On the birational geometry of some families of K3 surfaces

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1. Introduction

- This talk deals with some special families of complex projective K3 surfaces: in particular with Nikulin surfaces and their moduli.

- A K3 surface of genus g is a pair $(S, \mathcal{C})$ such that S is a K3, $\mathcal{C} \in \operatorname{Pic} S$ is big and nef and $< \mathcal{C}, \mathcal{C} >= 2g - 2$.

- $\mathcal{F}_g :=$ the moduli space of $(S, \mathcal{C})$. $\mathcal{F}_g$ is integral of dimension 19. A general point $[S, \mathcal{C}] \in \mathcal{F}_g$ satisfies the condition $\operatorname{Pic} S \cong \mathbb{Z}\mathcal{C}$.

- $\mathbb{L}$ lattice of rank ρ and signature $(1, \rho - 1)$, we have

$$\mathcal{F}_g^{\mathbb{L}} := \{[S, \mathcal{C}] \in \mathcal{F}_g \mid \mathrm{Pic}\, S \text{ contains a primitive copy of } \mathbb{L}\}$$

  Generically $\mathbb{L} = \mathrm{Pic}\, S$. $\mathcal{F}_g^{\mathbb{L}}$ has codimension $\rho - 1$.

- If $\rho = 2$ the loci $\mathcal{F}_g^{\mathbb{L}}$ are integral divisors, known as Noether - Lefschetz divisors. We are interested to the following one:

$$\mathcal{D}_g := \{[S, \mathcal{C}] \mid \exists\, \mathcal{M} \in \mathrm{Pic}\, S\ ,\ <\mathcal{M}, \mathcal{M}> = -4\ ,\ <\mathcal{M}, \mathcal{C}> = 0\}.$$ [1]

- To fix the notation

$$\mathcal{M} := \mathcal{O}_S(M)\ ,\ \mathcal{H} := \mathcal{C}(-M)\ ,\ \mathcal{A} := \mathcal{C}(-2M).$$

[1] Generically one has $h^0(\mathcal{M}) = 0$.

- Recent motivations to study $\mathcal{D}_g$ and some subloci:
- For general $[S, \mathcal{C}] \in \mathcal{D}_g$ and $g \geq 5$ we have embeddings

$$\mathbf{P}^{g-2} \xleftarrow{f_{\mathcal{H}}} S \xrightarrow{f_{\mathcal{C}}} \mathbf{P}^g$$

- For $C \in |\mathcal{C}|$ the map $f_{\mathcal{C}}/C$ is canonical and $f_{\mathcal{H}}/C$ is a paracanonical map. The latter is defined by $\omega_C \otimes \eta_C$ with $\eta_C := \mathcal{O}_C(-M)$.
- Moving $C \in |\mathcal{C}|$, then η_C can specialize to a torsion element.
- This is interesting in the general and in some extreme case. Let

$$T_n := \{C \in |\mathcal{C}| \mid h^0(\eta_C^{\otimes n}) \geq 1\}.$$

- In the general case: $|T_n| = \binom{2n^2-2}{g}$ and each $C \in T_n$ is smooth. [2]
- The generic Prym-Green conjecture concerns the moduli $\mathcal{R}_{g,n}$ of pairs (C, η) such that $\eta \in \mathrm{Pic}^0 C$ is non trivial of n-torsion.
- It says that the Koszul cohomology of the embedding defined by $\omega_C \otimes \eta$ is natural if (C, η) is general.
- Farkas-Kemeny: the property is true for (C, η_C), where $C \in T_n$ and $[S, \mathcal{C}] \in \mathcal{D}_g$ general, provided g is odd and $\binom{2n^2-2}{g} > 0$.
- Hence the conjecture follows in the same range.

[2] Cfr. Farkas-Kemeny and previous work by Barth- —].

