Counting points on Calabi-Yau threefolds - some computational aspects

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1 Introduction

Let $f_1, \ldots, f_k \in \mathbb{Q}[x_0, \ldots, x_N]$ be homogenous polynominals and consider the projective variety

$$X = \{(x_0 : \ldots : x_N) \in \mathbb{P}^N_{\mathbb{Q}};\ f_1(x_0, \ldots, x_N) = \ldots = f_k(x_0, \ldots, x_N) = 0\}.$$

Note that after clearing denominators we can assume that $f_1, \ldots, f_k \in \mathbb{Z}[x_0, \ldots, x_N]$. For each prime p we define

$$N_p := \#X(\mathbb{F}_p),$$

i.e. N_p is the number of points of X considered as an algebraic variety over the field $\mathbb{F}_p$. One can try and calculate N_p by means of a computer. The time needed to do this depends on the equations f_i, but in any case the difficulty grows immensely as p increases.

Let $\bar{X}_p = X(\mathbb{F}_p) \times_{\mathbb{F}_p} \bar{\mathbb{F}}_p$. Then there exists the Frobenius morphism $F_p : \bar{X}_p \to \bar{X}_p$ whose fixed point set is exactly $X(\mathbb{F}_p)$. We shall assume at this point that $\bar{X}_p$ is smooth. If $l \neq p$ is a different prime, then by the Lefschetz fixed point formula

$$N_p = \sum_{i=0}^{2 \dim X} (-1)^i \mathrm{Tr}(F_p^*; H^i_{ét}(\bar{X}_p, \mathbb{Q}_l)). \qquad (1)$$

The theme of this talk is that it is sometimes easier to control the right hand side of this formula than to compute N_p. In exceptional case one can compute the right hand side by computing the coefficients of some modular form or by computing N_p of a different and possibly easier to handle variety. We shall explain this in examples.

We would like to stress here that the theoretical background is well known to specialists. Our only claim to a genuine contribution is our contribution to the joint paper [HSvGvS] with B. van Geemen and D. van Straten in which we treat the case of the Barth-Nieto quintic and its relatives. It seemed to us worthwile, however, to advertise the computational aspects of the Weil conjectures to a wider audience which includes mathematicians who are interested in applications of algebraic geometry.

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2 Counting points on Calabi-Yau varieties

Let X be a smooth projective n-fold which is smooth and defined over the integers. The Frobenius morphism $F_p : \bar{X}_p \to \bar{X}_p$ defines endomorphisms

$$F_{i,p}^* : H^i_{\text{ét}}(X, \mathbb{Q}_l) \to H^i_{\text{ét}}(X, \mathbb{Q}_l).$$

(In order to simplify the notation we simply write $H^i_{\text{ét}}(X) = H^i_{\text{ét}}(X, \mathbb{Q}_l)$ for the l-adic cohomology.) Let

$$P_i(t) = P_{i,p}(t) = \det(1 - tF_{i,p}^*).$$

The *zeta-function* for the prime p is then defined by

$$Z_p(t) = \frac{P_1(t)P_3(t)\dots P_{2n-1}(t)}{P_0(t)P_2(t)\dots P_{2n}(t)}.$$

This is a priori a rational function with coefficients in $\mathbb{Q}_l$, but one can show that it is in fact contained in $\mathbb{Q}(t)$. One has $P_0(t) = 1-t$ and $P_{2n}(t) = 1-p^n t$. An important theorem of Deligne says that the $P_i(t)$ have integer coefficients and moreover

$$P_i(t) = \prod_j (1 - \alpha_{ij} t)$$

where the α_{ij} are algebraic integers with $|\alpha_{ij}| = p^{i/2}$. The most interesting part of the cohomology is the midde cohomology $H^n_{\text{ét}}(X)$. For each prime p we define the Euler factor of this prime by

