Rationality problems
joint with B. Hassett, A. Kresch, and A. Pirutka

Rationality

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- $\sqrt{3}, \ldots, \sqrt{17}$ – Theodorus of Cyrene (400 BC?, teacher of Plato): stopped at 17, because his algebra was weak

Plato's Republic, 380 BC

And surely you would not have the children of your ideal State, whom you are nurturing and educating – if the ideal ever becomes a reality – you would not allow the future rulers to be like $\alpha\lambda o\gamma\iota\ \gamma\rho\alpha\mu\mu\alpha\iota$ ("irrational lines"), having no reason in them, and yet to be set in authority over the highest matters?

Certainly not.

Then you will make a law that they shall have such an education as will enable them to attain the greatest skill in asking and answering questions?

Yes, he said, you and I together will make it.

Plato's Laws, 360 BC

ATHENIAN: But if some things are commensurable and others wholly incommensurable, and you think that all things are commensurable, what is your position in regard to them?
CLEINIAS: Clearly, far from good.
A: Concerning length and breadth when compared with depth, or breadth and length when compared with one another, are not all the Hellenes agreed that these are commensurable with one another in some way?
C: Quite true.
A: But if they are absolutely incommensurable, and yet all of us regard them as commensurable, have we not reason to be ashamed of our compatriots; is not this one of the things of which we were saying that not to know them is disgraceful, and of which to have a bare knowledge only is no great distinction?

Rationality: algebra

A field K/k is

(R) rational: if $K \simeq k(x_0, \ldots, x_n)$ for some n, i.e., if K is a purely-transcendental extension of k

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(U) unirational: if $K \subset k(x_0, \ldots, x_n)$, for some n

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$$(R) \Rightarrow (S) \Rightarrow (U)$$

Conics

All rational solutions (rational points) of

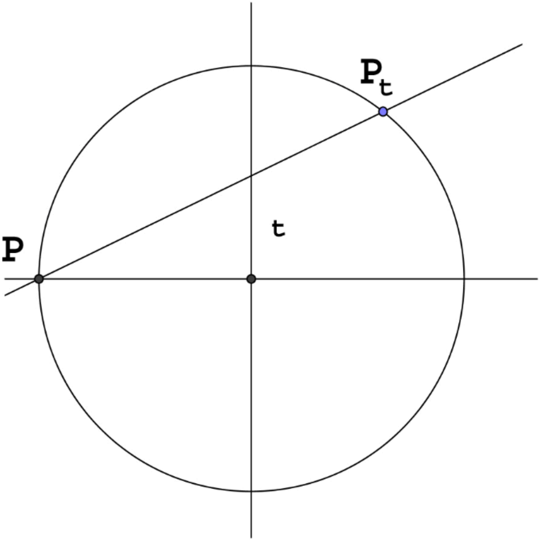
$$x^2 + y^2 = 1$$

are given by

$$x = \frac{1 - t^2}{1 + t^2}, \quad y = \frac{2t}{1 + t^2}.$$

Rational curve: parametrized by rational functions in one variable

Conics



Curves: Lüroth 1876

Beweis eines Satzes über rationale Curven.

Von J. Lüroth in Karlsruhe.

Wenn die Coordinaten eines Punktes einer Curve sich darstellen lassen als rationale Functionen eines Parameters λ, so entspricht stets jedem Werth von λ nur ein Punkt der Curve, dagegen braucht nicht immer jedem Punkt der Curve nur ein Werth von λ zu entsprechen, wie das Beispiel der Gleichungen $x = \lambda^2$, $y = \frac{1}{\lambda^2}$ zeigt.

Theorem

In dimension 1, rationality = stable rationality=unirationality.

Surfaces: Castelnuovo, Enriques

Theorem

In dimension 2, over $\mathbb{C}$, rationality $=$ stable rationality$=$unirationality.

Surfaces: Castelnuovo, Enriques

Theorem

In dimension 2, over $\mathbb{C}$, rationality = stable rationality=unirationality.

This can fail over nonclosed ground-fields k.

Approach via classification: consider del Pezzo surfaces X, i.e., $-K_X$ ample.

Surfaces

How to parametrize $x^3 + y^3 + z^3 + w^3 = 0$?

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\begin{aligned}
x &= -(s+r)t^2 + (s^2+2r^2)t - s^3 + rs^2 - 2r^2s - r^3 \\
y &= t^3 - (s+r)t^2 + (s^2+2r^2)t + rs^2 - 2r^2s + r^3 \\
z &= -t^3 + (s+r)t^2 - (s^2+2r^2)t + 2rs^2 - r^2s + 2r^3 \\
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\end{aligned}$$

What about $x^3 + y^3 + z^3 + 2w^3 = 0$?

