Counting points on Calabi-Yau threefolds  
- some computational aspects

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1 Introduction

Let $f_1, \ldots, f_k \in \mathbb{Q}[x_0, \ldots, x_N]$ be homogenous polynomials and consider the projective variety

$$X = \{(x_0 : \ldots : x_N) \in \mathbb{P}^N_{\mathbb{Q}}; \ f_1(x_0, \ldots, x_N) = \ldots = f_k(x_0, \ldots, x_N) = 0\}.$$ 

Note that after clearing denominators we can assume that $f_1, \ldots, f_k \in \mathbb{Z}[x_0, \ldots, x_N]$. For each prime $p$ we define

$$N_p := \#(X(\mathbb{F}_p)),$$

i.e. $N_p$ is the number of points of $X$ considered as an algebraic variety over the field $\mathbb{F}_p$. One can try and calculate $N_p$ by means of a computer. The time needed to do this depends on the equations $f_i$, but in any case the difficulty grows immensely as $p$ increases.

Let $\bar{X}_p = X(\mathbb{F}_p) \times_{\mathbb{F}_p} \bar{\mathbb{F}}_p$. Then there exists the Frobenius morphism $F_p : \bar{X}_p \to \bar{X}_p$ whose fixed point set is exactly $X(\mathbb{F}_p)$. We shall assume at this point that $\bar{X}_p$ is smooth. If $l \neq p$ is a different prime, then by the Lefschetz fixed point formula

$$N_p = \sum_{i=0}^{2\dim X} (-1)^i \text{Tr}(F_p^*; H^i_{cl}(\bar{X}_p, \mathbb{Q})),$$

(1)

The theme of this talk is that it is sometimes easier to control the right hand side of this formula than to compute $N_p$. In exceptional case one can compute the right hand side by computing the coefficients of some modular form or by computing $N_p$ of a different and possibly easier to handle variety. We shall explain this in examples.

We would like to stress here that the theoretical background is well known to specialists. Our only claim to a genuine contribution is our contribution to the joint paper [HSvGvS] with B. van Geemen and D. van Straten in which we treat the case of the Barth-Nieto quintic and its relatives. It seemed to us worthwhile, however, to advertise the computational aspects
of the Weil conjectures to a wider audience which includes mathematicians
who are interested in applications of algebraic geometry.

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2 Counting points on Calabi-Yau varieties

Let $X$ be a smooth projective $n$-fold which is smooth and defined over the
integers. The Frobenius morphism $F_p : \bar{X}_p \rightarrow \bar{X}_p$ defines endomorphisms

$$F_{i,p}^* : H^i_{\text{ét}}(X, \mathbb{Q}) \rightarrow H^i_{\text{ét}}(X, \mathbb{Q}).$$

(In order to simplify the notation we simply write $H^i_{\text{ét}}(X) = H^i_{\text{ét}}(X, \mathbb{Q})$ for
the $l$-adic cohomology.) Let

$$P_i(t) = P_{i,p}(t) = \det(1 - tF_{i,p}^*).$$

The zeta-function for the prime $p$ is then defined by

$$Z_p(t) = \frac{P_1(t)P_3(t) \ldots P_{2n-1}(t)}{P_0(t)P_2(t) \ldots P_{2n}(t)}.$$

This is a priori a rational function with coefficients in $\mathbb{Q}_p$, but one can show
that it is in fact contained in $\mathbb{Q}(t)$. One has $P_0(t) = 1 - t$ and $P_{2n}(t) = 1 - p^n t$.
An important theorem of Deligne says that the $P_i(t)$ have integer coefficients
and moreover

$$P_i(t) = \prod_j (1 - \alpha_{ij} t)$$

where the $\alpha_{ij}$ are algebraic integers with $|\alpha_{ij}| = p^{i/2}$. The most interesting
part of the cohomology is the middle cohomology $H^3_{\text{ét}}(X)$. For each prime $p$
we define the Euler factor of this prime by

$$L_p(H^3_{\text{ét}}(X), s) = \frac{1}{P_{n,p}(p^{-s})}$$

and the $L$-function by

$$L(H^3_{\text{ét}}(X), s) = \prod_p \frac{1}{P_{n,p}(p^{-s})}.$$
particular the action on $H^3_{\text{ét}}(X, \mathbb{Q}_l)$ defines a 2-dimensional representation of $\text{Gal}(ar{\mathbb{Q}}/\mathbb{Q})$.

