braid monodromy of curve singularities of Brieskorn Pham type

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Abstract

In [15] we defined braid monodromy invariants for equivalence classes of hypersurfaces singularities. The objective here is to determine the invariants in case of plane curve singularities of Brieskorn Pham type.

In particular we obtain a very natural presentation of the fundamental groups of their discriminant complements.

Introduction

In this paper we address the topic of discriminant complements of hypersurface singularities.

In the case of simple singularities such complements are identified as spaces of regular orbits for the Weyl group of the same type and are shown to be aspherical. Their fundamental groups are given by the Artin-Brieskorn groups of the same type with a natural presentation encoded by the corresponding Dynkin diagram. So there is a strong link to natural combinatorial structures.

Sadly enough only partial aspects can be generalized – especially to parabolic and hyperbolic singularities – but progress to arbitrary singularities has been slow since Brieskorn, in [4], listed some problems, which he intended for guidelines to the case of more general singularities. Among other he asked for the fundamental group and suggests to obtain these groups from a generic plane section using the theorem of Zariski and of van Kampen.

Here we want to present some new results using the braid monodromy invariants for discriminants introduced in [15, ch. 2].

To get a flavour of the basic set up, first recall that a holomorphic function, more precisely a holomorphic function germ is studied by means of versal unfoldings, e.g. given by a function

$$F(x,z,u) = f(x) - z + \sum b_i u_i.$$

In case of a semi universal unfolding the unfolding dimension is given by the Milnor number $\mu = \mu(f)$ and we get a diagram

$$\begin{array}{cccccc} z, u_1, \ldots, u_{\mu-1} & \mathbf{C}^\mu & \supset & \mathcal{D} & = & \{(z,u) | F(0,z,u) = 0 = \nabla F(0,z,u)\} \\ \downarrow & \downarrow & & & & \\ u_1, \ldots, u_{\mu-1} & \mathbf{C}^{\mu-1} & \supset & \mathcal{B} & = & \{u | F(\_,0,u) \text{ is not Morse}\} \end{array}$$

The restriction $p|_{\mathcal{D}}$ of the projection to the discriminant $\mathcal{D}$ is a finite map, such that the branch set coincides with the bifurcation set $\mathcal{B}$.

The key observation for the present work is, that a suitable restriction of p to a subset of $p^{-1}(\mathbf{C}^{\mu-1} \setminus \mathcal{B}) \setminus \mathcal{D}$ is a fibre bundle in a natural way. Its fibres are diffeomorphic to the μ-punctured disc and its isomorphism type depends only on the right equivalence class of f.

Thanks to Moishezon the study of complements of plane curves by the methods of Zariski and van Kampen has been revived [17], and has found a lot of applications. Conceptionally recast as braid monodromy theory it has been successfully used for projective surfaces and symplectic four-manifolds alike by investigating branch curves of finite branched maps to $\mathbf{P}^2$, [18].

The theory of braid monodromy has been generalized to the complements of hyperplane arrangements and it has found an interesting new interpretation in the theory of polynomial coverings by Hansen, [7, 11].

Based on this interpretation the fibre bundle obtained from $p|_{\mathcal{D}}$ naturally gives rise to a braid monodromy homomorphism, which is in fact given by the Lyashko Looijenga map up to an inner automorphism of Br_μ.

As in the case of plane curves the method of van Kampen leads to an explicit presentation of the fundamental group of the discriminant complement $\mathbf{C}^\mu \setminus \mathcal{D}$ in terms of generators and relations.

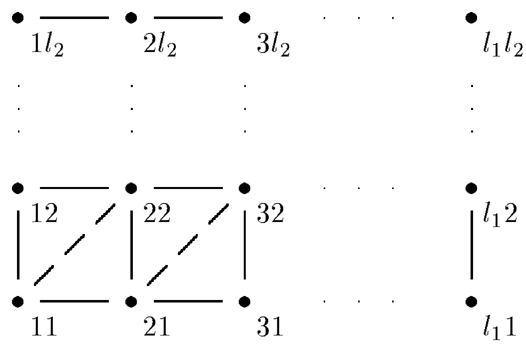
We address the problem to find the invariants and the group presentations for $\pi_1(\mathbf{C}^\mu \setminus \mathcal{D})$ in case of polynomial functions of the kind $f(x) = x_1^{l_1+1} + x_2^{l_2+1}$.

Pham [19] investigated functions of this type in arbitrary dimensions in the spirit of Lefschetz. He computed the homology of the regular fibre and then gave the global monodromy transformation thus generalizing the Picard Lefschetz situation $l_i = 1$.

Brieskorn exploited the same class of functions [5]. In response to one of his problems in [4] Hefez and Lazzeri computed the intersection lattice of f [12]. We owe them the description of a Milnor fibre and the choice of a natural geometrically distinguished path system.

Following common convention we call functions of this class Brieskorn Pham polynomials.

Our main results are most naturally stated referring to the geometrically distinguished Dynkin diagram associated to f by Pham, Gabrielov and Hefez & Lazzeri, [19, 9, 12].



The vertex set is $I = \{i_1 i_2 | 1 \le i_1 \le l_1, 1 \le i_2 \le l_2\}$ ordered lexicographically and the edge set $E = E_1 \cup E_{-1}$ is the union of edges of weight 1 resp. -1:

$$
\begin{aligned}
E_1 &= \{(i,j) |\, i_1 = j_1, |i_2 - j_2| = 1 \vee |i_1 - j_1| = 1, i_2 = j_2\} \\
E_{-1} &= \{(i,j) |\, i_1 = j_1 - 1, i_2 = j_2 - 1 \vee i_1 - 1 = j_1, i_2 - 1 = j_2\}
\end{aligned}
$$

Main Theorem *The braid monodromy group of a plane Brieskorn-Pham polynomial $x_1^{l_1+1} + x_2^{l_2+1}$ is generated by the following twist powers:*

$$
\begin{array}{rcl}
\sigma^2_{i_1 i_2, k_1 k_2} & : & (i,k) \notin E, \\
\sigma^3_{i_1 i_2, k_1 k_2} & : & (i,k) \in E, \\
\sigma^2_{i_1 i_2, j_1 j_2} \sigma^2_{i_1 i_2, k_1 k_2} \sigma^{-2}_{i_1 i_2, j_1 j_2} & : & (i,j), (j,k) \in E_1, (i,k) \in E_{-1}
\end{array}
$$

The most important corollary drawn from this theorem is a presentation of the fundamental group of the discriminant complement which can be computed by the Zariski van Kampen method.

Main Corollary *The fundamental group of the discriminant complement in a versal unfolding of a Brieskorn-Pham polynomial $x_1^{l_1+1} + x_2^{l_2+1}$ is presented by*

$$
\begin{array}{lll}
\langle t_i,\, i \in I \;\; | & t_i t_j = t_j t_i, & (i,k) \notin E, \\
& t_i t_j t_i = t_j t_i t_j, & (i,k) \in E, \\
& t_i t_j t_k t_i = t_j t_k t_i t_j, & (i,j), (j,k) \in E_1, (i,k) \in E_{-1}
\end{array}
$$

These presentations of fundamental groups are natural generalizations of the presentations of Artin Brieskorn groups associated to the simple singularities with a new flavour added by the fact that also triangles, i.e. 2-simplices of the Dynkin diagram, make their contribution to the relations of the presentation. As in the case of simple singularities they are determined by an intersection graph of f. Thus a further result has found an adequate generalization.

On one hand a major motivation for this paper was to make a contribution to the understanding of discriminant complements of unrestricted complexity and to give a solution to a problem posed by Brieskorn [4] three decades ago. But we were also interested for the following reasons:

First our results link Dynkin diagrams to presentations, so there is an implicit conjecture concerning all remaining hypersurface singularities and we hope to find an induction proof similar to Gabrielov's method [10] for the computation of intersection matrices.

The given presentations of fundamental groups π arrise in a natural setting generalising the standard presentations of Artin Brieskorn groups of finite type. There has recently been a surge of activities in combinatroial group theory thanks to the new ideas und techniques centering around the concept of Garside groups, [8]. In this framework the question should be addressed whether there exists a finite dimensional $K(\pi, 1)$. It could well prove to become a major ingredient to settle the question of asphericity of the discriminant complement, cf. Thom [22].

Finally our groups are the source of various monodromy homomorphisms, e.g. algebraic, geometric or the recently proposed symplectic monodromy, [1, 20]. A more detailed study of the kernel and of presentations for the image groups thus seems promissing.

Acknowledgment It is my pleasure to express my thanks to W. Ebeling, who introduced me to the beautiful topic of singularity theory, and to my colleges in Hannover for their interest and many fruitful discussions.

1 prerequisites

A natural approach to find the braid monodromy of a plane curve consists of two steps. First, identify the mapping class group of a generic reference fibre with the braid group and second, compute the parallel transport of the reference fibre along enough embedded simple loops enclosing just a single critical value.

In practise a local model is exploited to determine the conjugacy class associated to a simple loop – incidentally it is the class of $\sigma_1, \sigma_1^2, \sigma_1^3$ respectively in case of a simple tangency, an ordinary node or an ordinary cusp.

The difficult part is to get hold of the parallel transport, which needs very close inspection and a good grasp of the specific geometric situation. This was ingenously mastered by Moishezon [17] using an approximate description of the curve in case of essentially short distances.

In our case Brieskorn [4] suggested to restrict the projection p to a sufficiently generic line in its range, but we failed to find equations such that parallel transport became tractable.

Instead we extract all monodromy data from the unfolding of $f \in \mathbf{C}[x_1, x_2]$ given by

$$F(x, \alpha, z) \quad := \quad f(x) + z - \sum_{i=1}^{2} \alpha_i (l_i + 1) x_i,$$

which we call Hefez-Lazzeri unfolding, since it has been employed sucessfully in [12].

We gain an explicit discription for the discriminant divisor, a formal factorization with ξ_i a primitive l_i-th root of 1.

$$\prod_{1 \le \nu_1 \le l_1} \prod_{1 \le \nu_2 \le l_2} \left(-z + \sum_{i=1}^{2} l_i \xi_i^{\nu_i} \alpha_i^{\frac{l_i+1}{l_i}} \right).$$

On the other hand we loose the genericity of the local degenerations, so we may not compute the braid monodromy of the Hefez-Lazzeri unfolding. As a remedy we have to assign a set of braids to each degeneration reflecting the fact, that in a nearby generic family we would encounter several degenerations corresponding to several braids.

We will actually consider for small $\varepsilon_2 > 0$ the two families

$$f_\alpha(x_1, x_2) = x_1^{l_1+1} - \alpha(l_1 + 1)x_1 + x_2^{l_2+1} - \varepsilon_2(l_2 + 1)x_2$$
$$g_\alpha(x_1, x_2) = x_1^{l_1+1} - (l_1 + 1)x_1 + x_2^{l_2+1} - \alpha\varepsilon_2(l_2 + 1)x_2, \quad |\alpha| \le 1$$

Like in the generic case this suffices, for loops in their bases map to a generating set for the fundamental group of the bifurcation complement in the Hefez-Lazzeri base.

Such claims are put into rigorous framework by the concept of versal braid monodromy developed in [15, ch.4]. It will be cited in the appropriate places – notable in sections 8 and 9.

We will actually start this paper with an extensive computation of parallel transport in a line arrangement associated to f_α by a change of formal parameters. Since we want to transport only classes conjugate to some power of σ_1, we may as well transport embedded arcs between two punctures such that the mapping class can be chosen with support in a regular neighbourhood of the arc.

In section 8 we transfer the results on parallel transport to the discriminant family of f_α. For all its degenerations we give the local generators we have to assign and identify them with twists on arcs for which we know the parallel transport then.

We finish with a last section exploiting both families and their relation to arrive at aproof for bout our main claims.

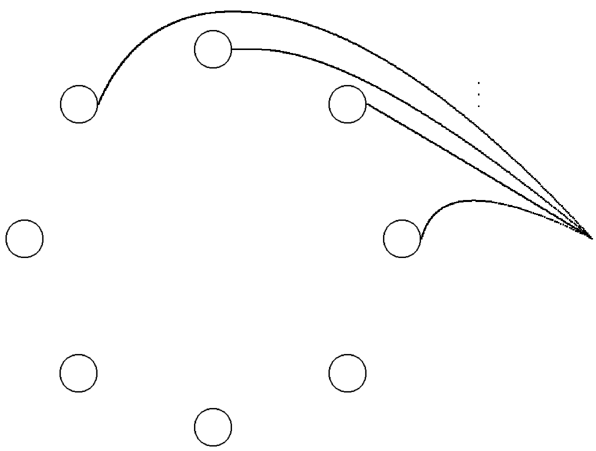
2 parallel transport in the model family

Let us consider the punctured disc bundle associated to the line arrangement

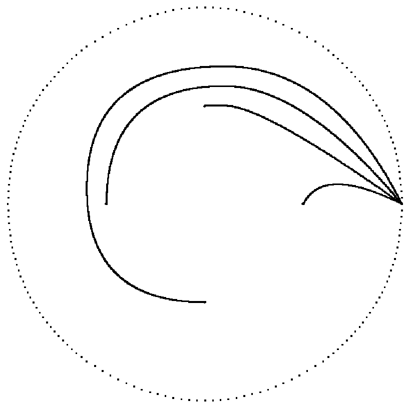
$$\prod_{\xi_1^{l_1},\xi_2^{l_2}=1} (z-\lambda\xi_1-\eta_2\xi_2) \quad = \quad 0.$$

which we call the *model discriminant family* associated to l_1, l_2, $\eta_2 \ll 1$.

The fibre at $\lambda = 1$ is a punctured disc for which Hefez and Lazzeri [12] have given a strongly distinguished system of paths $\omega_{i_1 i_2}$, $1 \leq i_1 \leq l_1, 1 \leq i_2 \leq l_2$, ordered lexicographically. Up to isotopy they can be obtained from two figures like the following in case $l_1 = 8, l_2 = 4$.



In the first figure a path has to be selected according to i_1. It terminates at a disc which should be replaced by the second figure.



The path selected in the second figure according to i_2 can be joint to the first to represent the isotopy class of $\omega_{i_1 i_2}$.

Accordingly indices from the set $\{i_1 i_2 \,|\, 1 \leq i_1 \leq l_1,\, 1 \leq i_2 \leq l_2\}$ are also assigned to the punctures in the fibre at $\lambda = 1$, to the lines of the arrangement and hence to any puncture in any fibre.

Our aim is to describe the parallel transport along radial paths and along circle segments with radius 1 or close to 0. We will find appropriate diffeomorphisms and obtain transported arcs.

Notation 2.1: We introduce polar coordinates $\lambda = te_\vartheta$, $e_\vartheta := e^{i\vartheta}$ of unit absolute value and $t \in \mathbf{R}^{\geq 0}$.

Definition 2.2: A parameter te_ϑ is called *critical*, if there is a pair $i_1 i_2, j_1 j_2$ of indices such that the corresponding lines meet at te_ϑ.
The pair may be specified and te_ϑ called *critical for the pair* $i_1 i_2, j_1 j_2$.

Let us first outline our general approach. For a family we first give a vector field on its total space. Next we check that the punctures form integral curves, so the corresponding flow preserves the punctures. Then we obtain some of the properties of the induced diffeomorphisms, to get finally the parallel transport of some geometric objects.
As most important technical tool we employ smooth *bump functions* $\chi, \chi_\varepsilon : \mathbf{C} \to \mathbf{R}$ for any real $\varepsilon > 0$:

$$\begin{aligned} \chi &: \quad 0 \leq \chi(z) = \chi(|z|) \leq 1, \chi(z) = 0 \text{ if } |z| \geq 1, \chi(z) = 1 \text{ if } |z| \leq \frac{1}{2}, \\ \chi_\varepsilon &: \quad \chi_\varepsilon(z) = \chi(z/\varepsilon), \end{aligned}$$

with support contained in the unit disc, resp. the disc of radius ε.

First we investigate the model discriminant family restricted to a radial path te_{ϑ_0}, $t \in [t_0, 1]$. We will consider the case only when this restriction has constant number of punctures, in which case we call it a *regular family.*
Our aim is to understand the corresponding parallel transport diffeomorphism mapping the initial fibre to the terminal fibre. Considered as an endomorphism of the plane it is seen to be supported on the set of points which are close enough to some puncture at some parameter, i.e. close enough to the union of their *traces*:

Definition 2.3: The *trace* of index $i_1 i_2$ in a family is the set of points z in the plane $\mathbf{C}$ such that z is a puncture of index $i_1 i_2$ for some parameter of the family base.