Threefolds

Consider Fano threefolds X, i.e., smooth X with $-K_X$ ample.

Lüroth's problem

Does unirationality imply rationality?

There were many unsuccessful attempts to find counterexamples.

Osservazioni sopra alcune varietà non razionali aventi tutti i generi nulli.

di GINO FANO.

In un lavoro pubblicato alcuni anni or sono negli " Atti „ di questa R. Accademia (1) ho dimostrato che la varietà del 4° ordine dello spazio S_4 priva di punti doppi, e la varietà M_3^6 di S_5 intersezione generale di una quadrica e di una varietà cubica di quest'ultimo spazio, pur avendo tutti i generi nulli, non sono razionali. La dimostrazione era fondata sull'impossibilità di soddisfare in pari tempo a certe condizioni, tutte necessarie per l'esistenza di sistemi omaloidici di superficie contenuti rispett. in quelle due varietà.

Unirational varieties in higher dimensions

Interesting classes of unirational varieties arise from quotients under actions of linear algebraic groups:

Let G be a group (e.g.., a finite group) and V a linear representation of G over k. Let

$$K := k(V)^G$$

be the field of invariants. Is K/k rational or stably rational?

Noether's problem

Let G be a finite group.

Yes for:

- $\mathfrak{S}_n$ in the standard representation
- abelian groups
- subgroups of $\mathfrak{S}_3, \mathfrak{S}_4$ (Noether)
- $\dim(V) \leq 3$

Unknown for:

- projective representations of $\mathfrak{S}_5, \mathfrak{S}_6, \mathfrak{A}_6, \mathfrak{A}_7 \hookrightarrow \mathrm{PGL}_4(\mathbb{C})$
- $\mathrm{SL}_2(\mathbb{F}_7) \subset \mathrm{SL}_4(\mathbb{C})$

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Counterexamples over $\mathbb{C}$ by Saltman for $|G| = \ell^9$, by Bogomolov for $|G| = \ell^6$, by Moravec for $|G| = \ell^5$ (2011).

Fields of invariants, over $\mathbb{C}$

- Katsylo: Rationality for quotients of $G = \mathrm{PGL}_2$
- Böhning–von Bothmer: rationality for quotients of $G = \mathrm{PGL}_3$
- Bogomolov-Böhning: stable rationality of V/G for various classical groups G

Counterexamples to Lüroth's problem

Major developments in 1971-72:

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- Iskovskikh-Manin: quartic in $\mathbb{P}^4$ via birational rigidity
- Clemens-Griffiths: cubic in $\mathbb{P}^4$ via intermediate Jacobians
- Artin-Mumford: conic bundles via Brauer groups

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- A smooth hypersurface of degree n in $\mathbb{P}^n$ is birationally rigid (de Fernex, 2013)

Intermediate Jacobians

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Implementation:

- Cubic threefolds (Clemens–Griffiths)
- Intersection of 3 quadrics and conic bundles (Beauville)
- Certain del Pezzo surface fibrations over $\mathbb{P}^1$ (Alexeev)

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Limitation: Does not detect failure of stable rationality

Specialization method

Idea (Clemens 1974): Let

$$\phi : \mathcal{X} \to B$$

be a family of Fano threefolds, with smooth generic fiber. Assume that there exists a point $b \in B$ such that the fiber

$$X := \phi^{-1}(b)$$

satisfies the following conditions:

(S) Singularities: X has at most rational double points

(O) Obstruction: the intermediate Jacobian $\mathrm{IJ}(\tilde{\mathcal{X}}_0)$ (of the resolution of singularities $\tilde{\mathcal{X}}_0$) is not a product of Jacobians of curves.

Then a general fiber $\mathcal{X}_b$ is not rational.

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Implementation (Beauville 1977): nonrationality of quartic and sextic double solids

Brauer group

Theorem (Artin-Mumford)

Let $X \to S$ be a conic bundle over a smooth projective rational surface with discriminant a smooth curve

$$D = \sqcup_{j=1}^{r} D_j \subset S,$$

and with $g(D_j) \geq 1$ for all j. Then

$$\mathrm{Br}(X) = (\mathbb{Z}/2)^{r-1}.$$

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Implementation: A special conic bundle over $\mathbb{P}^2$.