The determinant of $F_{3,p}^*$ is known to be $p^3$. This is a consequence of Poincaré duality (see also [Me, Lemma 4.4]). Hence

$$P_{3,p}(t) = 1 - a_p t + p^3 t^2$$

and the $L$-function is of the form

$$L(H^3_{\text{ét}}(X), s) = \prod_p \frac{1}{1 - a_p p^{-s} + p^3 p^{-2s}} = \sum_{k=1}^{\infty} a_k k^{-s}.$$ 

For a prime $p$ the coefficient $a_p = \text{Tr}(F_{3,p}^*)$ whereas for general $k$ the coefficient $a_k$ is a product of $a_p$’s where $p$ is a prime divisor of $k$.

**Conjecture (Fontaine/Mazur):** The $L$-function $L(H^3_{\text{ét}}(X), s)$ is (up to the factors associated to bad primes) the Mellin transform of a modular form $f$.

For a discussion of this see [FM, Conjecture 3]. The Mellin transform of a modular form $f(q) = \sum a_n q^n$ is defined as $\text{Mell}(f) = \sum a_n n^{-s}$.

On the other hand the numbers $a_p$ are closely related to the numbers $N_p$. In our case $H^1_{\text{ét}}(X) = H^5_{\text{ét}}(X) = 0$. Using Poincaré duality for $H^2_{\text{ét}}(X)$ and $H^1_{\text{ét}}(X)$ formula (1) becomes

$$N_p = 1 + (1 + p) \text{Tr} F_{2,p}^* + p^3 - a_p. \quad (2)$$

In some cases it is possible to determine $\text{Tr} F_{2,p}^*$. This is, for example, the case if $H^2(X, \mathbb{Z})$ is spanned by divisors which are defined over $\mathbb{Z}$. Then all eigenvalues of $\text{Tr} F_{2,p}^*$ are equal to $p$, hence $\text{Tr} F_{2,p}^* = b_2(X)p$. But even if this is not the case it is sometimes possible to compute $\text{Tr} F_{2,p}^*$ without too many difficulties (see [HSvGvS, Remark 3.3, resp. the proof of Theorem 1] for such an example).

Now, if one can prove that the $L$-function $L(H^3_{\text{ét}}(X), s)$ is modular, i.e. is the Mellin transform of a modular form $f$, then computing $N_p$ is under the above conditions equivalent to computing the coefficients $a_p$. Indeed, the latter can be much easier. Fortunately there is a technique which one can use to prove the modularity of $L(H^3_{\text{ét}}(X), s)$. The main point is a theorem of Serre based on work of Faltings and recast by Livné. To explain this we work with the prime $l = 2$. We have already remarked that the action of Frobenius defines a 2-dimensional representation $\rho_l : \text{Gal}(ar{\mathbb{Q}}/\mathbb{Q}) \to \text{Aut}(H^3_{\text{ét}}(X, \mathbb{Q}_l)) \cong \text{GL}(2, \mathbb{Q}_l)$. On the other hand, if $f$ is a new form of weight $k$, then by a theorem of Deligne [D] one can associate to $f$ a piece of the $l$-adic cohomology $H^{k+1}(X^{(k)}, \mathbb{Q}_l)$ where $X^{(k)}$ is the $k$-fold fibre product over $C$, of a universal
elliptic curve $X$ over a modular curve $C$. Which curve $C$ one has to take depends on the modular group for which $f$ is a modular form. In this way one can associate to $f$ another 2-dimensional representation $\rho_2 : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{GL}(2, \mathbb{Q}_2)$. The crucial theorem of Faltings, Serre and Livné is now the following

**Theorem 2.1** Let $\rho_1, \rho_2$ be two continuous 2-dimensional 2-adic representations of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ unramified outside a finite set $S$ of prime numbers. Let $\mathbb{Q}_S$ be the compositum of all quadratic extensions of $\mathbb{Q}$ which are unramified outside $S$. Let $T$ be a set of primes, disjoint from $S$, such that $\text{Gal}(\mathbb{Q}_S/\mathbb{Q}) = \{ F_p |_{\mathbb{Q}_s} ; p \in T \}$ where $F_p$ again denotes the Frobenius homomorphism. Suppose that