- The extreme case $T_n = |\mathcal{C}|$ is not excluded a priori, for instance

$$h^0(\mathcal{O}_S(nM)) = 1 \Longrightarrow h^0(\eta_C^{\otimes n}) = 1, \ \forall C \in |\mathcal{C}|.$$

- But then M defines a $\mathbb{Z}_n$-cover $\pi' : \tilde{S}' \to S$, branched at the unique $B \in |nM|$. Moreover the minimal model $\tilde{S}$ of $\tilde{S}'$ is a K3 surface.
- It follows that $\mathbb{Z}_n \subset \tilde{S}$ and $\tilde{S}/\mathbb{Z}_n \cong S$. By Nikulin this implies

$$2 \leq n \leq 8.$$

- One can also show (Garbagnati): $\mathcal{D}_g$ contains all the loci

$$\mathcal{D}_{g,n} := \{[S, \mathcal{C}] \in \mathcal{F}_g \mid S \cong \tilde{S}/\mathbb{Z}_n\},$$

where $\tilde{S}$ denotes a K3 surface and $\mathbb{Z}_n \subset \operatorname{Aut} \tilde{S}$.

- The loci $\mathcal{D}_{g,n}$ have been intensively studied by several authors[3]: to characterize lattice theoretically their irreducible components and to describe the projective models of the corresponding families.
- For every g, n an integral component exists such that, generically,

$$\mathrm{Pic}\, S = \mathbb{Z}\,\mathcal{C} \oplus <\mathcal{C}>^{\perp}\ .$$

- The birational geometry of the irreducible components of $\mathcal{D}_{g,n}$ is quite unknown: *Kodaira dimension, rationality questions in low genus, projective models or Mukai models and so on.*
- In this talk I am reporting some done work for $g = 8$ and lower, and some work in progress as well, in the case $n = 2$.

[3] E.g. Nikulin, Mukai [1988], Xiao [1996], van Geemen - Sarti [2006], Garbagnati - Sarti [2008], Garbagnati [2015]

2. Nikulin surfaces in low genus

- For $n = 2$ and $[S, \mathcal{C}] \in \mathcal{D}_{g,2}$ it is well known that

$$2M \sim N = N_1 + \cdots + N_8$$

where $N_1 \dots N_8$ are disjoint copies of $\mathbf{P}^1$. The irreducible components of $\mathcal{D}_{g,2}$ are 11-dimensional and well known.

- $\forall\ g$ a unique one exists so that, for a general point $[S, \mathcal{C}]$, one has

$$\mathrm{Pic}\, S = \mathbb{Z}\, \mathcal{C} \oplus < \mathcal{C} >^{\perp} .$$[4]

- **Definition** We denote such an irreducible component by

$$\mathcal{F}_g^N .$$

We say that $(S, \mathcal{C})$ is a **Nikulin surface** if $[S, \mathcal{C}] \in \mathcal{F}_g^N$.

[4] In general $< \mathcal{C} >^{\perp}$ is generated by $\mathcal{M}, \mathcal{O}_S(N_1) \dots \mathcal{O}_S(N_8)$. As an abstract lattice this is known as the Nikulin lattice.

- In low genus $\mathcal{F}_g^N$ sits in a fascinating system of geometric relations. Its unirationality is known for $g \leq 7$ [5]. We prove here:
- **Theorem 1** $\mathcal{F}_8^N$ *is rational.* [6]
- **Theorem 2** $\mathcal{D}_8$ *is birational to* $\mathbf{P}^{14} \times \mathcal{P}_6$.
- $\mathcal{P}_6$ is the moduli space of six unordered points of $\mathbf{P}^2$.
- A natural question arises: is $\mathcal{F}_g^N$ rational for $g \leq 7$?

[5] Farkas-Verra to appear in Advances of Math.

[6] — in *K3 surfaces and their moduli* Proceedings Schirmonnikoog 2014

- Let $[S, \mathcal{C}] \in \mathcal{D}_{g,2}$ general: $f_{\mathcal{H}}$ is an embedding and $f_{\mathcal{H}}(N_i)$ is a line. Moreover $f_{\mathcal{C}}$ is the contraction of N to an even set of 8 nodes.

$$S := f_{\mathcal{H}}(S) \subset \mathbf{P}^{g-2} \quad , \quad \overline{S} := f_{\mathcal{C}}(S) \subset \mathbf{P}^{g}.$$

- **Criterion** Let $[S, \mathcal{C}] \in \mathcal{D}_g$ and $\mathcal{H}$ very ample. Assume that

$$N_1 \ \ldots \ N_8 \subset S \subset \mathbf{P}^{g-2}$$

  are disjoint lines which are bisecant to A that is $AN_i = 2$.
  Then $A + N_1 + \cdots + N_8 \sim C$ and $[S, \mathcal{C}] \in \mathcal{D}_{g,2}$.
- For $g \equiv 0 \mod 4$ one has $\mathcal{F}_g^N = \mathcal{D}_{g,2}$. In this case the criterion characterizes $\mathcal{F}_g^N$.