Computing the Brauer group

Let $K = k(X)$ be the function field of a variety X over $k = \bar{k}$, $G_K := \mathrm{Gal}(\bar{K}/K)$ its Galois group, and

$$H^i(K) := H^i(G_K, \mathbb{Z}/n)$$

its i-th Galois cohomology. For every divisorial valuation ν of K we have a natural homomorphism

$$H^i(K) \xrightarrow{\partial_\nu} H^{i-1}(\kappa(\nu))$$

The group

$$H^i_{nr}(K) := \cap_\nu \mathrm{Ker}(\partial_\nu)$$

is a birational invariant; it vanishes for rational K. For smooth projective X we have

$$H^2_{nr}(K) = \mathrm{Br}(X)[n]$$

Computing the Brauer group: $K = k(V)^G$

Put

$$G^a := G/[G,G], \quad G^c := G/[[G,G],G]$$

and consider

$$1 \to Z \to G^c \to G^a \to 1.$$

We have a homomorphism

$$\wedge^2(G^a) \xrightarrow{\rho} Z, \quad (g,g') \mapsto [\tilde{g},\tilde{g}'] \in Z.$$

Then

$$H^2_{nr}(K) \simeq \mathrm{Ker}(\rho)/\langle \textit{commuting pairs} \rangle$$

Cycle-theoretic tools: CH_0

$\mathrm{CH}_0(X_k)$ is the abelian group generated by zero-dimensional subvarieties $x \in X$ (e.g., points $x \in X(k)$), modulo k-rational equivalence.

Assuming $X(k) \neq \emptyset$, there is a surjective degree homomorphism

$$\mathrm{CH}_0(X_k) \to \mathbb{Z}.$$

For which X is this an isomorphism?

Example

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- X a unirational or rationally-connected variety over $k = \mathbb{C}$.
- X a Kummer surface over $k = \bar{\mathbb{F}}_p$ (Bogomolov-T. 2005).

CH_0-triviality

A projective X/k is universally CH_0-trivial if for all k'/k

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For example, smooth k-rational varieties are universally CH_0-trivial. Unirational or rationally-connected varieties are not necessarily universally CH_0-trivial. Smooth projective X/k with $\mathrm{Br}(X) \neq \mathrm{Br}(k)$, or more generally, with nontrivial higher unramified cohomology, are not universally CH_0-trivial.

CH_0-triviality

A projective morphism

$$\beta : \tilde{X} \to X$$

of k-varieties is universally CH_0-trivial if for all k'/k

$$\beta_* : \mathrm{CH}_0(\tilde{X}_{k'}) \xrightarrow{\sim} \mathrm{CH}_0(X_{k'}).$$

CH_0-triviality

Theorem (Colliot-Thélène–Pirutka, 2015)

Let

$$\beta : \tilde{X} \to X$$

be a projective morphism such that for every scheme point x of X, the fiber $\beta^{-1}(x)$, considered as a variety over the residue field $\kappa(x)$, is universally CH_0-trivial. Then β is universally CH_0-trivial.

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For example,

$$\beta : \mathrm{Bl}_Z(X) \to X,$$

the blowup of a smooth variety X in a smooth subvariety Z, is universally CH_0-trivial.

Specialization method
Voisin 2014, Colliot-Thélène–Pirutka 2015

Let

$$\phi : \mathcal{X} \to B$$

be a flat projective morphism of complex varieties with smooth generic fiber. Assume that there exists a point $b \in B$ such that the fiber

$$X := \phi^{-1}(b)$$

satisfies the following conditions:

(S) Singularities: X admits a desingularization

$$\beta : \tilde{X} \to X$$

such that the morphism β is universally CH_0-trivial;

(O) Obstruction: the group $H^2_{nr}(\mathbb{C}(X), \mathbb{Z}/2)$ is nontrivial.

Then a very general fiber of ϕ is not stably rational.

Specialization method: First applications

Very general varieties below are not stably rational:

- Quartic double solids $X \to \mathbb{P}^3$ with ≤ 7 double points (Voisin 2014)
- Quartic threefolds (Colliot-Thélène–Pirutka 2014)
- Sextic double solids $X \to \mathbb{P}^3$ (Beauville 2014)
- Fano hypersurfaces of high degree (Totaro 2015)
- Cyclic covers $X \to \mathbb{P}^n$ of prime degree (Colliot-Thélène–Pirutka 2015)
- Cyclic covers $X \to \mathbb{P}^n$ of arbitrary degree (Okada 2016)

Conic bundles over rational surfaces

Theorem (Hassett-Kresch-T. 2015)

Let S be a smooth projective rational surface over k, an uncountable algebraically closed field of characteristic $\neq 2$. Let $\mathcal{L}$ be a linear system of effective divisors on S whose general member is smooth and irreducible. Let $\mathcal{M}$ be an irreducible component of the space of reduced nodal curves in $\mathcal{L}$ together with degree 2 étale covering. Assume that $\mathcal{M}$ contains a cover, nontrivial over every irreducible component of a reducible curve with smooth irreducible components. Then the conic bundle over S corresponding to a very general point of $\mathcal{M}$ is not stably rational.