This we can make explicit with a quick check:

Lemma 2.4 *Let $\varepsilon > 0$ be bounded from above by half the minimal distance between punctures in the fibres of the regular family over te_{ϑ_0}, $t \in [t_0, 1]$. Then the punctures form integral curves for the vector field*

$$v_\varepsilon(z,t) \quad = \sum_{\xi_1^{l_1}, \xi_2^{l_2}=1} \chi_\varepsilon(z - te_{\vartheta_0}\xi_1 - \eta_2\xi_2)e_{\vartheta_0}\xi_1,$$

and the corresponding diffeomorphisms are supported on the ε-neighbourhood of the union of all traces.

Hence parallel transport only affects small neighbourhoods of the punctures. Any arc will be changed only due to the movements of its endpoints and of the critical values which come close enough, to distances less than ε in fact. So we can imagine what happens to a given arc in the fibre at t_0:

Let the arc be a piece of rope. As the parameter t increases additional rope is laid out on the traces of both the critical values which form the ends of the arc. A critical value about to cross the arc will push it ahead and lay out a double rope behind forming a loop around its trace.

Likewise any time a critical value crosses a trace along which a multiple rope has previously laid down, it picks this rope up and pushes a multiple loop into it along its own trace.

So in the end the rope is lain down in an arbitrary small neighbourhood of the union of all traces, in fact the union can be restricted to that part of each trace traced after the corresponding critical value picked up rope for the first time.

We want to apply parallel transport to a very restricted set of arcs:

Definition 2.5: Given a critical parameter $t_0 e_{\vartheta_0}$ for the index pair $i_1 i_2, j_1 j_2$, an arc between the corresponding critical points in the fibre at $t_1 e_{\vartheta_1}$ is called *local v-arc* if

i) it is supported on the corresponding traces,

ii) the difference $t_1 - t_0$ is positive and small compared to the distances of critical parameters.

In case of $j_1 - i_1 = l_1/2$, $i_2 \neq j_2$, the traces of the corresponding critical points in a fibre $t_1 e_{\vartheta_1}$ meet only if $\vartheta_1 = \vartheta_0$. In this case we allow $\vartheta_1 \neq \vartheta_0$ nevertheless and concede that the local v-arc are supported on the traces except for a small part to join them.

Definition 2.6: Parallel transport of a local v-arc in a radial family by the differentiable flow to radius $t = 1$ yields an arc called *tangled v-arc*.

Definition 2.7: An arc in the fibre at $t_1 e_{\vartheta_1}$ is called *local w-arc* if

i) it connects punctures of indices $i_1 i_2, j_1 j_2$, $i_1^+ < j_1, i_2 = j_2$, by four line segments, $i_1^+ = i_1 + 1$,

ii) two segments are supported on the traces of the two punctures,

iii) the central pair forms a sharp wedge over the trace of the puncture of index $i_1^+ i_2$,

iv) its length and t_1 are small compared to the distance of critical parameters.

Definition 2.8: Parallel transport of a local w-arc in a radial family by the differentiable flow to radius $t = 1$ yields an arc called *tangled w-arc.*

To describe the local situation at a crossing of two or more critical points, we consider *tangled tails* of punctures. These one should imagine just as a piece of rope laid out by a critical point on its trace and tangled by subsequent critical points. Looking locally at the tail implies that it may decompose into several pieces.

Example 2.9: Imagine a crossing of just two traces, then the tangled tails look locally like

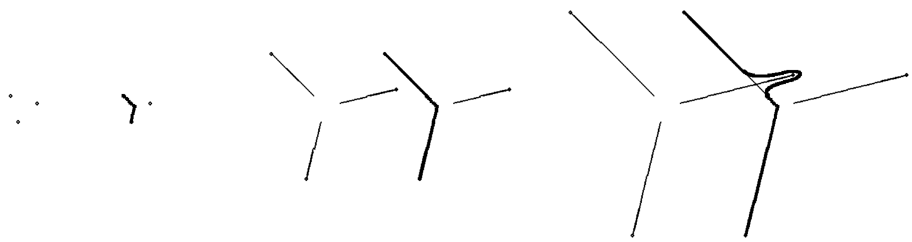
(The critical points pass from bottom to top, the first from left to right, the second from right to left.)

By construction a local v-arc is approximately supported on tails hence so is the transported arc throughout the radial family. In fact more is true. At each crossing of critical points, to which the transported arc comes close, it is approximately supported on the tails of the crossing punctures:

Lemma 2.10 *Locally at a crossing P all local components of a tangled v-arc can be assumed to be arbitrarily close approximations to one of the tangled tails of the punctures passing through P.*

Proof: All local components are laid out by a critical point which pushes them through P. Hence the smaller ε is, the better the approximation will be. $\square$

Example 2.11: For the family with $l_1 = 3, l_2 = 2$ and $\vartheta = \frac{\pi}{12}$ we have the sketches of a local v-arc at $.12 e_\vartheta$ and its parallel transports at $.56 e_\vartheta$ and e_ϑ together with the traces of all critical values.



In this example only one additional critical value is entangled.

Finally we observe, that for all but finitely many angles ϑ_1 and for all $t_1 > 0$ the family over the line segment from $t_1 e_{\vartheta_1}$ to e_{ϑ_1}

i) is a regular family, i.e. the segment does not pass a critical parameter,

ii) has no pair of distinct traces having more than one point in common.

Just note that distinct traces have at most one point in common, if and only if no trace contains a point $\eta_2\xi_2$, $\xi_2^{l_2} = 1$.

Similar to the case of radial families we can get hold of a diffeomorphism which represents the parallel transport over circular segments in the base. On particular subsets the map is in fact quite easily described.

Lemma 2.12 *Given the vector field*

$$v(z,\vartheta) \quad = \quad i\Big(z + \sum_{\xi_1^{l_1}=1} \chi_{2\eta_2}(z - e_\vartheta\xi_1)(e_\vartheta\xi_1 - z)\Big)$$

then

i) the punctures of the model family over the circle of radius $t = 1$ form integral curves,

ii) supposing $|z_0 - e_\vartheta\xi_1| \leq 2\eta_2$ the flow of v preserves the distance of $z_0(\vartheta)$ and $e_\vartheta\xi_1$.

iii) supposing $|z_0 - \xi_1| \geq 2\eta_2$ for all $\xi_1, \xi_1^{l_1} = 1$, $z_0(\vartheta) = z_0 e_\vartheta$ is an integral curve.

Proof: i) Each puncture forms a curve $e_\vartheta\xi_1 + \eta_2\xi_2$, $\xi_1^{l_1}, \xi_2^{l_2} = 1$, for which we can check the integrality condition:

$$\frac{d}{d\vartheta}(e_\vartheta\xi_1 + \eta_2\xi_2) \quad = \quad ie_\vartheta\xi_1 \quad = \quad i(e_\vartheta\xi_1 + \eta_2\xi_2 - \eta_2\xi_2) \quad = \quad v(e_\vartheta\xi_1 + \eta_2\xi_2, \vartheta).$$

ii) We have to show that the following complex numbers considered as real vectors are perpendicular for all ϑ:

$$(e_\vartheta\xi_1 - z_0(\vartheta)) \cdot \frac{d}{d\vartheta}(e_\vartheta\xi_1 - z_0(\vartheta)).$$

Both points move along integral curves, hence $\frac{d}{d\vartheta}e_\vartheta\xi_1 = ie_\vartheta\xi_1$ and

$$\begin{aligned}\frac{d}{d\vartheta}z_0(\vartheta) &= v(z_0(\vartheta),\vartheta)\\ &= iz_0(\vartheta)+i\chi_{2\eta_2}(z_0(\vartheta)-e_\vartheta\xi_1)(e_\vartheta\xi_1-z_0(\vartheta)).\end{aligned}$$

Since χ is a real valued function, the second function is a purely imaginary multiple of the first, hence they are orthogonal at all ϑ.
iii) Again we have only to check an integrality condition

$$\frac{d}{d\vartheta}z_0e_\vartheta \quad = \quad iz_0e_\vartheta \quad = \quad v(z_0e_\vartheta,\vartheta).$$

□

Let us rephrase the result of the lemma in more geometrical terms:

i) the flow realises parallel transport in the model family over circle segments of radius $t=1$,

ii) the $2\eta_2$-discs at points ξ_1, $\xi_1^{l_2}=1$ are mapped bijectively to $2\eta_2$-discs of the transported points preserving the distance,

iii) points outside these discs are mapped by a rigid rotation around the origin.

Lemma 2.13 *Given the vector field for $\varepsilon << \eta_2$*

$$v(z) \quad = \quad i\sum_{\xi_2^{l_2}=1}\chi_{4\varepsilon}(z-\eta_2\xi_2)(z-\eta_2\xi_2)$$

then

i) the punctures of the model family over the circle of radius $t=\varepsilon$ form integral curves,

ii) suppose $|z_0-\eta_2\xi_2|\leq 2\varepsilon$, $\xi_2^{l_2}=1$ then the curves $z_0(\vartheta)=(z_0-\eta_2\xi_2)e_\vartheta+\varepsilon_1\xi_2$ are integral for the flow of v,

iii) suppose $|z_0-\eta_2\xi_2|\geq 4\varepsilon$ for all $\xi_2,\xi_2^{\lambda_2}=1$, then $z_0(\vartheta)=z_0$ is an integral curve.

Proof: i) Since each puncture is on a curve $\varepsilon e_\vartheta\xi_1+\eta_2\xi_2$, the assertion follows from case ii).

ii) We check the integrality condition:

$$\frac{d}{d\vartheta}((z_0-\eta_2\xi_2)e_\vartheta+\eta_2\xi_2) \quad = \quad i(z_0-\eta_2\xi_2)e_\vartheta \quad = \quad v((z_0-\eta_2\xi_2)e_\vartheta+\eta_2\xi_2),\vartheta).$$

iii) Since the vector field vanishes at these points constant curves are integral curves. □

Again we restate these results in geometrical terms:

i) the flow realises parallel transport in the model family over circle segments of radius $t=\varepsilon$,

ii) the 2ε-discs at points $\eta_2\xi_2$, $\xi_2^{l_2}=1$, are rotated rigidly under parallel transport,

iii) points outside 4ε-discs of these points stay fix.

3 from tangled v-arcs to isosceles arcs

In this section we consider two different kinds of mapping classes in a fibre of large radius. Both kinds are twists on embedded arcs. So we may equally well investigate these arcs. Arcs of the first kind are called tangled v-arcs, they are obtained from local v-arcs by parallel transport along a radial path using the differentiable flow of the preceding section.

Arcs of the second kind are called *isosceles arcs*. They are supported on traces of two punctures and form the two sides of an approximate isosceles triangle. Again the degenerate case requires extra care. If two traces are parallel but close, an arc which is supported on these traces except for a small join between them is called a *straight isosceles arc*.

An isosceles arc is said to correspond to a tangled v-arc if it connects the same punctures. In general these two arcs are not isotopic. But we will define a group of mapping classes such that they belong to one orbit. In fact we will give some arcs, such that the group generated by the full twists on these arcs will do. They will be called *bisceles arcs* for the reason that they are supported on segments of two traces not necessarily of similar length.

Note that by this definition all isosceles arcs are subsumed under the notion of bisceles arcs except for the straight isosceles arcs.

We want to encode the isotopy class of a tangled v-arc into a planar diagram in the fibre at e_ϑ. This diagram will consist of all the traces each of which is directed from its *source point* – which is one of ξ_2, $\xi_2^{l_2} = 1$ – to its puncture.

Apart from the source points, there are only ordinary crossings, which are given by the mutual transversal intersection of several traces.

Crossings which are sufficiently close to the tangled v-arc are called *vertices* of the diagram. The segments of traces close to the tangled v-arc are called *essential traces*, they connect a vertex to a puncture.

At each vertex we put an order on the essential traces. The first or *dominant trace* is the one which passed last, which is incidentally the one such that the puncture end is closest. The other follow according to increasing distance to their puncture end. The order can be made explicit by labels assigned to the essential traces at each vertex. We can also make the dominant trace pass over by replacing the other traces by broken lines. Finally the lines are labeled at their ends by the index of the corresponding puncture.

We define the *essential diagram* to be obtained by discarding all lines except the essential traces and we notice that the tangled v-arc is still determined by this datum.

Definition 3.1: No essential diagram contains a directed cycle, hence the *height* function on vertices is well-defined by

$$\mathrm{ht}(P) \quad = \quad \max_{P'<P}(\mathrm{ht}(P'), 0) + 1.$$

where the maximum is taken over all vertices P' between P and a puncture on an essential trace. Each such vertex is called *subordinate* to P.

Given an essential diagram we consider *simple transformations* at vertices. We may change the crossing order at a vertex P if and only if all traces through P are dominant at each subordinate vertex. Note that on transformed diagrams we have to make the order explicit, since it can no longer be read off the distances to the punctures.

The first observation is that we can change an essential diagram by simple transformations only to get a diagram in which the traces of the v-arc punctures are dominant at all vertices they cross.

Lemma 3.2 *Given any vertex there is a composition of simple transformations which changes the crossing order at this vertex but nowhere else.*

Proof: If the vertex is of height one we can change it by a simple transformation. If not, a simple transformation can only be performed if the essential traces are dominant on subordinate vertices. But then we can argue inductively on the height of the vertex. All subordinate vertices are of less height, so by induction we may assume the existence of a composite transformation which makes the traces under consideration dominant there.

Then we can perform the simple transformation to change the local order. Finally we invoke the inverse of the composite transformation to put all other transformed vertices back to their initial state. $\square$

In particular, a series of simple transformations can be found such that the traces of the v-arc punctures become dominant.

The important step is to see, that for any simple transformation at a vertex P there is a choice of a mapping class such that

i) the mapping class is given by a product of full twists on bisceles arcs supported on the essential traces through P,

ii) a diagram transformed by a sequence of simple transformations encodes the isotopy class of the tangled v-arc transformed by the composition of the chosen mapping classes.

For the induction in the proof of the following lemma we need also a relation between tails at a vertex.

Definition 3.3: At a vertex a tail *dominates* another one, if it is isotopic to its trace up to an isotopy fixing the endpoints of both tails but not necessarily the punctures not involved.

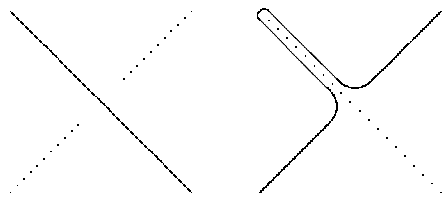
Lemma 3.4 *Given a diagram with orders at its vertices which are obtained by a composition of simple transformations from those of the essential diagram of a tangled v-arc. Then there is a diffeomorphism such that*

i) it represents a mapping class which is a product of full twists on bisceles arcs supported on essential traces,

ii) it is supported close to the essential traces, *iii) locally at every vertex the dominant trace is close to the image of the corresponding tail.*

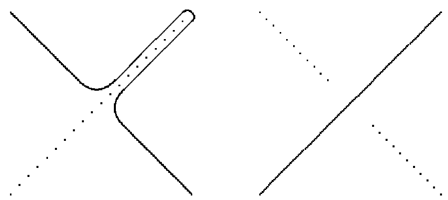
Proof: We assume in addition that each simple transformation reverses the order of consecutive traces and start an induction on the number of such transformations in the composite transformation.

So we consider a simple transformation. For simplicity we first assume that the vertex at which the order is changed is met by only two essential traces. By assumption these traces are dominant at subordinate vertices, hence we can depict the tangled tails of the two punctures involved as follows:



(The critical points pass from bottom to top, the first from left to right, the second from right to left.)

Now a full twist on the bisceles arc with the appropriate choice of orientation can be performed close to these traces to yield:



Hence our claim is true in this case.

The same applies if there are more essential traces and we want to reverse the order of the first two, since the corresponding tails are not effected by tails of lower order.

The situation changes drastically if our simple transformation reverses the order of traces none of which is dominant. Then the picture is modified by the essential traces of larger order pushing loops into the depicted tails.

But on the same time they push loops into the bisceles arc and hence into the support of the diffeomorphism we want to perform. Hence we need only to show that this pushed diffeomorphism will do.

Of course it has the second property. It also has the first property since the full twist on the modified bisceles arc is isotopic to the full twist on the bisceles arc conjugated by full twists on bisceles arcs with apex in the same vertex.

The third property is given, since the dominant traces and the corresponding tails are locally not changed except for the explicit case considered first, where the property can be simply checked.

Moreover for the induction process we should notice that any of our diffeomorphisms preserves domination of a tail over another one, except that it exchanges the role of the tails corresponding to the traces of which the order has been reversed.

To proceed our induction the first two properties are no obstacle. But we have to prove that the third property is preserved when performing an additional transformation.