- We start with some nice geometry of $\mathcal{F}_g^N$ for $g \leq 7$.[7]
- $g = 6$. Let $Q \subset \mathbf{P}^4$ be a smooth quadric, in the Grassmannian $G(1,4)$ consider the quadratic complex of lines

$$W := \{l \in G(1,4) \ / \ l \text{ is tangent to } Q\}.$$

- W is endowed with the quasi-étale double covering

$$\pi : \tilde{W} \to W$$

  branched on Sing W. This is the 2-Veronese embedding of $\mathbf{P}^3$.
- **Proposition** A general model $\overline{S}$ of genus 6 is a linear section of W. **Theorem 3** It follows that $\mathcal{F}_6^N$ is unirational.

[7] Farkas - —

- $g = 7$. $S \subset \mathbf{P}^5$ is the base locus of a net of quadrics. Fix 7 of the 8 lines, say $N_1 \dots N_8$. Then we have

$$C \sim C_o := R + N_1 + \dots + N_7$$

where R is a rational normal quintic and $RN_1 = \dots = RN_7 = 2$.

- Let $C'_o = R + N'_1 + \dots + N'_7$, where $N'_1 \dots N'_7$ are bisecant lines to R. Then C'_o is in the base locus S' of a unique net of quadrics. Moreover S' is a Nikulin surface endowed with the further line

$$N'_8 \sim 2C'_o - 2H' - N'_1 - \dots - N'_7.$$

- Let $\tilde{\mathcal{F}}^N_7$ be the quotient of the family of curves C'_o by $\operatorname{Aut} R$. Then $\tilde{\mathcal{F}}^N_7$ admits a natural dominant rational map of degree 8

$$f : \tilde{\mathcal{F}}^N_7 \to \mathcal{F}^N_7.$$

Theorem 4 *$\tilde{\mathcal{F}}^N_7$ is rational. In particular $\mathcal{F}^N_7$ is unirational.*

3. Nikulin surfaces of genus 8 and rational normal sextics

- Let $g = 8$ and $[S, \mathcal{C}] \in \mathcal{D}_8$ be general, we have an embedding

$$S \subset \mathbf{P}^6$$

with hyperplane sections $H \sim C - M$ of genus 6. We have

$$(C - 2M)^2 = -2 \text{ and } (C - 2M)H = 6.$$

- **Proposition** *For a general $[S, \mathcal{C}] \in \mathcal{F}_8^N$ the unique curve $A \in |C - 2M|$ is a smooth rational normal sextic spanning $\mathbf{P}^6$.*
- **Proposition** *For a general $[S, \mathcal{C}] \in \mathcal{F}_8^N$ the lines $N_1 \dots N_8$ are disjoint bisecant lines to A contained in S.*

- Mukai-Brill-Noether theory for $[X, \mathcal{O}_X(1)] \in \mathcal{F}_6$ [8]:
- CASE 1
  - If a smooth $H \in |\mathcal{O}_X(1)|$ is not trigonal nor biregular to a plane quintic, then H is generated by quadrics.
  - $\exists!$ H-stable rank 2 vector bundle $\mathcal{E}$ on X such that:
    - (i) *det* $\mathcal{E} \cong \mathcal{O}_X(1)$;
    - (ii) $h^0(\mathcal{E}) = 5$ *and* $h^i(\mathcal{E}) = 0$ *for* $i \geq 1$;
    - (iii) *det* $: \wedge^2 H^0(\mathcal{E}) \to H^0(\mathcal{O}_{\mathbf{P}^6}(1))$ *is surjective.*

[8] For simplicity we assume that $\mathcal{O}_X(1)$ is very ample

- Let us consider $\mathbf{P}^6 := \mathbf{P}H^0(\mathcal{O}_X(1))^*$ and the Plücker embedding

$$G(1,4) \subset \mathbf{P}^9 := \mathbf{P} \wedge^2 H^0(\mathcal{E})^*$$

of the Grassmannian of lines of $\mathbf{P}^4 := \mathbf{P}H^0(\mathcal{E})^*$.