$$L_p(H^n_{\text{ét}}(X), s) = \frac{1}{P_{n,p}(p^{-s})}$$

and the L-function by

$$L(H^n_{\text{ét}}(X), s) = \prod_p \frac{1}{P_{n,p}(p^{-s})}.$$

We shall now specialize to the case where X is a rigid Calabi-Yau threefold. By a Calabi-Yau threefold we mean a projective threefold X with $K_X = \mathcal{O}_X$ and $q(X) = h^1(\mathcal{O}_X) = 0$. By Serre duality it follows that also $h^2(\mathcal{O}_X) = 0$. The Calabi-Yau 3-fold X is rigid if $H^1(X, T_X) = H^1(X, \Omega^2_X) = 0$, i.e. if and only if $h^{12} = h^{21} = 0$. In this case $H^3_{\text{ét}}(X, \mathbb{Q}_l)$ is 2-dimensional. Note that the absolute Galois group $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ acts on the l-adic cohomology. In particular the action on $H^3_{\text{ét}}(X,\mathbb{Q}_l)$ defines a 2-dimensional representation of $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$.

The determinant of $F^*_{3,p}$ is known to be p^3. This is a consequence of Poincaré duality (see also [Me, Lemma 4.4].) Hence

$$P_{3,p}(t) = 1 - a_p t + p^3 t^2 \text{ where } a_p = \text{Tr}(F^*_{3,p})$$

and the L-function is of the form

$$L(H^3_{\text{ét}}(X), s) = \prod_p \frac{1}{1 - a_p p^{-s} + p^3 p^{-2s}} = \sum_{k=1}^{\infty} a_k k^{-s}.$$

For a prime p the coefficient $a_p = \text{Tr}(F^*_{3,p})$ whereas for general k the coefficient a_k is a product of a_p's where p is a prime divisor of k.

Conjecture (Fontaine/Mazur) : The L-function $L(H^3_{\text{ét}}(X), s)$ is (up to the factors associated to bad primes) the Mellin transform of a modular form f.

For a discussion of this see [FM, Conjecture 3]. The Mellin transform of a modular form $f(q) = \sum_n a_n q^n$ is defined as $\text{Mell}(f) = \sum_n a_n n^{-s}$.

On the other hand the numbers a_p are closely related to the numbers N_p. In our case $H^1_{\text{ét}}(X) = H^5_{\text{ét}}(X) = 0$. Using Poincaré duality for $H^2_{\text{ét}}(X)$ and $H^4_{\text{ét}}(X)$ formula (1) becomes

$$N_p = 1 + (1+p)\text{Tr}F^*_{2,p} + p^3 - a_p. \tag{2}$$

In some cases it is possible to determine $\text{Tr}F^*_{2,p}$ This is, for example, the case if $H^2(X,\mathbb{Z})$ is spanned by divisors which are defined over $\mathbb{Z}$. Then all eigenvalues of $\text{Tr}F^*_{2,p}$ are equal to p, hence $\text{Tr}F^*_{2,p} = b_2(X)p$. But even if this is not the case it is sometimes possible to compute $\text{Tr}F^*_{2,p}$ without too many difficulties (see [HSvGvS, Remark 3.3, resp. the proof of Theorem 1] for such an example).

Now, if one can prove that the L-function $L(H^3_{\text{ét}}(X), s)$ is modular, i.e. is the Mellin transform of a modular form f, then computing N_p is under the above conditions equivalent to computing the coefficients a_p. Indeed, the latter can be much easier. Fortunately there is a technique which one can use to prove the modularity of $L(H^3_{\text{ét}}(X), s)$. The main point is a theorem of Serre based on work of Faltings and recast by Livné. To explain this we work with the prime $l = 2$. We have already remarked that the action of Frobenius defines a 2-dimensional representation $\rho_1 : \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \text{Aut}(H^3_{\text{ét}}(X,\mathbb{Q}_2)) \cong \text{GL}(2,\mathbb{Q}_l)$. On the other hand, if f is a new form of weight k, then by a theorem of Deligne [D] one can associate to f a piece of the l-adic cohomology $H^{k+1}(X^{(k)},\mathbb{Q}_l)$ where $X^{(k)}$ is the k-fold fibre product over C, of a universal elliptic curve X over a modular curve C. Which curve C one has to take depends on the modular group for which f is a modular form. In this way one can associate to f another 2-dimensional representation $\rho_2 : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}(2, \mathbb{Q}_2)$. The crucial theorem of Faltings, Serre and Livné is now the following

