \prod_\alpha \mathcal{J}_{X_\alpha}^{1296m\cdot\mathrm{ram}_H(X_\alpha)} \otimes \prod_{I\in\mathcal{I}_2} \mathcal{J}_{D_I}^{24mn\cdot\mathrm{ram}_H(D_I)}\Big)$$

for all sufficiently big m, then the space of pluricanonical sections on $\widetilde{X}^{(2)}$ grows maximally.

Proof. It suffices to show that $H^0\Big(X^{(2)}, \mathcal{O}_{X^{(2)}}(m(K_{X^{(2)}} - L)) \otimes \iota_* \mathcal{J}^{m(1-\delta)}_{X^{(2^\circ)}_{\mathrm{nc}}}\Big)$ has a non–trivial global section for all sufficiently big m, because we can then use Proposition 7.4.

Given any non–trivial section as in the hypothesis of the proposition, we can multiply it with all its H–conjugates to obtain an H–invariant global section in

$$H^0\Big(X(n), \mathcal{O}_{X(n)}\Big(\widetilde{m}(K-L) - \widetilde{m}\sum_\alpha n\,\mathrm{ram}_H(D_\alpha)144D_\alpha\Big) \otimes \prod_\alpha \mathcal{J}_{X_\alpha}^{1296\widetilde{m}\cdot\mathrm{ram}_H(X_\alpha)} \otimes \prod_{I\in\mathcal{I}_2} \mathcal{J}_{D_I}^{24\widetilde{m}n\cdot\mathrm{ram}_H(D_I)}\Big),$$

where $\widetilde{m}$ is the product of m with the number of H–conjugates. This H–invariant section can be pushed down to give an element of $H^0\Big(X, \mathcal{O}_X(\widetilde{m}(K_X - L)\Big)$, since it vanishes at the the ramification divisor of the map $p : X(n) \to X$ as calculated in Proposition 7.5. Moreover, for every point $y \in X_\alpha$ we have

$$\mathfrak{m}_{X(n),y}^{1296\widetilde{m}\cdot\mathrm{ram}_H(X_\alpha)} \cap \mathcal{O}_{X(n),y}^{\mathrm{Stab}_H(y)} \subset \mathfrak{m}_{X,p(y)}^{\widetilde{m}(1-\delta)}\,. \tag{15}$$

Indeed, by Proposition 7.8 we have

$$1296\widetilde{m}\cdot\mathrm{ram}_H(X_\alpha) \geq \widetilde{m}\cdot k(\mathrm{Stab}_H(y)) \tag{16}$$

where $k(\mathrm{Stab}_H(y))$ is defined as in Section 7.2.2. We can thus apply Proposition 7.6 to obtain (15), where the $-\widetilde{m}\delta$ is dropped from the vanishing conditions upstairs to compensate for the constant N from the proposition as the inequality

$$\begin{aligned} &1296\widetilde{m} \cdot \mathrm{ram}_H(X_\alpha) \overset{(16)}{\geq} \widetilde{m} \cdot k(\mathrm{Stab}_H(y)) \\ &> \widetilde{m} \cdot k(\mathrm{Stab}_H(y)) \underbrace{-\widetilde{m} \cdot k(\mathrm{Stab}_H(y))\delta + N}_{<0 \text{ for } \widetilde{m}>>0} = k(\mathrm{Stab}_H(y))\widetilde{m}(1-\delta) + N \ , \end{aligned}$$

which holds for all sufficiently big $\widetilde{m} >> 0$, shows. The condition on $\widetilde{m}$ can be made precise and can be given in such a way that it only depends on the group H. We can therefore restrict w.l.o.g. to considering only the non–trivial sections from the hypothesis of the proposition for those m which are sufficiently big to satisfy the condition on $\widetilde{m}$.

Likewise using Proposition 7.10 we obtain for any point $y \in D_\alpha^\circ$ that

$$\mathfrak{m}_{X(n),y}^{144\widetilde{m}n\cdot \mathrm{ram}_H(D_\alpha)} \cap \mathcal{O}_{X(n),y}^{\mathrm{Stab}_H(y)} \subset \mathfrak{m}_{X,p(y)}^{\widetilde{m}(1-\delta)} \ . \tag{17}$$

For any point $y \in D_I = D_{i_1} \cap D_{i_2}$ lying on the intersection of two boundary divisors which does not lie on the intersection of three or more divisors we get by Proposition 7.11 that

$$\mathfrak{m}_{X(n),y}^{24\widetilde{m}n\cdot \mathrm{ram}_H(D_I)} \cap \mathcal{O}_{X(n),y}^{\mathrm{Stab}_H(y)} \subset \mathfrak{m}_{X,p(y)}^{\widetilde{m}(1-\delta)} \ . \tag{18}$$

Here we additionally used that, although the order of $\mathrm{Stab}_H(y)$ is bounded only by $24\,|\,\mathrm{Ram}_H(D_I)|$ which can be as big as $24n^2$, the value for $k(\mathrm{Stab}_H(y))$ is at most $24n\,\mathrm{ram}_H(D_I)$, which is no larger than $24n$. This is due to the fact that $\mathrm{Ram}_H(D_I)$ is contained in $\mathrm{Ram}_G(D_I) \cong \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ which has exponent n.

The inclusions in (15), (17) and (18) imply that we have in fact a section which vanishes at the non–canonical singularities in $X_{\mathrm{nc}}^{(2^\circ)} \subset X^{(2)}$ of order $\widetilde{m}(1-\delta)$ and thus defines a global section in $H^0\Big(X^{(2)}, \mathcal{O}_{X^{(2)}}(\widetilde{m}(K_{X^{(2)}} - L)) \otimes \iota_* \mathcal{J}_{X_{\mathrm{nc}}^{(2^\circ)}}^{\widetilde{m}(1-\delta)}\Big)$ when restricted to $X^{(2)}$. Since this argument works for all sufficiently big m (resp. $\widetilde{m}$), we can apply Proposition 7.4 to finish the proof. □

In order to show the existence of a non–trivial global section as in the above proposition, we will have to calculate the obstructions imposed by these vanishing conditions. The following proposition allows us to do this separately for each condition.

Proposition 7.13 *Let Γ be a subgroup of $\mathrm{Sp}(2g, \mathbb{Z})$ such that the space of pluri-canonical sections on $\widetilde{X}^{(2)}$ does not grow maximally. Then at least one of the following conditions is satisfied:*

(i) Γ *does not satisfy* $(*)$.

(ii)
$$\begin{aligned}&\dim H^0\Big(X(n), m(K-L)\Big)\\ &\qquad - \dim H^0\Big(X(n), m(K-L) - m\sum_\alpha n\,\mathrm{ram}_H(D_\alpha)144D_\alpha\Big)\\ &\succeq \frac{1}{4}\Big((1/6!)\,c_1(K_{X(n)}-L)^6m^6\Big) \qquad \textit{as } m\to\infty\,.\end{aligned}$$

(iii)
$$\begin{aligned}&\dim H^0\Big(X(n), m(K-L)\Big)\\ &\qquad - \dim H^0\Big(X(n), \mathcal{O}_{X(n)}\Big(m(K-L)\Big)\otimes\prod_\alpha \mathcal{J}_{X_\alpha}^{1296m\cdot\mathrm{ram}_H(X_\alpha)}\Big)\\ &\succeq \frac{1}{4}\Big((1/6!)\,c_1(K_{X(n)}-L)^6m^6\Big) \qquad \textit{as } m\to\infty\,.\end{aligned}$$

(iv)
$$\begin{aligned}&\dim H^0\Big(X(n), m(K-L)\Big)\\ &\qquad - \dim H^0\Big(X(n), \mathcal{O}_{X(n)}\Big(m(K-L)\Big)\otimes\prod_{I\in\mathcal{I}_2} \mathcal{J}_{D_I}^{24mn\cdot\mathrm{ram}_H(D_I)}\Big)\\ &\succeq \frac{1}{4}\Big((1/6!)\,c_1(K_{X(n)}-L)^6m^6\Big) \qquad \textit{as } m\to\infty\,.\end{aligned}$$

Proof. If (ii),(iii) and (iv) are all false, then

$$\begin{aligned}&\dim H^0\Big(X(n), m(K-L)\Big)\\ &\qquad - \dim H^0\Big(X(n), \mathcal{O}_{X(n)}\Big(m(K-L) - m\sum_\alpha n\,\mathrm{ram}_H(D_\alpha)144D_\alpha\Big)\\ &\qquad\qquad \otimes\prod_\alpha \mathcal{J}_{X_\alpha}^{1296m\cdot\mathrm{ram}_H(X_\alpha)}\otimes\prod_{I\in\mathcal{I}_2} \mathcal{J}_{D_I}^{24mn\cdot\mathrm{ram}_H(D_I)}\Big)\\ &\preceq \frac{3}{4}\Big((1/6!)\,c_1(K_{X(n)}-L)^6m^6\Big)\end{aligned}$$

as m tends to infinity. On the other hand, $\dim H^0\Big(X(n), m(K-L)\Big)$ grows like $(1/6!)\,c_1(K_{X(n)}-L)^6m^6$, since $(K-L)$ is ample for $n\geq 5$ by Theorem 2.11. So there must be a non–trivial global section in

$$\begin{aligned}&H^0\Big(X(n), \mathcal{O}_{X(n)}\Big(m(K-L) - m\sum_\alpha n\,\mathrm{ram}_H(D_\alpha)144D_\alpha\Big)\\ &\qquad \otimes\prod_\alpha \mathcal{J}_{X_\alpha}^{1296m\cdot\mathrm{ram}_H(X_\alpha)}\otimes\prod_{I\in\mathcal{I}_2} \mathcal{J}_{D_I}^{24mn\cdot\mathrm{ram}_H(D_I)}\Big)\end{aligned}$$

for all sufficiently big m. If (i) is also false, then Proposition 7.12 gives a contradiction, since it implies that the space of pluricanonical sections on $\widetilde{X}^{(2)}$ grows maximally. $\square$

7.3 Calculating obstructions

We will now calculate the obstructions coming from the singularities in the interior and the ones imposed by the boundary. This will allows us to conclude that the conditions given in Proposition 7.13 are only satisfied by finitely many subgroups Γ.

The reformulation of the required vanishing conditions carried out in the last section allows us to work on $X(n)$, which is smooth for $n \geq 3$ by Theorem 2.10. Recall from our discussion in Section 2.3 that the canonical divisor on this space is given by $K = K_{X(n)} = 4D - L$.

We start by calculating the obstructions from the boundary.

Proposition 7.14 *There is an integer n_0 such that for all $n \geq n_0$ and all subgroups $H < \mathrm{Sp}(6, \mathbb{Z}/n\mathbb{Z})$ the following inequality holds for all sufficiently big m:*

$$\begin{aligned}&\dim H^0\Big(X(n), m(K-L)\Big) \\ &\qquad - \dim H^0\Big(X(n), m(K-L) - m\sum_\alpha n\,\mathrm{ram}_H(D_\alpha)144D_\alpha\Big) \\ &< \frac{1}{7}\cdot 144^6 \cdot \Big(\frac{1}{\#\alpha}\sum_\alpha \mathrm{ram}_H(D_\alpha)\Big)\frac{m^6}{5!}\gamma(n)\ ,\end{aligned}$$

where $\gamma(n)$ is the order of $\mathrm{Sp}(6, \mathbb{Z}/n\mathbb{Z})$ as calculated in Section 3.1 and $\#\alpha$ denotes the number of boundary divisors as given in Lemma 3.24 (i).

Proof. We start with a rather strong estimate which allows us to consider the obstructions coming from each D_α individually:

$$\begin{aligned}&\dim H^0\Big(X(n), m(K-L)\Big) \\ &\qquad - \dim H^0\Big(X(n), m(K-L) - m\sum_\alpha n\,\mathrm{ram}_H(D_\alpha)144D_\alpha\Big) \\ &\leq \sum_\alpha \Big[\dim H^0\Big(X(n), m(K-L)\Big) \\ &\qquad - \dim H^0\Big(X(n), m(K-L) - mn\,\mathrm{ram}_H(D_\alpha)144D_\alpha\Big)\Big]\end{aligned}$$

The standard exact sequence associated to $D_\alpha \subset X(n)$ yields

$$\begin{aligned}&0 \to H^0\Big(X(n), m(K-L) - D_\alpha\Big) \\ &\qquad \to H^0\Big(X(n), m(K-L)\Big) \to H^0\Big(D_\alpha, m(K-L)\Big)\ .\end{aligned}$$

This can be generalized to

$$0 \to H^0\Big(X(n), m(K-L) - (j+1)D_\alpha\Big) \to H^0\Big(X(n), m(K-L) - jD_\alpha\Big) \to H^0\Big(D_\alpha, m(K-L) - jD_\alpha\Big)$$

for each integer j. This implies that

$$\begin{aligned}
&\dim H^0\Big(X(n), m(K-L)\Big) \\
&\qquad - \dim H^0\Big(X(n), m(K-L) - mn\,\mathrm{ram}_H(D_\alpha)144D_\alpha\Big) \\
&= \sum_{j=0}^{144mn\,\mathrm{ram}_H(D_\alpha)-1} \Big[\dim H^0\Big(X(n), m(K-L) - jD_\alpha\Big) \\
&\qquad\qquad - \dim H^0\Big(X(n), m(K-L) - (j+1)D_\alpha\Big)\Big] \\
&\le \sum_{j=0}^{144mn\,\mathrm{ram}_H(D_\alpha)-1} \dim H^0\Big(D_\alpha, m(K-L) - jD_\alpha\Big)\ .
\end{aligned}$$

We want to use Riemann–Roch and the Kodaira vanishing theorem to estimate $\dim H^0\Big(D_\alpha, m(K-L) - jD_\alpha\Big)$ for each integer $0 \le j \le 144mn\,\mathrm{ram}_H(D_\alpha) - 1$. For that we want to express $m(K-L) - jD_\alpha$ as the sum of an ample and a nef divisor on D_α.

We have that $m(K-L)$ is ample on $X(n)$ for $n \ge 5$ by Theorem 2.11. The same is true for its restriction to D_α and we can even subtract K_{D_α} if m is sufficiently big. So we have that $m(K_{X(n)} - L) - K_{D_\alpha}$ is ample on D_α.

The divisor $-D_\alpha$ might not be nef, but we can use the following relation for its normal bundle:

$$-D_\alpha\Big|_{D_\alpha} = \frac{1}{n}M(n) - \frac{1}{n}L\ ,$$

where L is just the restriction of the line bundle of modular forms to D_α and $M(n)$ is a bundle described in [Hul, p. 262]. It follows from Proposition C.1 in the appendix that the bundle $M(n)$ is nef.

The line bundle L is big and nef on $X(n)$. Hence mL has a non–trivial section for all sufficiently big m and we can estimate

$$\begin{aligned}
&\dim H^0\Big(D_\alpha, m(K-L) - jD_\alpha\Big) \\
&\le \dim H^0\Big(D_\alpha, m(K-L) - jD_\alpha + 144mL\Big) \\
&= \dim H^0\Big(D_\alpha, m(K-L) - j(D_\alpha - \frac{1}{n}L) + (144m - \frac{j}{n})L\Big)
\end{aligned}$$

for all $0 \le j \le 144mn\,\mathrm{ram}_H(D_\alpha) - 1$. Then

$$-j(D_\alpha - \frac{1}{n}L)\Big|_{D_\alpha} = \frac{j}{n}M(n)$$

is nef on D_α and

$$m(K_{X(n)} - L) - j(D_\alpha - \frac{1}{n}L) + (144m - \frac{j}{n})L - K_{D_\alpha}$$

is ample as the sum of ample and nef divisors. We can therefore apply the Kodaira vanishing theorem to obtain that

$$\begin{aligned} &H^i\Big(D_\alpha, m(K + 143L) - jD_\alpha\Big) \\ =&H^i\Big(D_\alpha, m(K - L) - j(D_\alpha - \frac{1}{n}L) + (144m - \frac{j}{n})L\Big) = 0 \quad \text{for all } i > 0 \,. \end{aligned}$$

Hence

$$\chi\Big(\mathcal{O}_{D_\alpha}(m(K + 143L) - jD_\alpha\Big) = \dim H^0\Big(D_\alpha, m(K + 143L) - jD_\alpha\Big) \,. \tag{19}$$

Let

$$\mathcal{L}_j := \mathcal{O}_{D_\alpha}\Big(m(K + 143L) - jD_\alpha\Big) \,.$$

Recall that the exponential Chern character $\mathrm{ch}(\mathcal{L}_j)$ of $\mathcal{L}_j$ is given by

$$\mathrm{ch}(\mathcal{L}_j) = 1 + c_1(\mathcal{L}_j) + \frac{1}{2}c_1(\mathcal{L}_j)^2 + \frac{1}{3!}c_1(\mathcal{L}_j)^3 + \frac{1}{4!}c_1(\mathcal{L}_j)^4 + \frac{1}{5!}c_1(\mathcal{L}_j)^5 \,.$$

Let $\mathcal{T}$ denote the tangent sheaf of D_α and $\mathrm{td}(\mathcal{T})$ its Todd class. We can now apply Hirzebruch–Riemann–Roch and obtain

$$\dim H^0(D_\alpha, \mathcal{L}_j) \overset{(19)}{=} \chi(\mathcal{L}_j) = \deg(\mathrm{ch}(\mathcal{L}_j).\,\mathrm{td}(\mathcal{T}))_5 \,,$$

where $(\ )_5$ denotes the component of degree 5 in $A(D_\alpha) \otimes \mathbb{Q}$ with $A(D_\alpha)$ the Chow ring of D_α.