- Then the projectivized dual of *det* defines a linear embedding

$$\delta : \mathbf{P}^6 \to \mathbf{P}^9.$$

- Moreover the next diagram is commutative

$$\begin{array}{ccc} \mathbf{P}^6 & \xrightarrow{\delta} & \mathbf{P}^9 \\ \uparrow & & \uparrow \\ X & \xrightarrow{f_{\mathcal{E}}} & G(1,4), \end{array}$$

where the vertical maps are inclusions and $f_{\mathcal{E}}$ is the embedding defined by $\mathcal{E}$: $x \in X \longrightarrow \mathcal{E}_x^* \subset H^0(\mathcal{E})^* \in G(1,4)$.

- We can assume $X \subset T := \mathbf{P}^6 \cap G(1,4) \subset \mathbf{P}^9$.
- Mukai theory in genus 6 says also that:

  (iv) *X is a quadratic section of T,*

- T is a smooth quintic Del Pezzo threefold if X is general.
- CASE 2:
  - Assume H is either trigonal or biregular to a plane quintic. Then H has Clifford index 1 and it follows that:
  - there exists an integral curve $D \subset X$ such that either $DH = 3$ and $D^2 = 0$ or $DH = 5$ and $D^2 = 2$.

- A general genus 8 Nikulin surface occurs in case (1), not in (2).

- **Proposition** *Let* $S \subset \mathbf{P}^6$ *be a general Nikulin surface of genus* 8 *embedded by* $f_{\mathcal{H}}$. *Then* S *is a quadratic section of a smooth* T. [9]

[9] PROOF Pic S is the orthogonal sum of rank 9 $\mathbb{Z}\mathcal{L} \oplus \mathbb{L}_S$, where $\mathbb{L}_S$ is the Nikulin lattice generated by $\mathcal{O}_S(M), \mathcal{O}_S(N_1) \dots \mathcal{O}_S(N_8)$. A standard computation we omit, shows that no divisor D exists such that $D^2 = 0$ and $DH = 3$ or $D^2 = 2$ and $DH = 5$. This excludes case (2).

- Let A and $S \subset T = \mathbf{P}^6 \cap G(1,4) \subset \mathbf{P}^9$ as above. Under the previous generality assumptions we study the restriction

$$\mathcal{E}_A := \mathcal{E} \otimes \mathcal{O}_A$$

  of the Mukai bundle $\mathcal{E}$ and discuss the possible cases. Of course we have $\mathcal{E}_A = \mathcal{O}_{\mathbf{P}^1}(m) \oplus \mathcal{O}_{\mathbf{P}^1}(n)$ with $m + n = 6$.

- **Theorem 5** *One has* $\mathcal{E}_A \cong \mathcal{O}_{\mathbf{P}^1}(3) \oplus \mathcal{O}_{\mathbf{P}^1}(3)$.

- Now we consider $\mathbb{P}_A := \mathbf{P}\mathcal{E}_A^*$ and the tautological map

$$u_A : \mathbb{P}_A \to \mathbf{P}^7 := \mathbf{P}H^0(\mathcal{E}_A)^*.$$

- This embeds $\mathbb{P}_A$ in $\mathbf{P}^7$ as a rational normal scroll of degree 6.
- The next standard exact sequence will be crucial:

$$0 \to \mathcal{E}(-A) \to \mathcal{E} \to \mathcal{E}_A \to 0.$$

- The associated long exact sequence is the following:

$$0 \to H^0(\mathcal{E}) \to H^0(\mathcal{E}_A) \xrightarrow{\partial_A} H^1(\mathcal{E}(-A)) \to 0.$$

- In particular one has
  - $h^0(\mathcal{E}) = 5$,
  - $h^0(\mathcal{E}_A) = 8$,
  - $h^1(\mathcal{E}(-A)) = 3$. [10]

- The coboundary map $\partial_A : H^0(\mathcal{E}_A) \to H^1(\mathcal{E}(-A))$ defines a plane

$$P_A := \mathbf{P}\, Im\ \partial_A^* \subset \mathbf{P}^7.$$

[10] PROOF Since $\mathcal{E}(-A)$ is H-stable and $H(H-2A) < 0$, it follows $h^0(\mathcal{E}(-A)) = 0$. Furthermore we know that $h^i(\mathcal{E}) = 0$ for $i \geq 1$ and we have $h^1(\mathcal{E}_A) = 0$ because $m, n \geq 0$. This implies the statement.