(a) $\text{Tr} \rho_1(F_p) = \text{Tr} \rho_2(F_p)$ for all $p \in T$,

(b) $\det \rho_1(F_p) = \det \rho_2(F_p)$ for all $p \in T$,

(c) $\text{Tr} \rho_1 \equiv \text{Tr} \rho_2 \equiv 0 \mod 2$ and $\det \rho_1 \equiv \det \rho_2 \mod 2$.

Then $\rho_1$ and $\rho_2$ have isomorphic semisimplifications, and hence $L(\rho_1, s) = L(\rho_2, s)$. In particular the good Euler factors of $\rho_1$ and $\rho_2$ coincide.

**Proof.** See [L, Theorem 4.3].

In particular examples condition (c) is often not difficult to check and in this case this theorem says that by checking finitely many numbers $a_p$ one can conclude the equality of almost all numbers $a_p$! This theorem has been used in a number of cases in exactly this way (see [SY],[Y],[Y]).

In this situation one can replace the computation of the numbers $N_p$ by the computation of the Fourier coefficients of a modular form. Another possible application is when one has two varieties which have the same $L$-function. Then one can choose the variety which is computationally easier to deduce the numbers $N_p$ for the other variety.

### 3 The Barth-Nieto quintic and its relatives

We consider the following three varieties.

(1) The **Barth-Nieto quintic** $N$ is given by the equations

\[
\begin{align*}
    x_0 & + \ldots + x_5 = 0 \\
    (N) \quad \frac{1}{x_0} & + \ldots + \frac{1}{x_5} = 0.
\end{align*}
\]

This defines a variety $N \subset \mathbb{P}^5$ contained in the hyperplane given by the first equation. Hence $N$ is a quintic threefold and it is singular along 20 lines
and has 10 isolated $A_1$-singularities. Barth and Nieto have shown that $N$ has a smooth Calabi-Yau model $Y$ with Euler number $e(Y) = 100$. Their construction shows that $Y$ is defined over $\mathbb{Z}$ and it can be checked that $Y$ has good reduction for $p \geq 5$.

(2) Let $\tilde{N}$ be the pullback of $N$ under the double cover of $\mathbb{P}^5$ branched along the union $S = \cup S_k$ of the 6 coordinate hyperplanes $S_k = \{x_k = 0\}$. In affine coordinates $(x_0 = 1)$ the variety $\tilde{N}$ is given by the equations

\[ y^2 = x_1 \ldots x_5 \]
\[ (\tilde{N}) \quad 1 + x_1 + \ldots + x_5 = 0 \]
\[ x_1 \ldots x_5 + x_2 \ldots x_5 + x_1 x_3 \ldots x_5 + \ldots + x_1 \ldots x_4 = 0. \]

The variety $\tilde{N}$ is singular and has a desingularization $\tilde{Y}$ which is defined over the integers and which has the property that there exists a morphism $\tilde{Y} \to Z$ which contracts 20 quadrics to lines. The variety $Z$ is a Calabi-Yau variety with Euler number $e(Z) = 80$. Since the morphism $\tilde{Y} \to Z$ comes from Mori theory we do not know whether $Z$ is defined over the integers. The map $\tilde{Y} \to Z$ induces an isomorphism $H^3(Z) \cong H^3(\tilde{Y})$ and we shall work with the $l$-adic cohomology group $H^3_\ell(\tilde{Y}, \mathbb{Q}_l)$. Again $\tilde{Y}$ has good reduction for $p \geq 5$. It is also worth noting that $\tilde{N}$ (and hence also $\tilde{Y}$ and $Z$) are birationally equivalent to the moduli of $(1,3)$-polarized abelian surfaces with a level-2 structure.