If the additional transformation does not affect a dominant trace, then neither does the diffeomorphism we perform. Since it also preserves the corresponding tail, we are done in this case.

So let us assume the additional transformation affects a dominant trace. Then the diffeomorphism we choose also affects both the trace and the tail. What we have to show is that the image of the tail which was second before and is first now has the claimed property.

By assumption this tail is only tangled along the essential traces through the vertex under consideration. Moreover we may assume that it dominates all tails through this vertex apart from the dominant one. Hence it is only tangled by the dominant trace and our diffeomorphism can be chosen to map it close to its trace as in the case depicted above. $\square$

Lemma 3.5 *Given a tangled v-arc there is a mapping class given by a composition of full twists on bisceles arcs supported on essential traces which maps the tangled v-arc to the isotopy class of the corresponding isosceles arc.*

Proof: By lemma 3.2 there is a composition of simple transformation changing vertex orders of the essential diagram of the given tangled v-arc in such a way that the traces of both puncture ends are dominant at each vertex.

Then by lemma 3.4 there is a diffeomorphism representing a mapping class as in the claim, which maps the tangled tails in such a way that locally at each vertex the dominant trace is close to its tail.

Thus the images of the tangled tails of both puncture ends may no longer deviate from the traces at any vertex. So they are isotopic to the traces and we conclude that the image arc is isotopic to the corresponding isosceles arc. $\square$

We did not bother to adjust our arguments explicitly for $j_1 - i_1 = l_1/2$, since we can choose $0 < \varepsilon \ll t_0|\vartheta_0 - \vartheta_1|$ small in comparison with the minimal diameter of local neighbourhoods of vertices.

We close this section with two observation, which will be used later:

Remark 3.6: All bisceles arcs supported on essential traces are – apart from the obvious one – not isosceles arcs, since one critical point has to pass after the other.

For the same reason, the *length* of each bisceles arc supported on essential traces of a tangled v-arc is bounded by the length of the corresponding isosceles arc.

The *length* is defined to be the maximum of the lengths of the two sides.

4 from bisceles arcs to coiled isosceles arcs

We stay in the same fibre as before, so we work in the same group of mapping classes. And we are still interested into orbits of subgroups generated by full twists on bisceles arcs.

We have accomplished so far, that we can express a tangled v-arc by means of an isosceles arc and twists on bisceles arcs. Now in a similar way we want to relate bisceles arcs and straight isosceles arcs to a third kind of arcs called *coiled isosceles arcs*. With straight isosceles arcs we will deal only at the end of the section.

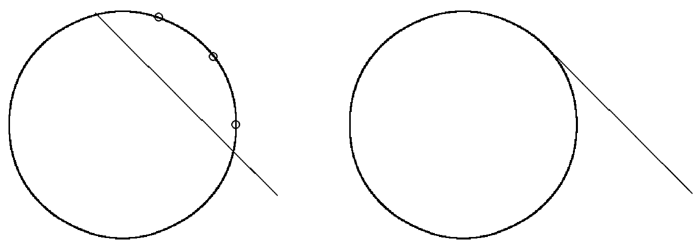
Again a bisceles arc and the associated coiled isosceles arc connect the same pair of punctures and – though not isotopic in general – belong to one orbit of a group generated by twists on specific bisceles arcs.

To make these statements precise, we first need to introduce some more geometric notions.

Definition 4.1: The *central core* is the disc of radius η_2 at the origin with all source points distributed on its boundary circle.

Definition 4.2: The *peripheral cores* are the discs of radius η_2 centred at the points $\xi_1 e_\vartheta, \xi_1^{l_1} = 1$. All critical points for $\lambda = e_\vartheta$ are distributed on their boundaries, the *peripheral circles*.

By looking at the following sketches we notice that a bisceles arc can take essentially two different positions relative to a peripheral core which contains one of its punctures.



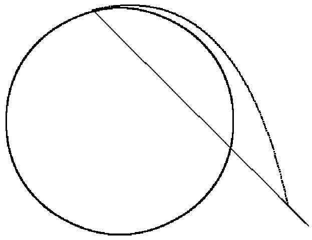
Definition 4.3: A bisceles arc is called *unobstructed* if it is isotopic to some arc supported outside the peripheral cores. It is called *obstructed* otherwise.

A bisceles arc of index pair $i_1 i_2, j_1 j_2$ is said to be *obstructed on the i-side*, if punctures of index $i_1 i_2'$ are obstacles to unobstructedness.

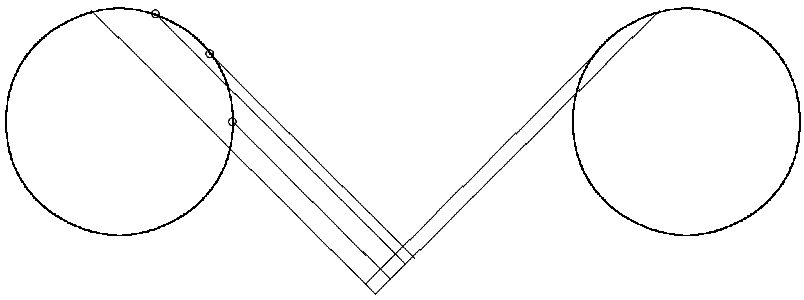
If a bisceles arc is obstructed then at least one side cuts through the corresponding peripheral circle and thus divides the set of critical points on the circle into two subsets.

Definition 4.4: If a bisceles arc is obstructed, then a set of critical points is called *obstructing set*, if the bisceles arc is unobstructed in the complement of the other punctures, i.e. isotopic to some arc supported outside the peripheral cores.

Since we may not isotopy arcs through punctures, we have to resort to changing the isotopy class by means of full twists on some suitable bisceles arcs. This has to be done in such a way, that up to isotopy the terminal part of the obstructed bisceles arc is simply replaced by a spiral segment coiled around the peripheral core.



To do so properly we choose a suitable obstructing set and employ twists on arcs which are supported on pairs of parallels to the sides of the bisceles arc and which connect a point of the obstructing set to another one or to a puncture of the bisceles arc.



By construction a bisceles arc bounds a well defined convex cone which we call the *inner cone* of the bisceles arc.

Thus given an obstructed bisceles arc, the critical points on its peripheral circles in the inner cone form a natural obstructing set and the parallels for this obstructing set are naturally called either *inner parallels* or *obstructing parallels* of the bisceles arc.

Next we choose a topological disc, which contains the obstructed bisceles arc and its inner parallels, but no further critical point. There is a natural way to identify the mapping class group of this disc with an abstract braid group:

Number all traces from left to right – supposing the cone opens upwards as in the sketch above. Let k' be the number of traces parallel to the first an let k be the total number of traces. If $\sigma_{i,j}$, $1 \leq i \leq k' < j \leq k$ is the class of the half twist on the parallel supported on the i^{th} and j^{th} trace, then we put

$$\begin{aligned} \sigma_{i,j} &= \sigma_{j,k}\sigma_{i,k}\sigma_{j,k}^{-1} \quad \text{if} \quad 1 \leq i < j \leq k', \\ \sigma_{i,j} &= \sigma_{1,i}\sigma_{1,j}\sigma_{1,i}^{-1} \quad \text{if} \quad k' < i < j \leq k. \end{aligned}$$

Then considering the elements $\sigma_{i,i+1}$ as the Artin generators of an abstract braid group yields the isomorphism, since it can be checked that arcs for the $\sigma_{i,i+1}$ can be chosen in such a way that they are disjoint outside the punctures.

Under this identification the full twists on obstructing parallels are given by

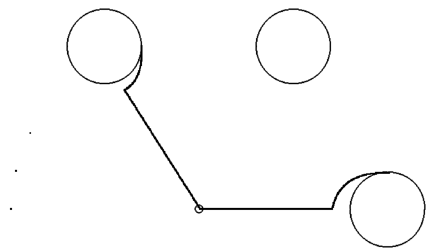
$$\sigma_{i,j}^2, \quad i \leq k' < j \leq k, \ (i,j) \neq (1,k).$$

We can now prove the result concerning the new kind of arcs we want to consider:

Definition 4.5: Any arc supported on two radial rays and two spiral segments in the η_2-neighbourhoods of peripheral cores is called a *coiled isosceles arc.*

Given a bisceles arc it is called the *associated coiled isosceles arc*, if both are isotopic to each other up to full twists on inner parallels.

Example 4.6: Naturally we imagine a coiled arc to spiral monotonously towards the peripheral cores. For $l_2 = 6$ and $l_2 = 4$ the given arc is a coiled isosceles arc.



Remark 4.7: By this definition an unobstructed bisceles arc is its own associated coiled isosceles arc.

Lemma 4.8 *Given a bisceles arc, there is an associated coiled isosceles arc unique up to isotopy.*

Proof: Due to the remark above in case of unobstructed bisceles arcs there is nothing to prove, because there are no inner parallels.

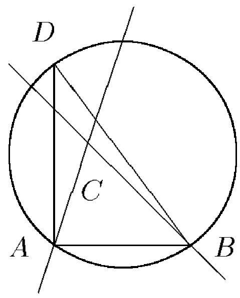
Otherwise, given a bisceles arc connecting punctures of indices $i_1 i_2, j_1 j_2$, an associated coiled isosceles arc – if it exists – must be isotopic to an arc supported in the topological disc considered above. But up to isotopy there is a unique arc in this disc which is supported in the complement of the peripheral cores and which connects the same pair of punctures. Hence the uniqueness claim is proved.

Then we consider the half twists corresponding to the bisceles arc and the arc just considered. They are identified with $\sigma_{1,k}$ and $\check{\sigma}_{i,k}$ (as defined on page 41). Since by A.5 they belong to an orbit under conjugation by full twists on the inner parallels, so do the corresponding arcs and existence of an associated coiled isosceles arc is shown. $\square$

From the simple observation that a side of a bisceles arc may only cut through either the central core or a peripheral core we can conclude that obstructing parallels are in fact bisceles arcs.

Lemma 4.9 *Each obstructing parallel is a bisceles arc.*

Proof: Suppose there is an obstructing parallel which is not a bisceles arc, then there is a source point D on one of its sides. If B, C denote the source points of the bisceles arc, we observe that the triangle BCD contains the apex A.



In particular A is contained in the central core and thus the considered bisceles arc is unobstructed. Hence the assumption lead to the contradiction that there is no obstructing parallel. □

We prefer to rephrase lemma 4.8 using lemma 4.9.

Proposition 4.10 *The class of a bisceles arc and the associated coiled isosceles arc belong to the same orbit for the action of full twists on bisceles arcs which are inner parallels.*

For the closing remark we come back to the topic of straight isosceles.

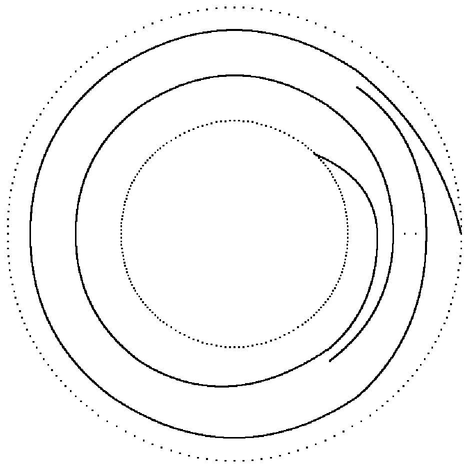
Remark 4.11: A straight isosceles arc only occurs for $j_1 - i_1 = l_1/2$ and by a short check we see, that the corresponding traces are directing in opposite ways. So they come close only if they pass the central core. Immediately we deduce, that a straight isosceles arc is isotopic to its associated coiled isosceles arc.

5 from coiled isosceles arcs to coiled twists

The aim of this section is to identify the isotopy class of the transported arc at $\lambda = 1$ in terms of the Hefez Lazzeri system of paths. In fact this system yields a well-defined identification of the mapping class group of the corresponding fibre with the abstract braid group, so we finally can even identify the twists on transported arcs with abstract braids.

We will see that a coiled isosceles arc transported along a circular segment at radius $t = 1$ is a coiled isosceles arc again, so we have to introduce notations and definitions in such a way, that we get hold of those geometric properties which eventually determine the braid associated to a coiled isosceles arc.

We get additional paths in the fibre at $\lambda = 1$ by a modification of the initial construction. Instead of a path segment as given in the second figure we may also join a path which spirals around the core n full times and then down to a puncture



Such a path is naturally selected by an index $i_2' = i_2 + nl_2$ and the notation $\omega_{i_1 i_2}$ is naturally extended to indices $i_1 i_2$ with i_2 an arbitrary integer.

Notation 5.1: Denote by $\omega^{(i_1)}$ the positive loop around all $\omega_{i_1 i_2}$, $1 \leq i_2 \leq l_2$.

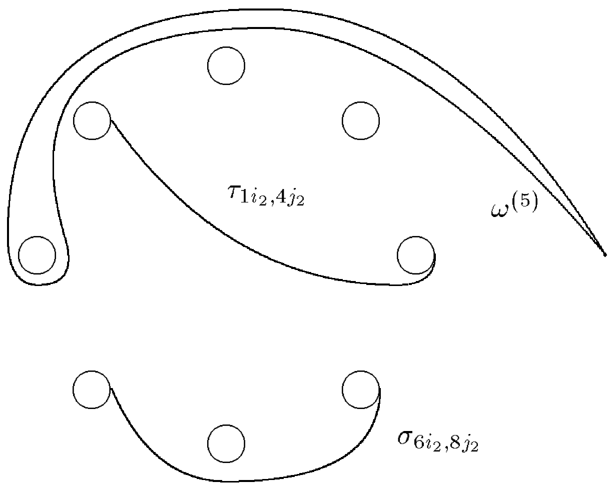
Remark 5.2: For $i_1 \neq j_1$ the paths $\omega_{i_1 i_2}, \omega_{j_1 j_2}$ do not intersect, whatever the integers i_2, j_2 are.
Moreover the loops $\omega^{(k_1)}$ can be chosen disjoint from both.

Now we can introduce twist braids corresponding to arcs which are determined by suitable joins of paths.

Notation 5.3: $\sigma_{i_1 i_2, j_1 j_2}$ is the $\frac{1}{2}$-twist on the union of $\omega_{i_1 i_2}$ with $\omega_{j_1 j_2}$.

Notation 5.4: $\tau_{i_1 i_2, j_1 j_2}$ is the $\frac{1}{2}$-twist on the union of $\omega_{i_1 i_2}$ and $\omega_{j_1 j_2}$ with the $\omega^{(k_1)}$, $i_1 \leq k_1 < j_1$ in between.

Example 5.5:



So far we have dwelled on the topology of the fibre at $\lambda = 1$. Now we extract the characteristic properties of the coiled isosceles.

Definition 5.6: The *winding angle* of a directed arc Γ in the plane of complex numbers with respect to a disjoint point z_0 is – in generalization of the winding number of a closed curve – defined by

$$\vartheta_\Gamma \quad := \quad \int_\Gamma \frac{-\mathrm{i}dz}{z - z_0}, \qquad (\mathrm{i}^2 = -1).$$

Notation 5.7: We introduce notation for some characteristic angles:

i) ϑ_1, ϑ_2, the angles between consecutive l_1^{th}, resp. l_2^{th} roots of unity,

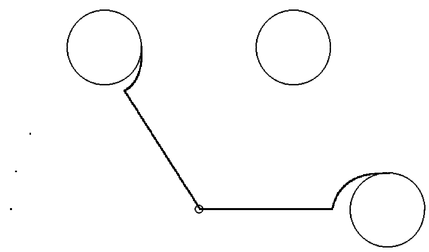
ii) $\vartheta^o := (j_1 - i_1)\vartheta_1$, the angle at the apex of the coiled isosceles arc with index pair $i_1 i_2, j_1 j_2$, note that $0 < \vartheta^o < 2\pi$,

iii) ϑ_i, ϑ_j, the winding angle of the i-side, resp. j-side starting at the apex, with respect to the center of the core of the corresponding peripheral circle,

iv) $\vartheta_j^o := \vartheta_j + \vartheta^o$, a useful shorthand.

The winding angle of a spiral is positive if it turns positively when approaching the peripheral core.

Example 5.8: In the example considered before, suppose the horizontal line supports the i-side then $\vartheta_i = -\frac{\pi}{2}, \vartheta_j = \frac{\pi}{3}$, otherwise $\vartheta_i = \frac{\pi}{3}, \vartheta_j = \frac{\pi}{6}$.



We want now to pin down some geometric properties shared by the coiled isosceles arcs associated to bisceles arcs or straight isosceles arcs.