We want to get an estimate for big m, so we only need to consider the coefficient of the highest power of m. Since only $c_1(\mathcal{L}_j)$ depends on m, this coefficient is coming from $c_1(\mathcal{L}_j)^5$, i.e.

$$\dim H^0(D_\alpha, \mathcal{L}_j) \sim \frac{1}{5!}c_1(\mathcal{L}_j)^5$$

for all sufficiently big m.

Instead of calculating this top–intersection on D_α, we use the fact that the bundle $\mathcal{L}_j$ is the restriction to D_α of a bundle on $X(n)$ to do the calculation on $X(n)$ where we know the intersection numbers thanks to a paper of van der Geer. We thus have

$$\begin{aligned} \frac{1}{5!}c_1(\mathcal{L}_j)^5 &= \frac{1}{5!}c_1\Big(\mathcal{O}_{D_\alpha}\Big(m(K + 143L) - jD_\alpha\Big)\Big)^5 \\ &= \frac{1}{5!}c_1\Big(\mathcal{O}_{X(n)}\Big(m(K + 143L) - jD_\alpha\Big)\Big)^5 .\, c_1\Big(\mathcal{O}_{X(n)}(D_\alpha)\Big) \,. \end{aligned} \tag{20}$$

To simplify notation we will from now on omit the Chern classes in the intersections, e.g. we write

$$\frac{1}{5!}\Big(m(K+143L)-jD_\alpha\Big)^5.\,D_\alpha \tag{21}$$

for the last term in (20).

Clearly this intersection number depends on j. Our next goal will be to find an uniform bound on this number which works for all j and all boundary divisors D_α.

We first expand the term in (21) and take the absolute values of the individual summands:

$$\begin{aligned}&\frac{1}{5!}\,\Big(m(K+143L)-jD_\alpha\Big)^5.\,D_\alpha\\ =&\frac{1}{5!}\left[\sum_{i=0}^{5}(-1)^i\binom{5}{i}\,m^{5-i}j^i(K+143L)^{5-i}.\,D_\alpha^{i+1}\right]\\ \leq&\frac{1}{5!}\left[\sum_{i=0}^{5}\binom{5}{i}\,m^{5-i}j^i\Big|(K+143L)^{5-i}.\,D_\alpha^{i+1}\Big|\right]\end{aligned} \tag{22}$$

Note that since $0<\mathrm{ram}_H(D_\alpha)\leq 1$ we have that

$$0\leq j\leq 144mn\,\mathrm{ram}_H(D_\alpha)-1<144mn\;. \tag{23}$$

Since the term in (22) has only positive summands and is thus increasing in j, we can use (23) to conclude that

$$\begin{aligned}&\frac{1}{5!}\;\Big(m(K+143L)-jD_\alpha\Big)^5.\,D_\alpha\\ \leq&\frac{m^5}{5!}\left[\sum_{i=0}^{5}\binom{5}{i}\,(144n)^i\Big|(K+143L)^{5-i}.\,D_\alpha^{i+1}\Big|\right]\;.\end{aligned}$$

Note that this intersection number is now independent of j but also independent of α, since all D_α are equivalent under the action of $\mathrm{Sp}(6,\mathbb{Z}/n\mathbb{Z})$. We can thus estimate

$$\begin{aligned}&\frac{1}{5!}\Big(m(K+143L)-jD_\alpha\Big)^5.\,D_\alpha\\ \leq&\frac{1}{\#\alpha}\frac{m^5}{5!}\sum_\alpha\left[\sum_{i=0}^{5}\binom{5}{i}\,(144n)^i\Big|(K+143L)^{5-i}.\,D_\alpha^{i+1}\Big|\right]\\ =&\frac{1}{\#\alpha}\frac{m^5}{5!}\left[\sum_{i=0}^{5}\binom{5}{i}\,(144n)^i\left|(K+143L)^{5-i}.\Big(\sum_\alpha D_\alpha^{i+1}\Big)\right|\right]\;.\end{aligned} \tag{24}$$

Note that $\sum_\alpha D_\alpha^k$ is a symmetric polynomial in the D_α. We denote the i–th elementary symmetric polynomial by Δ_i and can thus express $\sum_\alpha D_\alpha^k$ in terms of Δ_i. For example, $\sum_\alpha D_\alpha^3$ is given by

$$\sum_\alpha D_\alpha^3=\Delta_1^3+3\Delta_3-3\Delta_1\Delta_2\;.$$

Recalling that $K = K_{X(n)} = 4L - D = 4L - \Delta_1$, we can thus express the intersection given in (24) with L and Δ_i only. These intersection numbers have been calculated by van der Geer in [vdG2]. For instance, the term corresponding to $i = 2$ in (24) can be computed as follows:

$$\begin{aligned}(K+143L)^3.\Big(\sum_\alpha D_\alpha^3\Big) &= (147L - \Delta_1)^3.\Big(\Delta_1^3 + 3\Delta_3 - 3\Delta_1\Delta_2\Big)\\ &= \Big(147^3 \cdot \frac{1}{720}\cdot\frac{1}{n^3} - 147\cdot\frac{23}{80}\cdot\frac{1}{n^5} + \frac{215}{144}\cdot\frac{1}{n^6}\Big)\gamma(n)\ ,\end{aligned}$$

where $\gamma(n)$ is the order of $\mathrm{Sp}(6,\mathbb{Z}/n\mathbb{Z})$ as calculated in Section 3.1.

If we assume that n is sufficiently big, we only have to look at the highest power of n in this term which means that this intersection grows as

$$(K+143L)^3.\Big(\sum_\alpha D_\alpha^3\Big) \sim 147^3 \cdot \frac{1}{720}\cdot\frac{1}{n^3}\gamma(n)\ .$$

All the other terms in (24) can be computed by a similar calculation.

Putting all the individual calculations together we obtain that

$$\begin{aligned}&\frac{1}{5!}\Big(m(K+143L) - jD_\alpha\Big)^5.D_\alpha\\ &\leq \frac{1}{\#\alpha}\frac{m^5}{5!}\left[\sum_{i=0}^{5}\binom{5}{i}(144n)^i\left|(K+143L)^{5-i}.\Big(\sum_\alpha D_\alpha^{i+1}\Big)\right|\right]\\ &\sim \frac{1}{\#\alpha}\frac{m^5}{5!}\left[\Big(\frac{147^3}{72}\cdot\frac{\gamma(n)}{n^3}\Big) + 5\cdot 144\Big(\frac{147^3}{180}\cdot\frac{\gamma(n)}{n^2}\Big) + 10\cdot 144^2\Big(\frac{147^3}{720}\cdot\frac{\gamma(n)}{n}\Big)\right.\\ &\qquad\left. +10\cdot 144^3\Big(\frac{147\cdot 7}{120}\cdot\frac{\gamma(n)}{n^2}\Big) + 5\cdot 144^4\Big(\frac{147}{80}\cdot\frac{\gamma(n)}{n}\Big) + 144^5\Big(\frac{7}{144}\cdot\frac{\gamma(n)}{n}\Big)\right]\\ &\lesssim \frac{1}{\#\alpha}\frac{m^5}{5!}\cdot 144^5\cdot\frac{1}{7}\cdot\frac{\gamma(n)}{n} \qquad (25)\end{aligned}$$

for n sufficiently big. Note that this assumption on n can be made precise, i.e. we can explicitly calculate an integer n_0 such that for all $n \geq n_0$ the strict inequality is satisfied. Note also that the integer n_0 does neither depend on m nor α and is also independent of the subgroup H.

The estimate in (25) holds true for all integers j, so we can take the sum and have for all $n \geq n_0$ and all sufficiently big m

$$\begin{aligned}
&\sum_{j=0}^{144mn\,\mathrm{ram}_H(D_\alpha)-1} \dim H^0\Big(D_\alpha, m(K-L)-jD_\alpha\Big)\\
<&\sum_{j=0}^{144mn\,\mathrm{ram}_H(D_\alpha)-1} \left(\frac{1}{\#\alpha}\frac{m^5}{5!}\cdot 144^5\cdot\frac{1}{7}\cdot\frac{\gamma(n)}{n}\right)\\
=&\,144mn\,\mathrm{ram}_H(D_\alpha)\cdot\left(\frac{1}{\#\alpha}\frac{m^5}{5!}\cdot 144^5\cdot\frac{1}{7}\cdot\frac{\gamma(n)}{n}\right)\\
=&\,\frac{1}{7}\cdot 144^6\cdot\frac{\mathrm{ram}_H(D_\alpha)}{\#\alpha}\frac{m^6}{5!}\gamma(n)\ .
\end{aligned}$$

If we now consider all D_α and use the estimate from the beginning of this proof we can conclude that

$$\begin{aligned}
&\dim H^0\Big(X(n), m(K-L)\Big)\\
&\qquad - \dim H^0\Big(X(n), m(K-L) - m\sum_\alpha n\,\mathrm{ram}_H(D_\alpha)144D_\alpha\Big)\\
\leq&\sum_\alpha\sum_{j=0}^{144mn\,\mathrm{ram}_H(D_\alpha)-1} \dim H^0\Big(D_\alpha, m(K-L)-jD_\alpha\Big)\\
<&\,\frac{1}{7}\cdot 144^6\cdot\Big(\frac{1}{\#\alpha}\sum_\alpha \mathrm{ram}_H(D_\alpha)\Big)\frac{m^6}{5!}\gamma(n)
\end{aligned}$$

for all $n \geq n_0$ and all sufficiently big m as claimed. □

We can now use our results from Chapter 4 to show that condition (ii) of Proposition 7.13 is satisfied by at most finitely many groups Γ.

Lemma 7.15 *There are only finitely many subgroups Γ of $\mathrm{Sp}(6,\mathbb{Z})$ of finite index which satisfy*

$$\begin{aligned}
&\dim H^0\Big(X(n), m(K-L)\Big)\\
&\qquad - \dim H^0\Big(X(n), m(K-L) - m\sum_\alpha n\,\mathrm{ram}_H(D_\alpha)144D_\alpha\Big)\\
\succeq&\,\frac{1}{4}\Big((1/6!)\,c_1(K_{X(n)}-L)^6m^6\Big)
\end{aligned}$$

as m tends to infinity.

Proof. As we remarked in the beginning of this chapter we can assume that n is sufficiently big whenever necessary since there are only finitely many subgroups Γ of $\mathrm{Sp}(6,\mathbb{Z})$ which contain each $\Gamma(n)$.

Recall that the canonical divisor $K_{X(n)}$ on $X(n)$ is given by $4L - D$, so

$$K_{X(n)} - L = 3L - D\ .$$

The top–intersection $c_1(K_{X(n)} - L)^6 = c_1(3L - D)^6$ can be computed by using the tables in van der Geer's paper [vdG2]. However, for large n this intersection number is dominated by the highest power of n which comes from the top–intersection of $3L$ and is given by

$$c_1(K_{X(n)} - L)^6 \sim c_1(3L)^6 = \frac{3^6}{181440}\gamma(n)\ .$$

By Proposition 7.14 we have for all but finitely many subgroups Γ of $\mathrm{Sp}(6,\mathbb{Z})$ (disregarding those for which $n \le n_0$) that

$$\begin{aligned} &\dim H^0\Big(X(n), m(K-L)\Big) \\ &\qquad - \dim H^0\Big(X(n), m(K-L) - m\sum_\alpha n\,\mathrm{ram}_H(D_\alpha)144D_\alpha\Big) \\ &< \frac{1}{7}\cdot 144^6 \cdot \Big(\frac{1}{\#\alpha}\sum_\alpha \mathrm{ram}_H(D_\alpha)\Big)\frac{m^6}{5!}\gamma(n)\ . \end{aligned}$$

To prove the claim we just need to show that there are only finitely many subgroups Γ of $\mathrm{Sp}(6,\mathbb{Z})$ for which

$$\begin{aligned} &\frac{1}{7}\cdot 144^6 \cdot \Big(\frac{1}{\#\alpha}\sum_\alpha \mathrm{ram}_H(D_\alpha)\Big)\frac{m^6}{5!}\gamma(n) \\ &\ge \frac{1}{4}\Big((1/6!)\,c_1(K_{X(n)} - L)^6 m^6\Big) \sim \frac{1}{4}\cdot\frac{3^6}{181440}\cdot\frac{m^6}{6!}\gamma(n)\ . \end{aligned}$$

The latter condition is equivalent to

$$\Big(\frac{1}{\#\alpha}\sum_\alpha \mathrm{ram}_H(D_\alpha)\Big) \gtrsim \frac{1}{168}\cdot\frac{3^6}{1814400}\cdot\frac{1}{144^6}\ ,$$

so we have a lower bound for the ramification mean of H. By Theorem 4.3 there are only finitely many subgroups exceeding this lower bound which proves the claim. □

We will now proceed in a similar manner with the obstructions coming from the singularities in the interior. For that, recall the definitions of the involutions φ_α and their fix loci X_α as given in Section 5.2.

Proposition 7.16 *There is an integer n_0 such that for all $n \ge n_0$ and all subgroups $H < \mathrm{Sp}(6,\mathbb{Z}/n\mathbb{Z})$ the following inequality holds for all sufficiently big* *m:*

$$\begin{aligned} &\dim H^0\Big(X(n), m(K-L)\Big) \\ &\qquad\qquad - \dim H^0\Big(X(n), \mathcal{O}_{X(n)}\Big(m(K-L)\Big) \otimes \prod_\alpha \mathcal{J}_{X_\alpha}^{1296m\cdot \mathrm{ram}_H(X_\alpha)}\Big) \\ &< \frac{1296^5}{1728} \cdot \Big(\frac{1}{\#\alpha}\sum_\alpha \mathrm{ram}_H(X_\alpha)\Big)\frac{m^6}{5!}\gamma(n)\ , \end{aligned}$$

where $\gamma(n)$ is the order of $\mathrm{Sp}(6, \mathbb{Z}/n\mathbb{Z})$ as calculated in Section 3.1 and $\#\alpha$ denotes the number of components X_α as given in Corollary 5.14.