(3) The universal elliptic curve with a point of order 6 is given by the pencil of cubics

\[ t_0(x_0 + x_1 + x_2)(x_1 x_2 + x_2 x_0 + x_0 x_1) = t_1 x_0 x_1 x_2. \]

Let $W$ be the product of this pencil with itself over the base $\mathbb{P}^1 = \mathbb{P}^1(t_0, t_1)$. Then $W \subset \mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^2$ and using the affine coordinate $t$ it is given by the equations

\[ (W) \quad (x_0 + x_1 + x_2) \left( \frac{1}{x_0} + \frac{1}{x_1} + \frac{1}{x_2} \right) = t \]
\[ (y_0 + y_1 + y_2) \left( \frac{1}{y_0} + \frac{1}{y_1} + \frac{1}{y_2} \right) = t \]

$W$ is also singular and it is well known that it has a desingularisation $\tilde{W}$ which is a rigid Calabi-Yau variety.

**Theorem 3.1** The varieties $Y$ and $Z$ are rigid Calabi-Yau threefolds.

**Proof.** For the technical details we refer the reader to [HSvGvS]. The basic idea is a method which was first pioneered by B. van Geemen and which also uses a counting argument. One can show for $Y$ that the Néron-Severi group is generated by divisors which are defined over the integers. Hence all eigenvalues of $F^*_2 \cdot p$ are equal to $p$. We know that $e(Y) = 100$. Let
\[ a = h^{1,2}(Y) = h^{2,1}(Y). \] Then \( h^{1,1}(Y) = h^{2,2}(Y) = 50 + a. \) Moreover one can check easily that \( \det F_{3,p}^* = p^3. \) Hence by formula (2)

\[ a_p = \text{Tr}(F_{3,p}^*) = 1 + (a + 50)(p + p^2) + p^3 - N_p. \]

By the Riemann hypothesis

\[ |1 + (a + 50)(p + p^2) + p^3 - N_p| \leq b_3(Y)p^{3/2} = (2a + 2)p^{3/2}. \]

Computing \( N_{13}(Y) = 11260 \) by a computer then shows \( a = 0. \)

The situation for \( Z \) is more complicated. We first note that it is enough to prove \( h^{1,2}(Y) = h^{2,1}(Y) = 0. \) In this case it is no longer true that all eigenvalues of \( F_{2,p}^* \) are equal to \( p. \) One can, however, show [HSvGvS, Proposition 2.21] that this is still true for \( p \equiv 1 \mod 4. \) Then the same argument goes through where we can again work with the prime \( p = 13. \)

**Remark** We know that \( \text{Tr}(F_{2,p}^*) = 60p, \) if \( p \equiv 1 \mod 4 \) and conjecture that \( \text{Tr}(F_{2,p}^*) = 40p \) if \( p \equiv 3 \mod 4. \) We have checked this for all primes \( p \leq 59. \) In any case we know that all eigenvalues are \( \pm p. \)

The varieties described above are modular in the sense that their \( L \)-functions are (up to possibly the bad primes) the Mellin transform of a modular form. Let

\[ \Gamma_0(6) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z}); \ c \equiv 0 \mod 6 \right\} \]

and

\[ \Gamma_1(6) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(6); \ a \equiv 1 \mod 6 \right\}. \]

Note that both groups have the same images in \( \text{PSL}(2, \mathbb{Z}). \) Hence the corresponding modular curves \( X_{0}(6) \) and \( X_{1}(6) \) are isomorphic. They parametrize elliptic curves with a subgroup of order 6, resp. elliptic curves with a point of order 6. The space \( S_{4}(\Gamma_{0}(6)) = S_{4}(\Gamma_{1}(6)) \) of cusp forms of weight 4 has dimension 1. The form

\[
f(q) = [\eta(q)\eta(q^2)\eta(q^3)\eta(q^6)]^2 \\
= q \prod_{n=1}^{\infty} (1-q^n)^2(1-q^{2n})^2(1-q^{3n})^2(1-q^{6n})^2 \\
= q - 2q^2 - 3q^3 + 4q^4 + 6q^5 + 6q^6 - 16q^7 \\
- 8q^8 + 9q^9 - 12q^{10} + 12q^{11} - 12q^{12} + 38q^{13} + \ldots \\
= \sum b_n q^n
\]

is the normalized generator of this space. Here

\[ \eta(q) = q^{1/24} \prod_{n=1}^{\infty} (1-q^n) \]