Lemma 5.9 *The winding angle of a side of a coiled isosceles arc is in the open interval* $]-\frac{3\pi}{2}, \frac{3\pi}{2}[$.

Proof: The side of the bisceles arc is parallel to a side of the associated coiled isosceles arc. If the endpoint is on the half of the peripheral circle facing the origin, then the side is unobstructed and the winding angle is therefore in the range $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

Otherwise it may be obstructed and there are two ways to make it unobstructed depending on the other side. But in any case the absolute value of the winding angle does not exceed $\frac{3\pi}{2}$. $\square$

Lemma 5.10 *The following inclusions hold:*

$$\begin{aligned}
\textit{if } \vartheta^o \leq \pi : \quad \vartheta_i &\in \;]-\frac{3\pi}{2}, \frac{\pi}{2}],\\
\vartheta_j &\in \; [-\frac{\pi}{2}, \frac{3\pi}{2}[;\\
\textit{if } \vartheta^o > \pi : \quad \vartheta_i &\in \; [-\frac{\pi}{2}, \frac{3\pi}{2}[,\\
\vartheta_j &\in \;]-\frac{3\pi}{2}, \frac{\pi}{2}].
\end{aligned}$$

Proof: If the endpoint of the i-side is on the half circle facing the origin, then its winding angle is in $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

If the endpoint is on the opposite half circle, then the winding angle is in either $]-\frac{3\pi}{2}, -\frac{\pi}{2}[$ or $]\frac{\pi}{2}, \frac{3\pi}{2}[$ and the sign depends on the second endpoint. The sign is that of $\pi - \vartheta^o$ for the i-side and the opposite for the j-side. □

Moreover the considerations of this proof immediately yield the observation:

Lemma 5.11 *The j-side of the bisceles arc is disjoint from the central core if and only if*

$$\begin{aligned}
\textit{either} \quad \vartheta^o \leq \pi \quad \textit{and} \quad \vartheta_j &\in \;]\frac{\pi}{2}, \frac{3\pi}{2}[,\\
\textit{or} \quad \vartheta^o > \pi \quad \textit{and} \quad \vartheta_j &\in \;]-\frac{3\pi}{2}, -\frac{\pi}{2}[.
\end{aligned}$$

The next thing we have to exploit is the fact that at a parameter $\lambda = e_\vartheta$ bisceles do not exists for all index pairs.

Lemma 5.12 *A bisceles arc with index pair i_1i_2, j_1j_2 exists at $\lambda = e_\vartheta$ only if*

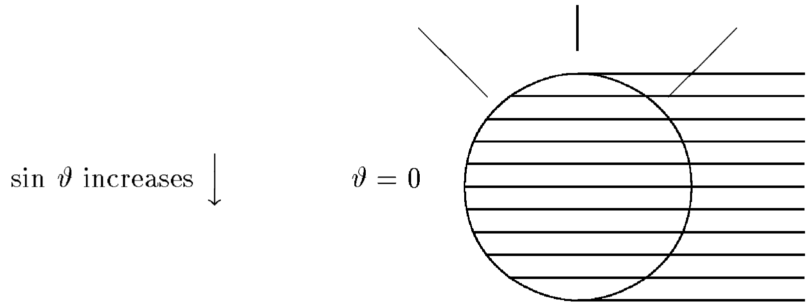
i) $i_2 = j_2$,

ii) $\vartheta^o \leq \pi$ *and* $\sin\vartheta_i \leq \sin\vartheta_j^o$, $\vartheta_j^o < \frac{3\pi}{2}$,

or

iii) $\vartheta^o > \pi$ *and* $\sin\vartheta_i \geq \sin\vartheta_j^o$, $\vartheta_j^o > \frac{\pi}{2}$.

Proof: In the first case the claim is obvious since the traces have their source points in common. So from now on we assume that the source points are distinct. Let us consider the case $\vartheta^o \leq \pi$ next. Then the possible traces for the index i_1i_2 are sketched near to the central core as well as the direction of possible traces with index j_1j_2. The second inequality is now read off easily, since $\sin\vartheta_i$ is the vertical component of the i-side and $\sin\vartheta_j^o$ the maximal vertical component of the j-side.



Suppose now ϑ_j^o exceeds $\frac{3\pi}{2}$, then ϑ_j exceeds $\frac{\pi}{2}$ and by 5.11 the j-side does not pass the central core. To be cut properly by the trace of the i-side its source point must be on the right hand half of the circle. But the horizontal component of ϑ_j^o is $\cos \vartheta_j^o$ which is not positive for $\vartheta_j^o \in [\frac{3\pi}{2}, \vartheta^o + \frac{3\pi}{2}]$.

The final case $\vartheta^o > \pi$ can be handled in strict analogy. $\square$

Since these better bounds hold obviously in the case of straight isosceles we get an improvement on the assertion of 5.10:

Lemma 5.13 *The following inclusions hold:*

$$
\begin{aligned}
\textit{if} \quad \vartheta^o \leq \pi: \quad \vartheta_i &\in \;]-\frac{3\pi}{2}, \frac{\pi}{2}], \\
\vartheta_j &\in \; [-\frac{\pi}{2}, \frac{3\pi}{2} - \vartheta^o[; \\
\textit{if} \quad \vartheta^o > \pi: \quad \vartheta_i &\in \; [-\frac{\pi}{2}, \frac{3\pi}{2}[, \\
\vartheta_j &\in \;]\frac{\pi}{2} - \vartheta^o, \frac{\pi}{2}].
\end{aligned}
$$

Now we combine the results to obtain a relation between the winding angles.

Lemma 5.14 *The winding angles are subject to*

$$\vartheta_i \leq \vartheta_j^o \leq \vartheta_i + 2\pi.$$

Proof: Suppose $\vartheta^o > \pi$. Then by lemma 5.13 $\vartheta_j^o \in]\frac{\pi}{2}, \frac{\pi}{2} + \vartheta^o]$ so

$$i) \quad \vartheta_i \leq \vartheta_j^o \qquad \text{or} \qquad ii) \quad \vartheta_i, \vartheta_j^o \in]\frac{\pi}{2}, \frac{3\pi}{2}[.$$

Also the conditions in the latter case imply $\vartheta_i \leq \vartheta_j^o$, since $\sin \vartheta_i \geq \sin \vartheta_j^o$ by lemma 5.12 and the sine function is decreasing in $]\frac{\pi}{2}, \frac{3\pi}{2}[$.

On the other hand by lemma 5.13

$$i) \quad \vartheta_i + 2\pi \geq \vartheta_j^o \qquad \text{or} \qquad ii) \quad \vartheta_i + 2\pi, \vartheta_j^o \in]\frac{3\pi}{2}, \frac{5\pi}{2}[.$$

and again the second case is a subcase of the first, since the sine function is increasing on $]\frac{3\pi}{2}, \frac{5\pi}{2}[$.

The case $\vartheta^o \leq \pi$ is done analogously. $\square$

Next we investigate the impact of parallel transport.

Lemma 5.15 *Under parallel transport along a circle segment of winding angle ϑ at radius $t = 1$ a coiled isosceles arc is mapped up to isotopy to a coiled isosceles arc with winding angles changed by $-\vartheta$.*

Proof: The line segments of the isosceles arc belong to the part which is rotated rigidly by the flow of the vector field in 2.12. The total rotation is of angle ϑ.
On the other hand the spirals are wound resp. unwound, since the endpoints are fixed relative to their peripheral centres, while the points on the boundary of the $2\eta_2$-discs are relatively rotated in opposite direction, hence the amount and sign of the change in the winding angles. $\square$

Remark 5.16: If we introduce $\vartheta'_i := \vartheta_i - \vartheta$, (similarly $\vartheta_j := \vartheta_j - \vartheta$), then ϑ_i is the i-side winding angle of the isosceles arc transported from angular parameter ϑ to $\lambda = 1$.

Lemma 5.17 *Suppose a coiled isosceles arc is associated to a bisceles arc or to an isosceles arc with index pair $i_1 i_2, j_1 j_2$, $i_1 < j_1$, $i_2 \neq j_2$, then the full twist on any of its parallel transports to $\lambda = 1$ along a circle segment of radius $t = 1$ is identified with one of the abstract braid elements*

$$\tau^2_{i_1 i'_2, j_1 j'_2},\ 1 \leq i_1 < j_1 \leq l_1,\ 1 \leq j'_2 - i'_2 < l_2.$$

Proof: The transported coiled isosceles arc at $\lambda = 1$ can be represented in a unique way by the join of loops $\omega^{(k_1)}, i_1 \leq k_1 < j_1$ and two paths $\omega_{i_1 i'_2}$, $\omega_{j_1 j'_2}$ with i'_2, j'_2 suitable chosen. Hence the corresponding half twist is identified with the abstract braid element $\tau_{i_1 i'_2, j_1 j'_2}$.

We note further that

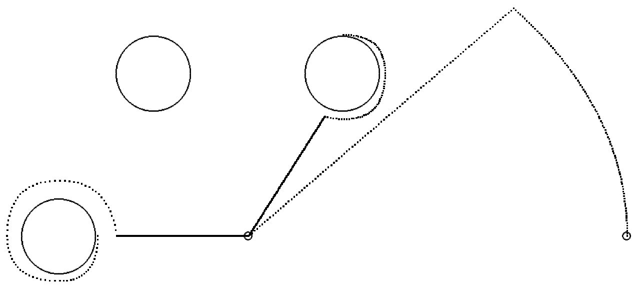
$$\begin{aligned} (j'_2 - 1)\vartheta_2 &= \vartheta'_j - \pi + (j_1 - 1)\vartheta_1, \\ (i'_2 - 1)\vartheta_2 &= \vartheta'_i - \pi + (i_1 - 1)\vartheta_1. \end{aligned}$$

Computing the difference using $\vartheta'_j - \vartheta'_i = \vartheta_j - \vartheta_i$ and $(j_1 - i_1)\vartheta_1 = \vartheta^o$ we get:

$$(j'_2 - i'_2)\vartheta_2 \;=\; \vartheta_j - \vartheta_i + \vartheta^o \;=\; \vartheta^o_j - \vartheta_i.$$

In case $j'_2 - i'_2 \leq 0$ this implies $\vartheta^o_j - \vartheta_i \leq 0$, in case $j'_2 - i'_2 \geq l_2$ we conclude $2\pi \leq \vartheta^o_j - \vartheta_i$, so both these cases contradict the assertion of lemma 5.14, since neither $\vartheta^o_j = \vartheta_i$ nor $\vartheta^o_j = \vartheta_i + 2\pi$ is possible under the assumption $i_2 \neq j_2$. Therefore we get $1 \leq j'_2 - i'_2 < l_2$, as claimed. $\square$

Example 5.18: Suppose the example from page 17 has been transported by an angle $\vartheta = \frac{5\pi}{3}$ along the circle arc of radius $t = 1$. Following the recipe above we get:



Hence assuming $l_1 = 6, l_2 = 4$ the associated abstract braid is $\tau_{22,45}$.

Let us call a coiled isosceles arc in the fibre at $\lambda = 1$ *associated* to a local v-arc, if it is obtained from the local v-arc by parallel transport along a radial segment, a transformation by full twists to get the associated coiled isosceles arc and parallel transport along a circle segment at $t = 1$.

We note then the following converse to 5.17:

Lemma 5.19 *Each element $\tau^2_{i_1 i_2, j_1 j_2}$, $1 \leq i_1 < j_1 \leq l_1$, $1 \leq j_2 - i_2 < l_2$, is the full twist on a coiled isosceles arc associated to a local v-arc.*

Proof: There is a local v-arc for each index pair $i_1 i_2, j_1 j_2$, $1 \leq i_1 < j_1 \leq l_1$, $1 \leq i_2, j_2 \leq l_2$. For each such local v-arc there is an associated coiled isosceles arc in the fibre at $\lambda = 1$, which determines some τ as above by 5.17. All others are then obtained by changing the winding angle of the circular path by suitable multiples of 2π. □

The case $i_2 = j_2$ requires extra care. We have analogues to lemma 5.17 and lemma 5.19.

Lemma 5.20 *Suppose a coiled isosceles arc is associated to a bisceles arc or to an isosceles arc with index pair $i_1 i_2, j_1 j_2$, $i_1 < j_1$, $i_2 = j_2$, then the full twist on any of its parallel transports to $\lambda = 1$ along a circle segment of radius $t = 1$ is identified with one of the abstract braid elements*

$$\tau^2_{i_1 i'_2, j_1 j'_2},\ 1 \leq i_1 < j_1 \leq l_1,\ j'_2 - i'_2 \in \{0, l_2\}.$$

Proof: With the same argument as in the proof of lemma 5.17 we can exclude the cases $j'_2 - i'_2 < 0$ and $j'_2 - i'_2 > l_2$. Since i_2, i'_2 and j_2, j'_2 may only differ by multiples of l_2 we are left with the two possibilities of the claim. □

Lemma 5.21 *Given an index pair $i_1 i_2, j_1 j_2$, $1 \leq i_1 < j_1 \leq l_1$, $j_2 - i_2 = l_2$, at least one of $\tau_{i_1 i_2, j_1 i_2}, \tau_{i_1 i_2, j_1 j_2}$ is the half twist on a coiled isosceles arc associated to a local v-arc.*

Proof: There is an local v-arc for the index pair $i_1 i_2, j_1 i_2$, $1 \leq i_1 < j_1 \leq l_1$, $1 \leq i_2 \leq l_2$. We know that for each such local v-arc there is an associated coiled isosceles arc in the fibre at $\lambda = 1$, which determines one of $\tau_{i_1 i'_2, j_1 i'_2}, \tau_{i_1 i'_2, j_1 j'_2}$ as above by 5.20. All others are then obtained by changing the winding angle of the circular path by suitable multiples of 2π. □

We close this section be identifying the twists unambiguously under special geometric assumptions.

Lemma 5.22 *Given any coiled isosceles arc with punctures of indices $i_1 i_2, j_1 j_2$, $i_1 < j_1$, $i_2 = j_2$, facing the origin, then the twist on any of its parallel transports to $\lambda = 1$ along a circle segment of radius $t = 1$ is identified with one of the abstract braid elements*

$$\tau_{i_1 i'_2, j_1 j'_2},\ 1 \leq i_1 < j_1 \leq l_1,$$

with $i'_2 = j'_2$ if $\vartheta^o \leq \pi$ and $i'_2 + l_2 = j'_2$ if $\vartheta^o > \pi$.

Proof: We run through the same consideration as in 5.17. But in the final step we are stuck since now $\vartheta_i = \vartheta_j^o \mod 2\pi$, so $\vartheta_i = \vartheta_j^o$ and $\vartheta_i = \vartheta_j^o - 2\pi$ are possible by 5.14. Since both punctures face the origin by hypothesis,

$$\vartheta_i \in [-\frac{\pi}{2}, \frac{\pi}{2}], \vartheta_j \in [-\frac{\pi}{2}, \frac{\pi}{2}].$$

In case $\vartheta_i = \vartheta_j^o$ which corresponds to $i_2' = j_2'$ we must have $\vartheta^o = \vartheta_i - \vartheta_j \leq \pi$. In case $\vartheta_i + 2\pi = \vartheta_j^o$ corresponding to $i_2' + l_2 = j_2'$ we conclude that $\vartheta^o = 2\pi + \vartheta_i - \vartheta_j \geq \pi$. □

Lemma 5.23 *Given any coiled isosceles arc associated to a bisceles arc with punctures of indices $i_1 i_2, j_1 j_2$, $i_1 < j_1$, $i_2 = j_2$, one of which exactly facing the origin, then the half twist on any of its parallel transports to $\lambda = 1$ along a circle segment of radius $t = 1$ is identified with one of the abstract braid elements*

$$\tau_{i_1 i_2', j_1 j_2'}, \ 1 \leq i_1 < j_1 \leq l_1, i_2' + l_2 = j_2',$$

under the assumption that $\vartheta^o \leq \pi$.

Proof: The claim is secured by similar considerations as in the proof of 5.22. We know that $\vartheta_i = \vartheta_j^o$ or $\vartheta_i = \vartheta_j^o + 2\pi$ and we imposed $\vartheta^o \leq \pi$.
If the puncture of index $i_1 i_2$ faces the origin, then from 5.10 and 5.11:

$$\vartheta_i \in [-\frac{\pi}{2}, \frac{\pi}{2}], \vartheta_j \in]\frac{\pi}{2}, \frac{3\pi}{2}[.$$

If the puncture of index $i_1 i_2$ does not face the origin, then we get:

$$\vartheta_i \in]-\frac{3\pi}{2}, -\frac{\pi}{2}[, \vartheta_j \in [-\frac{\pi}{2}, \frac{\pi}{2}].$$

In either case we can check that we are left with the possibility $\vartheta_i = \vartheta_j^o - 2\pi$. Hence $j_2' = i_2' + l_2$ holds in the index pair of the corresponding abstract braid. □

6 from local w-arcs to coiled twists

Having understood the parallel transport of local v-arcs sufficiently well, we can now consider the parallel transport of local w-arcs. They are only considered close to the degeneration at $\lambda = 0$ with $i_2 = j_2$.