Proof. We can proceed as in the proof of Proposition 7.14 and obtain

$$\begin{aligned} &\dim H^0\Big(X(n), \mathcal{O}_{X(n)}(m(K-L))\Big) \\ &\qquad\qquad - \dim H^0\Big(X(n), \mathcal{O}_{X(n)}\Big(m(K-L)\Big) \otimes \prod_\alpha \mathcal{J}_{X_\alpha}^{1296m\cdot \mathrm{ram}_H(X_\alpha)}\Big) \\ &\leq \sum_\alpha \Big[\dim H^0\Big(X(n), \mathcal{O}_{X(n)}\Big(m(K-L)\Big)\Big) \qquad\qquad (26) \\ &\qquad\qquad - \dim H^0\Big(X(n), \mathcal{O}_{X(n)}\Big(m(K-L)\Big) \otimes \mathcal{J}_{X_\alpha}^{1296m\cdot \mathrm{ram}_H(X_\alpha)}\Big)\Big] \end{aligned}$$

which allows us to consider each X_α separately. We blow up $X(n)$ along one such X_α and obtain the following diagram:

$$\begin{array}{ccc} E_\alpha & \subset & \widetilde{X}(n) \\ \big\downarrow & & \big\downarrow \pi \\ X_\alpha & \subset & X(n) \end{array}$$

where E_α is the exceptional divisor of this blow–up $\pi : \widetilde{X}(n) \to X(n)$. By [CEL, Lemma 3.3] we have for sufficiently big $m >> 0$

$$\begin{aligned} &\dim H^0\Big(X(n), \mathcal{O}_{X(n)}\Big(m(K-L)\Big) \otimes \mathcal{J}_{X_\alpha}^{1296m\cdot \mathrm{ram}_H(X_\alpha)}\Big) \\ &= \dim H^0\Big(\widetilde{X}(n), \mathcal{O}_{\widetilde{X}(n)}\Big(m\,\pi^*(K-L) - 1296m\,\mathrm{ram}_H(X_\alpha)\,E_\alpha\Big)\Big)\ . \end{aligned}$$

As in the proof of Proposition 7.14 we can now use the standard exact sequence associated to $E_\alpha \subset \widetilde{X}(n)$ to conclude that

$$\begin{aligned} &\dim H^0\Big(X(n), \mathcal{O}_{X(n)}\Big(m(K-L)\Big)\Big) \\ &\qquad\qquad - \dim H^0\Big(X(n), \mathcal{O}_{X(n)}\Big(m(K-L)\Big) \otimes \mathcal{J}_{X_\alpha}^{1296m\cdot \mathrm{ram}_H(X_\alpha)}\Big) \\ &\leq \sum_{j=0}^{1296m\cdot \mathrm{ram}_H(X_\alpha)-1} \dim H^0\Big(E_\alpha, \Big(\mathcal{O}_{\widetilde{X}(n)}(m\,\pi^*(K-L) - jE_\alpha)\Big)\Big|_{E_\alpha}\Big)\ . \qquad (27) \end{aligned}$$

E_α as the exceptional divisor of the blow–up of $X(n)$ along X_α is a fiber bundle over X_α which we denote by abuse of notation again by $\pi : E_\alpha \to X_\alpha$. It can be identified with the projectivization $\mathbb{P}(\mathcal{N}^\vee_{X_\alpha/X(n)})$ of the conormal bundle $\mathcal{N}^\vee_{X_\alpha/X(n)}$ of X_α in $X(n)$. (Recall our convention that for any vector bundle $\mathcal{E}$, the bundle $\mathbb{P}(\mathcal{E})$ is the projectivized bundle of lines in $\mathcal{E}$.) If $\zeta = \mathcal{O}_{\mathbb{P}(\mathcal{N}^\vee_{X_\alpha/X(n)})}(1)$ denotes the tautological bundle on $\mathbb{P}(\mathcal{N}^\vee_{X_\alpha/X(n)})$, we have that $-E_\alpha|_{E_\alpha} = \zeta$. With this notation the summands in (27) are the dimensions of the spaces of global sections of the bundles

$$\pi^*\Big(\mathcal{O}_{X_\alpha}(m(K-L))\Big) \otimes \zeta^{\otimes j} \tag{28}$$

on E_α.

By [Har1, Lemma 3.1] the higher direct images of this bundle vanish, i.e.

$$\mathrm{R}^i\pi_*\Big(\pi^*\Big(\mathcal{O}_{X_\alpha}(m(K-L))\Big) \otimes \zeta^{\otimes j}\Big) = 0$$

for all $i > 0$. Thus as a special case of the Leray spectral sequence, we obtain that

$$\begin{aligned} &H^i\Big(E_\alpha, \pi^*\Big(\mathcal{O}_{X_\alpha}(m(K-L))\Big) \otimes \zeta^{\otimes j}\Big) \\ =\, &H^i\Big(X_\alpha, \pi_*\Big(\pi^*\Big(\mathcal{O}_{X_\alpha}(m(K-L))\Big) \otimes \zeta^{\otimes j}\Big)\Big) \\ =\, &H^i\Big(X_\alpha, \mathcal{O}_{X_\alpha}\Big(m(K-L)\Big) \otimes \mathrm{S}^j\Big(\mathcal{N}^\vee_{X_\alpha/X(n)}\Big)\Big) \end{aligned} \tag{29}$$

for all $i \geq 0$ (cf. [Laz1, Proposition B.1.1]).

Our next step will be to use the following version of Griffiths vanishing theorem (cf. [Laz2, Theorem 7.3.1 and Variant 7.3.2]) on X_α:

Theorem 7.17 (Griffiths vanishing theorem) *Let X be a smooth complex irreducible projective variety. If $\mathcal{E}$ is a nef vector bundle, and L is any ample line bundle, then*

$$H^i\Big(X, \omega_X \otimes \mathrm{S}^m\mathcal{E} \otimes \det(\mathcal{E}) \otimes L\Big) = 0$$

for all $i > 0$ and all $m \geq 0$.

For that we need explicit knowledge of the conormal bundle $\mathcal{N}^\vee_{X_\alpha/X(n)}$. Recall that X_α is up to isomorphism the closure of $\mathcal{A}_1(n) \times \mathcal{A}_2(n) \subset \mathcal{A}_3(n)$ in $X(n)$. By a result of van der Geer this is just

$$X_\alpha \cong \mathcal{A}_1^{\mathrm{Vor}}(n) \times \mathcal{A}_2^{\mathrm{Vor}}(n) \ .$$

Let $\mathbb{E}_1$ and $\mathbb{E}_2$ denote the Hodge bundles on $\mathcal{A}_1^{\mathrm{Vor}}(n)$ and on $\mathcal{A}_2^{\mathrm{Vor}}(n)$ respectively. By explicitly calculating the transition functions we can show that

$$\mathcal{N}^\vee_{X_\alpha/X(n)} = \mathbb{E}_1 \boxtimes \mathbb{E}_2 := \mathrm{pr}_1^*\,\mathbb{E}_1 \otimes \mathrm{pr}_2^*\,\mathbb{E}_2 \ ,$$

where pr_1 and pr_2 denote the projections to $\mathcal{A}_1^{\mathrm{Vor}}(n)$ resp. $\mathcal{A}_2^{\mathrm{Vor}}(n)$.

The Hodge bundles $\mathbb{E}_1$ and $\mathbb{E}_2$ are positive semi-definite in the sense of Griffith (cf. [Gri]) by [Zuo, Theorem 1.2] and thus nef (cf. [DPS, Theorem 1.12]) on $\mathcal{A}_1^{\mathrm{Vor}}(n)$ resp. $\mathcal{A}_2^{\mathrm{Vor}}(n)$. (Recall that by definition a vector bundle $\mathcal{E}$ is said to be nef, if the Serre line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is a nef line bundle on the projectivized bundle $\mathbb{P}(\mathcal{E})$.) Hence their pullbacks $\mathrm{pr}_1^* \mathbb{E}_1$ and $\mathrm{pr}_2^* \mathbb{E}$ are nef and so is $\mathcal{N}_{X_\alpha/X(n)}^\vee$ as the tensor product of these pullbacks by [Laz2, Theorem 6.2.12 (iv)].

We also need the canonical divisor of $X_\alpha \cong \mathcal{A}_1^{\mathrm{Vor}}(n) \times \mathcal{A}_2^{\mathrm{Vor}}(n)$. Recall that the canonical divisors on $\mathcal{A}_1^{\mathrm{Vor}}(n)$ and $\mathcal{A}_2^{\mathrm{Vor}}(n)$ are given by

$$K_{\mathcal{A}_1^{\mathrm{Vor}}(n)} = 2L_1 - D_1 \quad \text{and} \quad K_{\mathcal{A}_2^{\mathrm{Vor}}(n)} = 3L_2 - D_2$$

respectively, where L_1 and L_2 denote the line bundles of modular forms and D_1 and D_2 the boundary of these spaces. Hence as the product of $\mathcal{A}_1^{\mathrm{Vor}}(n)$ and $\mathcal{A}_2^{\mathrm{Vor}}(n)$ the canonical divisor on X_α is just given by

$$K_{X_\alpha} = \mathrm{pr}_1^*(K_{\mathcal{A}_1^{\mathrm{Vor}}(n)}) + \mathrm{pr}_2^*(K_{\mathcal{A}_2^{\mathrm{Vor}}(n)}) = \mathrm{pr}_1^*(2L_1 - D_1) + \mathrm{pr}_2^*(3L_2 - D_2)\ . \qquad (30)$$

We need to know one more bundle to use Griffiths' vanishing theorem, namely the determinant bundle $\det(\mathcal{N}_{X_\alpha/X(n)}^\vee)$ of $\mathcal{N}_{X_\alpha/X(n)}^\vee$. Its first Chern class is given by

$$\begin{aligned} c_1(\det(\mathbb{E}_1 \boxtimes \mathbb{E}_2)) &= c_1\Big(\mathrm{pr}_1^* \mathbb{E}_1 \otimes \mathrm{pr}_2^* \mathbb{E}_2\Big) = 2c_1(\mathrm{pr}_1^* \mathbb{E}_1) + c_1(\mathrm{pr}_2^* \mathbb{E}_2) \\ &= 2\,\mathrm{pr}_1^* L_1 + \mathrm{pr}_2^* L_2\ . \end{aligned} \qquad (31)$$

If we consider the tensor product of this determinant bundle with the canonical bundle on X_α, we obtain

$$\omega_{X_\alpha} \otimes \Big(\mathcal{N}_{X_\alpha/X(n)}^\vee\Big) \overset{(30),(31)}{=} \mathcal{O}_{X_\alpha}\Big(\mathrm{pr}_1^*(4L_1 - D_1) + \mathrm{pr}_2^*(4L_2 - D_2)\Big) \qquad (32)$$

which is just the restriction of the line bundle on $X(n)$ given by $4L - D$ to X_α.

Note that the line bundle $(m-1)(3L - D) - L$ is ample on $X(n)$ for $n \geq 5$ and $m \geq 3$ (cf. [Hul, Theorem 0.2]) and so is its restriction to X_α. We can thus apply

Griffiths vanishing theorem and obtain

$$\begin{aligned}
&H^i\Big(X_\alpha, \mathcal{O}_{X_\alpha}\Big(m(K_{X(n)}-L)\Big)\otimes \mathrm{S}^j\Big(\mathcal{N}^\vee_{X_\alpha/X(n)}\Big)\Big)\\
=\ &H^i\Big(X_\alpha, \mathcal{O}_{X_\alpha}\Big(\Big[(3L-D)+L\Big]\\
&\qquad + \Big[(m-1)(3L-D)-L\Big]\Big)\otimes \mathrm{S}^j\Big(\mathcal{N}^\vee_{X_\alpha/X(n)}\Big)\Big)\\
\overset{(32)}{=}\ &H^i\Big(X_\alpha, \omega_{X_\alpha}\otimes \mathrm{S}^j\Big(\mathcal{N}^\vee_{X_\alpha/X(n)}\Big)\otimes \det(\mathcal{N}^\vee_{X_\alpha/X(n)})\\
&\qquad \otimes \mathcal{O}_{X_\alpha}\Big((m-1)(3L-D)-L\Big)\Big)\\
=\ &0
\end{aligned}$$

for all $i>0$. By (29) we have that the corresponding higher cohomology groups vanish upstairs on E_α and thus

$$\chi\Big(\pi^*\Big(\mathcal{O}_{X_\alpha}(m(K-L))\Big)\otimes\zeta^{\otimes j}\Big) = \dim H^0\Big(E_\alpha, \pi^*\Big(\mathcal{O}_{X_\alpha}(m(K-L))\Big)\otimes\zeta^{\otimes j}\Big)\ . \tag{33}$$

For each integer j we set

$$\mathcal{F}_j := \pi^*\Big(\mathcal{O}_{X_\alpha}(m(K-L))\Big)\otimes\zeta^{\otimes j}\ .$$

As in the proof of Proposition 7.14 we can now apply Hirzebruch–Riemann–Roch and obtain

$$\dim H^0(E_\alpha,\mathcal{F}_j)\overset{(33)}{=}\chi(\mathcal{F}_j)=\deg(\mathrm{ch}(\mathcal{F}_j).\,\mathrm{td}(\mathcal{T}))_5\ .$$

Since the Todd class $\mathrm{td}\,\mathcal{T}$ of the tangent sheaf $\mathcal{T}$ of E_α does not depend on m, we get that

$$\dim H^0(E_\alpha,\mathcal{F}_j)\sim\frac{1}{5!}c_1(\mathcal{F}_j)^5 \tag{34}$$

for all sufficiently big m.

We expand the term on the right hand side and obtain

$$\begin{aligned}
\frac{1}{5!}c_1(\mathcal{F}_j)^5 &= \frac{1}{5!}c_1(\pi^*\Big(\mathcal{O}_{X_\alpha}(m(K-L))\Big)\otimes\zeta^{\otimes j})^5\\
&=\frac{1}{5!}\sum_{k=0}^{5}\binom{5}{k}m^{5-k}j^k c_1\Big(\pi^*\mathcal{O}_{X_\alpha}(K-L)\Big)^{5-k}.\,c_1(\zeta)^k
\end{aligned}\tag{35}$$

As in the proof of Proposition 7.14, we will from now on simplify the notation by omitting the Chern classes in the notation and write

$$\frac{1}{5!}\sum_{k=0}^{5}\binom{5}{k}m^{5-k}j^k\pi^*(K-L)^{5-k}.\,\zeta^k \tag{36}$$

for the term in (35).

By [Ful, Remark 3.2.4] the tautological bundle ζ on E_α satisfies the relation

$$\zeta^2 + c_1(\pi^*\mathcal{N}^\vee_{X_\alpha/X(n)}).\,\zeta + c_2(\pi^*\mathcal{N}^\vee_{X_\alpha/X(n)}) = 0\ .$$

With this relation we can rewrite the sum in (36) and obtain after a straightforward calculation that

$$\begin{aligned}
&\frac{1}{5!}\quad \sum_{k=0}^{5}\binom{5}{k} m^{5-k}j^k\pi^*(K-L)^{5-k}.\,\zeta^k \\
=&\frac{1}{5!}\Bigg[\ \pi^*\Big[m^5(K-L)^5 - 10m^3j^2(K-L)^3.\,c_2(\mathcal{N}^\vee) \\
&\qquad + 10m^2j^3(K-L)^2.\,c_1(\mathcal{N}^\vee).\,c_2(\mathcal{N}^\vee) \\
&\qquad + 5mj^4(K-L)\,.\,\Big(c_2(\mathcal{N}^\vee)^2 - c_1(\mathcal{N}^\vee)^2.\,c_2(\mathcal{N}^\vee)\Big) \\
&\qquad + j^5\Big(c_1(\mathcal{N}^\vee)^3.\,c_2(\mathcal{N}^\vee) - 2c_1(\mathcal{N}^\vee).\,c_2(\mathcal{N}^\vee)^2\Big)\Big] \\
&+\pi^*\Big[5m^4j(K-L)^4 - 10m^3j^2(K-L)^3.\,c_1(\mathcal{N}^\vee) \\
&\qquad + 10m^2j^3(K-L)^2.\,\Big(c_1(\mathcal{N}^\vee)^2 - c_2(\mathcal{N}^\vee)\Big) \\
&\qquad + 5mj^4(K-L)\,.\,\Big(2c_1(\mathcal{N}^\vee).\,c_2(\mathcal{N}^\vee) - c_1(\mathcal{N}^\vee)^3\Big) \\
&\qquad + j^5\Big(c_2(\mathcal{N}^\vee)^2 + c_1(\mathcal{N}^\vee)^4 - 3c_1(\mathcal{N}^\vee)^2.\,c_2(\mathcal{N}^\vee)\Big)\Big].\,\zeta\Bigg]\ ,
\end{aligned}$$

where $\mathcal{N}^\vee := \mathcal{N}^\vee_{X_\alpha/X(n)}$. Note that all the terms in the first of the two summands consists of pullbacks of intersections coming from X_α. Consequently, they all have to vanish, since X_α is only 4–dimensional. For the second summand, we have pullbacks of top–dimensional intersections on X_α intersected with the tautological bundle, which means that we can calculate these terms on X_α. Hence

$$\begin{aligned}
\frac{1}{5!}c_1(\mathcal{F}_j)^5 = \frac{1}{5!}\Big[&5m^4j(K-L)^4 - 10m^3j^2(K-L)^3.\,c_1(\mathcal{N}^\vee) \\
&+ 10m^2j^3(K-L)^2.\,\Big(c_1(\mathcal{N}^\vee)^2 - c_2(\mathcal{N}^\vee)\Big) \\
&+ 5mj^4(K-L)\,.\,\Big(2c_1(\mathcal{N}^\vee).\,c_2(\mathcal{N}^\vee) - c_1(\mathcal{N}^\vee)^3\Big) \\
&+ j^5\Big(c_2(\mathcal{N}^\vee)^2 + c_1(\mathcal{N}^\vee)^4 - 3c_1(\mathcal{N}^\vee)^2.\,c_2(\mathcal{N}^\vee)\Big)\Big]\ ,
\end{aligned} \tag{37}$$

where the intersection on the right hand side is now on X_α.