6
is the Dedekind $\eta$-function. The $L$-function of $f$ is the Mellin transform

$$L(f, s) = \sum_{n=1}^{\infty} b_n n^{-s}.$$  

Since $b_n = O(n^{3/2})$ for $n \to \infty$ it converges for $Re(s) > 5/2$. It has an analytic continuation to an entire function. Furthermore, there is a functional equation relating $L(f, s)$ and $L(f, 4 - s)$. Since $f$ is a Hecke eigenform its $L$-function is an Euler product

$$L(f, s) = \prod_{p \text{ prime}} L_p(f, s)$$

with Euler factors

$$L_p(f, s) = \frac{1}{1 - b_p p^{-s} + p^3 \cdot p^{-2s}} \text{ for } p \geq 5$$

and $L_p(f, s) = (1 + p \cdot p^{-s})^{-1}$ for $p < 5$. Recall also that by Deligne [D] $L(f, s) = L(p_f, s)$ where $p_f : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{Aut}(H^3_{et}(\mathcal{W}, \mathbb{Q}_p)) \cong \text{GL}(2, \mathbb{Q}_p)$ and where $\mathcal{W}$ is a small resolution of the second fibre product of the universal elliptic curve $S_1(6) \to X_1(6)$ over the base.

**Theorem 3.2** The varieties $Y, Z$ and $\mathcal{W}$ are modular. More precisely for the $L$-functions $L(H^3_{et}(Y), s) \overset{\circ}{=} L(H^3_{et}(Z), s) \overset{\circ}{=} L(H^3_{et}(\mathcal{W}), s) \overset{\circ}{=} L(f, s)$ where $\overset{\circ}{=}$ means that the Euler factors for $p \geq 5$ coincide.

**Proof.** The modularity of $\mathcal{W}$ goes, of course, back to Deligne [D]. A proof for $Y$ and $Z$ can be found in [HSvGvS, Theorem 3.2]. See also the historic remarks here. The crucial point is the application of Theorem 2.1. For this one can take the set $T = \{5, 7, 11, 13, 17, 19, 23, 73\}$. Using slightly more theory Meyer [Me] showed that it already suffices to compute $a_p$ and $N_p$ for $p$ in $T = \{5, 7, 11, 13, 17, 19, 23\}$. By a conjecture of Tate this result should imply the existence of correspondences between these varieties inducing an isomorphism of the middle cohomology groups. This is easy for $Y$ and $Z$ since $Z$ is (rationally) a 2:1 cover of $Y$. In [HSvGvS, Theorem 4.1] we found an explicit birational equivalence, which is defined over $\mathbb{Z}$, between $Y$ and $Z$. We also found [HSvGvS, Theorem 4.3] a birational equivalence to Verrill’s Calabi-Yau variety $V$ (see [V]) which has the same $L$-series and is given by the equation

$$(V) \quad (1 + x + xy + xyz)(1 + z + yz + xyz) = \frac{(t + 1)^2}{t}xyz.$$  

We now have 5 series of numbers, namely $\#W(p), \#Y(p), \#Z(p), \#V(p)$ and $a_p$ and knowing one of these numbers determines the others (in the
case of \( Z(p) \) one has to assume \( p \equiv 1 \mod 4 \), although one can probably prove with some more effort that this assumption is unnecessary). The computation of the numbers \( \#Y(p) \) etc. falls into two parts of which one is theoretical and the other uses a computer. We shall explain this here for \( Y \), the other cases are similar. The variety \( N \) given by the equations \( (N) \) is singular along 20 lines and 10 isolated nodes, namely the “Segre” points given by the orbit of \((1:1:1:-1:-1:-1)\) under the permutation group \( S_6 \). One obtains the non-singular model \( Y \) from \( N \) as follows. One first blows up \( \mathbb{P}^1 = \{x_0 + \cdots + x_5 = 0\} \) in the 15 points \( P_{klmn} = \{x_k = x_l = x_m = x_n = 0\} \) and then in the strict transforms of the 20 lines \( L_{klm} = \{x_k = x_l = x_m = 0\} \). Finally one replaces the 10 nodes by \( \mathbb{P}^1 \)’s. All of this can be done over the integers. We have already introduced the hyperplanes \( S_k = \{x_k = 0\} \). Note that

\[
S_k \cap N = \bigcup_{l=0}^{5} F_{kl} \quad \text{where} \quad F_{kl} = \{x_k = x_l = 0\}.
\]