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Example: A very general conic bundle $X \to \mathbb{P}^2$, with discriminant a curve of degree ≥ 6, is not stably rational.

Del Pezzo fibrations

Theorem (Hassett-T. 2016)

A very general fibration $\pi : \mathcal{X} \to \mathbb{P}^1$ *in quartic del Pezzo surfaces which is not rational and not birational to a cubic threefold is not stably rational.*

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Remark: These are fibrations $\mathcal{X}$ of height

$$h(\mathcal{X}) := \deg(c_1(\omega_\pi)^3) = -2\deg(\pi_* \omega_\pi^{-1}) \geq 14.$$

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Ideal of proof: Pick a section, blow up, obtain a cubic surface with a line, the corresponding conic bundle over $\mathbb{P}^1 \times \mathbb{P}^1$ has discriminant D of genus $h(\mathcal{X}) - 4$ and admitting a decomposition $D = D_1 \cup D_2$ satisfying the previous theorem.

Del Pezzo fibrations

Our original motivation to study quartic del Pezzo fibrations

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Theorem (Hassett-T. 2013)

If $h(\mathcal{X}) \leq 12$ then π admits a space of sections mapping birationally to the intermediate Jacobian $\mathrm{IJ}(\mathcal{X})$.

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Over finite ground fields the theorem implies that for this class of quartic del Pezzo surfaces over $\mathbb{F}_q(t)$ the Brauer-Manin obstruction to the existence of rational points is the only one (conjectured by Colliot-Thélène).

Fano threefolds

Theorem (Hassett-T. 2016)

A very general nonrational Fano threefold X over $k = \mathbb{C}$ which is not rational and not birational to a cubic threefold is not stably rational.

Fano threefolds: idea and implementation

Find suitable degenerations with mild singularities and birational to conic bundles.

Nonrational Fano threefolds with

$$\mathrm{Pic}(V) = -K_V\mathbb{Z} \quad \text{and} \quad d = d(V) = -K_V^3 :$$

- $d = 2$ sextic double solid
- $d = 4$ quartic
- $d = 6$ intersection of a quadric and a cubic
- $d = 8$ intersection of three quadrics
- $d = 10$ section of $\mathrm{Gr}(2, 5)$ by two linear forms and a quadric
- $d = 14$ birational to a cubic threefold

Fano threefolds: idea and implementation

Nonrational Fano threefolds of index 2:

- $d = 1 \cdot 8$ double cover of $\mathbb{P}(1,1,1,2)$ ramified in a cubic
- $d = 2 \cdot 8$ quartic double solid
- $d = 3 \cdot 8$ cubic threefold

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Nonrational Fano threefolds of higher Picard rank:

- double cover of $\mathbb{P}^1 \times \mathbb{P}^2$ ramified in D of bi-degree $(2,4)$
- divisor in $\mathbb{P}^2 \times \mathbb{P}^2$ of bi-degree $(2,2)$
- double cover of $\mathrm{Bl}_p(\mathbb{P}^3)$
- double cover of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ ramified in D of degree $(2,2,2)$

Fano threefolds: degenerations

From general quartic del Pezzo $\mathcal{X} \to \mathbb{P}^1$ to Fano threefolds V:

- $d = 2$: $h(\mathcal{X}) = 22 \Rightarrow$ sextic double solid V with 32+4 nodes
- $d = 4$: $h(\mathcal{X}) = 20 \Rightarrow$ quartic threefold with 16 nodes
- $d = 6$: $h(\mathcal{X}) = 18 \Rightarrow$ quadric $\cap$ cubic with 8 nodes
- $d = 8$: $h(\mathcal{X}) = 16 \Rightarrow$ intersection of three quadrics with 4 nodes
- $d = 10$: $h(\mathcal{X}) = 14 \Rightarrow$ specialization of a V with 2 nodes

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The other families of Fano threefolds are conic bundles, but not very general, as in the theorem above. Additional work is needed.

Fano threefolds and del Pezzo fibrations: example

Consider the intersection of two $(1,2)$-hypersurfaces in $\mathbb{P}^1 \times \mathbb{P}^4$:

$$sP_1 + tQ_1 = sP_2 + tQ_2 = 0.$$

Let $v_1, \ldots, v_{16} \in \mathbb{P}^4$ denote the solutions to

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- Projection onto the second factor gives a quartic threefold

  $$V := \{P_1 Q_2 - Q_1 P_2 = 0\} \subset \mathbb{P}^4$$

  with 16 nodes $v_1, \ldots, v_{16}$.