Let us first look at parallel transport along circular segments of very small radius.

Lemma 6.1 *Under parallel transport along circle segments of radius $\varepsilon << \eta_2$ local w-arcs are mapped to local w-arcs.*

Proof: This is immediate, for the parallel transport can be realised by the flow of the vector field in 2.13, which is rigid rotation for the support of the local w-arcs. □

Next local w-arcs are transported along a radial segment. We get then tangled w-arcs in the fibre e_ϑ. The simpler arcs, to which we want to compare them, are called *isosceles w-arcs* and they relate to isosceles arcs as local w-arcs relate to local v-arcs.

The isosceles w-arc of index pair i_1i_2, j_1i_2 can be best understood from the isosceles arcs of index pairs $i_1i_2, i_1^+i_2$ and $i_1^+i_2, j_1i_2$, which are called its *constituents*. It is isotopic to the first constituent acted on by a positive half twist on the second. It can be chosen to be composed of four line segments, two of which are supported on the traces of the punctures i_1i_2 and j_1i_2, while the middle pair forms a sharp wedge over the trace of the puncture $i_1^+i_2$, cf. the example below.

An isosceles w-arc is called *corresponding* to a given tangled w-arc, if both connect the same pair of punctures.

The same methods as in the case of tangled v-arcs can now be employed to relate tangled and isosceles w-arcs.

Lemma 6.2 *Up to conjugation by full twists on bisceles arcs of shorter length a tangled w-arc is isotopic to the corresponding isosceles w-arc.*

We now make an observation which will help us to be concerned mostly with isosceles w-arcs which are supported outside the peripheral circles except for an arbitrarily small neighbourhood of the puncture $i_1^+i_2$. They shall be referred to as *unobstructed isosceles w-arcs.*

Lemma 6.3 *Any local w-arc for the index pair i_1i_2, j_1i_2 can be transported to radius $t = 1$ along a circle segment of radius $t = \varepsilon$ and a radial segment such that the corresponding isosceles w-arc is unobstructed, except in case of $j_1 - i_1 = (l_1 + 1)/2$.*

Proof: In the cases under consideration either $j_1 - i_1 \leq l_1/2$ or $j_1 - i_1^+ > l_1/2$. We choose $\vartheta = (i_2 - 1)\vartheta_2 - (j_1 - 1)\vartheta_1 \mp \frac{\pi}{2}$ respectively, so we get

$$|(i_2 - 1)\vartheta_2 - (k_1 - 1)\vartheta_1 - \vartheta)| \geq \frac{\pi}{2} \quad \text{for } k_1 = i_1, i_1^+, j_1.$$

Therefore at $\lambda = e_\vartheta$ the punctures with index pairs $i_1i_2, i_1^+i_2, j_1i_2$ are all situated on the halfs of their peripheral circles facing the origin. Accordingly the isosceles w-arc corresponding to the transported local w-arc is unobstructed. $\square$

In the remaining case we can only arrange that the punctures of the i-side and of the j-side face the origin.

Lemma 6.4 *If $j_1 - i_1 = (l_1 + 1)/2$, then the local w-arc can be transported to radius $t = 1$ along a circle segment of radius $t = \varepsilon$ and a radial segment such that*

i) only the wedge of the corresponding isosceles w-arc is obstructed,

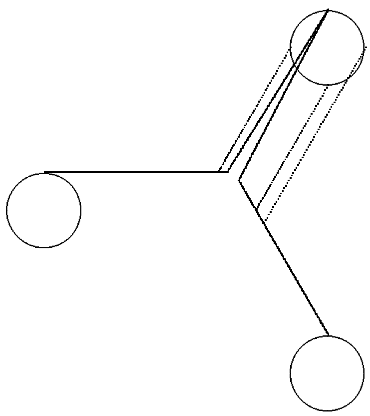
ii) every critical point of its peripheral circle belongs to the inner cone of either of the constituents or is of index $i_1^+i_2$.

Proof: We choose $\vartheta = (i_2-1)\vartheta_2 - (j_1-1)\vartheta_1 + \frac{\pi}{2}$ and get

$$\begin{aligned} |(i_2-1)\vartheta_2 - (k_1-1)\vartheta_1 - \vartheta)| &\geq \frac{\pi}{2} \qquad \text{for } k_1 = i_1, j_1, \\ |(i_2-1)\vartheta_2 - (i_1^+-1)\vartheta_1 - \vartheta)| &< \frac{\pi}{2}. \end{aligned}$$

So the punctures of the i-side and of the j-side of the corresponding isosceles w-arc face the origin as before, but the wedge is obstructed. Since both inner angles are less than π, all critical points on the corresponding peripheral circle belong to an inner cone, except the puncture of index $i_1^+ i_2$. $\square$

Example 6.5: An isosceles w-arc with obstructed wedge is obtained in case of $l_1 = 3, l_2 = 4$, $i_2 = 2$:



The final parallel transport of an isosceles w-arc along a circular segment at radius $t = 1$ can be understood using its constituents.

Lemma 6.6 *Parallel transport along a circle segment of radius $t = 1$ of an unobstructed isosceles w-arc yields an arc isotopic to the parallel transport of its first constituent acted upon by a positive half twist on the parallel transport of its second constituent.*

Proof: The relation between an isosceles w-arc and its constituents is preserved under parallel transport. $\square$

Lemma 6.7 *Up to full twists on bisceles arcs of shorter length an isosceles w-arc obstructed on its wedge only is isotopic to the coiled isosceles arc associated to its first constituent acted upon by a positive half twist on the coiled isosceles arc associated to its second constituent.*

Proof: The same full twists on inner parallels which map the constituents to the isotopy classes of their associated coiled isosceles arcs also maps the isosceles w-arc to the isotopy class of the arc obtained from the associated coiled isosceles arcs. $\square$

Let us finally summarize the results of this section:

Lemma 6.8 *Local w-arcs in a fibre close to the origin and twists among the following elements with* $1 \leq i_1 < j_1 \leq l_1$, $i_2' = i_2 - l_2$,

$$\begin{array}{ll} \tau^{-1}_{i_1 i_2', i_1^+ i_2'} \tau^2_{i_1^+ i_2', j_1 i_2'} \tau_{i_1 i_2', i_1^+ i_2'}, & j_1 - i_1 \leq l_1/2, \\ \tau^{-1}_{i_1 i_2', i_1^+ i_2'} \tau^2_{i_1^+ i_2', j_1 i_2} \tau_{i_1 i_2', i_1^+ i_2'}, & j_1 - i_1 \geq l_1/2 + 1, \\ \tau^{-1}_{i_1 i_2', i_1^+ i_2} \tau^2_{i_1^+ i_2', j_1 i_2} \tau_{i_1 i_2', i_1^+ i_2}, & j_1 - i_1 = l_1/2 + 1/2. \end{array}$$

correspond in such a way that

i) *each local w-arc can be transported along a circle arc of radius* ε *and a radial segment to* $t = 1$, *such that the twist on the corresponding isosceles w-arc transports to* $\lambda = 1$ *along the circle of radius* $t = 1$ *to yield one of the given twists up to conjugation by full twists on obstructing parallels to its constituents.*

ii) *each given twist can be obtained from a local w-arc as in i).*

Proof: If $j_1 - i_1 \leq l_1/2$ then $(j_1 - i_1^+)\vartheta_1, (i_1^+ - i_1)\vartheta_1 \leq \pi$, hence by 5.22 and 6.6 we can get an element of the first row, since by the braid relation it does not matter if we transform the first constituent by a positive full twist on the second or if we transform the second by a negative full twist on the first.

Similarly if $j_1 - i_1^+ > l_1/2$ then $(i_1^+ - i_1)\vartheta_1 \leq \pi$ but $(j_1 - i_1^+)\vartheta_1 > \pi$, so we get a twist of the second row, again with 5.22 and 6.6.

In the final case we argue along the same line with 5.23 and 6.7, so also in case $j_1 - i_1 = l_1/2 + 1/2$ we get twists among the given ones.

As in the similar cases proved before, we get all twist this way as we can transport around the circle at $t = 1$ as many times as necessary. $\square$

7 the length of bisceles arcs

In this section we want to compare the length of bisceles arcs to a real number we assign to index pairs.

Definition 7.1: The modulus of a pair $i_1 i_2, j_1 j_2$ of indices is given by

$$\eta_2 \left| \frac{\sin(\pi \frac{i_2 - j_2}{l_2})}{\sin(\pi \frac{i_1 - j_1}{l_1})} \right|.$$

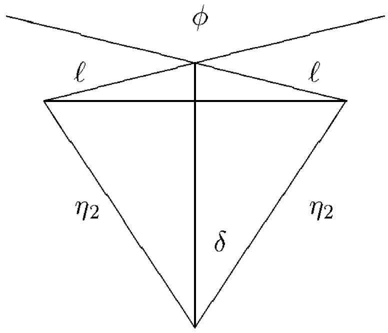
In this way a modulus is assigned to all objects with an index pair.

Since modulus is in some way complementary to length, we introduce it also for bisceles arcs.

Definition 7.2: The modulus of a bisceles arc is the shorter of the two distances from the apex to both source points.

Lemma 7.3 *The modulus* t_0 *of a critical parameter* $t_0 e_{\vartheta_0}$ *for the pair* $i_1 i_2, j_1 j_2$ *coincides with the modulus for that index pair.*

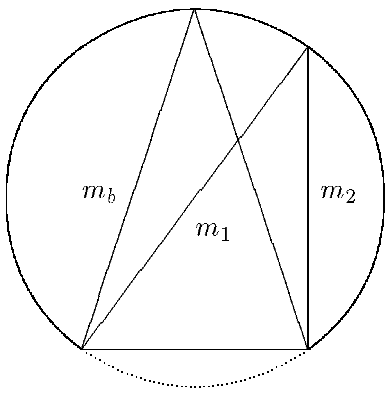
Proof: Given the traces at angle ϑ_0 the pair corresponding to $i_1 i_2, j_1 j_2$ meet at an apex which forms an isosceles triangle with both source points on the circle of radius η_2. So with $\delta = \pm\pi \frac{i_2 - j_2}{l_2}$, $\phi = \pm 2\pi \frac{i_1 - j_1}{l_1}$ the length of the sides equals the modulus as can be seen from the following sketch.



□

Lemma 7.4 *The modulus of a bisceles arc bounds the modulus of the corresponding index pair from below. Equality holds only in the case that the bisceles arc is an isosceles arc.*

Proof: The apex of the bisceles arc which depends on the parameter angle ϑ determines a triangle over the base given by the two source points. The base and the angle over it are independent of ϑ, whereas the length of the shorter side is the bisceles arc modulus. The modulus of the pair is the length of a side if both sides are equal which happens for a specific ϑ.
The claim is now obvious from the following sketch, $m_b \geq m_2 = \min(m_1, m_2)$:



The algebraic argument reads as follows: By the cosine formula

$$m_1^2 + m_2^2 - 2m_1 m_2 \cos(apex) = 2m_b^2 - 2m_b^2 \cos(apex).$$

We can get a lower bound for the l.h.s. assuming w.l.o.g. $m_1 \leq m_2$:

$$\begin{aligned} m_1^2 + m_2^2 - 2m_1 m_2 \cos(apex) &= (m_1 - m_2)^2 + m_1 m_2 (2 - 2\cos(apex)) \\ &\geq m_1^2 (2 - 2\cos(apex)) \end{aligned}$$

Then the conclusion $m_b \geq m_1$ is immediate. □

Now we compare the modulus of arcs we encountered in preceding sections.

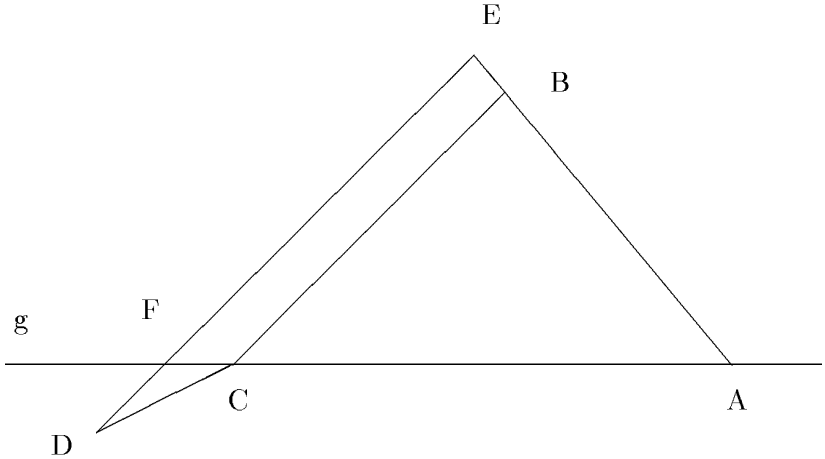
Lemma 7.5 *The modulus of a bisceles arc supported on the essential traces of a tangled v-arc is strictly larger than the modulus of the corresponding isosceles arc.*

Proof: This claim follows from lemma 7.4 above and the remark on page 14. □

Lemma 7.6 *An obstructing parallel to a bisceles arc is of strictly larger modulus.*

Proof: Let us consider first the case that the obstructing parallel has a side in common with the obstructed bisceles arc:
We have thus a triangle ABC formed by the source points A, C of the traces of obstructed bisceles arc and its apex B. Similarly we have a triangle AED formed by the source points A, D of the obstructing parallel and its apex E. We have $g_{AE} = g_{AB}$ and $B \in \overline{AE}$. Moreover $g_{DE} || g_{BC}$ and D is separated from A by g_{BC}. Denote by F the intersection of g_{DE} and g_{AC}. Then g_{DE} is divided into rays bounded by E resp. F and the finite segment $\overline{EF}$.

Now D may not be on the ray bounded by E, since then B is in the interior of ACD, hence in the central core contrary to the assumption on obstructedness. Neither may D belong to $\overline{EF}$ since otherwise $\overline{AD}$ cuts $\overline{BC}$ which is impossible since $\overline{BC}$ is on the obstructed side of the bisceles arc and may hence not be cut by the chord $\overline{AD}$ of the central core. So F is on $\overline{DE}$.



Obviously now we have $|AB| < |AE|$ and $|BC| < |EF| < |DE|$. In case $|AB| \leq |BC|$ we get $|DE| > |EF| \geq |AE| > |AB|$ by proportionality. In case $|BC| \leq |AB|$ we similarly have $|AE| > |AB| \geq |BC|$ and $|DE| > |BC|$. So in any case we get

$$\min(|DE|, |AE|) \quad > \quad \min(|AB|, |BC|).$$

which is the claim.

Suppose now that the obstructing parallel has no side in common with the obstructed bisceles arc, then there is an intermediate obstructing parallel which has a side in common with each. So the full result is obtained in two steps as above. □

Lemma 7.7 *The full twist on a bisceles arc which is not an isosceles arc transported along $t = 1$ to $\lambda = 1$ is in the group generated by all twists τ^2 of modulus larger than the modulus of the bisceles arc.*

Proof: The obstructing parallels are bisceles arcs of strictly larger modulus. Hence we may as well assume the bisceles arc to be unobstructed. Its parallel transport is then isotopic to an arc defining some τ of larger modulus, which is strictly larger in case the bisceles arc is no isosceles arc. $\square$

8 the discriminant family

In this section we will work with the discriminant family of the families of function we consider. In order to compute the versal braid monodromy in the next section, we have to find the locally assigned groups. Moreover we need to compare the parallel transport in the discriminant family to parallel transport in the model family.