Let

\[
U = N \setminus \bigcup F_{kl}.
\]

The resolution \( Y \to N \) affects \( U \) only in the last step where we replace the 10 nodes by \( \mathbb{P}^1 \)’s. Since these \( \mathbb{P}^1 \)’s are defined over the integers we have to add 10\( p \) to the number of points. To compute the number \( \#U(p) \) before this last step in the resolution we use a computer. What happens outside \( U \) can be controlled by hand. Blowing up the points \( P_{klmn} \) introduces Cayley cubics, i.e. a \( \mathbb{P}^2 \) blown up in 6 points. The exceptional locus which results from blowing up the strict transforms \( L_{klm}^{(i)} \) of the lines \( L_{klm} \) is a union of quadrics and the strict transforms of the planes \( F_{kl} \) are again Cayley cubics. In each of these case we can count the number of points \( \mod p \) by hand for all primes \( p \).

We used a Maple programme to compute the number of points on the various varieties. All computations were done on a Duron Processor 700 MHz with 64 kB RAM. The variety \( W \) is easier to handle than the other varieties, since it is the product (over the base) of a pencil of plane cubics with itself. To compute the number of points on \( Z \) we count the number of points in \( U(p) \) such that \( u = x_1 \ldots x_5 \) is a square modulo \( p \). This is the case if and only if \( u^q \equiv 1 \mod p \) where \( q = (p - 1)/2 \). We give two times for each of the calculations. The first column gives for each variety the time needed (in seconds) using a naive programme which simply runs through all possibilities. The second programme makes use of the symmetries of the equations. In either case the time needed is of order \( O(n^3) \) for \( W \) and of order \( O(n^4) \) for the other varieties. Our use of the symmetries gives us roughly a factor of 2 for \( W \) and \( V \) and a factor of 20 for the other varieties, but is still not optimal. Meyer [Me] has developed a more subtle approach for the variety \( Y \). He gains the following factors where the primes are given in brackets \( 46(p = 37), 43(p = 47), 39(p = 59), 33(p = 67) \) and \( 26(p = 97) \).
Another way to speed up the computations is to write a C++ programme instead of using Maple (this was done in [Me]). Running these programmes on our machine we found an improvement of a factor ??.

The computation of the Fourier expansion of \( f \) can be done in more than one way, at least in this case. The naive approach is to make use of the fact that the form \( f \) has a product expansion and to simply expand it. This is still faster than counting points on any of the varieties with the exception of \( W \) which needs roughly the same time. This method has the disadvantage that one soon encounters integers which produce an error message in MAPLE because they are too large. On the other hand there is the package HECKE developed by W. A. Stein [S]. This programme enables one to calculate a basis of a space of modular forms e.g. for the groups \( \Gamma_0(N) \) for given level, weight and character. The problem is reduced to computing a basis for the space of newforms. These spaces are spanned by eigenforms with respect to the Hecke operator. Using modular symbols and theoretical work of Manin [Ma] the computation of the coefficients of the Fourier expansion of a basis consisting of eigenform can thus be reduced to a linear algebra problem. Note that in our case the form \( f \) is a newform and the space of cusp forms \( S_1(\Gamma_0(6)) \) has dimension 1.

Comparing counting points and the computation of the Fourier coefficients one should be aware of the following difference. Counting points is done for each prime \( p \) separately, whereas the programmes computing the Fourier coefficients produce the numbers \( a_p \) simultaneously up to a given prime. Hence we produce two tables. In the first table we give the times needed to compute the numbers \( N(p) \) for the various varieties for a given prime \( p \). In the second table we compare the times needed to compute all numbers \( N(p) \) and the Fourier coefficients \( a_p \) up to a fixed prime.

The final result is that the computation of the Fourier coefficients \( a_p \) using HECKE is much faster than any of the counting methods.
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