Rationality in families: Hassett-Pirutka-T. 2016

There exist smooth families of projective rationally connected fourfolds $\mathcal{X} \to B$ over $k = \mathbb{C}$ such that:

- For every $b \in B$ the fiber X_b is a quadric surface bundle over a rational surface S;
- For very general $b \in B$ the fiber $\mathcal{X}_b$ is not stably rational;
- The set of $b \in B$ such that $\mathcal{X}_b$ is rational is dense in B.

Two difficulties:

- Construction of special X satisfying (**O**) and (**S**)

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Two difficulties:

- Construction of special X satisfying (**O**) and (**S**)
- Rationality constructions

Rationality in families: idea

Consider a quadric surface bundle

$$\pi : \mathcal{Q} \to \mathbb{P}^2,$$

with smooth generic fiber. Let $D \subset \mathbb{P}^2$ be the degeneration curve; assume that D is smooth. Then $\mathcal{Q}$ is characterized by:

- the double cover $T \to \mathbb{P}^2$ with ramification in D
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When $\deg(D) \geq 6$, $\mathrm{Pic}(T)$ and $\mathrm{Br}(T)$ can change as we vary D.

Rationality in families: implementation

We consider bi-degree $(2, 2)$ hypersurfaces

$$X \subset \mathbb{P}^2 \times \mathbb{P}^3.$$

Projection onto the first factor gives a quadric bundle over $\mathbb{P}^2$, its degeneration divisor $D \subset \mathbb{P}^2$ is an octic curve.

Note: Cubic fourfolds containing a plane give rise to quadric surface bundles with degeneration curve of degree 6.

Special fiber

Let

$$X \subset \mathbb{P}^2_{[x:y:z]} \times \mathbb{P}^3_{[s:t:u:v]}$$

be a bi-degree $(2,2)$ hypersurface given by

$$yzs^2 + xzt^2 + xyu^2 + F(x,y,z)v^2 = 0,$$

where

$$F(x,y,z) := x^2 + y^2 + z^2 - 2xy - 2yz - 2xz.$$

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The discriminant curve for the projection $X \to \mathbb{P}^2$ is given by

$$x^2y^2z^2F(x, y, z) = 0.$$

Special fiber

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Special fiber

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- Desingularization: by hand; the singular locus is a union of 6 conics, intersecting transversally

Rationality

It suffices to produce Hodge classes in $H^{2,2}(X,\mathbb{Z})$ intersecting the class of the fiber of $\pi : X \to \mathbb{P}^2$ in odd degree. Then the quadric over the function field $\mathbb{C}(\mathbb{P}^2)$ has a section, and X is rational.

The corresponding Noether-Lefschetz locus is dense in the usual topology of the moduli space.

Special cubic fourfolds

Addington–Hassett–T.–Várilly-Alvarado 2016

The locus of rational cubic fourfolds in $\mathcal{C}_{18}$ – special cubic fourfolds of discriminant 18 – is dense.

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Idea: Every $X \in \mathcal{C}_{18}$ admits a fibration $X \to \mathbb{P}^2$ with general fiber a degree 6 Del Pezzo surface. A multisection of degree coprime to 3 forces rationality. The locus of such cubics is dense in $\mathcal{C}_{18}$.

Summary

- The specialization method of Voisin, further developed by Colliot-Thélène–Pirutka, has triggered new advances in the study of rationality properties of higher-dimensional varieties.

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Summary

- The specialization method of Voisin, further developed by Colliot-Thélène–Pirutka, has triggered new advances in the study of rationality properties of higher-dimensional varieties.
- Understanding unramified cohomology of conic and quadric bundles over higher-dimensional bases is of major importance.
- Rationality and stable rationality of cubic hypersurfaces remain a challenge.

Plato's Laws

ATHENIAN: A man who is good for anything ought to be able, when he thinks, to distinguish the natures of commensurable and incommensurable quantities in their relation to one another; and different persons should compete with one another in asking questions, which will be a far better and more graceful way of passing their time than the old man's game of checkers.

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CLEINIAS: I dare say; and these pastimes are not so very unlike a game of checkers.
ATHENIAN: And these, as I maintain, Cleinias, are the studies which our youth ought to learn, for they are innocent and not difficult; the learning of them will be an amusement, and they will benefit the state. If any one is of another mind, let him say what he has to say.
CLEINIAS: Certainly.