Theorem 2.1 *Let ρ_1, ρ_2 be two continuous 2-dimensional 2-adic representations of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ unramified outside a finite set S of prime numbers. Let $\mathbb{Q}_S$ be the compositum of all quadratic extensions of $\mathbb{Q}$ which are unramified outside S. Let T be a set of primes, disjoint from S, such that $\mathrm{Gal}(\mathbb{Q}_S/\mathbb{Q}) = \{F_p|_{\mathbb{Q}_S}; p \in T\}$ where F_p again denotes the Frobenius homomorphism. Suppose that*

(a) $\mathrm{Tr}\,\rho_1(F_p) = \mathrm{Tr}\,\rho_2(F_p)$ *for all* $p \in T$,

(b) $\det \rho_1(F_p) = \det \rho_2(F_p)$ *for all* $p \in T$,

(c) $\mathrm{Tr}\,\rho_1 \equiv \mathrm{Tr}\,\rho_2 \equiv 0 \bmod 2$ *and* $\det \rho_1 \equiv \det \rho_2 \bmod 2$.

Then ρ_1 and ρ_2 have isomorphic semisimplifications, and hence $L(\rho_1, s) = L(\rho_2, s)$. In particular the good Euler factors of ρ_1 and ρ_2 coincide.

Proof. See [L, Theorem 4.3]. □

In particular examples condition (c) is often not difficult to check and in this case this theorem says that by checking finitely many numbers a_p one can conclude the equality of almost all numbers a_p! This theorem has been used in a number of cases in exactly this way (see [SY],[V],[Y]).

In this situation one can replace the computation of the numbers N_p by the computation of the Fourier coefficients of a modular form. Another possible application is when one has two varieties which have the same L-function. Then one can choose the variety which is computationally easier to deduce the numbers N_p for the other variety.

3 The Barth-Nieto quintic and its relatives

We consider the following three varieties.

(1) The *Barth-Nieto quintic* N is given by the equations

$$(N) \quad \begin{array}{ccccc} x_0 & + \ldots + & x_5 & = & 0 \\ \dfrac{1}{x_0} & + \ldots + & \dfrac{1}{x_5} & = & 0. \end{array}$$

This defines a variety $N \subset \mathbb{P}^5$ contained in the hyperplane given by the first equation. Hence N is a quintic threefold and it is singular along 20 lines and has 10 isolated A_1-singularities. Barth and Nieto have shown that N has a smooth Calabi-Yau model Y with Euler number $e(Y) = 100$. Their construction shows that Y is defined over $\mathbb{Z}$ and it can be checked that Y has good reduction for $p \geq 5$.

(2) Let $\tilde{N}$ be the pullback of N under the double cover of $\mathbb{P}^5$ branched along the union $S = \cup S_k$ of the 6 coordinate hyperplanes $S_k = \{x_k = 0\}$. In affine coordinates $(x_0 = 1)$ the variety $\tilde{N}$ is given by the equations

$$(\tilde{N}) \qquad \begin{aligned} &y^2 = x_1 \dots x_5 \\ &1 + x_1 + \dots + x_5 = 0 \\ &x_1 \dots x_5 + x_2 \dots x_5 + x_1 x_3 \dots x_5 + \dots + x_1 \dots x_4 = 0. \end{aligned}$$

The variety $\tilde{N}$ is singular and has a desingularization $\tilde{Y}$ which is defined over the integers and which has the property that there exists a morphism $\tilde{Y} \to Z$ which contracts 20 quadrics to lines. The variety Z is a Calabi-Yau variety with Euler number $e(Z) = 80$. Since the morphism $\tilde{Y} \to Z$ comes from Mori theory we do not know whether Z is defined over the integers. The map $\tilde{Y} \to Z$ induces an isomorphism $H^3(Z) \cong H^3(\tilde{Y})$ and we shall work with the l-adic cohomology group $H^3_{\text{ét}}(\tilde{Y}, \mathbb{Q}_l)$. Again $\tilde{Y}$ has good reduction for $p \geq 5$. It is also worth noting that $\tilde{N}$ (and hence also $\tilde{Y}$ and Z) are birationally equivalent to the moduli of (1,3)-polarized abelian surfaces with a level-2 structure.