Lemma 8.1 *The discriminant and the model discriminant family over the punctured parameter bases have a common unramified cover.*

Proof: The equation for the discriminant family has a formal factorisation

$$\prod_{\xi_1^{l_1},\xi_2^{l_2}=1} (z-\alpha^{\frac{l_1+1}{l_1}}\xi_1-\varepsilon_2^{\frac{l_2+1}{l_2}}\xi_2) \quad = \quad \prod_{\xi_1^{l_1}=1} ((z-\alpha^{\frac{l_1+1}{l_1}}\xi_1)^{l_2}-\varepsilon_2^{l_2+1}) \quad = \quad 0.$$

as opposed to the equation for the model discriminant family:

$$\prod_{\xi_1^{l_1},\xi_2^{l_2}=1} (z-\lambda\xi_1-\eta_2\xi_2) \quad = \quad \prod_{\xi_1^{l_1}=1} ((z-\lambda\xi_1)^{l_2}-\eta_2^{l_2}) \quad = \quad 0.$$

These equations coincide for $\eta_2^{l_2}=\varepsilon_2^{l_2+1}$ and $\lambda^{l_1}=\alpha^{l_1+1}$. Hence the family parameterized by β

$$\prod_{\xi_1^{l_1}=1} ((z-\beta^{l_1+1}\xi_1)^{l_2}-\varepsilon_2^{l_2+1}) \quad = \quad 0.$$

is isomorphic to the pull backs of the discriminant family and the model discriminant family by the covering map $\beta\mapsto\alpha=\beta^{l_1}$ resp. $\beta\mapsto\lambda=\beta^{l_1+1}$, if $\varepsilon_2^{l_2+1}=\eta_2^{l_2}$.

In this way we can understand polar coordinates of the bases of the two discriminant families as different coordinates of the universal cover of the bases punctured at the origin.

So with polar coordinates r and θ in the base of the discriminant family we can immediately compare parallel transport in the two families:

Lemma 8.2 *Parallel transport in the discriminant family and in the model discriminant family coincides if $r_0^{l_1+1}=t_0^{l_1}$ and $\theta(l_1+1)=\vartheta l_1$*

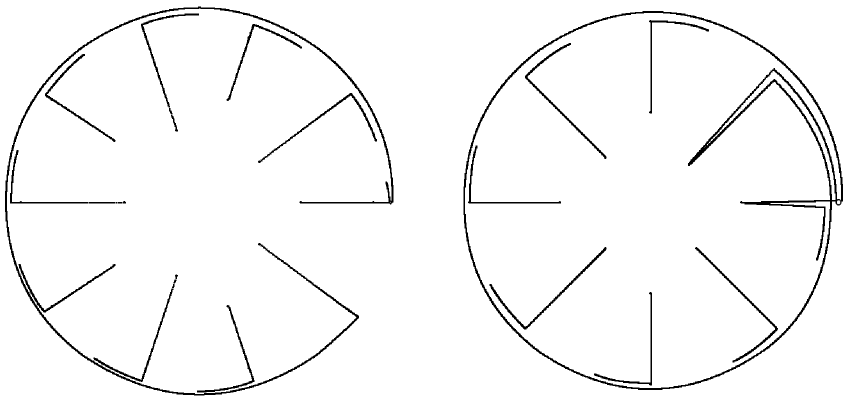
i) along radial paths te_ϑ, $t\in[t_0,1]$ and $re_\theta, r\in[r_0,1]$,

ii) along circular paths of radius 1 of winding angles ϑ and θ respectively.

We can now define *standard paths* in the bases of both families by asking them to be supported on radial segments and circular segments as in the lemma.

And for each standard path in one base we get another one in the other with the same parallel transport.

Example 8.3: A system of standard paths for the discriminant family associated to $l_1 = 4, l_2 = 2$ is thus related to standard paths in the base of the model discriminant family:



To get the versal braid monodromy of the discriminant family, we therefore need to transfer the locally assigned groups from local Milnor fibres of the discriminant family to local Milnor fibres of the model discriminant family and transport them along all possible standard paths.

We assign a group to a local Milnor fibre in the model discriminant using the fact that the fibre is isomorphic to a local Milnor fibre in the discriminant family by way of the two finite covering maps.

Lemma 8.4 *The group assigned to a Milnor fibre at a regular parameter $t_1 e_{\vartheta_1}$, sufficiently close to a singular parameter $t_0 e_{\vartheta_0} \neq 0$ with $t_1 - t_0 > 0$, is generated by full twists on local v-arcs.*

Proof: The singular fibre corresponds to a function with non-degenerate critical points only, cf. the proofs in [15, L. 5.10,5.11]. So by definition the locally assigned group is generated by mapping classes fixing all punctures and supported on small discs each of which is a Milnor fibre for just one multiple puncture.

By close inspection we can see that the local v-arcs are supported on such discs and the full twists on local v-arcs generate the group of all mapping classes of each disc which preserve the punctures. $\square$

Lemma 8.5 *The group assigned to a Milnor fibre at a regular parameter $t_1 e_{\vartheta_1}$, sufficiently close to a singular parameter $\lambda = 0$, is generated by full twists on local w-arcs and $\frac{3}{2}$-twists on local v-arcs with index pair $i_1 i_2, i_1^+ i_2$.*

Proof: The singular fibre corresponds to a function which has l_2 critical points of type A_{l_1} with distinct critical values. So by definition the group locally assigned to each disc, which is a local Milnor fibre of a multiple puncture, is generated by the mapping classes of the braid monodromy of the singular function germ it corresponds to.

Each of the critical points of type A_{l_1} is unfolded linearly, so the local Milnor fibre can be naturally identified with the Milnor fibre encountered in [15, L. 4.7]. And in combination with [15, L. 4.6] we conclude that local generators are given by the $\frac{3}{2}$-twists on v-arcs with index pairs $i_1 i_2, i_1^+ i_2$ and full twists on arcs winding positively from a puncture of index $i_1 i_2$ to a puncture of index $j_1 i_2$, $j_2 > i_1^+ = i_1 + 1$, around all v-arcs.

By lemma A.3 we can see that instead we can use the twists of the claim to generate the same group. □

To summarize the preceding discussion we should note:

Remark 8.6: The versal braid monodromy of the family of functions

$$x_1^{l_1+1} - \alpha(l_1+1)x_1 + x_2^{l_2+1} - \varepsilon_2(l_2+1)x_2$$

is generated by the parallel transport of the appropriate twists as given by lemma 8.4 and lemma 8.5 along all standard paths in the model discriminant family.

9 conclusion

In the progress of the subsequent argument we have to replace generating sets for subgroups of Br_n at several points. The key lemma to justify such transitions is

Lemma 9.1 *Given two finitely filtered sets of elements of a group*

$$S = S_n \supset S_{n-1} \ldots \supset S_1, \qquad T_n \supset T_{n-1} \ldots \supset T_1$$

Then S and T generate the same subgroup if

i) $T_1 = S_1$,

ii) given $t \in T_k - T_{k-1}$ there is $s \in S_k$, such that t is equal to s up to conjugation by elements in $\langle S_{k-1} \rangle$,

iii) given $s \in S_k - S_{k-1}$ there is $t \in T_k$, such that s is equal to t up to conjugation by elements in $\langle S_{k-1} \rangle$.

The last hypothesis may be replaced by

iii') given $s \in S_k - S_{k-1}$ there is $t \in T_k$ such that s is equal to t up to conjugation by elements in $\langle T_{k-1}, S_{k-1} \rangle$.

Proof: We show $\langle T_k\rangle = \langle S_k\rangle$. So $i)$ starts the induction. Then $\langle T_k\rangle \subset \langle S_k\rangle$ since by induction $\langle T_{k-1}\rangle \subset \langle S_{k-1}\rangle \subset \langle S_k\rangle$ and by $ii)$ $t \in T_k - T_{k-1}$ implies $t \in \langle S_k\rangle$.

On the other hand by induction $\langle S_{k-1}\rangle \subset \langle T_{k-1}\rangle$, therefore $s \in S_k - S_{k-1}$ implies $s \in \langle T_k, S_{k-1}\rangle \subset \langle T_k\rangle$ if $iii)$ holds resp. if $iii')$ holds. In either case we get $\langle S_k\rangle \subset \langle T_k\rangle$. □

Its first application is in the proof of the following claim:

Proposition 9.2 *The versal braid monodromy the family of functions*

$$x_1^{l_1+1} - \alpha(l_1+1)x_1 + x_2^{l_2+1} - \varepsilon_2(l_2+1)x_2$$

is generated by twists ($i_1^+ = i_1 + 1$, $i_2' = i_2 - l_2$):

$$\begin{array}{rl}
\tau^2_{i_1 i_2, j_1 j_2}, & 1 \le i_1 < j_1 \le l_1, 1 \le j_2 - i_2 < l_2,\\
\tau^3_{i_1 i_2', i_1^+ i_2}, & 1 \le i_1 < i_1^+ \le l_1, 1 \le i_2 \le l_2,\\
\tau^{-1}_{i_1 i_2', i_1^+ i_2}\tau^2_{i_1^+ i_2', j_1 i_2}\tau_{i_1 i_2', i_1^+ i_2}, & 1 < i_1^+ < j_1 \le l_1, 1 \le i_2 \le l_2.
\end{array}$$

Proof: The versal braid monodromy of a one parameter family can by definition be computed from their locally assigned groups of mapping classes and the parallel transport of these groups along a distinguished system of paths in the associated discriminant family, cf. [15, L. 5.7]

The locally assigned groups in the discriminant family were given in lemma 8.4 and lemma 8.5 to be twists on local v-arcs and local w-arcs.

So by the closing remark of the last section parallel transport of local v-arcs and local w-arcs along all possible standard paths in the base of the model discriminant family generate the versal braid monodromy.

Note that the length of the circular part is not necessarily restricted to $[0, 2\pi[$.

We denote by T the set of braid generators obtained by parallel transport and identification using the Hefez Lazzeri path system in the fibre at $\lambda = 1$.

T is divided into subsets according to the index pair of the punctures connected by the corresponding arc.

The given set S of braid group elements is also divided into subsets according to the modulus the index pairs of each element, which is unambiguous since we note immediately that the modulus of all index pairs occurring in the second and third row is zero.

Since the moduli of elements in T and S form a finite descending sequence $m_1 > \ldots > m_n = 0$, we can impose finite filtrations

$$T_k := \{\tau \in T | m(\tau) \ge m_k\}, \quad S_k := \{\tau \in S | m(\tau) \ge m_k\}.$$

To prove our claim, we are thus left to check the hypotheses of lemma 9.1:

Since $l_2 > 1$, the maximal modulus m_1 is positive. Hence S_1 only contains twists on parallel transports of local v-arcs. The local v-arcs of highest modulus get not tangled when transported along a radial arc, since entangling bisceles arcs have to be of larger modulus 7.5. The isosceles thus obtained are unobstructed, since obstructing parallels would be of larger modulus, 7.6. By 5.17 each element of T_1 is in S_1. Conversely by 5.19, each element in S_1 is an element in T of equal modulus, hence in T_1.

Given an element in T of modulus $m_k > 0$, which is the parallel transport of an local v-arc, then there is an element in S obtained from the same local v-arc transported along the same path, but conjugated by twists which are the parallel transports of bisceles arcs of strictly larger modulus, 7.5, 7.6. So the second hypothesis of lemma 9.1 holds for elements in $T_k - T_{k-1}$ of positive modulus.

Conversely each full twist in S of positive modulus is obtained by parallel transport from an local v-arc of equal modulus up to twists by entangled and obstructing bisceles arcs, 5.19. So due to 7.5, 7.6 again the third hypothesis holds for the twists obtained from local v-arcs of positive modulus.

We are left with elements of modulus $m_n = 0$ and consider local w-arcs first. Though we have to transport along a standard path to get full twist elements in $T_n - T_{n-1}$, we have to rely on a result which makes use of a different kind of paths. We recall that in section 6 we transported a local w-arc along a circular arc of small radius, then along a radial segment and finally along a circular segment of radius $t = 1$.

Each standard path can be coupled with a path of the second kind in such a way, that the closed path obtained as their join has winding number zero with respect to the origin. Hence parallel transport from $\lambda = 1$ along this closed path amounts to conjugation by a composition of full twists of positive modulus in T_{n-1}.

We deduce that the elements of T obtained by parallel transport along standard paths yield the elements given in lemma 6.8 up to conjugation by full twists in T_{n-1} and S_{n-1}. By lemma A.7 they are even conjugate to the full twists in $S_n - S_{n-1}$.

This relation can obviously be reversed in the sense, that for each full twist in S_n of modulus zero we have an element in T equal up to conjugation by elements in S_{n-1} and T_{n-1}.

Finally we have to address the $\frac{3}{2}$-twists in T a nd S. A $\frac{3}{2}$-twist in T is obtained by the parallel transport of a local v-arc with index pair $i_1 i_2, i_1^+ i_2$. We conclude that up to full twists elements in S_{n-1} and T_{n-1} the $\frac{3}{2}$-twists in T are among the elements given in lemma 5.20. By lemma A.6 they are among the elements in S even up to full twists in S_{n-1} and T_{n-1}.

Conversely the pairs of twists considered in lemma 5.21 correspond bijectively to the $\frac{3}{2}$-twists of S and are both equal up to conjugation by full twists of positive modulus by lemma A.6. We deduce that also each $\frac{3}{2}$-twist of S is an element of T up to conjugation by full twists in S_{n-1} and T_{n-1}. □

We modify the generating set for the families of type f_α:

Lemma 9.3 *The versal braid monodromy of the family*

$$x_1^{l_1+1} - \alpha(l_1+1)x_1 + x_2^{l_2+1} - \varepsilon_2(l_2+1)x_2$$

is generated by twists

$$\begin{array}{ll} \tau^2_{i_1 i_2, j_1 j_2}, & 1 \le i_1 < j_1 \le l_1, 1 \le j_2 - i_2 < l_2, \\ \sigma^3_{i_1 i_2, i_1^+ i_2}, & 1 \le i_1 < i_1^+ \le l_1, 1 \le i_2 \le l_2, \\ \sigma^2_{i_1 i_2, j_1 i_2}, & 1 < i_1^+ < j_1 \le l_1, 1 \le i_2 \le l_2. \end{array}$$

Proof: By definition $\tau^3_{i_1 i_2', i_1^+ i_2} = \sigma^3_{i_1 i_2, i_1^+ i_2}$, $1 \le i_1 < i_1^+ \le l_1, 1 \le i_2 \le l_2$. Moreover the two generating sets of lemma A.3 generate the same subgroups via the homomorphisms

$$\psi_{i_2} : \sigma_{i_1, j_1} \mapsto \sigma_{i_1 i_2, j_1 i_2}.$$

Hence generators

$$\sigma^3_{i_1 i_2, i_1^+ i_2}, \sigma^2_{i_1 i_2, j_1 i_2}, \quad 1 \le i_1 < i_1^+ < j_1 \le l_1.$$

may be replaced by

$$\sigma^3_{i_1 i_2, i_1^+ i_2}, \psi_{i_2}(\sigma^{-2}_{i,i^+} \check{\sigma}^2_{i_1, j_1} \sigma^2_{i,i^+}), \quad 1 \le i_1 < i_1^+ < j_1 \le l_1.$$

Since the latter elements coincide with $\tau^{-1}_{i_1 i_2', i_1^+ i_2} \tau^2_{i_1^+ i_2', j_1 i_2} \tau_{i_1 i_2', i_1^+ i_2}$ we are done. $\square$

Definition 9.4: The *cable twist* in $\mathrm{Br}_{l_1 l_2}$ is defined to be the element

$$\delta_\phi := \prod_{i_1}^{l_1} \prod_{i_2}^{l_2 - 1} \sigma_{i_1 i_2, i_1 i_2^+}.$$

(They were already considered in the disguise of turning the peripheral circles.)

Lemma 9.5 *The versal braid monodromy of a family of functions $f_\alpha(x_1, x_2)$ is generated by the elements*

$$\begin{array}{ll} \sigma^3_{i_1 i_2, j_1 j_2}, & i_2 = j_2,\ i_1^+ = j_1, \\ \sigma^2_{i_1 i_2, j_1 j_2}, & i_2 = j_2,\ i_1^+ < j_1, \\ \sigma^2_{i_1 i_2, j_1 j_2} & \textit{with } 1 \le i_1 < j_1 \le l_1, 1 \le i_2 - j_2 < l_2. \end{array}$$

Proof: We have to show that the elements in 9.5 and those in 9.3 generate the same subgroup of $\mathrm{Br}_{l_1 l_2}$. Since both generator sets have the elements with equal second index component in common, it suffices to prove that the remaining elements of each set generate the same braid subgroup.