- Let $\mathbf{P}^4 := \mathbf{P}H^0(\mathcal{E})^*$ and $\mathbb{P}_S := \mathbf{P}\mathcal{E}^*$. Dualizing the sequence and projectivizing we define the linear projection of center P_A:

$$\alpha_A : \mathbf{P}^7 \to \mathbf{P}^4.$$

- Furthermore we have the commutative diagram

$$\begin{array}{ccc} \mathbf{P}^7 & \xrightarrow{\alpha_A} & \mathbf{P}^4 \\ {\scriptstyle u_A}\uparrow & & \uparrow{\scriptstyle u_S} \\ \mathbb{P}_A & \xrightarrow{i} & \mathbb{P}_S \end{array}$$

  where the vertical arrows are the tautological maps. One expects that $\alpha_A(R)$ has exactly six apparent double points.

- As we will see, this is actually true holds for a general $[S, \mathcal{C}] \in \mathcal{D}_8$. Not along the Nikulin locus!

4. Nikulin surfaces of genus 8 and symmetric cubic threefolds

- A symmetric cubic threefold is just a cubic hypersurface

$$V := \{det(a_{ij}) = 0\} \subset \mathbf{P}^4,$$

where $a_{ij} = a_{ji}$ are linear forms.

- We assume $\dim < a_{11} \dots a_{33} >= 5$ so that

$$V = Sec\ B,$$

with B a rational normal quartic curve.

- The family of bisecant lines to B is $\mathbf{P}^2$. It is a congruence of lines

$$W \subset G(1,4) \subset \mathbf{P}^9$$

of class $(3,6)$, embedded in $\mathbf{P}^9$ by the 3-Veronese map.

- Since $\mathcal{E}_A$ is balanced then $\mathbb{P}_A = \mathbf{P}^1 \times \mathbf{P}^1$ and its tautological embedding is defined by $|\mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^1}(1,3)|$. We have

$$R := u_A(\mathbb{P}_A) \subset \mathbf{P}^7 = \mathbf{P} H^0(\mathcal{E}_A)^*.$$

- R is a rational normal sextic scroll: we fix it once at all.
- Restricting to R the top arrow of the diagram

$$\begin{array}{ccc} \mathbf{P}^7 & \xrightarrow{\alpha_A} & \mathbf{P}^4 \\ {\scriptstyle u_A}\uparrow & & {\scriptstyle u_S}\uparrow \\ \mathbb{P}_A & \xrightarrow{i} & \mathbb{P}_S \end{array}$$

- we obtain a linear projection

$$\alpha_A : R \to \mathbf{P}^4.$$

- α_A is a finite morphism of degree 1 onto its image [11]. Let

$$Z \subset R$$

  be the subscheme of points where α_A is not an embedding.

- Then $\ell(Z) = 12$ by double point formula.
- In other words R has six apparent ordinary double points if α_A is a sufficiently general projection in $\mathbf{P}^4$.

[11] Since A is reduced!

- If S is a Nikulin surface this is actually not the case for simple geometric reasons: A has 8 bisecant lines

$$N_1 \dots N_8 \subset S \subset G(1,4).$$

- On the other hand it follows from the construction that

$$\mathbb{P}_A := \{(x,l) \in \mathbf{P}^4 \times A \ / \ x \in l\} \subset \mathbf{P}^4 \times G(1,4),$$

  and that $\alpha_A(R)$ is the projection of it in $\mathbf{P}^4$.

- But the fibres of $\mathbb{P}_A$ at the two points of $A \cap N_i$ are projected in $\mathbf{P}^4$ to lines of the pencil N_i. So they intersect at its center n_i. Hence:

$$\operatorname{Sing} \alpha_A(R) \supseteq \{n_1 \dots n_8\} \ !$$

- **Theorem 6**
  - $\operatorname{Sing} \alpha_A(R)$ *is a rational normal quartic* B*,*
  - $\alpha_A(R)$ *is a degenerate genus 4 K3 surface,*
  - *let* $V = Sec\ B$ *then* $\exists!\ Q \in |\mathcal{I}_{B/\mathbf{P}^4}(2)|$ *so that*