(3) The universal elliptic curve with a point of order 6 is given by the pencil of cubics

$$t_0(x_0 + x_1 + x_2)(x_1 x_2 + x_2 x_0 + x_0 x_1) = t_1 x_0 x_1 x_2.$$

Let W be the product of this pencil with itself over the base $\mathbb{P}^1 = \mathbb{P}^1(t_0, t_1)$. Then $W \subset \mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^2$ and using the affine coordinate t it is given by the equations

$$(W) \qquad \begin{aligned} (x_0 + x_1 + x_2)\left(\tfrac{1}{x_0} + \tfrac{1}{x_1} + \tfrac{1}{x_2}\right) &= t \\ (y_0 + y_1 + y_2)\left(\tfrac{1}{y_0} + \tfrac{1}{y_1} + \tfrac{1}{y_2}\right) &= t \end{aligned}$$

W is also singular and it is well known that it has a desingularisation $\widehat{W}$ which is a rigid Calabi-Yau variety.

Theorem 3.1 *The varieties Y and Z are rigid Calabi-Yau threefolds.*

Proof. For the technical details we refer the reader to [HSvGvS]. The basic idea is a method which was first pioneered by B. van Geemen and which also uses a counting argument. One can show for Y that the Néron-Severi group is generated by divisors which are defined over the integers. Hence all eigenvalues of $F^*_{2,p}$ are equal to p. We know that $e(Y) = 100$. Let

$a = h^{1,2}(Y) = h^{2,1}(Y)$. Then $h^{1,1}(Y) = h^{2,2}(Y) = 50 + a$. Moreover one can check easily that $\det F^*_{3,p} = p^3$. Hence by formula (2)

$$a_p = \mathrm{Tr}(F^*_{3,p}) = 1 + (a + 50)(p + p^2) + p^3 - N_p.$$

By the Riemann hypothesis

$$|1 + (a + 50)(p + p^2) + p^3 - N_p| \leq b_3(Y)p^{3/2} = (2a + 2)p^{3/2}.$$

Computing $N_{13}(Y) = 11260$ by a computer then shows $a = 0$.

The situation for Z is more complicated. We first note that it is enough to prove $h^{1,2}(\tilde{Y}) = h^{2,1}(\tilde{Y}) = 0$. In this case it is no longer true that all eigenvalues of $F^*_{2,p}$ are equal to p. One can, however, show [HSvGvS, Proposition 2.21] that this is still true for $p \equiv 1 \bmod 4$. Then the same argument goes through where we can again work with the prime $p = 13$. □

Remark We know that $\mathrm{Tr}(F^*_{2,p}) = 60p$, if $p \equiv 1 \bmod 4$ and conjecture that $\mathrm{Tr}(F^*_{2,p}) = 40p$ if $p \equiv 3 \bmod 4$. We have checked this for all primes $p \leq 59$. In any case we know that all eigenvalues are $\pm p$.

The varieties described above are modular in the sense that their L-functions are (up to possibly the bad primes) the Mellin transform of a modular form. Let

$$\Gamma_0(6) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z});\ c \equiv 0 \bmod 6 \right\}$$

and

$$\Gamma_1(6) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(6);\ a \equiv 1 \bmod 6 \right\}.$$

Note that both groups have the same images in $PSL(2,\mathbb{Z})$. Hence the corresponding modular curves $X_0(6)$ and $X_1(6)$ are isomorphic. They parametrize elliptic curves with a subgroup of order 6, resp. elliptic curves with a point of order 6. The space $S_4(\Gamma_0(6)) = S_4(\Gamma_1(6))$ of cusps forms of weight 4 has dimension 1. The form

$$\begin{aligned} f(q) &= [\eta(q)\eta(q^2)\eta(q^3)\eta(q^6)]^2 \\ &= q \prod_{n=1}^{\infty} (1-q^n)^2(1-q^{2n})^2(1-q^{3n})^2(1-q^{6n})^2 \\ &= q - 2q^2 - 3q^3 + 4q^4 + 6q^5 + 6q^6 - 16q^7 \\ &\quad -8q^8 + 9q^9 - 12q^{10} + 12q^{11} - 12q^{12} + 38q^{13} + \ldots \\ &= \sum b_n q^n \end{aligned}$$