Notice that both sets are filtered by level, $j_1 - i_1$, which is underlined:

$$\begin{aligned}
S_1 &= \{\sigma^2_{i_1 i_2, j_1 j_2} | 1 \le i_1 < j_1 \le l_1, \underline{j_1 - i_1 = 1}, 1 \le i_2 - j_2 < l_2\} \\
S_2 &= S_1 \cup \{\sigma^2_{i_1 i_2, j_1 j_2} | 1 \le i_1 < j_1 \le l_1, \underline{j_1 - i_1 = 2}, 1 \le i_2 - j_2 < l_2\} \\
&\vdots \\
S_{l_1} &= \{\sigma^2_{i_1 i_2, j_1 j_2} | 1 \le i_1 < j_1 \le l_1, 1 \le i_2 - j_2 < l_2\} \\
T_1 &= \{\tau^2_{i_1 i_2, j_1 j_2} | 1 \le i_1 < j_1 \le l_1, \underline{j_1 - i_1 = 1}, 1 < j_2 - i_2 \le l_2\} \\
T_2 &= T_1 \cup \{\tau^2_{i_1 i_2, j_1 j_2} | 1 \le i_1 < j_1 \le l_1, \underline{j_1 - i_1 = 2}, 1 < j_2 - i_2 \le l_2\} \\
&\vdots \\
T_{l_1} &= \{\tau^2_{i_1 i_2, j_1 j_2} | 1 \le i_1 < j_1 \le l_1, 1 < j_2 - i_2 \le l_2\}.
\end{aligned}$$

For the proof we need therefore to check the hypotheses of 9.1 only: The first, $S_1 = T_1$, is immediate, since elements of level one coincide almost by definition

$$\sigma_{i_1 i_2, i_1^+ j_2} = \tau_{i_1 i_2', i_1^+ j_2}, \quad i_2' + l_2 = i_2.$$

For the inductive hypothesis lemma A.8 yields, that elements $\tau^2_{i_1 0, k_1 k_2}$ and $\sigma^2_{i_1 l_2, k_1 k_2}$ with $1 \le i_1 < k_1 \le l_1, 1 \le k_2 < l_2$ are equal up to conjugation by elements in $S_{k_1 - i_1 - 1} \cup T_{k_1 - i_1 - 1}$, i.e. by elements of smaller level.

To extend this result to the remaining elements we consider the action of overall conjugation by δ_ϕ. Since this conjugation is level preserving, we get, that

$$\sigma^2_{i_1 i_2, k_1 j_2} = \delta_\phi^{i_2'} \sigma^2_{i_1 l_2, k_1 k_2} \delta_\phi^{-i_2'}, \tau^2_{i_1 i_2', k_1 j_2} = \delta_\phi^{i_2'} \tau^2_{i_1 0, k_1 k_2} \delta_\phi^{-i_2'}$$

are equal up to conjugation by elements of smaller level since $\sigma^2_{i_1 l_2, k_1 k_2}, \tau^2_{i_1 0, k_1 k_2}$ are. The hypotheses are hence met, for each generator $\sigma^2_{i_1 i_2, j_1 j_2}$ or $\tau^2_{i_1 i_2, j_1 j_2}$ is in the conjugation orbit of a $\sigma^2_{i_1 l_2, k_1 k_2}$ resp. $\tau^2_{i_1 0, k_1 k_2}$ by δ_ϕ. □

Lemma 9.6 *The versal braid monodromy G_2 of the family $g_\alpha(x_1, x_2)$ restricted to the unit disc is generated by the elements*

$$\begin{aligned}
&\sigma^3_{i_1 i_2, i_1 j_2}, \quad 1 \le i_2 = j_2 - 1 < l_2, 1 \le i_1 \le l_1, \\
&\sigma^2_{i_1 i_2, i_1 j_2}, \quad 1 \le i_2 < j_2 - 1 < l_2, 1 \le i_1 \le l_1.
\end{aligned}$$

Proof: The only critical parameter in the disc $|\alpha| \le 1$ is $\alpha = 0$. The corresponding critical function is $x_1^{l_1+1} - (l_1+1)x_1 + x_2^{l_2+1}$, which has l_1 critical point of type $A(l_2)$ with distinct critical values.

The bifurcation divisors of the families of functions parameterized by α,

$$-l_1 \xi + x_2^{l_2+1} + \varepsilon_2 \alpha (l_2 + 1) x_2, \quad \xi^{l_1} = 1,$$

embed into the bifurcation divisor of g_α, and the corresponding embeddings of punctured discs induce embeddings of mapping class groups which correspond under the standard identifications with the braid groups $\mathrm{Br}_{l_1 l_2}$ and Br_{l_2} by the Hefez Lazzeri choice of a strongly distinguished system of paths to the embeddings

$$\phi_{i_1} : \sigma_{i_2,j_2} \mapsto \sigma_{i_1 i_2, i_1 j_2}, \quad 1 \le i_1 \le l_1$$

The versal braid monodromies of the families above can be identified with the braid monodromy of [15, L. 4.6] generated by $\sigma_{i_2}^3, \sigma_{i_2,j_2}^3, |i-j| \ge 2$ and yield then the versal braid monodromy of the family g_α as claimed. □

Remark 9.7: The subcable twists on indices $i_1 i_2$ to $i_1 j_2$ are defined as

$$\delta_{i_1 i_2, i_1 j_2} := \prod_{k_2=i_2}^{j_2-1} \sigma_{i_1 k_2, i_1 k_2^+}.$$

$\delta_{i_1 i_2, i_1 j_2}^{j_2-i_2+1}$ is central and $\delta_{i_1 i_2, i_1 j_2}^{j_2-i_2+2}$ is in G_2, and so is $\delta_\phi^{l_2+1}$.

Lemma 9.8 *For given $i_1 < j_1$ the same braid subgroup is generated by the elements*

$$\sigma_{i_1 i_2, j_1 j_2}^2, \quad 1 \le i_2 - j_2 < l_2,$$

and by suitably chosen G_2-conjugates of elements

$$\begin{array}{ll} \sigma_{i_1 i_2, j_1 j_2}^2 & 1 \le i_2, j_2 \le l_2, i_2, i_2^+ \ne j_2, \\ \sigma_{i_1 i_2, j_1 i_2}^2 \sigma_{i_1 i_2, j_1 j_2}^2 \sigma_{i_1 i_2, j_1 i_2}^{-2} & 1 < i_2^+ = j_2 \le l_2, \\ \sigma_{i_1 i_2, i_1 j_2}^2 \sigma_{i_1 i_2, j_1 j_2}^2 \sigma_{i_1 i_2, i_1 j_2}^{-2} & 1 < i_2^+ = j_2 \le l_2. \end{array}$$

Proof: We introduce filtrations $T = T_2 \supset T_1$ and $S = S_2 \supset S_1$ on the two sets of elements by

$$\begin{array}{rcl} S_1 & = & \{\sigma_{i_1 i_2, j_1 j_2}^2 | 1 \le i_2, j_2 \le l_2, i_2, i_2^+ \ne j_2\} \\ T_1 & = & \{\sigma_{i_1 i_2, j_1 j_2}^2 | i_2, j_2 \not\equiv 0 \mod (l_2+1)\} \end{array}$$

Since $\delta_\phi^{l_2+1} \in G_2$, we may conjugate the elements of S_1 by all powers $\delta_\phi^{m(l_2+1)}$, similarly $\delta_{j_1 1, j_1 l_2}^{l_2+1}$ is an element of G_2, hence all elements

$$\begin{array}{rcl} \delta_\phi^{m(l_2+1)} (\sigma_{i_1 i_2, j_1 j_2}^2) \delta_\phi^{-m(l_2+1)}, & & \\ \delta_\phi^{m(l_2+1)} (\delta_{j_1 1, j_1 l_2}^{l_2+1} \sigma_{i_1 i_2, j_1 j_2}^2 \delta_{j_1 1, j_1 l_2}^{-l_2-1}) \delta_\phi^{-m(l_2+1)} & = & \delta_\phi^{m(l_2+1)} \sigma_{i_1 i_2, j_1 j_2'}^2 \delta_\phi^{-m(l_2+1)}, \end{array}$$

with $j_2' = j_2 - l_2 - 1$, are G_2-conjugates of elements in S_1. In fact each $t \in T_1$ is such a conjugate and each $s \in S_1$ has a conjugate in T_1.

To complete the proof we observe that the braids $\delta_{j_1 i_2^+, j_1 l_2}^{l_2-i_2+1}$, resp. $\delta_{i_1 1, i_1 j_2^-}^{j_2}$ are elements of G_2 again. Hence we may invoke 9.9 to show that the elements in $S_2 - S_1$ have G_2-conjugates which are equal up to conjugation by elements in T_1 to elements

$$\sigma_{i_1 i_2, j_1 0}^2, \sigma_{i_1 l_2^+, j_1 j_2}^2, \quad 1 \le i_2 < l_2, 1 < j_2 \le l_2,$$

which in turn are contained in $T_2 - T_1$.

Because conjugation by powers $\delta_\phi^{m(l_2+1)}$ yields all elements of $T_3 - T_2$ and preserves the set T_2, all elements in $T_2 - T_1$ up to conjugation by elements in T_1 are G_2-conjugates of elements in $S_2 - S_1$, so we are done. □

Lemma 9.9 *If $1 \le i_2 < l_2$ and $j_2 = 0$ then*

$$\sigma_{i_1 i_2, j_1 0}, \quad (\delta^{l_2 - i_2 + 1}_{j_1 i_2^+, j_1 l_2})^{-1} \sigma^2_{i_1 i_2, j_1 i_2} \underline{\sigma_{i_1 i_2, j_1 i_2^+}} \sigma^{-2}_{i_1 i_2, j_1 i_2} \delta^{l_2 - i_2 + 1}_{j_1 i_2^+, j_1 l_2},$$

are equal up to conjugation by twists $\sigma_{i_1 i_2, j_1 k_2}$ with $k_2 \neq 0$, $i_2 - l_2 < k_2 < i_2$.

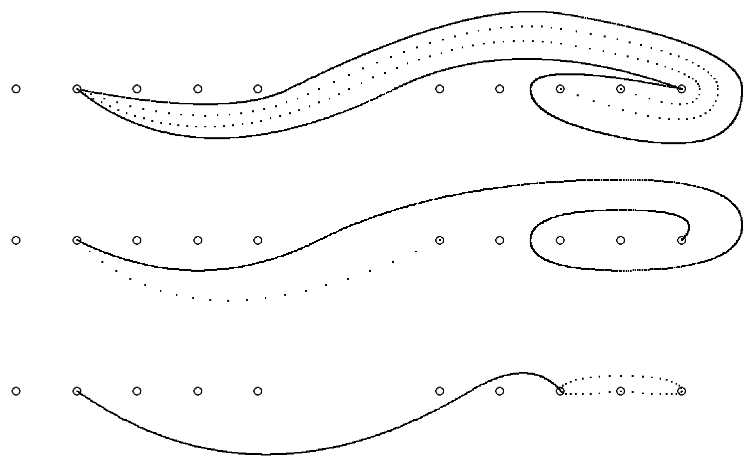


Figure 1: $\sigma_{i_1 2, j_1 0}$ being conjugated to $\sigma^2_{i_1 2, j_1 2} \underline{\sigma_{i_1 2, j_1 3}} \sigma^{-2}_{i_1 2, j_1 2}$

Proof: We claim that conjugation by $(\prod_{k_2=1}^{i_2^-} \sigma^2_{i_1 i_2, j_1 k_2})^{-1} (\prod_{k_2 = i_2^+ - l_2}^{-1} \sigma^2_{i_1 i_2, j_1 k_2})$ will do. On $\sigma_{i_1 i_2, j_1 0}$ conjugation by the second factor equals conjugation by $(\delta^{l_2 - i_2}_{j_1 i_2^+, j_1 l_2})$ and $\delta_{j_1 i_2^+, j_1 l_2}$ commutes with the first factor. Hence we are left to show that

$$\begin{aligned}
& (\prod_{k_2=1}^{i_2^-} \sigma^2_{i_1 i_2, j_1 k_2})^{-1} (\delta_{j_1 i_2^+, j_1 l_2}) \underline{\sigma_{i_1 i_2, j_1 0}} (\delta_{j_1 i_2^+, j_1 l_2})^{-1} (\prod_{k_2=1}^{i_2^-} \sigma^2_{i_1 i_2, j_1 k_2}) \\
= \; & \sigma^2_{i_1 i_2, j_1 i_2} \underline{\sigma_{i_1 i_2, j_1 i_2^+}} \sigma^{-2}_{i_1 i_2, j_1 i_2}
\end{aligned}$$

Since the arcs of $(\delta_{j_1 i_2^+, j_1 l_2}) \underline{\sigma_{i_1 i_2, j_1 0}} (\delta_{j_1 i_2^+, j_1 l_2})^{-1}$ and $\sigma_{i_1 i_2, j_1 i_2^+}$ can be chosen to bound a disc which contains the puncture of indices $j_1 1$ to $j_1 i_2$ we can conclude with A.4 that

$$(\delta_{j_1 i_2^+, j_1 l_2}) \underline{\sigma_{i_1 i_2, j_1 0}} (\delta_{j_1 i_2^+, j_1 l_2})^{-1} \;=\; (\prod_{k_2=1}^{i_2} \sigma^2_{i_1 i_2, j_1 k_2}) \underline{\sigma_{i_1 i_2, j_1 i_2^+}} (\prod_{k_2=1}^{i_2} \sigma^2_{i_1 i_2, j_1 k_2})^{-1}$$

from which we deduce the claim. □

Proof of the main theorem: We have in 9.6 that the braid monodromy group G_2 is generated by

$$\begin{array}{ll}
\sigma^3_{i,k} & i_1 = k_1, i_2^+ = k_2, \\
\sigma^2_{i,k} & i_1 = k_1, i_2^+ < k_2.
\end{array}$$

To get generators for the total braid monodromy we have – by [15, Prop.5.14] – to add the elements of 9.5. Of course we replace those of the third row using 9.8, so we add instead

$$\begin{array}{rcl} \sigma^3_{i_1i_2,j_1j_2} & : & i_2=j_2,\ i_1^+=j_1, \\ \sigma^2_{i_1i_2,j_1j_2} & : & i_2=j_2,\ i_1^+<j_1, \\ \sigma^2_{i_1i_2,j_1j_2} & : & 1\le i_2,j_2\le l_2, i_2, i_2^+\neq j_2, \\ \sigma^2_{i_1i_2,j_1j_2}\sigma^2_{i_1i_2,k_1k_2}\sigma^{-2}_{i_1i_2,j_1j_2} & : & j_1\in\{i_1,k_1\},\ i_2^+=k_2,\ i_2\le j_2\le k_2. \end{array}$$

Since elements in the last row can also be given as $\sigma^{-2}_{j_1j_2,k_1k_2}\sigma^2_{i_1i_2,k_1k_2}\sigma_{j_1j_2,k_1k_2}$ we can discard the conjugating twists except for the cases $i<j<k$, $i_1^+=k_1$, $i_2^+=k_2$.

By a check on the indices we see, that in order to get the claim we have express the additional elements $\sigma^3_{i,k}$, $i_1^+=k_1$, $i_2^+=k_2$ in terms of other ones.

Let $j:=i_1^+i_2$ so $\sigma^3_{i,j}$, $\sigma^3_{j,k}$ and $\sigma^2_{i,j}\sigma^2_{i,k}\sigma^{-2}_{i,j}$ are among the generators. By

$$\begin{aligned} \sigma_{i,k} &= \sigma_{j,k}\sigma_{i,k}\sigma_{j,k}\sigma^{-1}_{i,k}\sigma^{-1}_{j,k} \\ &= \sigma_{j,k}\sigma^2_{i,k}\sigma_{j,k}\sigma_{i,k}\sigma^{-1}_{j,k}\sigma^{-2}_{i,k}\sigma^{-1}_{j,k} \\ &= \sigma^3_{j,k}\sigma^{-2}_{j,k}\sigma^2_{i,k}\sigma^2_{j,k}\sigma_{i,j}\sigma^{-2}_{j,k}\sigma^{-2}_{i,k}\sigma^2_{j,k}\sigma^{-3}_{j,k} \\ &= \sigma^3_{j,k}(\sigma^2_{i,j}\sigma^2_{i,k}\sigma^{-2}_{i,j})\sigma_{i,j}(\sigma^2_{i,j}\sigma^{-2}_{i,k}\sigma^{-2}_{i,j})\sigma^{-3}_{j,k} \end{aligned}$$

we conclude, that also $\sigma^3_{i,k}$ is an element of the braid monodromy. □

A presentation of the fundamental group of the discriminant complement can now be obtained from generators β_j of the braid monodromy according to the method of Zariski van Kampen.

$$\pi_1 \;\cong\; \langle t_1,...,t_\mu |\, t_i^{-1}\beta_j(t_i)\rangle.$$

First we note that the choice of the generators β_j and of generators of the free group does not matter. To reduce the number of relations for the proof of the main corollary, we observe the following consequence.