$$\alpha_A(R) = Q \cap V.$$

- So far $A \subset G(1,4)$ is defined by a special embedding: what is the special feature of it?
- A general embedding $A \subset G(1,4)$ has six bisecant lines contained in $G(1,4)$ because of enumerative reasons.
- In our case there is a family B_A of bisecant lines in $G(1,4)$.
- Each $n \in B$ belongs to two lines of $\alpha_A(R)$. This defines a pencil of lines N_n which is a bisecant line to A contained in $G(1,4)$:

$$B_A = \{N_n \ , \ n \in B\}.$$

- A defines a scroll S_A in $G(1,4)$: $S_A = \cup N_n,\ n \in B$.
- S_A is a degenerate K3 surface of genus 6:

$$A \in |\mathcal{I}_{A/T}(2)|\ ,\ \mathrm{Sing}\, S_A = A.$$

- **Theorem 7** *A smooth $S' \in |\mathcal{I}_{A/T}(2)|$ is a genus 8 Nikulin surface.*
- Proof: $S' = Q' \cap T$ for some $Q' \in \mathcal{I}_T(2)|$. On the other hand

$$Q' \cdot S_A = 2A + N'_1 + \cdots + N'_8.$$

Since N'_i is a bisecant line to A then S' is a Nikulin surface.

- The family of special sextics A modulo Aut $G(1,4)$ is dominated by

$$\mathbf{P}^5 := |\mathcal{I}_{B|V}(2)|.$$

- Moreover it is birational to the rational surface

$$\Sigma := \mathbf{P}^5 / \operatorname{Aut} B.$$

- On $\mathbf{P}^5$ we have a $\mathbf{P}^9$-bundle

$$\pi : \mathbb{P} \to \mathbf{P}^5$$

with fibre $|\mathcal{I}_{A/T}(2)|$ at $A \in \mathbf{P}^5$.

- With some more elaboration:
  - The natural map $\mathbb{P}/Aut\ B \to \mathcal{F}_8^N$ is birational.
  - $\mathbb{P}/Aut\ B$ is birational to $\mathbf{P}^9 \times \Sigma$.
- We have sketched the proof of

Theorem 1 *The moduli space of genus 8 Nikulin surfaces is rational.*

- **Remark 1** *The Mukai construction for* $\overline{S}$:

  Let $f : T \to \mathbf{P}^9$ be defined by $|\mathcal{I}_{A/T}(2)|$. The birational image of f is a singular Fano 3-fold of degree 14 $\overline{T} = \mathbf{P}^9 \cap G(1,5)$. f contracts S_A to a rational normal octic curve B_A. *Sing* $\overline{T} = B_A$.

- **Remark 2** *Mukai's analogies between genus 12 and 8*:

  Counting dimensions the moduli of Nikulin surfaces of genus 8 containing a general hyperplane section C are expected to be of dimension zero. Instead their dimension is one. This parallels the case of genus 12 for a general K3 surface. [12]

[12] Lelli Chiesa - Knutsen - — Work in progress

5. Rational normal sextics, 6-nodal cubic 3-folds and $\mathcal{D}_8$

- The construction considered works for $[S, \mathcal{C}] \in \mathcal{D}_8$

$$A \subset S \subset T = \mathbf{P}^6 \cap G(1,4) \subset \mathbf{P}^9,$$

- but this time the commutative diagram

$$\begin{array}{ccc} \mathbf{P}^7 & \xrightarrow{\alpha_A} & \mathbf{P}^4 \\ {\scriptstyle u_A}\uparrow & & \uparrow{\scriptstyle u_S} \\ \mathbf{P}\mathcal{E}_A^* & \xrightarrow{i} & \mathbf{P}\mathcal{E}^* \end{array}$$

defines a *generic* linear projection, of center the plane P_A,

$$\alpha_A : R \to \mathbf{P}^4.$$

- We have:
  - $\operatorname{Sing}\alpha_A(R) = \{o_1 \dots o_6\}$, six general points [13],
  - A has exactly six bisecant lines contained in $G(1,4)$.
- Modulo $PGL(5)$ it is not restrictive to fix the set

$$O := \{o_1 \dots o_6\} \subset \mathbf{P}^4.$$

  The stabilizer of O is the symmetric group $\mathfrak{s}_6 \subset PGL(5)$.
- $\mathfrak{s}_6$ acts on the family

$$\mathcal{R} := \{\text{sextic rational scrolls } \overline{R} \ / \ \operatorname{Sing}\overline{R} = O\}$$

  $\mathcal{R}$ dominates the moduli of general projections $\alpha_A : R \to \mathbf{P}^4$.