Recall from (31) that the first Chern class of the conormal bundle $\mathcal{N}^\vee$ is given by

$$c_1(\mathcal{N}^\vee) = 2\,\mathrm{pr}_1^*\,L_1 + \mathrm{pr}_2^*\,L_2\ .$$

The second Chern class of the product $\mathcal{N}^\vee = \mathbb{E}_1 \boxtimes \mathbb{E}_2$ can be expressed in terms of the Chern classes of $\mathbb{E}_1$ and $\mathbb{E}_2$ as follows:

$$\begin{aligned} c_2(\mathcal{N}^\vee) &= c_2(\mathrm{pr}_2^* \mathbb{E}_2) + c_1(\mathrm{pr}_1^* \mathbb{E}_1) \,.\, c_1(\mathrm{pr}_2^* \mathbb{E}_2) + \underbrace{c_1(\mathrm{pr}_1^* \mathbb{E}_1)^2}_{=0} \\ &= \frac{1}{2} c_1(\mathrm{pr}_2^* \mathbb{E}_2)^2 + c_1(\mathrm{pr}_1^* \mathbb{E}_1) \,.\, c_1(\mathrm{pr}_2^* \mathbb{E}_2) =: \frac{1}{2} \mathrm{pr}_2^* L_2^2 + \mathrm{pr}_1^* L_1 \,.\, \mathrm{pr}_2^* L_2 \,, \end{aligned}$$

where we used the identity $2\,c_2(\mathbb{E}_2) = c_1(\mathbb{E}_2)^2$ on $\mathcal{A}_2^{\mathrm{Vor}}(n)$ (cf. [vdG2, §2]) and the fact that $c_1(\mathbb{E}_1)^2 \equiv 0$ on the 1–dimensional space $\mathcal{A}_1^{\mathrm{Vor}}(n)$.

We then have

$$\begin{aligned} c_1(\mathcal{N}^\vee) &= 2\,\mathrm{pr}_1^* L_1 + \mathrm{pr}_2^* L_2 = L + \mathrm{pr}_1^* L_1 \\ c_1(\mathcal{N}^\vee)^2 - c_2(\mathcal{N}^\vee) &= \frac{3}{2} L^2 + 2\,\mathrm{pr}_1^* L_1 \,.\, \mathrm{pr}_2^* L_2 \\ 2c_1(\mathcal{N}^\vee) \,.\, c_2(\mathcal{N}^\vee) - c_1(\mathcal{N}^\vee)^3 &= -2\,\mathrm{pr}_1^* L_1 \,.\, \mathrm{pr}_2^* L_2^2 \\ c_2(\mathcal{N}^\vee)^2 + c_1(\mathcal{N}^\vee)^4 - 3c_1(\mathcal{N}^\vee)^2 \,.\, c_2(\mathcal{N}^\vee) &= 0 \,, \end{aligned}$$

where we used that $\mathrm{pr}_1^* L_1 + \mathrm{pr}_2^* L_2$ coincides with the restriction of the line bundle L on $X(n)$ to X_α and the fact that all terms involving $\mathrm{pr}_1^* L_1^2$ or $\mathrm{pr}_2^* L_2^4$ vanish.

Thus the intersection in (37) can be rewritten in terms of K, L, $\mathrm{pr}_1^* L_1$, and $\mathrm{pr}_2^* L_2$ as follows:

$$\begin{aligned} \frac{1}{5!} c_1(\mathcal{F}_j)^5 = \frac{1}{5!} \Big[& 5m^4 j (K-L)^4 - 10 m^3 j^2 (K-L)^3 . \Big(L + \mathrm{pr}_1^* L_1 \Big) \\ & + 10 m^2 j^3 (K-L)^2 . \Big(\frac{3}{2} L^2 + 2\,\mathrm{pr}_1^* L_1 \,.\, \mathrm{pr}_2^* L_2 \Big) \\ & - 10 m j^4 (K-L) \,.\, \Big(\mathrm{pr}_1^* L_1 \,.\, \mathrm{pr}_2^* L_2^2 \Big) \Big] \,. \end{aligned}$$

As in the proof of Proposition 7.14 we can get an estimate for $\frac{1}{5!} c_1(\mathcal{F}_j)^5$ which is independent of j by first taking absolute values and then using that $0 \le j \le 1296m \cdot \mathrm{ram}_H(X_\alpha) - 1 < 1296m$. This gives us

$$\begin{aligned} \frac{1}{5!} c_1(\mathcal{F}_j)^5 \le \frac{m^5}{5!} \Big[& 5 \cdot 1296 \Big| (K-L)^4 \Big| + 10 \cdot 1296^2 \Big| (K-L)^3 . \Big(L + \mathrm{pr}_1^* L_1 \Big) \Big| \\ & + 10 \cdot 1296^3 \Big| (K-L)^2 . \Big(\frac{3}{2} L^2 + 2\,\mathrm{pr}_1^* L_1 \,.\, \mathrm{pr}_2^* L_2 \Big) \Big| \\ & + 10 \cdot 1296^4 \Big| (K-L) \,.\, \Big(\mathrm{pr}_1^* L_1 \,.\, \mathrm{pr}_2^* L_2^2 \Big) \Big| \Big] \end{aligned} \tag{38}$$

for all j.

We will now calculate each of the above summands. Using that $K = K_{X(n)} = 4L - D$, the third summand reads

$$\begin{aligned}
&(3L-D)^2.\Big(\frac{3}{2}L^2 + 2\operatorname{pr}_1^* L_1 \,.\, \operatorname{pr}_2^* L_2\Big)\\
= \;& \frac{27}{2}L^4 + 18L^2.\operatorname{pr}_1^* L_1 \,.\, \operatorname{pr}_2^* L_2 - 9L^3.D - 12L\,.\operatorname{pr}_1^* L_1 \,.\, \operatorname{pr}_2^* L_2 \,.\, D\\
&+ \frac{3}{2}L^2.D^2 + 2\operatorname{pr}_1^* L_1 \,.\, \operatorname{pr}_2^* L_2 \,.\, D^2
\end{aligned}$$

In principal we could calculate all these terms. However, if n is sufficiently big, we only need to consider the terms with the highest power of n involved. Since every boundary divisor D contributes a factor of $1/n$ these are exactly the ones containing only the classes L, $\operatorname{pr}_1^* L_1$, and $\operatorname{pr}_2^* L_2$. Thus

$$(3L-D)^2.\Big(\frac{3}{2}L^2 + 2\operatorname{pr}_1^* L_1 \,.\, \operatorname{pr}_2^* L_2\Big) \sim \frac{27}{2}L^4 + 18L^2.\operatorname{pr}_1^* L_1 \,.\, \operatorname{pr}_2^* L_2$$

for all sufficiently big n.

Note that

$$\begin{aligned}
L^2.\operatorname{pr}_1^* L_1 \,.\, \operatorname{pr}_2^* L_2 &= (\operatorname{pr}_1^* L_1 + \operatorname{pr}_2^* L_2)^2.\operatorname{pr}_1^* L_1 \,.\, \operatorname{pr}_2^* L_2\\
&= \underbrace{\operatorname{pr}_1^* L_1^3.\operatorname{pr}_2^* L_2}_{=0} + 2\underbrace{\operatorname{pr}_1^* L_1^2.\operatorname{pr}_2^* L_2^2}_{=0} + \operatorname{pr}_1^* L_1 \,.\, \operatorname{pr}_2^* L_2^3\\
&= \operatorname{pr}_1^* L_1 \,.\, \operatorname{pr}_2^* L_2^3\,,
\end{aligned}$$

where we used again that the class $\operatorname{pr}_1^* L_1^2$ vanish. Comparing this with

$$\begin{aligned}
L^4 &= (\operatorname{pr}_1^* L_1 + \operatorname{pr}_2^* L_2)^4\\
&= \underbrace{\operatorname{pr}_1^* L_1^4}_{=0} + 4\underbrace{\operatorname{pr}_1^* L_1^3.\operatorname{pr}_2^* L_2}_{=0} + 6\underbrace{\operatorname{pr}_1^* L_1^2.\operatorname{pr}_2^* L_2^2}_{=0} + 4\operatorname{pr}_1^* L_1 \,.\, \operatorname{pr}_2^* L_2^3 + \underbrace{\operatorname{pr}_2^* L_2^4}_{=0}\\
&= 4\operatorname{pr}_1^* L_1 \,.\, \operatorname{pr}_2^* L_2^3
\end{aligned}$$

we get that

$$L^2.\operatorname{pr}_1^* L_1 \,.\, \operatorname{pr}_2^* L_2 = \operatorname{pr}_1^* L_1 \,.\, \operatorname{pr}_2^* L_2^3 = \frac{1}{4}L^4 \tag{39}$$

on X_α. Hence

$$(3L-D)^2.\Big(\frac{3}{2}L^2 + 2\operatorname{pr}_1^* L_1 \,.\, \operatorname{pr}_2^* L_2\Big) \sim 18L^4$$

for all sufficiently big n. The other terms in (38) can be estimated analogously and we obtain for all sufficiently big n and all integers j

$$\begin{aligned}
\frac{1}{5!}c_1(\mathcal{F}_j)^5 &\lesssim \frac{m^5}{5!}\Big[5\cdot 1296\cdot 81\,|L^4| + 10\cdot 1296^2\cdot\frac{135}{4}\,|L^4|\\
&\qquad + 10\cdot 1296^3\cdot 18\,|L^4| + 10\cdot 1296^4\cdot\frac{3}{4}\,|L^4|\Big]\\
&< \frac{m^5}{5!}\cdot 10\cdot 1296^4\,|L^4|\,.
\end{aligned}$$

The fact that L is just the restriction of the line bundle L on $X(n)$ to X_α allows us to calculate this intersection number on $X(n)$, namely

$$\frac{1}{5!}c_1(\mathcal{F}_j)^5 \lesssim \frac{m^5}{5!}\cdot 10\cdot 1296^4\,|L^4.\,[X_\alpha]|\ , \tag{40}$$

where $[X_\alpha]$ denotes the class of X_α in the Chow ring of $X(n)$.

Since all components X_α are equivalent under the action of $\mathrm{Sp}(6,\mathbb{Z}/n\mathbb{Z})$, we can rewrite this intersection as follows:

$$L^4.\,[X_\alpha] = \frac{1}{\#\alpha}L^4.\,[Y]\ , \tag{41}$$

where $Y = \sum_\alpha X_\alpha$ and $\#\alpha$ denotes the number of components.

Note that Y is just the pullback of the product $\mathcal{A}_1^{\mathrm{Vor}} \times \mathcal{A}_2^{\mathrm{Vor}} \subset \mathcal{A}_3^{\mathrm{Vor}} = X(1)$ to $X(n)$. According to [vdG2, Proposition 3.2] its class in the Chow ring of $X(n)$ is thus given by

$$[Y] = \frac{21}{2}L^2 - \frac{5}{2}nL\,.\,D + \frac{1}{8}n^2D^2 + \frac{1}{24}n^2\Delta_2\ ,$$

where Δ_2 is the second elementary symmetric polynomial in the D_i. Using the tables of van der Geer (cf. [vdG2]), we can now calculate the intersection in (41) and obtain

$$\begin{aligned} L^4.\,[X_\alpha] &= \frac{1}{\#\alpha}L^4.\,\Big(\frac{21}{2}L^2 - \frac{5}{2}nL\,.\,D + \frac{1}{8}n^2D^2 + \frac{1}{24}n^2\Delta_2\Big) \\ &= \frac{1}{\#\alpha}\Big(\frac{21}{2}\cdot\frac{1}{181440} - 0 + 0 + 0\Big)\,\gamma(n) = \frac{1}{\#\alpha}\cdot\frac{1}{17280}\,\gamma(n)\ . \end{aligned}$$

It now follows from (40) and (41) that we have for all j and all sufficiently big n that

$$\frac{1}{5!}c_1(\mathcal{F}_j)^5 \lesssim \frac{1}{\#\alpha}\cdot\frac{m^5}{5!}\cdot 1296^4\cdot\frac{1}{1728}\,\gamma(n)\ .$$

Note that we can explicitly give an integer n_0 such that the strict inequality holds in the above statement for all $n \geq n_0$. Moreover, this integer is independent of m and α and also does not depend on the subgroup H. By our observations in (28) and (34) at the beginning of this proof, we thus have for all $n \geq n_0$ and all sufficiently big m that

$$\dim H^0\Big(E_\alpha, \big(\mathcal{O}_{\widetilde{X}(n)}(m\,\pi^*(K-L) - jE_\alpha)\big)\Big|_{E_\alpha}\Big) < \frac{1}{\#\alpha}\cdot\frac{m^5}{5!}\cdot\frac{1296^4}{1728}\,\gamma(n)\ . \tag{42}$$

Since this estimate holds for all integers j, we can take the sum and obtain that

$$\begin{aligned}
&\dim H^0\Big(X(n),\mathcal{O}_{X(n)}\big(m(K-L)\big)\Big)\\
&\qquad\qquad -\dim H^0\Big(X(n),\mathcal{O}_{X(n)}\big(m(K-L)\big)\otimes\mathcal{J}_{X_\alpha}^{1296m\cdot\mathrm{ram}_H(X_\alpha)}\Big)\\
&\overset{(27)}{\leq}\sum_{j=0}^{1296m\cdot\mathrm{ram}_H(X_\alpha)-1}\dim H^0\Big(E_\alpha,\big(\mathcal{O}_{\widetilde{X}(n)}(m\,\pi^*(K-L)-jE_\alpha)\big)\Big|_{E_\alpha}\Big)\\
&\overset{(42)}{<}\Big[1296m\cdot\mathrm{ram}_H(X_\alpha)\Big]\cdot\frac{1}{\#\alpha}\cdot\frac{m^5}{5!}\cdot\frac{1296^4}{1728}\,\gamma(n)\\
&=\frac{1}{\#\alpha}\,\mathrm{ram}_H(X_\alpha)\frac{m^6}{5!}\cdot\frac{1296^5}{1728}\gamma(n)\ .
\end{aligned}$$

Summing over all α now gives the claim together with (26). □

We proceed as in Lemma 7.15 to use this result to conclude that condition (iii) of Proposition 7.13 is satisfied by at most finitely many groups Γ.

Lemma 7.18 *There are only finitely many subgroups* Γ *of* $\mathrm{Sp}(6,\mathbb{Z})$ *of finite index which satisfy*

$$\begin{aligned}
&\dim H^0\big(X(n),m(K-L)\big)\\
&\qquad\qquad -\dim H^0\Big(X(n),\mathcal{O}_{X(n)}\big(m(K-L)\big)\otimes\prod_\alpha\mathcal{J}_{X_\alpha}^{1296m\cdot\mathrm{ram}_H(X_\alpha)}\Big)\\
&\succeq\frac{1}{4}\Big((1/6!)\,c_1(K_{X(n)}-L)^6m^6\Big)
\end{aligned}$$

as m tends to infinity.