is the normalized generator of this space. Here

$$\eta(q) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)$$

is the Dedekind η-function. The L-function of f is the Mellin transform

$$L(f,s) = \sum_{n=1}^{\infty} b_n n^{-s}.$$

Since $b_n = O(n^{3/2})$ for $n \to \infty$ it converges for $Re(s) > 5/2$. It has an analytic continuation to an entire function. Furthermore, there is a functional equation relating $L(f,s)$ and $L(f,4-s)$. Since f is a Hecke eigenform its L-function is an Euler product

$$L(f,s) = \prod_{p\,\text{prime}} L_p(f,s)$$

with Euler factors

$$L_p(f,s) = \frac{1}{1 - b_p p^{-s} + p^3 \cdot p^{-2s}} \quad \text{for}\, p \geq 5$$

and $L_p(f,s) = (1 + p \cdot p^{-s})^{-1}$ for $p < 5$. Recall also that by Deligne [D] $L(f,s) = L(\rho_f, s)$ where $\rho_f : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Aut}(H^3_{\text{ét}}(\widehat{W}, \mathbb{Q}_2)) \cong \mathrm{GL}(2, \mathbb{Q}_2)$ and where $\widehat{W}$ is a small resolution of the second fibre product of the universal elliptic curve $S_1(6) \to X_1(6)$ over the base.

Theorem 3.2 *The varieties Y, Z and $\widehat{W}$ are modular. More precisely for the L-functions $L(H^3_{\text{ét}}(Y), s) \mathrel{\overset{\circ}{=}} L(H^3_{\text{ét}}(Z), s) \mathrel{\overset{\circ}{=}} L(H^3_{\text{ét}}(\widehat{W}), s) \mathrel{\overset{\circ}{=}} L(f,s)$ where $\overset{\circ}{=}$ means that the Euler factors for $p \geq 5$ coincide.*

Proof. The modularity of $\widehat{W}$ goes, of course, back to Deligne [D]. A proof for Y and Z can be found in [HSvGvS, Theorem 3.2]. See also the historic remarks here. The crucial point is the application of Theorem 2.1. For this one can take the set $T = \{5, 7, 11, 13, 17, 19, 23, 73\}$. Using slightly more theory Meyer [Me] showed that it already suffices to compute a_p and N_p for p in $T' = \{5, 7, 11, 13, 17, 19, 23\}$. □

By a conjecture of Tate this result should imply the existence of correspondences between these varieties inducing an isomorphism of the middle cohomology groups. This is easy for Y and Z since Z is (birationally) a 2:1 cover of Y. In [HSvGvS, Theorem 4.1] we found an explicit birational equivalence, which is defined over $\mathbb{Z}$, between Y and Z. We also found [HSvGvS, Theorem 4.3] a birational equivalence to Verrill's Calabi-Yau variety V (see [V]) which has the same L-series and is given by the equation

$$\text{(V)} \qquad (1 + x + xy + xyz)(1 + z + yz + xyz) = \frac{(t+1)^2}{t} xyz.$$

We now have 5 series of numbers, namely $\#W(p), \#Y(p), \#Z(p), \#V(p)$ and a_p and knowing one of these numbers determines the others (in the case of $Z(p)$ one has to assume $p \equiv 1 \bmod 4$, although one can probably prove with some more effort that this assumption is unnecessary). The computation of the numbers $\#Y(p)$ etc. falls into two parts of which one is theoretical and the other uses a computer. We shall explain this here for Y, the other cases are similar. The variety N given by the equations (N) is singular along 20 lines and 10 isolated nodes, namely the "Segre" points given by the orbit of (1:1:1: -1:-1:-1) under the permutation group S_6. One obtains the non-singular model Y from N as follows. One first blows up $\mathbb{P}^4 = \{x_0 + \cdots + x_5 = 0\}$ in the 15 points $P_{klmn} = \{x_k = x_l = x_m = x_n = 0\}$ and then in the strict transforms of the 20 lines $L_{klm} = \{x_k = x_l = x_m = 0\}$. Finally one replaces the 10 nodes by $\mathbb{P}^1$'s. All of this can be done over the integers. We have already introduced the hyperplanes $S_k = \{x_k = 0\}$. Note that

$$S_k \cap N = \bigcup_{l=0}^{5} F_{kl} \quad \text{where } F_{kl} = \{x_k = x_l = 0\}.$$