Lemma 9.10 *Suppose $\sigma=\beta\sigma_1^l\beta^{-1}\in \mathrm{Br}_n$, then the normal subgroup generated by $t_i^{-1}\sigma(t_i)$, $i=1,...,n$, is equal to the normal subgroup generated by*

$$\beta(\underbrace{t_1^{-1}t_2^{-1}\cdots}_{l\ factors}\underbrace{t_*\cdots t_1t_2}_{l\ factors}).$$

Proof: We replace the generators t_i by generators $\beta(t_i)$, so the normal subgroup is generated by

$$(\beta(t_i))^{-1}\sigma(\beta(t_i)) = \beta(t_i^{-1})\beta(\sigma_1^l(t_i)) = \beta(t_i^{-1}\sigma_1^l(t_i)).$$

This yields trivial relations for $i>2$ and for $i=1,2$ conjugate relators, given explicitly using the Artin action of twists on free generators:

$$\beta(t_1^{-1}\underbrace{t_2^{-1}t_1^{-1}\cdots}_{l-1\text{ factors}}\underbrace{t_*\cdots t_1t_2}_{l\text{ factors}}),\beta(t_2^{-1}\underbrace{t_2^{-1}t_1^{-1}\cdots}_{l\text{ factors}}\underbrace{t_*\cdots t_1t_2}_{l\text{ factors}}),$$

Hence the claim follows. □

Proof of the main corollary: We start with generators $t_i, i \in I$ in bijection to the simple loops of the Hefez Lazzeri system. Relations are determined from the generators of the main theorem. To apply the preceding lemma we note that a generator of the first two rows can be factored as $\beta\sigma_1^2\beta^{-1}$ and $\beta\sigma_1^3\beta^{-1}$ respectively in such a way that

$$\beta(t_1) = t_i, \quad \beta(t_2) = t_j.$$

and similarly generators of the last row can be conjugated such that

$$\beta(t_1) = t_i, \quad \beta(t_2) = t_j t_k t_j^{-1}.$$

The sufficient set of relators is then given by:

$$\begin{array}{rl} t_i^{-1}t_j^{-1}t_i^{-1}t_jt_it_j & \text{in case } (i,j) \in E \\ t_i^{-1}t_j^{-1}t_it_j & \text{in case } (i,j) \notin E \\ t_i^{-1}t_jt_k^{-1}t_j^{-1}t_it_jt_kt_j^{-1} & \text{in case } (i,j),(j,k) \in E_1, (i,k) \in E_{-1}. \end{array}$$

So we are done as soon as we conjugate the relators of length 8 by t_jt_i to get:

$$\begin{aligned} & (t_jt_i)^{-1}t_i^{-1}t_jt_k^{-1}t_j^{-1}t_it_jt_kt_j^{-1}t_jt_i \\ = \; & t_j^{-1}t_i^{-1}t_j^{-1}t_jt_k^{-1}t_j^{-1}t_it_jt_kt_i \\ = \; & t_j^{-1}t_i^{-1}t_k^{-1}t_j^{-1}t_it_jt_kt_i. \end{aligned}$$

□

A braid computations

This appendix is designed to serve several purposes. First the progress in the text is eased if some of the computational obstacles are hidden in this appendix. Second the arguments are often similar and it is easier to get used to them, if they are used in one place instead of being scattered throughout.

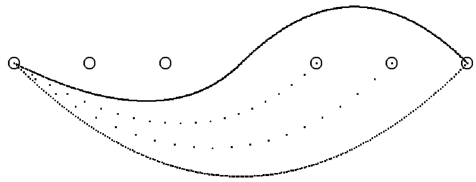
Since all index sets we use are ordered, we can always denote by i^+ the immediate successor of i in some index set. The same notation applies also to single components of multiindices.

Moreover we underline a conjugated element to make the structure more transparent.

Remark A.1: The twist $\check{\sigma}_{i,k} := (\prod_{i<j<k} \sigma_{i,j}^2)\underline{\sigma_{i,k}}(\prod_{i<j<k} \sigma_{i,j}^2)^{-1}$ is the twist on the horizontal arc from i to k passing behind all intermediate punctures (as opposed to the arc of $\sigma_{i,k}$ which passes in front).

Remark A.2: The half twist on the arc from i to k passing in front up to j and behind from $j+1$ onwards can be given as

$$(\prod_{j<k'<k} \sigma_{i,k'}^2)\underline{\sigma_{i,k}}(\prod_{j<k'<k} \sigma_{i,k'}^2)^{-1}.$$



Lemma A.3 *The braid subgroup* $\mathrm{Br}(A_n) \subset \mathrm{Br}_n$ *is generated by elements*

$$\begin{array}{ll} \sigma_{i,i+1}^3 & 1 \le i < n \\ \sigma_{i,i^+}^{-2} \check\sigma_{i,j}^2 \sigma_{i,i^+}^2 & 1 < i, i^+ < j \le n. \end{array}$$

Proof: Consider the following two filtered sets of elements of Br_n.

$$\begin{array}{ll} S_1 := \{\sigma_{i,i+1}^3\} & S_k := S_1 \cup \{\sigma_{i,j}^2 | 1 < j - i \le k\}, \\ T_1 := \{\sigma_{i,i+1}^3\} & T_k := T_1 \cup \{\sigma_{i,i^+}^{-2} \check\sigma_{i,j}^2 \sigma_{i,i^+}^2 | 1 < j - i \le k\} \end{array}$$

By the first remark we get the relation

$$\sigma_{i,i^+}^{-2} \check\sigma_{i,j}^2 \sigma_{i,i^+}^2 = \Big(\prod_{i^+ < j' < j} \sigma_{i,j'}^2 \Big) \underline{\sigma_{i,j}} \Big(\prod_{i^+ < j' < j} \sigma_{i,j'}^2 \Big)^{-1}.$$

so $S_2 = T_2$ and the other hypotheses of lemma 9.1 hold as well. Therefore the assertion is proved, since S_n is known to generate $\mathrm{Br}(A_n)$. □

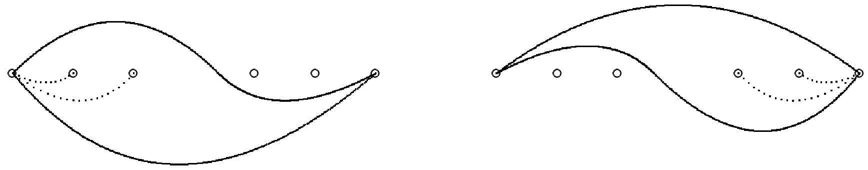
Lemma A.4 *Suppose in a punctured disc two otherwise distinct arcs meet in the punctures* p, q *thus defining a inner disc. If there is a system of arcs such that*

i) each puncture in the inner disc is connected by an arc with either p *or* q*,*

ii) apart from p, q *all arcs have no points in common,*

then the twists on outer arcs coincide up to conjugation by full twists on inner arcs.

Proof: We may identify the mapping class group of neighbourhood of the inner disc with the abstract braid group, such that the twists on inner arcs correspond to $\sigma_{1,j}^2, 1 < j \le m$ and $\sigma_{j,n}^2, m < j < n$ and the twist on the outer arcs correspond to $\sigma_{1,n}$ and $\check\sigma_{1,n}$. The claim then follows from

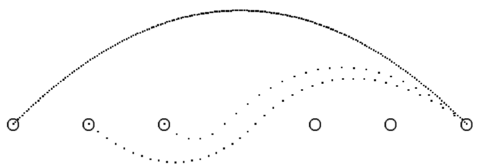


$$\Big(\prod_{j=2}^{m} \sigma_{1,j}^2\Big) \underline{\sigma_{1,n}} \Big(\prod_{j=2}^{m} \sigma_{1,j}^2\Big)^{-1} = \Big(\prod_{j=m+1}^{n-1} \sigma_{j,n}^2\Big) \underline{\check\sigma_{1,n}} \Big(\prod_{j=m+1}^{n-1} \sigma_{j,n}^2\Big)^{-1}.$$

□

Lemma A.5 *For any j, $i < j \leq k$ the twist $\check{\sigma}_{i,k}$ can be given as $\sigma_{i,k}$ suitably conjugated by braids $\sigma_{i',j'}$, $i \leq i' < j \leq j' \leq k$.*

Proof: First note that $\sigma_{i,k}$ and $\check{\sigma}_{i,k}$ are twists on arc which meet the initial hypothesis of lemma A.4. By the second remark above, the full twists on arcs from the puncture of index i' passing in front up to $j-1$ and behind from j onwards is in the group generated by elements $\sigma^2_{i',k}, \sigma^2_{i',j'}$, $i < i' < j \leq j' < k$.

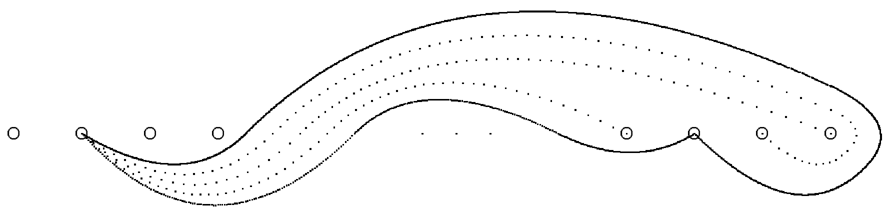


On the other hand these arcs and the arcs to which the $\sigma^2_{i,j'}$, $j \leq j' < k$ are associated can be chosen simultaneously to meet the remaining hypotheses of lemma A.4. So we get our claim. $\square$

Lemma A.6 *The elements $\tau_{i_1 i'_2, j_1 i'_2}$ and $\tau_{i_1 i'_2, j_1 i_2}$ are equal up to conjugation by twists $\tau^2_{i_1 i_2, j_1 j_2}$, $1 \leq j_2 - i'_2 < l_2$ for all $1 \leq i_1 < j_1 \leq l_1$, $i'_2 = i_2 - l_2$.*

Proof: In fact we have

$$\Big(\prod_{i'_2 < j_2 < i_2} \tau^2_{i_1 i'_2, j_1 j_2}\Big)^{-1} \underline{\tau_{i_1 i'_2, j_1 i'_2}} \Big(\prod_{i'_2 < j_2 < i_2} \tau^2_{i_1 i'_2, j_1 j_2}\Big) = \tau_{i_1 i'_2, j_1 i_2}.$$



Again the claim can also be proved by checking the hypotheses of lemma A.4. $\square$

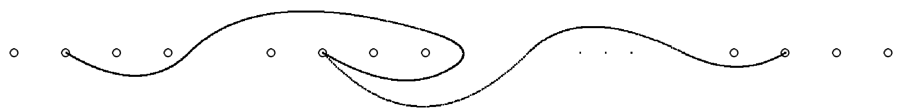
Lemma A.7 *Up to conjugation by twists $\tau^2_{i_1 i_2, j_1 j_2}$, $1 \leq i_1 < j_1 \leq l_1$, $1 \leq j_2 - i_2 < l_2$, elements ($i'_2 := i_2 - l_2$)*

$$\begin{array}{ll}
\tau^{-1}_{i_1 i'_2, i_1^+ i'_2} \tau^2_{i_1^+ i'_2, j_1 i'_2} \tau_{i_1 i'_2, i_1^+ i'_2} & \textit{if} \quad j_1 - i_1 \leq l_1/2, \\
\tau^{-1}_{i_1 i'_2, i_1^+ i'_2} \tau^2_{i_1^+ i'_2, j_1 i_2} \tau_{i_1 i'_2, i_1^+ i'_2} & \textit{if} \quad j_1 - i_1 \geq l_1/2 + 1, \\
\tau^{-1}_{i_1 i'_2, i_1^+ i_2} \tau^2_{i_1^+ i'_2, j_1 i_2} \tau_{i_1 i'_2, i_1^+ i_2} & \textit{if} \quad j_1 - i_1 = l_1/2 + 1/2.
\end{array}$$

correspond bijectively to the elements

$$\tau^{-1}_{i_1 i'_2, i_1^+ i_2} \tau^2_{i_1^+ i'_2, j_1 i_2} \tau_{i_1 i'_2, i_1^+ i_2} \qquad 1 < i_1 + 1 = i_1^+ < j_1 \leq l_1, 1 \leq i_2 \leq l_2.$$

Proof: The elements of the third row already have the claimed factorisation. In the other cases we have to conjugate in such a way that central twist and conjugating twist are conjugated simultaneously to the claimed twists.
For elements of the second row, it is only the conjugating twist which has not the claimed form.

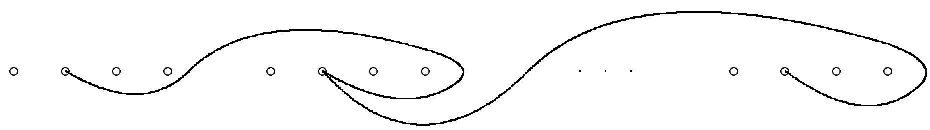


The proof of A.6 shows, that we can get

$$(\prod_{i_2'<j_2<i_2} \tau^2_{i_1 i_2', j_1 j_2})^{-1} \underline{\tau_{i_1 i_2', i_1^+ i_2'}} (\prod_{i_2'<j_2<i_2} \tau^2_{i_1 i_2', j_1 j_2}) = \tau_{i_1 i_2', i_1^+ i_2}.$$

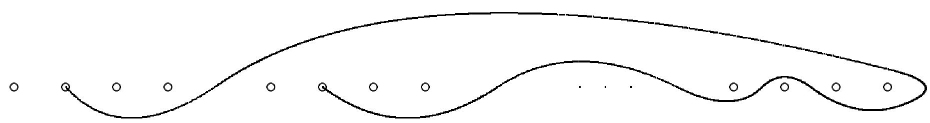
Moreover we check at once, that the conjugating factor commutes with $\tau_{i_1^+ i_2', j_1 i_2}$, hence an overall conjugation of $\tau^{-1}_{i_1 i_2', i_1^+ i_2'} \tau^2_{i_1^+ i_2', j_1 i_2} \tau_{i_1 i_2', i_1^+ i_2'}$ with $(\prod_{i_2'<j_2<i_2} \tau^2_{i_1 i_2', j_1 j_2})$ yields $\tau^{-1}_{i_1 i_2', i_1^+ i_2} \tau^2_{i_1^+ i_2', j_1 i_2} \tau_{i_1 i_2', i_1^+ i_2}$.

In case of elements of the first row, neither twist is in the right shape:



We first take care of the middle factor $\tau_{i_1^+ i_2', j_1 i_2'}$, as before just by conjugation with $(\prod_{i_2'<j_2<i_2} \tau^2_{i_1^+ i_2', j_1 j_2})$.

But since we have to conjugate overall, also the conjugating factors are conjugated, and they are not unaffected:



But we can now conjugate by $(\prod_{i_2'<j_2<i_2} \tau^2_{i_2+i_1', j_1 j_2})$ and $(\prod_{i_2'<j_2<i_2} \tau^2_{i_1 i_2', i_1^+ j_2})$, which are twists on arcs disjoint to that of $\tau_{i_1^+ i_2', j_1 i_2}$, hence commutating with it. $\square$

Lemma A.8 *Given* $1 \le i_1 < k_1 \le l_1, 1 \le k_2 < l_2$ *the elements*

$$\tau^2_{i_1 0, k_1 k_2} \quad \textit{and} \quad \sigma^2_{i_1 l_2, k_1 k_2}$$

coincide up to conjugation by elements

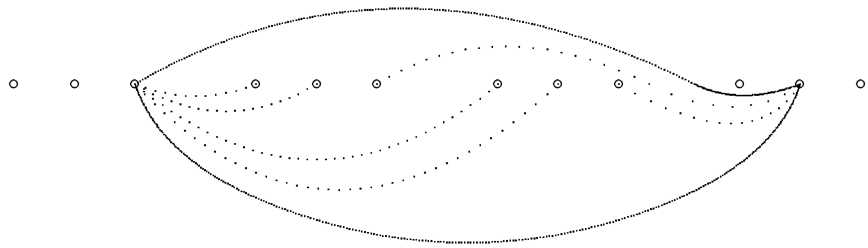
$$\begin{aligned} &\sigma^2_{1 l_2, j_1 j_2}, \quad i_1 < j_1 < k_1, 1 \le j_2 < l_2, \\ &\tau^2_{j_1 0, k_1 k_2}, \quad i_1 < j_1 < k_1. \end{aligned}$$

Proof: It suffices to check that arcs to which the given twists are associated can be chosen simultaneously in such a way that

i) they are confined to the disc with boundary given by the arcs corresponding to $\tau_{i_1 0, k_1 k_2}$ and $\sigma_{i_1 l_2, k_1 k_2}$,

ii) they are distinct outside the punctures of indices $i_1 l_2$ and $k_1 k_2$,

iii) all punctures in the disc are joint by some arc with either the puncture of index $i_1 l_2$ or that of index $k_1 k_2$.



So by lemma A.4 we may conclude that the assertion holds. □

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