Proof. We can proceed exactly as in the proof of Lemma 7.15 and obtain with the estimate in Proposition 7.16 that it suffices to show that there are only finitely many subgroups Γ of $\mathrm{Sp}(6,\mathbb{Z})$ of finite index satisfying

$$\begin{aligned}
&\frac{1296^5}{1728}\cdot\Big(\frac{1}{\#\alpha}\sum_\alpha\mathrm{ram}_H(X_\alpha)\Big)\frac{m^6}{5!}\,\gamma(n)\\
&\geq\frac{1}{4}\Big((1/6!)\,c_1(K_{X(n)}-L)^6m^6\Big)\sim\frac{1}{4}\cdot\frac{3^6}{181440}\cdot\frac{m^6}{6!}\,\gamma(n)\ .
\end{aligned}$$

We can rewrite this condition as

$$\Big(\frac{1}{\#\alpha}\sum_\alpha\mathrm{ram}_H(X_\alpha)\Big)\gtrsim\frac{1}{840}\cdot\frac{1}{432^5}\ .$$

Now Theorem 5.16 proves the claim. □

To finish the proof of the main theorem, we still have to deal with the obstructions coming from the intersection of two boundary divisors D_I, $I\in\mathcal{I}_2$, in $X(n)$.

Proposition 7.19 *There is an integer n_0 such that for all $n \geq n_0$ and all subgroups $H < \mathrm{Sp}(6, \mathbb{Z}/n\mathbb{Z})$ the following inequality holds for all sufficiently big m:*

$$\begin{aligned} &\dim H^0\Big(X(n), m(K-L)\Big) \\ &\qquad - \dim H^0\Big(X(n), \mathcal{O}_{X(n)}\Big(m(K-L)\Big) \otimes \prod_{I\in\mathcal{I}_2} \mathcal{J}_{D_I}^{24mn\cdot \mathrm{ram}_H(D_I)}\Big) \\ &< 24^6 \cdot \Big(\frac{1}{\#\mathcal{I}_2} \sum_{I\in\mathcal{I}_2} \mathrm{ram}_H(D_I)\Big) \frac{m^6}{5!}\, \gamma(n)\ , \end{aligned}$$

where $\gamma(n)$ is the order of $\mathrm{Sp}(6, \mathbb{Z}/n\mathbb{Z})$ and $\#\mathcal{I}_2$ denotes the number of intersections of two boundary divisors as given in Section 6.1.

Proof. As before we can consider each component D_I separately by estimating

$$\begin{aligned} &\dim H^0\Big(X(n), \mathcal{O}_{X(n)}(m(K-L))\Big) \\ &\qquad - \dim H^0\Big(X(n), \mathcal{O}_{X(n)}\Big(m(K-L)\Big) \otimes \prod_{I\in\mathcal{I}_2} \mathcal{J}_{D_I}^{24mn\cdot \mathrm{ram}_H(D_I)}\Big) \\ &\leq \sum_{I\in\mathcal{I}_2} \Big[\dim H^0\Big(X(n), \mathcal{O}_{X(n)}\Big(m(K-L)\Big)\Big) \qquad (43) \\ &\qquad - \dim H^0\Big(X(n), \mathcal{O}_{X(n)}\Big(m(K-L)\Big) \otimes \mathcal{J}_{D_I}^{24mn\cdot \mathrm{ram}_H(D_I)}\Big)\Big]\ . \end{aligned}$$

We consider the blow–up of $X(n)$ along one such D_I and obtain

$$\begin{array}{ccc} E_I & \subset & \widetilde{X}(n) \\ \downarrow & & \downarrow \pi \\ D_I & \subset & X(n) \end{array}$$

where E_I denotes the exceptional divisor of this blow–up. As in the proof of Proposition 7.16 we have by [CEL, Lemma 3.3] that

$$\begin{aligned} &\dim H^0\Big(X(n), \mathcal{O}_{X(n)}\Big(m(K-L)\Big) \otimes \mathcal{J}_{D_I}^{24mn\cdot \mathrm{ram}_H(D_I)}\Big) \\ &= \dim H^0\Big(\widetilde{X}(n), \mathcal{O}_{\widetilde{X}(n)}\Big(m\,\pi^*(K-L) - 24mn\,\mathrm{ram}_H(D_I)\,E_I\Big)\Big)\ . \end{aligned}$$

for all sufficiently big m, which then implies by the standard exact sequences associated to $E_I \subset \widetilde{X}(n)$ that

$$\begin{aligned}
&\dim H^0\Big(X(n), \mathcal{O}_{X(n)}\big(m(K-L)\big)\Big) \\
&\qquad - \dim H^0\Big(X(n), \mathcal{O}_{X(n)}\big(m(K-L)\big) \otimes \mathcal{J}_{D_I}^{24mn\cdot \mathrm{ram}_H(D_I)}\Big) \\
&\leq \sum_{j=0}^{24mn\cdot \mathrm{ram}_H(D_I)-1} \dim H^0\Big(E_I, \big(\mathcal{O}_{\widetilde{X}(n)}(m\,\pi^*(K-L) - jE_I)\big)\Big|_{E_I}\Big) . \qquad (44)
\end{aligned}$$

As before we can consider E_I as a fiber bundle $\pi : E_I \to D_I$ which can be identified with the projectivization $\mathbb{P}(\mathcal{N}^\vee_{D_I/X(n)})$ of the conormal bundle $\mathcal{N}^\vee_{D_I/X(n)}$ of D_I in $X(n)$. We are now in exactly the same situation as in the proof of Proposition 7.16 where we applied Griffiths vanishing theorem.

For any inclusion $X \subset Y \subset Z$ of varieties we have the exact sequence

$$0 \to \mathcal{N}_{X/Y} \to \mathcal{N}_{X/Z} \to \mathcal{N}_{Y/Z}|_X \to 0$$

which implies that the normal bundle of an intersection of two varieties is the sum of their normal bundles. In our situation we thus have

$$\mathcal{N}_{D_I/X(n)} = \mathcal{N}_{D_{i_1}/X(n)}|_{D_I} \oplus \mathcal{N}_{D_{i_2}/X(n)}|_{D_I}$$

where $I = (i_1, i_2) \in \mathcal{I}_2$. For the conormal bundle we obtain

$$\mathcal{N}^\vee_{D_I/X(n)} = \mathcal{N}^\vee_{D_{i_1}/X(n)} \oplus \mathcal{N}^\vee_{D_{i_2}/X(n)} \qquad (45)$$

where we simplified the notation by omitting the restrictions to D_I of the conormal bundles $\mathcal{N}^\vee_{D_{i_k}/X(n)}$.

Instead of looking at the conormal bundle $\mathcal{N}^\vee_{D_I/X(n)}$, we will consider

$$\begin{aligned}
\mathcal{N}^\vee_{D_I/X(n)} \otimes \mathcal{O}_{D_I}(\tfrac{1}{n}L) &= \Big(\mathcal{N}^\vee_{D_{i_1}/X(n)} \oplus \mathcal{N}^\vee_{D_{i_2}/X(n)}\Big) \otimes \mathcal{O}_{D_I}(\tfrac{1}{n}L) \\
&= \Big(\mathcal{N}^\vee_{D_{i_1}/X(n)} \otimes \mathcal{O}_{D_I}(\tfrac{1}{n}L)\Big) \oplus \Big(\mathcal{N}^\vee_{D_{i_2}/X(n)} \otimes \mathcal{O}_{D_I}(\tfrac{1}{n}L)\Big) .
\end{aligned}$$

Since we know from the proof of Proposition 7.14 that the two bundles $\mathcal{N}^\vee_{D_{i_k}/X(n)} \otimes \mathcal{O}_{D_I}(\frac{1}{n}L)$ are nef, we can conclude that $\mathcal{N}^\vee_{D_I/X(n)} \otimes \mathcal{O}_{D_I}(\frac{1}{n}L)$ as their direct sum is nef, too.

As before, we obtain that the canonical bundle on D_I is given by

$$\omega_{D_I} = \omega_{X(n)}|_{D_I} \otimes \det(\mathcal{N}^\vee_{D_I/X(n)}) . \qquad (46)$$

To apply Griffiths vanishing theorem we will need the identities

$$\mathrm{S}^j\Big(\mathcal{N}^\vee_{D_I/X(n)} \otimes \mathcal{O}_{D_I}(\tfrac{1}{n}L)\Big) = \mathrm{S}^j\Big(\mathcal{N}^\vee_{D_I/X(n)}\Big) \otimes \mathcal{O}_{D_I}(\tfrac{j}{n}L) \qquad (47)$$

and

$$\det\left(\mathcal{N}^{\vee}_{D_I/X(n)} \otimes \mathcal{O}_{D_I}(\tfrac{1}{n}L)\right) = \mathcal{O}_{D_I}(\tfrac{2}{n}L) \otimes \det\left(\mathcal{N}^{\vee}_{D_I/X(n)}\right), \tag{48}$$

where we used that $\mathcal{N}^{\vee}_{D_I/X(n)}$ is a rank 2 bundle.

Note that $(m-1)(3L-D)-(1+2/n)L$ is ample on $X(n)$ for $n>4$ and all sufficiently big m by [Hul, Theorem 0.2]. The same holds for the restriction of this divisor to D_I. If m is sufficiently big we can furthermore subtract $2\det(\mathcal{N}^{\vee}_{D_I/X(n)})$ and still get something ample on D_I. We thus have that

$$\mathcal{O}_{D_I}\Big((m-1)\big(3L-D\big)-\big(1+\tfrac{2}{n}\big)L\Big) \otimes \det(\mathcal{N}^{\vee}_{D_I/X(n)})^{\otimes(-2)}$$

is ample on D_I if $n>4$ and m is sufficiently big.

For $0 \le j \le 24mn \cdot \mathrm{ram}_H(D_I) - 1 < 24mn$ the bundle $(24m-(j/n))L$ is nef and we still have that

$$\begin{aligned}
&\mathcal{O}_{D_I}\Big((m-1)K_{X(n)} + \big(23m - \tfrac{j+2}{n}\big)L\Big) \otimes \det(\mathcal{N}^{\vee}_{D_I/X(n)})^{\otimes(-2)}\\
=\,&\mathcal{O}_{D_I}\Big((m-1)\big(3L-D\big) - \big(1+\tfrac{2}{n}\big)L + \big(24m-\tfrac{j}{n}\big)L\Big) \otimes \det(\mathcal{N}^{\vee}_{D_I/X(n)})^{\otimes(-2)}
\end{aligned}$$

is ample for m sufficiently big and $n>4$.

Using this and the fact that $\mathcal{N}^{\vee}_{D_I/X(n)} \otimes \mathcal{O}_{D_I}(\frac{1}{n}L)$ is nef as we have seen in our above discussion we obtain by Griffiths vanishing theorem that

$$\begin{aligned}
& H^i\Big(D_I, \mathcal{O}_{D_I}\big(m(K_{X(n)}+23L)\big) \otimes \mathrm{S}^j\big(\mathcal{N}^{\vee}_{D_I/X(n)}\big)\Big)\\
\overset{(47)}{=}\ & H^i\Big(D_I, \mathcal{O}_{D_I}\big(mK_{X(n)} + (23m-\tfrac{j}{n})L\big) \otimes \mathrm{S}^j\big(\mathcal{N}^{\vee}_{D_I/X(n)} \otimes \mathcal{O}_{D_I}(\tfrac{1}{n}L)\big)\Big)\\
=\ & H^i\Big(D_I, \big[\mathcal{O}_{D_I}(K_{X(n)}) \otimes \det(\mathcal{N}^{\vee}_{D_I/X(n)})\big] \otimes \big[\mathcal{O}_{D_I}(\tfrac{2}{n}L) \otimes \det(\mathcal{N}^{\vee}_{D_I/X(n)})\big]\\
&\quad \otimes \big[\mathcal{O}_{D_I}\big((m-1)K_{X(n)} + \big(23m-\tfrac{j+2}{n}\big)L\big) \otimes \det(\mathcal{N}^{\vee}_{D_I/X(n)})^{\otimes(-2)}\big]\\
&\quad \otimes \mathrm{S}^j\big(\mathcal{N}^{\vee}_{D_I/X(n)} \otimes \mathcal{O}_{D_I}(\tfrac{1}{n}L)\big)\Big)\\
\overset{(46),(48)}{=}\ & H^i\Big(D_I, \omega_{D_I} \otimes \mathrm{S}^j\big(\mathcal{N}^{\vee}_{D_I/X(n)} \otimes \mathcal{O}_{D_I}(\tfrac{1}{n}L)\big) \otimes \det\big(\mathcal{N}^{\vee}_{D_I/X(n)} \otimes \mathcal{O}_{D_I}(\tfrac{1}{n}L)\big)\\
&\quad \otimes \mathcal{O}_{D_I}\big((m-1)K_{X(n)} + \big(23m-\tfrac{j+2}{n}\big)L\big) \otimes \det(\mathcal{N}^{\vee}_{D_I/X(n)})^{\otimes(-2)}\Big)\\
=\ & 0
\end{aligned}$$

for all $i>0$.

As in the proof of Proposition 7.16 we can translate this into a corresponding statement on E_I using the Leray spectral sequence and the vanishing of the higher direct images and obtain

$$\begin{aligned}&\chi\Big(\pi^*\Big(\mathcal{O}_{D_I}(m(K_{X(n)}+23L))\Big)\otimes\zeta^{\otimes j}\Big)\\&=\dim H^0\Big(E_I,\pi^*\Big(\mathcal{O}_{D_I}(m(K_{X(n)}+23L))\Big)\otimes\zeta^{\otimes j}\Big)\ ,\end{aligned}$$

where $\zeta=\mathcal{O}_{\mathbb{P}(\mathcal{N}^\vee_{D_I/X(n)})}(1)$ denotes the tautological bundle on E_I.

Comparing this with the summands in (44), we get for each integer j the estimate

$$\begin{aligned}&\dim H^0\Big(E_I,\Big(\mathcal{O}_{\widetilde{X}(n)}(m\,\pi^*(K_{X(n)}-L)-jE_I)\Big)\Big|_{E_I}\Big)\\&=\dim H^0\Big(E_I,\pi^*\Big(\mathcal{O}_{D_I}(m(K_{X(n)}-L))\Big)\otimes\zeta^{\otimes j}\Big)\\&\le\dim H^0\Big(E_I,\pi^*\Big(\mathcal{O}_{D_I}(m(K_{X(n)}+23L))\Big)\otimes\zeta^{\otimes j}\Big)\\&=\chi\Big(\pi^*\Big(\mathcal{O}_{D_I}(m(K_{X(n)}+23L))\Big)\otimes\zeta^{\otimes j}\Big)\ ,\end{aligned}\tag{49}$$

where we used the fact that adding multiples of the line bundle L does not decrease the dimension of the corresponding space of global sections.

To simplify notation we set for each integer j

$$\mathcal{F}_j:=\pi^*\Big(\mathcal{O}_{D_I}(m(K_{X(n)}+23L))\Big)\otimes\zeta^{\otimes j}\ .$$

We can now apply Hirzebruch–Riemann–Roch to get that

$$\dim H^0(E_I,\mathcal{F}_j)\overset{(49)}{=}\chi(\mathcal{F}_j)=\deg(\mathrm{ch}(\mathcal{F}_j).\,\mathrm{td}(\mathcal{T}))_5\ .$$

We only need an estimate for $\dim H^0(E_I,\mathcal{F}_j)$ for $m>>0$ and can thus ignore the Todd class $\mathrm{td}\,\mathcal{T}$ of the tangent sheaf $\mathcal{T}$ of E_I since it does not contribute any factors of m, i.e. we have

$$\dim H^0(E_I,\mathcal{F}_j)\sim\frac{1}{5!}c_1(\mathcal{F}_j)^5\tag{50}$$

for all sufficiently big m.