Let

$$U = N \setminus \cup F_{kl}.$$

The resolution $Y \to N$ affects U only in the last step where we replace the 10 nodes by $\mathbb{P}^1$'s. Since these $\mathbb{P}^1$'s are defined over the integers we have to add $10p$ to the number of points. To compute the number $\#U(p)$ before this last step in the resolution we use a computer. What happens outside U can be controlled by hand. Blowing up the points P_{klmn} introduces Cayley cubics, i.e. a $\mathbb{P}^2$ blown up in 6 points. The exceptional locus which results from blowing up the strict transforms $L^{(1)}_{klm}$ of the lines L_{klm} is a union of quadrics and the strict transforms of the planes F_{kl} are again Cayley cubics. In each of these case we can count the number of points $\bmod p$ by hand for all primes p.

We used a Maple programme to compute the number of points on the various varieties. All computations were done on a Duron Processor 700 MHz with 64 kB RAM. The variety W is easier to handle than the other varieties, since it is the product (over the base) of a pencil of plane cubics with itself. To compute the number of points on Z we count the number of points in $U(p)$ such that $u = x_1 \ldots x_5$ is a square modulo p. This is the case if and only if $u^q \cong 1 \bmod p$ where $q = (p-1)/2$. We give two times for each of the calculations. The first column gives for each variety the time needed (in seconds) using a naive programme which simply runs through all possibilities. The second programme makes use of the symmetries of the equations. In either case the time needed is of order $O(n^3)$ for W and of order $O(n^4)$ for the other varieties. Our use of the symmetries gives us roughly a factor of 2 for W and V and a factor of 20 for the other varieties, but is still not optimal. Meyer [Me] has developed a more subtle approach for the variety Y. He gains the following factors where the primes are given in brackets $46(p=37), 43(p=47), 39(p=59), 33(p=67)$ and $26(p=97)$.

Another way to speed up the computations is to write a C++ programme instead of using Maple (this was done in [Me]). Running these programmes on our machine we found an improvement of a factor ??.

The computation of the Fourier expansion of f can be done in more than one way, at least in this case. The naive approach is to make use of the fact that the form f has a product expansion and to simply expand it. This is still faster than counting points on any of the varieties with the exception of W which needs roughly the same time. This method has the disadvantage that one soon encounters integers which produce an error message in MAPLE because they are too large. On the other hand there is the package HECKE developed by W. A. Stein [S]. This programme enables one to calculate a basis of a space of modular forms e.g. for the groups $\Gamma_0(N)$ for given level, weight and character. The problem is reduced to computing a basis for the space of newforms. These spaces are spanned by eigenforms with respect to the Hecke operator. Using modular symbols and theoretical work of Manin [Ma] the computation of the coefficients of the Fourier expansion of a basis consisting of eigenform can thus be reduced to a linear algebra problem. Note that in our case the form f is a newform and the space of cusp forms $S_4(\Gamma_0(6))$ has dimension 1.

Comparing counting points and the computation of the Fourier coefficients one should be aware of the following difference. Counting points is done for each prime p separately, whereas the programmes computing the Fourier coefficients produce the numbers a_p simultaneously up to a given prime. Hence we produce two tables. In the first table we give the times needed to compute the numbers $N(p)$ for the various varieties for a given prime p. In the second table we compare the times needed to compute all numbers $N(p)$ and the Fourier coefficients a_p up to a fixed prime.

The final result is that the computation of the Fouriere coefficients a_p using HECKE is much faster than any of the counting methods.