[13] In the unique open $PGL(5)$-orbit

- We have:
- **Theorem 8** $\mathcal{D}_8$ is birational to a $\mathbf{P}^9 \times \mathcal{R}/\mathfrak{s}_6$.
- **Theorem 9** $\mathcal{R}/\mathfrak{s}_6$ is birational to $\mathbf{P}^5 \times \mathcal{P}_6$.
- $\mathcal{P}_6$ is the moduli space of six unordered points in $\mathbf{P}^2$. Theorem 2 relies on six nodal cubic 3-folds and on the classical work of Corrado Segre on singular cubic 3-folds.

- Proof of theorem 9:

  Consider the 4-dimensional linear system of 6-nodal cubic 3-folds

$$\mathbb{I} := |\mathcal{I}^2_{\mathcal{O}}(3)|.$$

- Let $\overline{R} \in \mathcal{R}$ be general then

$$\exists\ !\ V \in \mathbb{I} := |\mathcal{I}^2_{\mathcal{O}}(3)|\ /\ V \supset \overline{R}.$$

- The assignement $\overline{R} \longrightarrow V$ defines a dominant map

$$p : \mathcal{R}/\mathfrak{s}_6 \to \mathbb{I}/\mathfrak{s}_6.$$

- What is the fibre at $V \mod \mathfrak{s}_6$ of $p : \mathcal{R}/\mathfrak{s}_6 \to \mathbb{I}/\mathfrak{s}_6$?

- The disjoint union of two 5-dimensional linear systems of sextic scrolls of V. Why? To explain this one has to study the map

$$f : \mathbf{P}^4 \to \mathbb{I}^* = \mathbf{P}^4$$

defined by $\mathbb{I}$. This map is very classical:

- $f(\mathbf{P}^4)$ is the Segre cubic 3-fold Σ and the fibres of f are the rational normal quartics through O.

- Hence $f(V)$ is a cubic surface: a hyperplane section of Σ.

- $f(V)$ has a distinguished double six: this means two nets

$$L \text{ and } \overline{L}$$

of skew cubics defining a Schlaefli double six. [14]

- This means two contractions to sets of six points:

$$f_L : f(V) \to \mathbf{P}^2 \text{ and } f_{\overline{L}} : f(V) \to \mathbf{P}^2$$

conjugated in the very well known way. [15]

- Modulo PGL(3), this means two elements of $\mathcal{P}_6$ conjugated by Schlaefli involution of $\mathcal{P}_6$.

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[14] Σ has exactly 15 planes. They define 15 lines on $f(V)$. The remaining 12 define the double six.

[15] $\overline{L} \sim 5L - 2e_1 - \cdots - 2e_6$.

- $\mathbb{I}/\mathfrak{s}_6$ is the moduli space of Schlaefli double sixers and actually the weighted projective space of dimension four

$$\mathbb{P}[1,2,3,4,5].$$

Therefore it admits a natural double cover

$$\pi : \mathcal{P}_6 \to \mathbb{P}[1,2,3,4,5].$$

- Notice also that:

Theorem 10 *The Fano surface of lines $F(V)$ splits in three irreducible components birationally parametrized by*

$$|L|\ ,\ |\overline{L}|\ ,\ f(V).^{16}$$

[16] Proof: use the six nodes of V

- The ruling of lines of each six nodal sextic scroll $\overline{R} \subset V$ is a conic of skew cubics in $|L|$ or in $|\overline{L}|$.

- Since $\mathcal{P}_6$ is the moduli space of contractions $f_L : f(V) \to \mathbf{P}^2$, the construction defines a $\mathbf{P}^5$-bundle structure

$$r : \mathcal{R}/\mathfrak{s}_6 \longrightarrow \mathcal{P}_6$$

over a non empty open set of $\mathcal{P}_6$.

- The fibre of r at the moduli point of f_L is $|2L|$. Hence:

$$\mathcal{R}/\mathfrak{s}_6 \cong \mathbf{P}^5 \times \mathcal{P}_6.$$