This is exactly the same situation as in (34) and we can use again the relation on the tautological bundle and the fact that all other bundles are coming from D_I to conclude that as in (37)

$$\begin{aligned}\frac{1}{5!}c_1(\mathcal{F}_j)^5=\frac{1}{5!}\Big[&5j\Big(m(K_{X(n)}+23L)\Big)^4-10j^2\Big(m(K_{X(n)}+23L)\Big)^3.\,c_1(\mathcal{N}^\vee)\\&+10j^3\Big(m(K_{X(n)}+23L)\Big)^2.\,\Big(c_1(\mathcal{N}^\vee)^2-c_2(\mathcal{N}^\vee)\Big)\\&+5j^4\Big(m(K_{X(n)}+23L)\Big).\,\Big(2c_1(\mathcal{N}^\vee).\,c_2(\mathcal{N}^\vee)-c_1(\mathcal{N}^\vee)^3\Big)\\&+j^5\Big(c_2(\mathcal{N}^\vee)^2+c_1(\mathcal{N}^\vee)^4-3c_1(\mathcal{N}^\vee)^2.\,c_2(\mathcal{N}^\vee)\Big)\Big]\ ,\end{aligned}\tag{51}$$

where the intersection on the right hand side takes place on D_I and $\mathcal{N}^\vee$ denotes the conormal bundle $\mathcal{N}^\vee_{D_I/X(n)}$ of D_I in $X(n)$.

As we observed in (45) the conormal bundle $\mathcal{N}^\vee$ is given by

$$\mathcal{N}^\vee_{D_I/X(n)} = \mathcal{N}^\vee_{D_{i_1}/X(n)} \oplus \mathcal{N}^\vee_{D_{i_2}/X(n)}$$

where $I = (i_1, i_2) \in \mathcal{I}_2$. Using the Whitney sum formula, we can calculate the first and second Chern classes of this direct sum and obtain that

$$c_1(\mathcal{N}^\vee) = D_{i_1} + D_{i_2} \quad \text{and} \quad c_2(\mathcal{N}^\vee) = D_{i_1} \,.\, D_{i_2} \,.$$

Note carefully that the right hand sides of these equations are intersections on $X(n)$ which we have to restrict to D_I. But, in fact, now all the intersections in (51) can be expressed as restrictions of intersections on $X(n)$. Hence we can do the calculation on $X(n)$ and obtain

$$\begin{aligned}\frac{1}{5!}c_1(\mathcal{F}_j)^5 = \frac{1}{5!}\Big[&5j\Big(m(K_{X(n)} + 23)\Big)^4 - 10j^2\Big(m(K_{X(n)} + 23L)\Big)^3 . \Big(D_{i_1} + D_{i_2}\Big)\\ &+ 10j^3\Big(m(K_{X(n)} + 23L)\Big)^2 . \Big(D_{i_1}^2 + D_{i_1} \,.\, D_{i_2} + D_{i_2}^2\Big)\\ &- 5j^4\Big(m(K_{X(n)} + 23L)\Big) . \Big(D_{i_1}^3 + D_{i_1}^2 \,.\, D_{i_2} + D_{i_1} \,.\, D_{i_2}^2 + D_{i_2}^3\Big)\\ &+ j^5\Big(D_{i_1}^4 + D_{i_1}^3 \,.\, D_{i_2} + D_{i_1}^2 \,.\, D_{i_2}^2 + D_{i_1} \,.\, D_{i_2}^3 + D_{i_2}^4\Big)\Big] . [D_I] \,,\end{aligned} \tag{52}$$

where $[D_I]$ denotes the class of $D_I = D_{i_1} \cap D_{i_2}$ in $X(n)$.

To get an estimate for $\frac{1}{5!}c_1(\mathcal{F}_j)^5$ which is independent of j, we first take absolute values of the terms in (52) and use that $0 \le j \le 24mn \cdot \mathrm{ram}_H(D_I) - 1 < 24mn$. For instance, the second term in (52) can be estimated as follows:

$$\begin{aligned}&- 10j^2\Big[\Big(m(K_{X(n)} + 23L)\Big)^3 . \Big(D_{i_1} + D_{i_2}\Big)\Big] . [D_I]\\ &\le 10j^2\Big(m^3\Big|(K_{X(n)} + 23L)^3 . (D_{i_1} + D_{i_2}) \,.\, [D_I]\Big|\Big)\\ &< 10 \cdot 24^2\, m^5 n^2\Big(\Big|(K_{X(n)} + 23L)^3 . (D_{i_1} + D_{i_2}) \,.\, [D_I]\Big|\Big)\end{aligned}$$

Note that this estimate is not only independent of j but also of $I = (i_1, i_2) \in \mathcal{I}_2$ since all D_I are equivalent under the action of $\mathrm{Sp}(6, \mathbb{Z}/n\mathbb{Z})$. We thus can take the sum over all D_I and obtain

$$\begin{aligned}&- 10j^2\Big[\Big(m(K_{X(n)} + 23L)\Big)^3 . \Big(D_{i_1} + D_{i_2}\Big)\Big] . [D_I]\\ &< \frac{1}{\#\mathcal{I}_2} \sum_{I=(i_1,i_2)\in\mathcal{I}_2} 10 \cdot 24^2\, m^5 n^2\Big(\Big|(K_{X(n)} + 23L)^3 . (D_{i_1} + D_{i_2}) \,.\, [D_I]\Big|\Big)\\ &= \frac{1}{\#\mathcal{I}_2}\, 10 \cdot 24^2\, m^5 n^2\Big(\Big|(K_{X(n)} + 23L)^3 . \sum_{I=(i_1,i_2)\in\mathcal{I}_2}\Big((D_{i_1} + D_{i_2}) \,.\, [D_I]\Big)\Big|\Big)\end{aligned} \tag{53}$$

Using that the class of $D_I = D_{i_1} \cap D_{i_2}$ in $X(n)$ is just given by $D_{i_1} \,.\, D_{i_2}$, we can consider the term $\sum_{I=(i_1,i_2)\in\mathcal{I}_2}\Big((D_{i_1}+D_{i_2})\,.\,[D_I]\Big)$ as a symmetric polynomial in the D_{i_k}. If we denote the k–th elementary symmetric polynomial by Δ_k, we get that

$$\sum_{I=(i_1,i_2)\in\mathcal{I}_2}\Big((D_{i_1}+D_{i_2})\,.\,D_{i_1}\,.\,D_{i_2}\Big) = \sum_{I=(i_1,i_2)\in\mathcal{I}_2}\Big(D_{i_1}^2\,.\,D_{i_2}+D_{i_1}\,.\,D_{i_2}^2\Big) = \Delta_1\,.\,\Delta_2 - 3\Delta_3\,.$$

Recalling that $K_{X(n)} = 4L - D$, we can thus express the intersection given in (53) in terms of L and boundary components D and Δ_k only. Since every boundary divisor contributes a factor of $(1/n)$, it suffices for sufficiently big n to consider the terms with the highest power of L involved. This gives us the following estimate

$$\begin{aligned}
&- 10j^2\Big[\Big(m(K_{X(n)}+23L)\Big)^3.\Big(D_{i_1}+D_{i_2}\Big)\Big].\,[D_I]\\
&\lesssim \frac{1}{\#\mathcal{I}_2}\,10\cdot 24^2\,m^5n^2\Big(27^3\,\Big|L^3.(\Delta_1\,.\,\Delta_2-3\Delta_3)\Big|\Big)
\end{aligned}$$

for all sufficiently big n and all integers $0 \le j < 24mn$.

We can proceed analogously with the other terms in (52) and get that

$$\begin{aligned}
&\frac{1}{5!}c_1(\mathcal{F}_j)^5\\
&\lesssim \frac{m^5}{5!}\cdot\frac{1}{\#\mathcal{I}_2}\Big[5\cdot 24\cdot 27^4\,n\Big|L^4.\,\Delta_2\Big| + 10\cdot 24^2\cdot 27^3\,n^2\Big|L^3.(\Delta_1\,.\,\Delta_2-3\Delta_3)\Big|\\
&\quad + 10\cdot 24^3\cdot 27^2\,n^3\Big|L^2.(\Delta_1^2.\,\Delta_2-3\Delta_1\,.\,\Delta_3-\Delta_2^2+6\Delta_4)\Big|\\
&\quad + 5\cdot 24^4\cdot 27\,n^4\Big|L\,.\Big(\Delta_1^3.\,\Delta_2-3\Delta_1^2.\,\Delta_3+6\Delta_1\,.\,\Delta_4\\
&\qquad\qquad -2\Delta_1\,.\,\Delta_2^2+4\Delta_2\,.\,\Delta_3-10\Delta_5\Big)\Big|\\
&\quad + 24^5\,n^5\Big|\Delta_1^4.\,\Delta_2-3\Delta_1^3.\,\Delta_3+6\Delta_1^2.\,\Delta_4-3\Delta_1^2.\,\Delta_2^2-10\Delta_1\,.\,\Delta_5\\
&\qquad\qquad +8\Delta_1\,.\,\Delta_2\,.\,\Delta_3-3\Delta_3^2+\Delta_2^3-7\Delta_2\,.\,\Delta_4+15\Delta_6\Big|\Big]
\end{aligned}\tag{54}$$

for all sufficiently big n and all integers $0 \le j < 24mn$. Here we used the following identities of symmetric polynomials:

$$\sum_{I=(i_1,i_2)\in\mathcal{I}_2} D_{i_1}\,.\,D_{i_2} = \Delta_2$$

$$\sum_{I=(i_1,i_2)\in\mathcal{I}_2} \Big(D_{i_1}^3.\,D_{i_2} + D_{i_1}^2.\,D_{i_2}^2 + D_{i_1}\,.\,D_{i_2}^3\Big) = \Delta_1^2.\,\Delta_2 - 3\Delta_1\,.\,\Delta_3 - \Delta_2^2 + 6\Delta_4$$

$$\sum_{I=(i_1,i_2)\in\mathcal{I}_2} \Big(D_{i_1}^4.\,D_{i_2} + D_{i_1}^3\,.\,D_{i_2}^2 + D_{i_1}^2.\,D_{i_2}^3 + D_{i_1}\,.\,D_{i_2}^4\Big)$$
$$= \Delta_1^3.\,\Delta_2 - 3\Delta_1^2.\,\Delta_3 + 6\Delta_1\,.\,\Delta_4 - 2\Delta_1\,.\,\Delta_2^2 + 4\Delta_2\,.\,\Delta_3 - 10\Delta_5$$

$$\sum_{I=(i_1,i_2)\in\mathcal{I}_2} \Big(D_{i_1}^5.\,D_{i_2} + D_{i_1}^4\,.\,D_{i_2}^2 + D_{i_1}^3\,.\,D_{i_2}^3 + D_{i_1}^2.\,D_{i_2}^4 + D_{i_1}\,.\,D_{i_2}^5\Big)$$
$$= \Delta_1^4.\,\Delta_2 - 3\Delta_1^3.\,\Delta_3 + 6\Delta_1^2.\,\Delta_4 - 3\Delta_1^2.\,\Delta_2^2 - 10\Delta_1\,.\,\Delta_5$$
$$+ 8\Delta_1\,.\,\Delta_2\,.\,\Delta_3 - 3\Delta_3^2 + \Delta_2^3 - 7\Delta_2\,.\,\Delta_4 + 15\Delta_6\ .$$

The intersection numbers in (54) are given by

$$L^4.\,\Delta_2 = 0$$
$$L^3.(\Delta_1\,.\,\Delta_2 - 3\Delta_3) = 0$$
$$L^2.(\Delta_1^2.\,\Delta_2 - 3\Delta_1\,.\,\Delta_3 - \Delta_2^2 + 6\Delta_4) = 0$$
$$L\,.(\Delta_1^3.\,\Delta_2 - 3\Delta_1^2.\,\Delta_3 + 6\Delta_1\,.\,\Delta_4 - 2\Delta_1\,.\,\Delta_2^2 + 4\Delta_2\,.\,\Delta_3 - 10\Delta_5) = -\frac{1}{12}\cdot\frac{\gamma(n)}{n^5}$$
$$\Delta_1^4.\,\Delta_2 - 3\Delta_1^3.\,\Delta_3 + 6\Delta_1^2.\,\Delta_4 - 3\Delta_1^2.\,\Delta_2^2 - 10\Delta_1\,.\,\Delta_5$$
$$+ 8\Delta_1\,.\,\Delta_2\,.\,\Delta_3 - 3\Delta_3^2 + \Delta_2^3 - 7\Delta_2\,.\,\Delta_4 + 15\Delta_6 = \frac{5}{16}\cdot\frac{\gamma(n)}{n^6}$$

as can be computed from the results of van der Geer (cf. [vdG2]). Hence

$$\frac{1}{5!}c_1(\mathcal{F}_j)^5$$
$$\lesssim \frac{m^5}{5!}\cdot\frac{1}{\#\mathcal{I}_2}\Big[5\cdot 24^4\cdot 27\,n^4\Big(\frac{1}{12}\cdot\frac{\gamma(n)}{n^5}\Big) + 24^5\,n^5\Big(\frac{5}{16}\cdot\frac{\gamma(n)}{n^6}\Big)\Big]$$
$$< \frac{m^5}{5!}\cdot\frac{1}{n}\cdot\frac{1}{\#\mathcal{I}_2}24^5\,\gamma(n)$$

for all sufficiently big n. This statement can be made precise, i.e. there is an integer n_0 such that in the above estimate the strict inequality holds for all $n \geq n_0$. Note that the integer n_0 can be chosen independently of m and $I = (i_1, i_2)$.

By (50) we thus have for all $n \geq n_0$ and all sufficiently big m that

$$\dim H^0\Big(E_I, \Big(\mathcal{O}_{\widetilde{X}(n)}(m\,\pi^*(K-L) - jE_I)\Big)\Big|_{E_I}\Big) < \frac{m^5}{5!}\cdot\frac{1}{n}\cdot\frac{1}{\#\mathcal{I}_2}24^5\,\gamma(n)\ . \quad (55)$$

This estimate holds for all integers $0 \leq j \leq 24mn \cdot \mathrm{ram}_H(D_I) - 1$, so we can take the sum over all j and obtain

$$\begin{aligned}
&\dim H^0\Big(X(n), \mathcal{O}_{X(n)}\big(m(K-L)\big)\Big) \\
&\qquad - \dim H^0\Big(X(n), \mathcal{O}_{X(n)}\big(m(K-L)\big) \otimes \mathcal{J}_{D_I}^{24mn\cdot\mathrm{ram}_H(D_I)}\Big) \\
&\overset{(44)}{\leq} \sum_{j=0}^{24mn\cdot\mathrm{ram}_H(D_I)-1} \dim H^0\Big(E_I, \big(\mathcal{O}_{\widetilde{X}(n)}(m\,\pi^*(K-L) - jE_I)\big)\Big|_{E_I}\Big) \\
&\overset{(55)}{<} \Big[24mn \cdot \mathrm{ram}_H(D_I)\Big] \cdot \frac{m^5}{5!} \cdot \frac{1}{n} \cdot \frac{1}{\#\mathcal{I}_2} 24^5\,\gamma(n) \\
&= \frac{1}{\#\mathcal{I}_2}\,\mathrm{ram}_H(D_I) \cdot \frac{m^6}{5!} \cdot 24^6\,\gamma(n)\ .
\end{aligned}$$

Summing now over all D_I as in (43) gives the desired result. □

The above result can again be used to prove that condition (iv) of Proposition 7.13 is satisfied by at most finitely many subgroups Γ.