prime	W		Y		Z		V	
5	.000	.000	.010	.000	.010	.000	.009	.000
7	.000	.010	.070	.000	.051	.011	.050	.020
11	.010	.010	.519	.030	.560	.059	.380	.211
13	.021	.010	1.089	.090	1.210	.091	.781	.420
17	.060	.060	3.669	.229	3.900	.271	2.531	1.429
19	.100	.080	5.949	.351	6.351	.399	4.100	2.381
23	.199	.099	13.561	.830	14.429	.851	9.451	5.229
29	.390	.261	36.610	1.980	38.490	2.140	26.010	13.850
31	.510	.269	48.679	2.550	51.281	2.771	34.760	18.400
37	.870	.519	102.520	5.241	108.809	5.659	74.091	38.801
41	1.210	.669	157.819	7.991	168.750	8.479	114.851	60.011
43	1.409	.789	192.470	9.620	207.481	10.391	140.850	73.630
47	1.879	1.060	279.380	13.799	303.420	14.800	204.491	106.930
53	2.750	1.510	461.419	22.591	506.821	24.090	338.819	176.979
59	3.870	2.120	761.260	34.911	828.101	37.500	560.809	282.249
61	4.531	2.361	892.759	40.720	954.661	43.540	641.710	334.149
67	6.180	3.240	1320.119	60.930	1411.191	65.731	950.260	500.380
71	7.359	3.929	1683.510	77.611	1813.141	83.179	1214.620	628.411
73	7.859	4.390	1892.270	86.869	2022.961	93.600	1361.969	716.341
79	10.281	5.529	2622.090	120.491	2811.570	129.059	1889.000	993.580
83	11.989	6.540	3218.420	146.820	3448.640	157.800	2301.000	1214.101
89	14.851	8.109	4245.909	194.840	4578.831	209.641	3096.210	1615.359
97	19.690	10.470	6135.649	276.121	6573.690	297.010	4404.560	2300.080
101	22.381	12.270	7242.140	324.990	7526.910	349.619	5140.050	2721.281
103		12.940		352.359		379.471		2940.879
107		14.561		412.059		443.450		3442.550
109		15.450		443.270		480.140		3699.270
113		17.370		511.790		554.081		4211.829

prime	W		Y		Z		V		Cusp	Hecke
5	.000	.000	.010	.000	.010	.000	.009	.000	.000	
7	.000	.010	.080	.000	.061	.011	.059	.020	.009	
11	.010	.020	.599	.030	.621	.070	.439	.231	.050	
13	.031	.030	1.688	.120	1.831	.161	1.220	.651	.051	
17	.091	.090	5.357	.349	5.731	.432	3.751	2.080	.169	.0
19	.191	.170	11.306	.700	12.082	.831	7.851	4.461	.230	
23	.390	.269	24.867	1.530	26.511	1.682	17.302	9.690	.420	
29	.780	.530	61.477	3.510	65.001	3.822	43.312	23.540	.921	
31	1.290	.799	110.156	6.060	116.282	6.593	78.072	41.940	1.200	
37	2.160	1.318	212.676	11.301	225.091	12.252	152.163	80.741	2.080	
41	3.370	1.987	370.495	19.292	393.841	20.731	267.014	140.752	3.039	
43	4.779	2.776	562.965	28.912	601.322	31.122	407.864	214.382	3.620	.3
47	6.658	3.836	842.345	42.711	904.742	45.922	612.355	321.312	5.470	
53	9.408	5.346	1303.764	65.302	1411.563	70.012	951.174	498.291	8.030	
59	13.278	7.466	2065.024	100.213	2239.664	107.512	1511.983	780.540	11.561	
61	17.809	9.827	2957.783	140.933	3194.325	151.052	2153.693	1114.689	12.620	
67	23.989	13.067	4277.902	201.863	4605.516	216.783	3103.953	1615.069	18.490	
71	31.348	16.996	5961.412	279.474	6418.657	299.962	4318.573	2243.480	24.239	
73	39.207	21.386	7853.682	366.343	8441.618	393.562	5680.542	2959.821	28.301	1.4
79	49.488	26.915	10475.772	486.834	11253.188	522.621	7569.542	3953.401	ERROR	
83	61.477	33.455	13694.192	633.654	14701.828	680.421	9870.542	5167.502		
89	76.328	41.564	17940.101	828.494	19280.659	890.062	12966.752	6782.861		
97	96.018	52.034	24075.750	1104.615	25854.349	1187.072	17371.312	9082.941		
101	118.399	64.304	31317.890	1429.605	33381.259	1536.691	22511.362	11804.222		2.5
103		77.244		1781.964		1916.162		14745.101		2.9
107		91.805		2194.023		2359.612		18187.651		3.0
109		107.255		2637.293		2839.752		21886.921		3.2
113		124.625		3149.083		3393.833		26098.750		3.6

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