Lemma 7.20 *There are only finitely many subgroups* Γ *of* $\mathrm{Sp}(6,\mathbb{Z})$ *of finite index which satisfy*

$$\begin{aligned}
&\dim H^0\big(X(n), m(K-L)\big) \\
&\qquad - \dim H^0\Big(X(n), \mathcal{O}_{X(n)}\big(m(K-L)\big) \otimes \prod_{I\in\mathcal{I}_2} \mathcal{J}_{D_I}^{24mn\cdot\mathrm{ram}_H(D_I)}\Big) \\
&\succeq \frac{1}{4}\Big((1/6!)\, c_1(K_{X(n)} - L)^6 m^6\Big)
\end{aligned}$$

as m *tends to infinity.*

Proof. This proof is completely analogous to the ones given in Lemma 7.15 and Lemma 7.18. We can use the estimate in Proposition 7.19 to get a lower bound on $\frac{1}{\#\mathcal{I}_2}\sum_{I\in\mathcal{I}_2} \mathrm{ram}_H(D_I)$ and then use Theorem 6.3 to finish the proof. □

7.4 Proof of the Main Theorem

We are now ready to prove the main theorem of this thesis:

Theorem 7.14 *There are only finitely many subgroups* Γ *of* $\mathrm{Sp}(6,\mathbb{Z})$ *of finite index such that the space of pluricanonical sections on* $\left(\widetilde{\mathcal{A}}_\Gamma^{\mathrm{Vor}}\right)^{(2)}$ *does not grow maximally.*

Proof. If Γ is a subgroup of $\mathrm{Sp}(6,\mathbb{Z})$ for which the space of pluricanonical sections on $\left(\widetilde{\mathcal{A}}_\Gamma^{\mathrm{Vor}}\right)^{(2)}$ does not grow maximally, then one of the four conditions (i) to (iv) of Proposition 7.13 has to be satisfied. However in Proposition 7.2 and Remark 7.3 we have seen that condition (i) is only satisfied by a finite number of subgroups. The same is true for conditions (ii), (iii) and (iv) by Lemma 7.15, Lemma 7.18 and Lemma 7.20 respectively. □

Appendix A

The group $\mathbf{S_{2g-1}}$

In this section we will collect various results on the group S_{2g-1}, the stabilizer of V_{2g-1} in $\mathrm{Sp}(2g, \mathbb{Z}/p\mathbb{Z})$. They play an important role in the proof of the main theorem of Chapter 4 (cf. Lemma 4.8).

For any prime p we consider the symplectic group $G := \mathrm{Sp}(2g, \mathbb{Z}/p\mathbb{Z})$ and its action on the vector space $V = (\mathbb{Z}/p\mathbb{Z})^g$. More precisely, we are interested in the subspace

$$V_{2g-1} := (*, \dots, *, 0) \subset V$$

and its stabilizer in G which is given by

$$S_{2g-1} := \left\{ \begin{pmatrix} A & 0 & B & m_3 \\ m_1^T & u & m_2^T & m_4 \\ C & 0 & D & m_5 \\ 0 & 0 & 0 & u^{-1} \end{pmatrix};\ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2g-2, \mathbb{Z}/p\mathbb{Z}), \right.$$
$$\left. \begin{array}{l} u \in (\mathbb{Z}/p\mathbb{Z})^*, m_1, m_2, m_3, m_5 \in (\mathbb{Z}/p\mathbb{Z})^{g-1}, m_4 \in \mathbb{Z}/p\mathbb{Z}, \\ \quad A \cdot m_2 - B \cdot m_1 = u \cdot m_3\,, \\ \quad C \cdot m_2 - D \cdot m_1 = u \cdot m_5 \end{array} \right\}.$$

as can be easily seen by a short calculation. We will first compute the order of this group.

Proposition A.1 *The group S_{2g-1} has order*

$$|S_{2g-1}| = p^{g^2} \cdot (p-1) \cdot \prod_{i=1}^{g-1} (p^{2i} - 1)$$

and its index in $\mathrm{Sp}(2g, \mathbb{Z}/p\mathbb{Z})$ *is given by*

$$[\mathrm{Sp}(2g, \mathbb{Z}/p\mathbb{Z}) : S_{2g-1}] = \frac{p^{2g} - 1}{p - 1}\ .$$

Proof. It suffices to note that for any choice of u, m_3 and m_5 together with A, B, C, D the vectors m_1 and m_2 are uniquely determined and then use the known formula for $\mathrm{Sp}(2g-2, \mathbb{Z}/p\mathbb{Z})$. Comparing the order of S_{2g-1} with the order of $\mathrm{Sp}(2g, \mathbb{Z}/p\mathbb{Z})$ gives the result on the index. □

Consider the set of subspaces $W_g \subset V_{2g-1}$ which are isomorphic to $(\mathbb{Z}/p\mathbb{Z})^g$. The group S_{2g-1} acts on this set, however, this action is not transitive. This means that we have more than one orbit under this action. We will be interested in the orbit containing the canonical choice for a subspace $W_g \cong (\mathbb{Z}/p\mathbb{Z})^g$, namely the subspace $V_g \subset V_{2g-1}$ given by

$$V_g := (\underbrace{*, \ldots, *}_{g \text{ times}}, \underbrace{0, \ldots, 0}_{g \text{ times}}) \subset V \; .$$

The stabilizer in S_{2g-1} of V_g can be computed to be

$$\mathrm{Stab}_{S_{2g-1}}(V_g) = \left\{ \left(\begin{array}{cc|cc} A & 0 & B & m_3 \\ m_1^T & u & m_2^T & m_4 \\ \hline 0 & 0 & A^{-T} & m_5 \\ 0 & 0 & 0 & u^{-1} \end{array} \right) ; \; A \in \mathrm{GL}(g-1, \mathbb{Z}/p\mathbb{Z}), \right.$$
$$\left. \begin{array}{l} B \in \mathrm{Mat}(g-1, \mathbb{Z}/p\mathbb{Z}), AB^T = BA^T, u \in (\mathbb{Z}/p\mathbb{Z})^*, \\ m_1, m_2, m_3, m_5 \in (\mathbb{Z}/p\mathbb{Z})^{g-1}, m_4 \in \mathbb{Z}/p\mathbb{Z}, \\ m_1 = -u \cdot A^T \cdot m_5, \; A \cdot m_2 - B \cdot m_1 = u \cdot m_3 \end{array} \right\} ,$$

where $A^{-T} = \left(A^T\right)^{-1}$.

Proposition A.2 *The group* $\mathrm{Stab}_{S_{2g-1}}(V_g)$ *has order*

$$\left|\mathrm{Stab}_{S_{2g-1}}(V_g)\right| = p^{g^2} \cdot (p-1) \cdot \prod_{i=1}^{g-1} (p^i - 1) \; .$$

Proof. Note that given $A \in \mathrm{GL}(g-1, \mathbb{Z}/p\mathbb{Z})$ there are exactly $p^{g(g-1)/2}$ choices for B such that the relation $AB^T = BA^T$ is satisfied. Together with u, m_3 and m_5 these two matrices determine m_1 and m_2 uniquely. Using this observation together with the well–known formula for the order of $\mathrm{GL}(g-1, \mathbb{Z}/p\mathbb{Z})$ this proves the claim. □

Comparing this with the order of S_{2g-1} computed in Proposition A.1, we obtain as an immediate consequence the size of the orbit of V_g in S_{2g-1}.

Corollary A.3 *The order of the orbit of V_g under the action of S_{2g-1} is given by*

$$\left|\mathrm{orb}_{S_{2g-1}}(V_g)\right| = \prod_{i=1}^{g-1} (p^i + 1) \; .$$

Let e_1 and e_g denote the first and the g-th vector of the canonical basis of $V = (\mathbb{Z}/p\mathbb{Z})^{2g}$ respectively. By calculating the stabilizers of these two vectors under the action of S_{2g-1}, we can calculate the orders of their orbits which are given by

$$\left|\mathrm{orb}_{S_{2g-1}}(e_1)\right| = p(p^{2g-2}-1)\ , \quad \left|\mathrm{orb}_{S_{2g-1}}(e_g)\right| = p-1\ .$$

Comparing this with the order of V_{2g-1} we obtain that

$$\left\{0\right\} \dot\cup \left\{\mathrm{orb}_{S_{2g-1}}(e_g)\right\} \dot\cup \left\{\mathrm{orb}_{S_{2g-1}}(e_1)\right\} = V_{2g-1}\ .$$

This implies that

$$\bigcup_{W_g \in \mathrm{orb}_{S_{2g-1}}(V_g)} W_g = V_{2g-1}\ ,$$

i.e. V_g sweeps out all of V_{2g-1} under the action of S_{2g-1}. Or to state it in a different way, every primitive vector in $V_{2g-1}^* := V_{2g-1} \setminus \{0\}$ is contained in at least one subspace $W_g \in \mathrm{orb}_{S_{2g-1}}(V_g)$. However, in Chapter 4 we will need a more precise statement.

Proposition A.4 *Let v be any primitive vector in V_{2g-1}^*.*

(i) If v is lying in the orbit of e_1 under the action of S_{2g-1}, it is contained in exactly $\prod_{i=1}^{g-2}(p^i+1)$ subspaces $W_g \in \mathrm{orb}_{S_{2g-1}}(V_g)$

(ii) If v is lying in the orbit of e_g under the action of S_{2g-1}, it is contained in exactly $\prod_{i=1}^{g-1}(p^i+1)$ subspaces $W_g \in \mathrm{orb}_{S_{2g-1}}(V_g)$

Proof. It suffices to prove the results for e_1 and e_g. Consider the set of pairs (v, W_g) where $(\mathbb{Z}/p\mathbb{Z})^g \cong W_g \subset V_{2g-1}$ and $v \in W_g^*$. The group S_{2g-1} acts on this set of pairs and so does its subgroup S_{e_1}, the stabilizer of e_1 in S_{2g-1}. The stabilizer of the standard pair (e_1, V_g) under the action of S_{e_1} can be computed by intersecting the stabilizer of V_g in S_{2g-1} with S_{e_1}. Its order is given by

$$\left|\mathrm{Stab}_{S_{e_1}((e_1,V_g))}\right| = p^{g^2-1} \cdot (p-1) \cdot \prod_{i=1}^{g-2}(p^i-1)\ .$$

By comparing this to the order of S_{e_1}, which can be easily computed to be

$$|S_{e_1}| = \left|\mathrm{Stab}_{S_{2g-1}(e_1)}\right| = p^{g^2-1} \cdot (p-1) \cdot \prod_{i=1}^{g-2}(p^{2i}-1)\ ,$$

we can conclude that the orbit of the standard pair (e_1, V_g) in S_{e_1} has size

$$\left|\mathrm{orb}_{S_{e_1}}((e_1, V_g))\right| = \prod_{i=1}^{g-2}(p^i+1)\ .$$

This means that the primitive vector e_1 is contained in exactly $\prod_{i=1}^{g-2}(p^i + 1)$ subspaces $W_g \in \mathrm{orb}_{S_{2g-1}}(V_g)$ as claimed. A similar calculation for the orbit of the pair (e_g, V_g) yields the corresponding result for e_g. □

As an easy consequence we obtain

Corollary A.5 *Every vector $v \in V_{2g-1}^*$ is contained in at least $\prod_{i=1}^{g-2}(p^i + 1)$ different subspaces $W_g \in \mathrm{orb}_{S_{2g-1}}(V_g)$.*

Appendix B

Combinatorics and number theoretic computations

In this section we will provide a combinatorial result which is frequently used in the counting arguments in the proofs of Chapters 4 and 6. Moreover, we will do some number theoretic computations which are needed in the proof of the main result of Chapter 5.

We often have to deal with certain means over finite sets, e.g. the ramification mean given in Definition 4.2. There we usually have the situation that we know the mean, but not the individual values. Nevertheless, we want to conclude that we are guaranteed to have a sufficiently big number of individual values which exceed a given lower bound. This can be done by a counting argument as given by the following proposition:

Proposition B.1 *Let S be a finite set together with a function $v : S \to [0,1]$. Let μ denote the mean of S with respect to v, i.e.*

$$\mu := \frac{1}{|S|} \sum_{s \in S} v(s) \ .$$

Given $0 \leq \varepsilon < 1$ there are at least

$$\frac{\mu - \varepsilon}{1 - \varepsilon} \cdot |S|$$

different $s \in S$ with $v(s) > \varepsilon$.

Proof. Let γ denote the number of $s \in S$ with the property that $v(s) > \varepsilon$. We can then estimate the mean of S with respect to v as follows:

$$\mu \cdot |S| = \sum_{s \in S} v(s) \leq \gamma \cdot 1 + \Big(|S| - \gamma\Big) \cdot \varepsilon$$

This is equivalent to

$$(\mu - \varepsilon) \cdot |S| \leq \gamma(1 - \varepsilon)$$

which proves the claim. □

To estimate the orders of subgroups of $\mathrm{Sp}(2g, \mathbb{Z}/n\mathbb{Z})$ (or any other matrix group with coefficients in $\mathbb{Z}/n\mathbb{Z}$), one often has to know something about the greatest common divisors of the coefficients of the elements of these subgroups with n. To get lower bounds for the orders of such groups, one is interested in identifying those elements which have a small gcd. For that, the knowledge of the combinatorics involved is required which is provided by the following two propositions:

Proposition B.2 *Let $n = p^t$ be a prime power and $0 \leq s \leq t$. Then there are exactly $(n/p^s)^k$ different $(\alpha_1, \ldots, \alpha_k) \in (\mathbb{Z}/n\mathbb{Z})^k$ with*

$$\gcd(\alpha_1, \ldots, \alpha_k, n) \geq p^s .$$

Proof. This statement is easy to check for $k = 1$. Then it suffices to note that, since $n = p^t$, we have $\gcd(\alpha_1, \ldots, \alpha_k, n) \geq p^s$ if and only if $\gcd(\alpha_i, n) \geq p^s$ for all $i = 1, \ldots, k$. □

While this proposition told us something about the greatest common divisors of all coefficients, the following deals with a certain sum:

Proposition B.3 *Let $n = p^t$ be a prime power and $0 \leq s \leq t$. Then there are for every $(\alpha_1, \ldots, \alpha_k) \in (\mathbb{Z}/n\mathbb{Z})^k$ with $\gcd(\alpha_1, \ldots, \alpha_k, n) = 1$ exactly (n^k/p^s) different $(\beta_1, \ldots, \beta_k) \in (\mathbb{Z}/n\mathbb{Z})^k$ satisfying*

$$\gcd(\alpha_1\beta_1 + \cdots + \alpha_k\beta_k, n) \geq p^s .$$

Proof. Since $\gcd(\alpha_1, \ldots, \alpha_k, n) = 1$, at least one α_i satisfies $\gcd(\alpha_i, n) = 1$. W.l.o.g. we can assume that this is the case for α_1.

For any $(\beta_2, \ldots, \beta_k) \in (\mathbb{Z}/n\mathbb{Z})^{k-1}$ and any given $\gamma \in \mathbb{Z}/n\mathbb{Z}$ there is exactly one $\beta_1 \in \mathbb{Z}/n\mathbb{Z}$ such that

$$\gamma = \alpha_1\beta_1 + \alpha_2\beta_2 + \cdots + \alpha_k\beta_k ,$$

since α_1 is invertible in $\mathbb{Z}/n\mathbb{Z}$. This implies that for each $(\beta_2, \ldots, \beta_k) \in (\mathbb{Z}/n\mathbb{Z})^{k-1}$ there are exactly (n/p^s) different choices for $\beta_1 \in \mathbb{Z}/n\mathbb{Z}$ such that

$$\gcd(\alpha_1\beta_1 + \cdots + \alpha_k\beta_k, n) \geq p^s .$$

Since there are n^{k-1} different $(\beta_2, \ldots, \beta_k) \in (\mathbb{Z}/n\mathbb{Z})^{k-1}$ this proves the claim. □

Appendix C

Geometry of the boundary components

In this section, we will describe the structure of the boundary divisors D_α of $\mathcal{A}_3^{\mathrm{Vor}}(n)$. This description is given in detail in Section 3 of [Hul] and goes back to results of Nakamura ([Nak]) and Tsushima ([Tsu]). We will use this description to show the nefness of a certain line bundle which we will need in Chapter 7.

Recall that each boundary divisor D_α of $\mathcal{A}_3^{\mathrm{Vor}}(n)$ can be considered as the closure of the preimage of a top–dimensional component $\mathcal{A}_2^\alpha(n)$ of the Satake compactification. The fibration $\pi : D_\alpha^\circ \to \mathcal{A}_2^\alpha(n) \cong \mathcal{A}_2(n)$ is the universal family of abelian surfaces with a level–n structure if $n \geq 3$ (cf. [Mum2]). This can be extended to a flat family $\pi : D_\alpha \to \mathcal{A}_2^{\mathrm{Vor}}(n)$ of surfaces. The fibers over the boundary of $\mathcal{A}_2^{\mathrm{Vor}}(n)$ are degenerate abelian surfaces (cf. [Nak] and [Tsu]). To describe the fibers in more detail we have to recall that every boundary component of $\mathcal{A}_2^{\mathrm{Vor}}(n)$ is isomorphic to a Shioda modular surface $S(n) \to \mathcal{A}_1^{\mathrm{Vor}}$. The type of a point $P \in \mathcal{A}_2^{\mathrm{Vor}}(n)$ can then be defined as follows:

P has type I $\iff P \in \mathcal{A}_2(n)$

P has type II $\iff P$ lies on a smooth fiber of a boundary component $S(n)$

P has type IIIa $\iff P$ is a smooth point on a singular fiber of $S(n)$

P has type IIIb $\iff P$ is a singular point of an n–gon in $S(n)$.

The structure of the fiber $A_P = \pi^{-1}(P)$ in each case is described in [Hul, Proposition 3.1], a result which goes back to Nakamura and Tsushima. The fibers are smooth abelian surfaces and cycles of n elliptic ruled surfaces in cases I and II respectively, and consist of n^2 copies of $\mathbb{P}^1 \times \mathbb{P}^1$ or $2n^2$ copies of $\mathbb{P}^2$ and n^2 copies of $\mathbb{P}^2$ blown–up in 3 points in cases IIIa and IIIb respectively.

With the above description we will now show that a certain line bundle is nef. This result will be useful when we calculate the obstructions coming from the boundary divisors in Chapter 7. Instead of just looking at the conormal bundle of D_α, we consider the line bundle

$$M(n) = -nD_\alpha|_{D_\alpha} + L$$

on D_α which has been introduced in [Hul, p. 262], where L is just the restriction of the line bundle of modular forms to D_α. I am indebted to Prof. K. Hulek for providing the following result.

Proposition C.1 *The line bundle $M(n)$ on D_α is nef for $n \geq 3$.*

Proof. We first recall that the restriction of $M(n)$ to each fibre A_P is an even multiple of a (degenerate) theta divisor and thus ample (cf. [Ale2, Theorem 5.3]). Hence if C is a curve such that $\pi(C)$ is a point, then $M(n)\,.\,C > 0$. Note that this applies in particular to curves contained in a fiber of type IIIb.

We still have to consider the case where C is a curve in D_α such that $\pi(C)$ is again a curve. We consider also the projection $\pi^{\mathrm{Sat}} : D_\alpha \to \mathcal{A}_2^{\mathrm{Sat}}(n)$ and distinguish three cases

(1) $\pi^{\mathrm{Sat}}(C) \cap \mathcal{A}_2(n) \neq \emptyset$,

(2) $\pi^{\mathrm{Sat}}(C) \subset \mathcal{A}_1^{\mathrm{Sat}}(n)$, but $\pi^{\mathrm{Sat}}(C) \cap \mathcal{A}_1(n) \neq \emptyset$,

(3) $\pi^{\mathrm{Sat}}(C) \subset \mathcal{A}_0(n)$.

We shall prove the result by showing that for a given irreducible curve C one can find a section $s \in H^0(D_\alpha, M(n)^{\otimes k})$ for some $k > 0$ such that s does not vanish identically on the curve C. Recall that we can use theta functions to construct explicit sections of $M(n)^{\otimes k}$ (cf. [Hul, Proposition 3.2]).

Case (1): Here the result can be deduced immediately from [Hul, Proposition 4.1]. In fact, this proposition gives a much stronger result since it does also give a bound on the vanishing order of s along the boundary. Alternatively, one can write down explicit theta functions which generate $M(n)^{\otimes k}$ restricted to a given smooth fiber A_P.

Case (2): For a general point $Q \in C$ the fiber $A_{\pi(Q)}$ is a cycle of n elliptic ruled surfaces. We consider one such surface and denote it by Y. The restriction of $M(n)$ to Y has degree $2n$ on the base curve and degree 2 on the fibers. Again, one can write down explicit theta functions which define sections of $M(n)$ and whose restrictions to Y form a basis of $H^0(Y, M(n))$. For details of this standard computation, we refer to the analogous computations in [HKW, Part II, Proposition 5.7] and [HW, Lemma 4.1.4].

By [EP, Theorem 5] $M(n)|_Y$ is globally generated and by [EP, Theorem 7] $M(n)^{\otimes 2}|_Y$ is very ample. In particular, we can find a section of $M(n)$ which does not vanish identically on C.

Case (3): We have already mentioned that we can assume that C is not contained in a fiber of type IIIb. Hence C intersects fibers of type IIIa which are unions of n^2 copies of $\mathbb{P}^1 \times \mathbb{P}^1$. We can then repeat the arguments from Case (2). For analogous computations see [HKW, Part II, Section 5].

Alternatively, we can consider one irreducible component $\mathbb{P}^1 \times \mathbb{P}^1$ of a type IIIb fiber which intersects with C. We can use the standard theta function $\theta_{00}(\tau, z)$ to construct a section of $M(n)$ which is nonzero on all torus orbits of $\mathbb{P}^1 \times \mathbb{P}^1$ (cf. [HKW, Part II, Proposition 5.22]). One can then use the torus action to move C away from the zero set of this particular section. Since the torus action is continuous this does not change the intersection number and we can again conclude that $M(n) \,.\, C \geq 0$. $\square$

Bibliography

[Ale1] V. Alexeev: Complete moduli in the presence of semiabelian group action. *Ann. of Math.* **155** (2002), 611 – 708

[Ale2] V. Alexeev: Compactified Jacobians and Torelli Map. *Publ. RIMS, Kyoto Univ.* **40** (2004), 1241 – 1265

[AMRT] A. Ash, D. Mumford, M. Rapoport and Y. Tai: *Smooth Compactification of Locally Symmetric Varieties.* Math. Sci. Press, Brookline, Mass., 1975

[Bai] W. L. Baily: Satake's compactification of V_n. *Am. J. Math.* **80** (1958), 348 – 364

[Bak] H. F. Baker: *A locus with 25,920 linear self–transformations*, volume 39. Cambridge University Press, 1946

[BGAL] Ch. Birkenhake, V. González-Aguilera and H. Lange: Automorphisms of 3-dimensional abelian varieties. *Contemp. Math.* **240** (1999), 25–47

[BLS] H. Bass, M. Lazard and J.-P. Serre: Sous-groupes d'indices finis dans $\mathrm{SL}(n, \mathbf{Z})$. *Bull. AMS* **70** (1964), 385 – 392

[BMS] H. Bass, J. Milnor and J.-P. Serre: Solution of the congruence subgroup problem for SL_n $(n \geq 3)$ and Sp_{2n} $(n \geq 2)$. *Pub. Math. IHES* **33** (1967), 59 – 137

[Bor] L. A. Borisov: A finiteness theorem for subgroups of $\mathrm{Sp}(4, \mathbf{Z})$. *J. Math. Sci. (New York)* **94** (1999), no. 1, 1073–1099

[Car1] H. Cartan: Quotient d'un Espace Analytique par un Groupe d'Automorphismes. In *Algebraic Geometry and Topology, A Symposium in Honor of S. Lefschetz*, 90 – 102, 1957

[Car2] H. Cartan: Quotients of complex analytic spaces. In *Contributions to function theory*, 1 – 15, Bombay, 1960

[CEL] S. D. Cutkosky, L. Ein and R. Lazarsfeld: Positivity and complexity of ideal sheaves. *Math. Ann.* **321** (2001), 213–234

[Cle] H. Clemens: Double solids. *Adv. in Math.* **47** (1983), 107–230

[DO] I. Dolgachev and D. Ortland: *Point sets in projective spaces and theta functions*, volume 165. Soc. Math. France, Paris, 1988

[Don] R. Donagi: The unirationality of $\mathcal{A}_5$. *Ann. of Math.* **119** (1984), 269–307

[DPS] J.-P. Demailly, Th. Peternell and M. Schneider: Compact complex manifolds with numerically effective tangent bundles. *J. Algebraic Geometry* **3** (1994), 295–345

[EP] E. Esteves and M. Popa: Effective very ampleness for generalized theta divisors. *Duke Math J.* **123** (2004), 429–444

[ER1] R. M. Erdahl and S. S. Ryshkov: The empty sphere. *Can. J. Math.* **XXXIX** (1987), 794–824

[ER2] R. M. Erdahl and S. S. Ryshkov: The empty sphere, part II. *Can. J. Math.* **XL** (1988), 1058–1073

[Fre] E. Freitag: *Siegelsche Modulfunktionen*. Grundlehren der mathematischen Wissenschaften **254**. Springer–Verlag, Berlin, 1983

[Fro] F. G. Frobenius: Theorie der linearen Formen mit ganzen Coeffizienten. *Journal reine angew. Math.* **86**,**88** (1879,1880), 146–208 bzw. 96–116

[Ful] W. Fulton: *Intersection theory*, volume 2 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge.* Springer-Verlag, Berlin, second edition, 1998

[vG] B. van Geemen: The moduli space of curves of genus 3 with level 2 structure is rational. Preprint

[vdG1] G. van der Geer: On the geometry of a Siegel modular threefold. *Math. Ann.* **260** (1982), 317–350

[vdG2] G. van der Geer: The Chow ring of the moduli space of abelian threefolds. *J. Algebraic Geom.* **7** (1998), 753–770

[vdG3] G. van der Geer: Cycles on the moduli space of abelian varieties. In *Moduli of curves and abelian varieties*, volume E33 of *Aspects Math.*, 65–89, Braunschweig, 1999. Vieweg

[Gri] P. Griffiths: Hermitian differential geometry, Chern classes, and positive vector bundles. In *Global Analysis (Papers in Honor of K. Kodaira)*, 185–251. Univ. Tokyo Press, Tokyo, 1969

[Hal] M. Hall, Jr.: A topology for free groups and related groups. *Ann. of Math.* **52** (1950), 127–139

[Har1] R. Hartshorne: Ample vector bundles. *Publications mathématiques de l'I.H.É.S.* **29** (1966), 63–94

[Har2] R. Hartshorne: *Algebraic Geometry.* Graduate texts in mathematics **52**. Springer–Verlag, New York, 1977

[Hir1] H. Hironaka: Resolution of singularities of an algebraic variety over a field of characteristic zero. *Ann. of Math.* **79** (1964), 109–326

[Hir2] F. Hirzebruch: Elliptische Differentialoperatoren auf Mannigfaltigkeiten. In *Festschrift zur Gedchtnisfeier fr Karl Weierstrass*, 583–606, 1965

[HKW] K. Hulek, C. Kahn and S. H. Weintraub: *Moduli Spaces of Abelian Surfaces: Compactification, Degenerations, and Theta Functions.* Expositions in Mathematics **12**. de Gruyter, Berlin, 1993

[HS] K. Hulek and G.K. Sankaran: The Geometry of Siegel Modular Varieties. In *Higher dimensional birational geometry*, Adv. Stud. Pure Math. **35**, 89 – 156, Kyoto, 1997. Math. Soc. Japan

[Hul] K. Hulek: Nef divisors on moduli spaces of abelian varieties. In *Complex analysis and algebraic geometry*, 255–274. de Gruyter, Berlin, 2000

[HW] K. Hulek and S. H. Weintraub: The principal polarizations of abelian surfaces. *Math. Ann.* **286** (1990), 281–307

[Igu1] J.-I. Igusa: Arithmetic variety of moduli for genus two. *Ann. of Math.* **72** (1960), 612–649

[Igu2] J.-I. Igusa: A Desingularization Problem in the Theory of Siegel Modular Functions. *Math. Ann.* **168** (1967), 228–260

[Igu3] J.-I. Igusa: *Theta Functions.* Grundlehren der mathematischen Wissenschaften **194**. Springer–Verlag, Berlin, 1972

[Kat] P. Katsylo: Rationality of the moduli variety of curves of genus 3. *Comment. Math. Helv.* **71** (1996), 507–524

[Laz1] R. Lazarsfeld: *Positivity in algebraic geometry. I*, volume 48 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge.* Springer-Verlag, Berlin, 2004

[Laz2] R. Lazarsfeld: *Positivity in algebraic geometry. II*, volume 49 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge.* Springer-Verlag, Berlin, 2004

[LB] H. Lange and Ch. Birkenhake: *Complex Abelian Varieties.* Grundlehren der mathematischen Wissenschaften **302**. Springer–Verlag, Berlin, 1992

[MM] S. Mori and S. Mukai: *The uniruledness of the moduli space of curves of genus* 11. Lecture Notes in Math. **1016**. Springer–Verlag, Berlin, 1983

[MS] D. R. Morrison and G. Stevens: Terminal Quotient Singularities in Dimension Three and Four. *Proc. of the Amer. Math. Soc.* **90** (1984), no. 1, 15–20

[Mum1] D. Mumford: *Abelian Varieties*. Oxford University Press, Bombay, 1970

[Mum2] D. Mumford: *On the Kodaira dimension of the Siegel modular variety.* Lecture Notes in Math. **997**. Springer–Verlag, Berlin, 1983

[Nak] I. Nakamura: On moduli of stable quasi abelian varieties. *Nagoya Math. J.* (1975), no. 58, 149–214

[Nam] Y. Namikawa: *Toroidal Compactification of Siegel Spaces.* Lecture Notes in Math. **812**. Springer–Verlag, Berlin, 1980

[Oda] T. Oda: *Convex Bodies and Algebraic Geometry.* Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, Bd. **15**. Springer–Verlag, Berlin, 1988

[Pri] D. Prill: Local classification of quotients of complex manifolds by discontinuous groups. *Duke Math. Journal* **34** (1967), 375 – 386

[Rei] M. Reid: Young Person's Guide to Canonical Singularities. *Proc. of Symp. in Pure Math.* **46** (1987), 345–414

[Sat] I. Satake: On the compactification of the Siegel space. *J. Indian Math. Soc.* **20** (1956), 259–281

[Sch] D. Schmidt: *Automorphismen 2– und 3–dimensionaler abelscher Varietäten.* Doktorarbeit, Universität Erlangen–Nürnberg, 1997

[Tai] Y. Tai: On the Kodaira Dimension of the Moduli Space of Abelian Varieties. *Invent. math.* **68** (1982), 425–439

[Tho] J. G. Thompson: A finiteness theorem for subgroups of PSL(2, **R**) which are commensurable with PSL(2, **Z**). In *The Santa Cruz Conference on Finite Groups (Univ. California, Santa Cruz, Calif., 1979)*, volume 37 of *Proc. Sympos. Pure Math.*, 533–555. Amer. Math. Soc., Providence, R.I., 1980

[Tod] J. A. Todd: Some types of rational quartic primal in four dimensions. *Proc. Lond. Math. Soc.* **42** (1936), 316–323

[Tsu] R. Tsushima: A formula for the dimension of spaces of Siegel cusp forms of degree three. *Amer. J. Math.* **102** (1980), no. 5, 937–977

[Ver] A. Verra: A short proof of unirationality of $\mathcal{A}_5$. *Indagationes Math.* **46** (1984), 339–355

[Zuo] K. Zuo: On the negativity of kernels of Kodaira-Spencer maps on Hodge bundles and applications. *Asian J. Math.* **4** (2000), no. 1, 279–302

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Publikationen

The index of $\Gamma(d_1, d_2)$ *in* $\mathrm{Sp}(4, \mathbb{Z})$, Anhang zu *Monodromy of Picard–Fuchs differential equations for Calabi–Yau threefolds* by Y.–H. Chen, Y. Yang, N. Yui, erscheint im Journal für reine und angewandte Mathematik

Some intersection numbers of divisors on toroidal compactifications of $\mathcal{A}_g$, arXiv:math.AG/07071274 (mit S. Grushevsky, K. Hulek)

Intersection theory of toroidal compactifications of $\mathcal{A}_4$, Bull. London Math. Soc. 38 (2006), no. 3, 396–400. (mit S. Grushevsky, K. Hulek)

A new family of rational surfaces in $\mathbb{P}^4$, Journal of Symbolic Computation 39 (2005), 51–60. (mit H.–Ch. Graf v. Bothmer, K. Ludwig)

The Kodaira dimension of certain moduli spaces of Abelian surfaces, Mathematische Nachrichten 274–275 (2004), no. 1, 32–39.